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VARIATIONAL FORMULATION FOR THE STURM-LIOUVILLE PROBLEM OF FRACTIONAL DIFFERENTIAL EQUATION WITH GENERALIZED (P, Q)-LAPLACIAN OPERATOR*

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Abstract In this paper, the Sturm-Liouville boundary value problem is studied for fractional differential equation with generalized (p, q)-Laplacian operator. By imposing mild assumptions on nonlinearity f, several new existence results of at least one or two nontrivial weak solutions are established through variational methods and critical point theorems. Furthermore, the criteria is also investigated for the nonexistence result.

Keywords Fractional differential equation, Sturm-Liouville boundary value problem, variational method.

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1. Introduction

Fractional calculus operators are nonlocal and have time memory, which are due to the fact that convolution integrals are included in the definitions of fractional integral and differential operators (For details, please refer to Res. [9,17]). Based on such characteristics, fractional differential equations (FDEs for short) are ideal tools to describe various complex materials and physical phenomena. Hence, there are widely applications in interdisciplinary subjects such as viscoelastic mechanics, control theory and fluid mechanics, etc [1,5,8]. However, it is very difficult to obtain the explicit expressions of analytical solutions of FDEs due to the lack of mature methods. In view of this, studying the existence of solutions for FDEs becomes more important and necessary. Variational methods and critical point theory are

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powerful tools on investigating the existence of solutions for FDEs, rich results have been achieved in recent years [2, 22, 23].

In 2017, Tian and Nieto considerd the following Sturm-Liouville boundary value problem of discontinuous fractional differential equation for the first time [19]

$$\begin{cases} -\frac{d}{dt}[\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))] = \lambda f(t, u(t)), \text{ a.e. } t \in [0, T], \\ au(0) - b(\frac{1}{2}{}_0D_t^{-\beta}(u'(0)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(0))) = 0, \\ cu(T) + d(\frac{1}{2}{}_0D_t^{-\beta}(u'(T)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(T))) = 0, \end{cases} \tag{1.1}$$

where $0 \leq \beta < 1$, $a, c > 0$, $b, d \geq 0$, $\lambda > 0$, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$. By using critical point theorems, the variational structure and existence results were established. Moreover, the authors pointed out that the topological degree is not adapted since the equivalent integral equation of (1.1) not able to be solved.

With the p-Laplacian operator [10] widely applies in many theoretical models, such as nonlinear elastic mechanics and non-Newtonian fluid theory, the study for p-Laplacian problems attracts considerable attention(see [11, 15, 16]). In 2019, Nyamoradi and Tersian [16] studied a class of fractional Sturm-Liouville boundary value problem with p-Laplacian operator

$$\begin{cases} {}_tD_T^{\alpha}\left(\frac{1}{(h(t))^{p-2}}\Phi_p(h(t){}_0^CD_t^{\alpha}u(t))\right) + a(t)\Phi_p(u(t)) = \lambda f(t, u(t)), \\ \alpha_1\Phi_p(u(0)) - \alpha_2{}_tD_T^{\alpha-1}(\Phi_p({}_0^CD_t^{\alpha}u(0))) = 0, \\ \beta_1\Phi_p(u(T)) + \beta_2{}_tD_T^{\alpha-1}(\Phi_p({}_0^CD_t^{\alpha}u(T))) = 0, \end{cases} \tag{1.2}$$

for $t \in [0, T]$, where $\Phi_p(s) = |s|^{p-2}s$ $(p > 0)$, $h(t) \in L^{\infty}([0, T], \mathbb{R})$ and $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$. Through the Mountain pass theorem, some interesting existence results were obtained under the super p-order Ambrosetti-Rabinowitz (AR for short) condition. In 2021, Min et al. [15] obtained the multiplicity of solutions for a class of p-Laplacian type fractional Sturm-Liouville boundary value problem via variational methods.

With the emergence of the coexistence problem of p and q-Laplacian operators, the (p, q)-Laplacian problems gain more importance ([3, 4, 12]). In [12] by means of the Mountain pass theorem, the authors discussed the existence of at least one nontrivial solution for an impulsive fractional coupled system with generalized (p, q)-Laplacian operator. In [4] by using variational methods, the existence of at least one nonnegative weak solution was obtained in space $D^{1,p}(\mathbb{R}^N) \bigcap D^{1,q}(\mathbb{R}^N)$. In practical applications, (p, q)-Laplacian operator has related physical explanation. It originates from the general reaction-diffusion equation $u_t = div(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u) + \phi(x, u)$. This kind of system is widely used in physics, chemistry, biology, etc. [6, 13].

Inspired by above work, we consider a class of fractional Sturm-Liouville boundary value problem with $(p,\ q)$-Laplacian operator in this paper

$$(P_\lambda)\begin{cases} {}_tD_T^{\alpha}\Phi_p({}_0^CD_t^{\alpha}u(t))+\Phi_p(u(t))=\lambda D_uf(t,u(t),v(t)),\ \text{a.e. } t\in[0,T],\\ {}_tD_T^{\beta}\Phi_q({}_0^CD_t^{\beta}v(t))+\Phi_q(v(t))=\lambda D_vf(t,u(t),v(t)),\ \text{a.e. } t\in[0,T],\\ a_1\Phi_p(u(0))-b_{1t}D_T^{\alpha-1}(\Phi_p({}_0^CD_t^{\alpha}u(0)))=0,\\ a_2\Phi_q(v(0))-b_{2t}D_T^{\beta-1}(\Phi_q({}_0^CD_t^{\beta}v(0)))=0,\\ a_1'\Phi_p(u(T))+b'_{1t}D_T^{\alpha-1}(\Phi_p({}_0^CD_t^{\alpha}u(T)))=0,\\ a_2'\Phi_q(v(T))+b'_{2t}D_T^{\beta-1}(\Phi_q({}_0^CD_t^{\beta}v(T)))=0, \end{cases}$$

where $\lambda>0$, a_i, a_i', b_i, b_i' are positive constants, $i=1,2$, $1<p\le q<\infty$, ${}_0^CD_t^{\alpha}$, ${}_0^CD_t^{\beta}$ and ${}_tD_T^{\alpha}$, ${}_tD_T^{\beta}$ are the left Caputo and right Riemann-Liouville fractional derivatives with order $\alpha\in(\frac{1}{p},1]$ and $\beta\in(\frac{1}{q},1]$, respectively; Φ_p and Φ_q are p and q-Laplacian operators with $\Phi_k(s)=|s|^{k-2}s$ $(s\ne 0)$, $\Phi_k(0)=0$, $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous with respect to t, for all $(x,y)\in\mathbb{R}^2$, continuously differentiable with respect to x and y for almost every $t\in[0,T]$, $D_xf(t,x)$ denotes the derivative of $f(t,x)$ with respect to x.

To the best of our knowledge, there is almost no relevant work for fractional Sturm-Liouville boundary value problems with generalized p and q-Laplacian operators simultaneously. Differ from [15,16,19], we focus on $(p,\ q)$-Laplacian operator with $1<p\le q<\infty$ in the paper, which brings some difficulties in theoretical analysis owing to the lack of homogeneity of $(p,\ q)$-Laplacian operator, such as establishing variational structure and applying usual methods of critical point theory. Additionally, the existence of at least one weak solution was obtained under the super p-order AR condition in [16]. While, the existence, multiplicity and nonexistence results are established in our work based on some looser conditions than the AR condition. The main goal of this paper is to extend and supplement some existing results in [12,15,16,19].

Throughout the paper, the nonlinearity f is imposed on following hypotheses:

(A_0) There exists a constant $M_0>0$ such that

$$\lim_{|(x,y)|\to\infty}\frac{f(t,x,y)}{\mid x\mid^p+\mid y\mid^q}<M_0,\ \forall\ x,y\in\mathbb{R},\ a.e.\ t\in[0,T];$$

(A_0') There exist constants $M_1>0$, $0\le\gamma_1<p$ and $0\le\gamma_2<q$, such that

$$\lim_{|(x,y)|\to\infty}\frac{f(t,x,y)}{\mid x\mid^{\gamma_1}+\mid y\mid^{\gamma_2}}<M_1,\ \forall\ x,y\in\mathbb{R},\ a.e.\ t\in[0,T];$$

(A_1) $\lim\limits_{|(x,y)|\to\infty}\frac{f(t,x,y)}{|x|^q+|y|^q}=+\infty$ uniformly in $(x,y)\in\mathbb{R}\times\mathbb{R}$, $t\in[0,T]$;

(A_2) There exists a constant L, such that $\lim\limits_{x\to0,y\to0}\frac{f(t,x,y)}{|x|^p+|y|^q}<L$ uniformly in $(x,y)\in\mathbb{R}\times\mathbb{R}$, $t\in[0,T]$;

(A_3) For any $x,y\in\mathbb{R}$ and $0<r\le1$, there exists a constant $d\ge0$ such that

$$G(t,rx,ry)\le G(t,x,y)+d,$$

where $G(t,x,y)=D_xf(t,x,y)x+D_yf(t,x,y)y-qf(t,x,y)$;

(A_4) There exists a constant μ, such that $0<D_xf(t,x,y)x+D_yf(t,x,y)y\le\mu f(t,x,y)$, for any $(x,y)\in\mathbb{R}\times\mathbb{R}$.

Next, we state our main theorems as following:

Theorem 1.1. *If (A_0) holds and $\lambda \in (0, \min\{\frac{1}{pM_0T(C_{(\alpha,p)})^p}, \frac{1}{qM_0T(C_{(\beta,q)})^q}\})$, where $C_{(\alpha,p)}$ and $C_{(\beta,q)}$ are introduced in Lemma 2.2. Then, the problem (P_λ) exists at least one weak solution.*

Corollary 1.1. *If (A_0') holds and $\lambda \in (0, +\infty)$. Then, the problem (P_λ) exists at least one weak solution.*

Theorem 1.2. *If (A_1), (A_2), (A_3) hold. Then, the problem (P_λ) exists at least two weak solutions.*

Theorem 1.3. *If (A_2) and (A_4) hold. Then, the problem (P_λ) does not exist any nontrivial weak solution.*

Example 1.1. Let $p = 3, q = 4$. Choose $\gamma_1 = 2, \gamma_2 = 3$, $M_1 > T$ and the function $f(t, x, y) = te^{-|x|}x^2 + te^{-|y|}y^3$. It is easily to observe that

$$\lim_{|(x,y)|\to\infty} \frac{te^{-|x|}x^2 + te^{-|y|}y^3}{|\, x\,|^2 + |\, y\,|^3} \leq T < M_1,\ \forall\ x, y \in \mathbb{R},\ a.e.\ t \in [0, T].$$

Hence, (A_0') holds with $\lambda \in (0, +\infty)$. Then, the problem (P_λ) exists at least one weak solution based on Corollary 1.1.

Example 1.2. Define the function

$$f(t, x, y) = |x|^m \ln|x| - \frac{1}{m}|x|^m + |y|^n \ln|y| - \frac{1}{n}|y|^n,\ t \in [0, T],\ (x, y) \in \mathbb{R} \times \mathbb{R},$$

where $m, n \geq q$ are two constants. By a direct computation, we have $D_x f(t, x, y) = m|x|^{m-2}x \ln|x|$ and $D_y f(t, x, y) = n|y|^{n-2}y \ln|y|$. Then

$$\begin{aligned}
&D_x f(t, rx, ry)rx + D_y f(t, rx, ry)ry - qf(t, rx, ry) \\
=&(m - q)|rx|^m \ln|rx| + \frac{q}{m}|rx|^m + (n - q)|ry|^n \ln|ry| + \frac{q}{n}|ry|^n \\
\leq& m|x|^m \ln|x| - q|x|^m \ln|x| + \frac{q}{m}|x|^m + n|y|^n \ln|y| - q|y|^n \ln|y| + \frac{q}{n}|y|^n \\
=&D_x f(t, x, y)x - D_y f(t, x, y)y + qf(t, x, y).
\end{aligned}$$

Hence, (A_3) is satisfied. Obviously, (A_1) and (A_2) hold, and we can easily verify that the function f satisfies (A_3) but does not satisfy the AR condition.

2. Preliminaries and Lemmas

In this section, some important lemmas and theorems are introduced for establishing our main results.

Definition 2.1 ([9, 17]). Let $n - 1 \leq \gamma < n$, $n \in \mathrm{N}$, function $u(t)$ is defined on $[0, T]$.
(i) Define the left and right Riemann-Liouville fractional derivatives ${}_0D_t^\gamma u(t)$ and ${}_tD_T^\gamma u(t)$ of order γ for function $u(t)$ by

$${}_0D_t^\gamma u(t) = \frac{d^n}{dt^n}{}_0D_t^{\gamma-n}u(t) = \frac{1}{\Gamma(n - \gamma)}\frac{d^n}{dt^n}\int_0^t (t - \zeta)^{n-\gamma-1}u(\zeta)d\zeta,$$

$$ {}_tD_T^{\gamma}u(t) = (-1)^n \frac{d^n}{dt^n} {}_tD_T^{\gamma-n}u(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_t^T (\zeta - t)^{n-\gamma-1}u(\zeta)d\zeta. $$

(ii) For any $u(t) \in AC([0,T],\mathbb{R})$, define the left and right Caputo fractional derivatives ${}_0^CD_t^{\gamma}$ and ${}_t^CD_T^{\gamma}$ of order γ for function $u(t)$ by

$$ {}_0^CD_t^{\gamma}u(t) = {}_0D_t^{\gamma-n}u^{(n)}(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-\zeta)^{n-\gamma-1}u^{(n)}(\zeta)d\zeta, $$
$$ {}_t^CD_T^{\gamma}u(t) = (-1)^n {}_tD_T^{\gamma-n}u^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \int_t^T (\zeta - t)^{n-\gamma-1}u^{(n)}(\zeta)d\zeta. $$

Lemma 2.1 ([9]). *For $f \in L^p([0,T],\mathbb{R})$, $g \in L^q([0,T],\mathbb{R})$ and $p \geq 1, q \geq 1, \frac{1}{p}+\frac{1}{q} \leq 1+\gamma$ or $p \neq 1, q \neq 1, \frac{1}{p}+\frac{1}{q} = 1+\gamma$. Then*

$$ \int_0^T ({}_0D_t^{-\gamma}f(t))g(t)dt = \int_0^T ({}_tD_T^{-\gamma}g(t))f(t)dt, \qquad \gamma > 0. $$

Let $1 < \mathfrak{m} < \infty$, $\frac{1}{\mathfrak{m}} < \gamma \leq 1$. We define the fractional derivative space

$$ X_0^{\gamma,\mathfrak{m}} = \{u(t) \in AC([0,T],\mathbb{R}) : {}_0^CD_t^{\gamma}u(t) \in L^{\mathfrak{m}}([0,T],\mathbb{R})\} $$

as the closure of $C^{\infty}([0,T],\mathbb{R})$ endowed with the weighted norm

$$ \|u\|_{(\gamma,\mathfrak{m})} := \left(\int_0^T |\, u(t) \,|^{\mathfrak{m}} \, dt + \int_0^T |\, {}_0^CD_t^{\gamma}u(t) \,|^{\mathfrak{m}} \, dt \right)^{\frac{1}{\mathfrak{m}}}. \tag{2.1} $$

$X_0^{\gamma,\mathfrak{m}}$ is proved to be a reflexive and separable Banach space in [19].

Lemma 2.2 ([16]). *For any $u(t) \in X_0^{\gamma,\mathfrak{m}}$, we have $\|u\|_{\infty} \leq C_{(\gamma,\mathfrak{m})}\|u\|_{(\gamma,\mathfrak{m})}$, where $C_{(\gamma,\mathfrak{m})} = \max\left\{\frac{T^{\gamma-\frac{1}{\mathfrak{m}}}}{\Gamma(\gamma)((\gamma-1)\mathfrak{m}'+1)^{\frac{1}{\mathfrak{m}'}}}, 1\right\} + \left[\frac{2^{\mathfrak{m}-1}}{T}\max\left\{1, \left(\frac{T^{\gamma}}{\Gamma(\gamma+1)}\right)^{\mathfrak{m}}\right\}\right]^{\frac{1}{\mathfrak{m}}}$, $1 < \mathfrak{m}, \mathfrak{m}' < \infty$ with $\frac{1}{\mathfrak{m}} + \frac{1}{\mathfrak{m}'} = 1$.*

Lemma 2.3 ([7]). *Let $1 < \mathfrak{m} < \infty$, $\frac{1}{\mathfrak{m}} < \gamma \leq 1$. $X_0^{\gamma,\mathfrak{m}}$ is compactly embedded in $C([0,T],\mathbb{R})$.*

Here, we define the fractional derivative space $X_{(p,q)}$ as $X_0^{\alpha,p} \times X_0^{\beta,q}$ with the weighted norm

$$ \|(u,v)\|_{X_{(p,q)}} = \|u\|_{(\alpha,p)} + \|v\|_{(\beta,q)}, \forall\ u(t) \in X_0^{\alpha,p}, v(t) \in X_0^{\beta,q}, \tag{2.2} $$

where $\frac{1}{p} < \alpha \leq 1$, $\frac{1}{q} < \beta \leq 1$, $1 < p \leq q < \infty$, the definitions of $\|\cdot\|_{(\alpha,p)}$ and $\|\cdot\|_{(\beta,q)}$ are same with $\|\cdot\|_{(\gamma,\mathfrak{m})}$. Obviously, $X_{(p,q)}$ also is a reflexive and separable Banach space. The following consequences are immediately from Lemma 2.2

$$ \|u\|_{\infty} \leq C_{(\alpha,p)}\|u\|_{(\alpha,p)}, \ \|v\|_{\infty} \leq C_{(\beta,q)}\|v\|_{(\beta,q)}, \tag{2.3} $$
$$ \|(u,v)\|_{\infty} = \max_{t\in[0,T]} |u(t)| + \max_{t\in[0,T]} |v(t)| \leq \max\{C_{(\alpha,p)}, C_{(\beta,q)}\}\|(u,v)\|_{X_{(p,q)}}. \tag{2.4} $$

Lemma 2.4. *We say $(u,v) \in X_{(p,q)}$ is the weak solution of the problem (P_{λ}), if the following equation holds:*

$$ \int_0^T \Phi_p({}_0^CD_t^{\alpha}u(t)){}_0^CD_t^{\alpha}x(t) + \Phi_p(u(t))x(t) + \Phi_q({}_0^CD_t^{\beta}v(t)){}_0^CD_t^{\beta}y(t) + \Phi_q(v(t))y(t)dt $$

$$+\frac{a_1}{b_1}\Phi_p(u(0))x(0)+\frac{a_1'}{b_1'}\Phi_p(u(T))x(T)+\frac{a_2}{b_2}\Phi_q(v(0))y(0)+\frac{a_2'}{b_2'}\Phi_q(v(T))y(T)$$
$$=\lambda\int_0^T D_uf(t,u(t),v(t))x(t)+D_vf(t,u(t),v(t))y(t)dt,\ \forall\ (x,y)\in X_{(p,q)}. \tag{2.5}$$

Proof. Since ${}_tD_T^\alpha u(t)=-\frac{d}{dt}{}_tD_T^{\alpha-1}u(t)$, based on Lemma 2.1 and the boundary conditions in the problem (P_λ), we have

$$\begin{aligned}&\int_0^T {}_tD_T^\alpha\Phi_p({}_0^CD_t^\alpha u(t))\cdot x(t)dt=-\int_0^T x(t)d[{}_tD_T^{\alpha-1}\Phi_p({}_0^CD_t^\alpha u(t))] \qquad (2.6)\\ =&{}_tD_T^{\alpha-1}\Phi_p({}_0^CD_t^\alpha u(0))x(0)-{}_tD_T^{\alpha-1}\Phi_p({}_0^CD_t^\alpha u(T))x(T)\\ &+\int_0^T {}_tD_T^{\alpha-1}\Phi_p({}_0^CD_t^\alpha u(t))x'(t)dt\\ =&\frac{a_1}{b_1}\Phi_p(u(0))x(0)+\frac{a_1'}{b_1'}\Phi_p(u(T))x(T)+\int_0^T\Phi_p({}_0^CD_t^\alpha u(t)){}_t^CD_T^\alpha x(t)dt.\end{aligned}$$

The same results can be easily get that

$$\begin{aligned}&\int_0^T {}_tD_T^\beta\Phi_q({}_0^CD_t^\beta v(t))\cdot y(t)dt \qquad (2.7)\\ =&\frac{a_2}{b_2}\Phi_q(v(0))y(0)+\frac{a_2'}{b_2'}\Phi_q(v(T))y(T)+\int_0^T\Phi_q({}_0^CD_t^\beta v(t)){}_t^CD_T^\beta y(t)dt.\end{aligned}$$

Substituting $x(t)$ and $y(t)$ into the equations of the problem (P_λ) and integrating on both sides from 0 to T, then, from (2.6) and (2.7), we can obtain (2.5). □

Consider the functional $J_\lambda: X_{(p,q)}\to\mathbb{R}$ with

$$\begin{aligned}J_\lambda(u,v):=&\int_0^T\frac{1}{p}(\mid {}_0^CD_t^\alpha u(t)\mid^p+\mid u(t)\mid^p)+\frac{1}{q}(\mid {}_0^CD_t^\beta v(t)\mid^q+\mid v(t)\mid^q)dt \qquad (2.8)\\ &+\frac{a_1}{pb_1}|u(0)|^p+\frac{a_1'}{pb_1'}|u(T)|^p+\frac{a_2}{qb_2}|v(0)|^q\\ &+\frac{a_2'}{qb_2'}|v(T)|^q-\lambda\int_0^T f(t,u(t),v(t))dt.\end{aligned}$$

Owing to the continuity of f, we can obtain that $J_\lambda\in C^1(X_{(p,q)},\mathbb{R})$ and

$$\begin{aligned}J_\lambda'(u,v)(x,y)=&\int_0^T\Phi_p({}_0^CD_t^\alpha u(t)){}_0^CD_t^\alpha x(t)+\Phi_q({}_0^CD_t^\beta v(t)){}_0^CD_t^\beta y(t) \qquad (2.9)\\ &+\Phi_p(u(t))x(t)+\Phi_q(v(t))y(t)dt+\frac{a_1}{b_1}\Phi_p(u(0))x(0)\\ &+\frac{a_1'}{b_1'}\Phi_p(u(T))x(T)+\frac{a_2}{b_2}\Phi_q(v(0))y(0)+\frac{a_2'}{b_2'}\Phi_q(v(T))y(T)\\ &-\lambda\int_0^T D_uf(t,u(t),v(t))x(t)+D_vf(t,u(t),v(t))y(t)dt.\end{aligned}$$

Notice that, the critical point of J_λ is the weak solution of the problem (P_λ).

Definition 2.2 ([20]). Let X be a Banach space and $J\in C^1(X,\mathbb{R})$. Functional J satisfies the Palais-Smale condition if each sequence $\{u_k\}_{k=1}^\infty\subset X$ such that $\{J(u_k)\}$ is bounded and $\lim\limits_{k\to\infty}J'(u_k)=0$ possesses strongly convergent subsequence in X.

Theorem 2.1 ([14]). *Let X be a real reflexive Banach space. If the functional $J : X \to \mathbb{R}$ is weakly lower semi-continuous and coercive, i.e., $\lim_{\|\nu\|\to\infty} J(\nu) = +\infty$, then there exists $\nu^* \in X$ such that $J(\nu^*) = \inf_{\nu\in X} J(\nu)$. If J is Fréchet differentiable in X, then $J'(\nu^*) = 0$.*

Theorem 2.2 ([18]). *Let X be a Banach space. Functional $J \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition and $J(0) = 0$. Assume that*
(i) There exist $\rho > 0$ and $\sigma > 0$ such that $J|_{\partial\Omega_\rho(0)} \geq \sigma$;
(ii) There exists $\nu_0 \in X/\overline{\Omega}_\rho(0)$ such that $J(\nu_0) \leq 0$.
Then, J has one critical value $c \geq \sigma$ with $c = \inf_{g\in\Upsilon} \max_{\nu\in[0,1]} J(g(\nu))$, where $\Omega_\rho(0)$ is an open ball in X of radius ρ centered at 0, $\Upsilon = \{g \in C([0, 1], X) \mid g(0) = 0, g(1) = \nu_0\}$.

Remark 2.1. The Mountain pass theorem still holds if the Palais-Smale condition replace with the Cerami condition [20].

Theorem 2.3 (Theorem 38.A, [21]). *For the functional $J : \Omega \subseteq X \to \mathbb{R}$ with $\Omega \neq \emptyset$, $\min_{x\in\Omega} J(x) = c$ exists a solution in case the following assumptions hold:*
(i) X is a real reflexive Banach space;
(ii) Ω is a bounded and weakly sequentially closed set;
(iii) J is sequentially weakly lower semi-continuous on Ω.

3. Proof of theorems

In this section, the existence, multiplicity and nonexistence results are discussed for the problem (P_λ) relying on Theorem 2.1, Theorem 2.2 and Theorem 2.3.

3.1. Proof of Theorem 1.1.

Proof. Let $\{(x_k, y_k)\}_{k=1}^{\infty}$ is a weakly convergent sequence to (x, y) in $X_{(p,q)}$, then from the continuity of f and Lemma 2.3, we have

$$\begin{aligned}
&\liminf_{k\to\infty}\{\frac{1}{p}\|x_k\|^p_{(\alpha,p)} + \frac{1}{q}\|y_k\|^q_{(\beta,q)} + \frac{a_1}{pb_1}|x_k(0)|^p + \frac{a_1'}{pb_1'}|x_k(T)|^p \\
&\quad + \frac{a_2}{qb_2}|y_k(0)|^q + \frac{a_2'}{qb_2'}|y_k(T)|^q - \lambda\int_0^T f(t, x_k(t), y_k(t))dt\} \\
\geq& \frac{1}{p}\|x\|^p_{(\alpha,p)} + \frac{1}{q}\|y\|^q_{(\beta,q)} + \frac{a_1}{pb_1}|x(0)|^p + \frac{a_1'}{pb_1'}|x(T)|^p \\
&\quad + \frac{a_2}{qb_2}|y(0)|^q + \frac{a_2'}{qb_2'}|y(T)|^q - \lambda\int_0^T f(t, x(t), y(t))dt,
\end{aligned}$$

hence, J_λ is sequentially weakly lower semi-continuous.

From (A_0), there exists a constant $C_0 > 0$ such that

$$f(t, x, y) \leq M_0(|\ x\ |^p + |\ y\ |^q) + C_0,\ \forall\ (x, y) \in \mathbb{R}\times\mathbb{R},\ a.e.\ t \in [0, T]. \tag{3.1}$$

Then, due to (2.8) and (3.1) yields

$$J_\lambda(x, y) \geq \frac{1}{p}\|x\|^p_{(\alpha,p)} + \frac{1}{q}\|y\|^q_{(\beta,q)} - \lambda\int_0^T M_0(|\ x\ |^p + |\ y\ |^q) + C_0 dt$$

$$\geq \frac{1}{p}\|x\|^p_{(\alpha,p)} + \frac{1}{q}\|y\|^q_{(\beta,q)} - \lambda M_0 T(\|x\|^p_\infty + \|y\|^q_\infty) - \lambda C_0 T$$
$$\geq \left(\frac{1}{p} - \lambda M_0 T(C_{(\alpha,p)})^p\right)\|x\|^p_{(\alpha,p)} + \left(\frac{1}{q} - \lambda M_0 T(C_{(\beta,q)})^q\right)\|y\|^q_{(\beta,p)} - \lambda C_0 T,$$

since $\lambda \in (0, \min\{\frac{1}{pM_0T(C_{(\alpha,p)})^p}, \frac{1}{qM_0T(C_{(\beta,q)})^q}\})$, so that J_λ is coercive. Hence, the problem (P_λ) exists at least one weak solution in $X_{(p,q)}$ under Theorem 2.1. □

3.1.1. Proof of Corollary 1.1.

Proof. (A_0') implies that there exists a constant $C_0' > 0$ such that $f(t,x,y) \leq M_1(|\, x\,|^{\gamma_1} + |\, y\,|^{\gamma_2}) + C_0'$, then

$$\begin{aligned} J_\lambda(x,y) \geq & \frac{1}{p}\|x\|^p_{(\alpha,p)} - \lambda M_1 T(C_{(\alpha,p)})^{\gamma_1}\|x\|^{\gamma_1}_{(\alpha,p)} + \frac{1}{q}\|y\|^q_{(\beta,q)} \\ & - \lambda M_1 T(C_{(\beta,q)})^{\gamma_2}\|y\|^{\gamma_2}_{(\beta,q)} - \lambda C_0' T, \end{aligned}$$

notice that $0 \leq \gamma_1 < p$ and $0 \leq \gamma_2 < q$, so that $J_\lambda(x,y) \to \infty$ as $\|(x,y)\|_{X_{(p,q)}} \to \infty$, i.e., J_λ is coercive. Similar discussion with the Proof of Theorem 1.1, we can get the similar existence result. □

3.2. Proof of Theorem 1.2.

Proof. We divide the proof into three steps.
Step 1: J_λ satisfies the Cerami condition. Indeed, suppose that $\{(x_n,y_n)\} \subset X_{(p,q)}$ is the Cerami sequence associated with J_λ, i.e., $\{J_\lambda(x_n,y_n)\}$ is bounded and $\|J_\lambda'(x_n,y_n)\|(1+\|(x_n,y_n)\|_{X_{(p,q)}}) \to 0 (n\to\infty)$, which means that

$$J_\lambda(x_n,y_n) + o(1) = K, \quad J_\lambda'(x_n,y_n)(x_n,y_n) = o(1), \tag{3.2}$$

where $K > 0$ is a constant, and $o(1) \to 0$ as $n \to \infty$.

Assume that $\|(x_n,y_n)\|_{X_{(p,q)}} \to \infty (n\to\infty)$. Denote $H_n(t) = (H_n^1(t), H_n^2(t)) = \frac{(x_n(t),y_n(t))}{\|(x_n,y_n)\|_{X_{(p,q)}}}$. Obviously, $H_n(t) \in X_{(p,q)}$ and $\|H_n\|_{X_{(p,q)}} = 1$, which implies that $(H_n^1(t), H_n^2(t)) \rightharpoonup (H_0^1(t), H_0^2(t))$ on $X_{(p,q)}$ (up to subsequences). It follows from Lemma 2.3 that $(H_n^1(t), H_n^2(t)) \to (H_0^1(t), H_0^2(t))$ on $t \in [0,T]$. Define $D_0 = \{t \in [0,T] \mid (H_n^1(t), H_n^2(t)) \neq (0,0)\}$, then $\lim\limits_{n\to\infty} \frac{(x_n(t),y_n(t))}{\|(x_k,y_k)\|_{X_{(p,q)}}} \neq (0,0)$, for $t \in D_0$. In view of (A_1) that $\lim\limits_{n\to\infty} \frac{f(t,x_n(t),y_n(t))}{|(x_n(t),y_n(t))|^q} = +\infty$, a.e. in D_0, namely

$$\lim_{n\to\infty} \frac{f(t,x_n(t),y_n(t))}{\|(x_n,y_n)\|^q_{X_{(p,q)}}} = \lim_{n\to\infty} \frac{f(t,x_n(t),y_n(t))}{|\,(x_n(t),y_n(t))\,|^q} \,|\, H_n(t)\,|^q = +\infty, \text{ a.e. in } D_0. \tag{3.3}$$

Based on (2.8), (3.2), (3.3) and $\|\,(x_n,y_n)\,\|_{X_{(p,q)}} \to \infty (n\to\infty)$, we obtain

$$\begin{aligned} & \lim_{n\to\infty} \frac{\lambda \int_0^T f(t,x_n,y_n)dt}{\|(x_n,y_n)\|^q_{X_{(p,q)}}} \\ \leq & \lim_{n\to\infty} \left\{ \frac{\frac{1}{p}\|x_n\|^p_{(\alpha,p)} + \frac{1}{q}\|y_n\|^q_{(\beta,q)} + \frac{a_1}{pb_1}|x_n(0)|^p}{\|(x_n,y_n)\|^q_{X_{(p,q)}}} \right. \end{aligned}$$

$$+\frac{\frac{a_1'}{pb_1'}|x_n(T)|^p+\frac{a_2}{qb_2}|y_n(0)|^q+\frac{a_2'}{qb_2'}|y_n(T)|^q-K+o(1)}{\|(x_n,y_n)\|^q_{X_{(p,q)}}}\Big\}$$
$$\leq\frac{1}{p}+\frac{1}{q}+\frac{a_1(C_{(\alpha,p)})^p}{pb_1}+\frac{a_1'(C_{(\alpha,p)})^p}{pb_1'}+\frac{a_2(C_{(\beta,q)})^q}{qb_2}+\frac{a_2'(C_{(\beta,q)})^q}{qb_2'},$$

which contradicts with (3.3). So that $(H_0^1(t),H_0^2(t))=(0,0)$ a.e. $t\in[0,T]$.

For each $n\in$ N, consider a continuous function $\theta\in[0,1]\to J_\lambda(\theta x_n,\theta y_n)$, there exists a sequence $\theta_n\in[0,1]$ such that

$$J_\lambda(\theta_n x_n,\theta_n y_n)=\max_{0\leq\theta\leq1}J(\theta x_n,\theta y_n). \tag{3.4}$$

Since $\|(x_n,y_n)\|_{X_{(p,q)}}\to\infty(n\to\infty)$, there exist $\xi>0$ and n_0 large enough such that $\frac{\xi}{\|(x_n,y_n)\|_{X_{(p,q)}}}\in(0,1)$ for all $n\geq n_0$. Judging by (2.8) and (3.4), we have

$$\begin{aligned}J_\lambda(\theta_n x_n,\theta_n y_n)\geq&J_\lambda\left(\frac{\xi}{\|(x_n,y_n)\|_{X_{(p,q)}}}x_n,\frac{\xi}{\|(x_n,y_n)\|_{X_{(p,q)}}}y_n\right)\\
=&\frac{\xi^p}{p}\|H_n^1\|^p_{(\alpha,p)}+\frac{\xi^q}{q}\|H_n^2\|^q_{(\beta,q)}+\frac{a_1\xi^p}{pb_1}|H_n^1(0)|^p+\frac{a_1'\xi^p}{pb_1'}|H_n^1(T)|^p\\
&+\frac{a_2\xi^q}{qb_2}|H_n^2(0)|^q+\frac{a_2'\xi^q}{qb_2'}|H_n^2(T)|^q-\lambda\int_0^T f(t,\xi H_n^1(t),\xi H_n^2(t))dt\\
\geq&\min\{\frac{\xi^p}{p},\frac{\xi^q}{q}\}(\|H_n^1\|^q_{(\alpha,p)}+\|H_n^2\|^q_{(\beta,q)})-\lambda\int_0^T f(t,\xi H_n^1(t),\xi H_n^2(t))dt\\
\geq&2^{-q}\min\{\frac{\xi^p}{p},\frac{\xi^q}{q}\}-\lambda\int_0^T f(t,\xi H_n^1(t),\xi H_n^2(t))dt,\end{aligned}$$

for all $n\geq n_0$. Because $(H_0^1(t),H_0^2(t))=(0,0)$ a.e. $t\in[0,T]$, according to Lebesgue theorem, we can choose a large number n_1 with $n_1>n_0$ such that

$$\int_0^T f(t,\xi H_n^1(t),\xi H_n^2(t))dt<\frac{1}{2\lambda}\cdot\frac{\min\{\frac{\xi^p}{p},\frac{\xi^q}{q}\}}{2^q},\forall n\geq n_1.$$

Hence, we can obtain that $J_\lambda(\theta_n x_n,\theta_n y_n)\geq\frac{\min\{\frac{\xi^p}{p},\frac{\xi^q}{q}\}}{2^{q+1}}$, for each $n>n_1$. It follows from the arbitrariness of ξ that

$$J_\lambda(\theta_n x_n,\theta_n y_n)\to+\infty\quad\text{as}\quad n\to\infty. \tag{3.5}$$

Taking into account of (3.4) that $\frac{d}{d\theta}J_\lambda(\theta x_n,\theta y_n)\mid_{\theta=\theta_n}=0$, namely

$$J_\lambda'(\theta_n x_n,\theta_n y_n)(\theta_n x_n,\theta_n y_n)=0. \tag{3.6}$$

Then, consider (2.8), (2.9), (3.6), (A_3) and (3.2), we derive

$$\begin{aligned}qJ_\lambda(\theta_n x_n,\theta_n y_n)=&qJ_\lambda(\theta_n x_n,\theta_n y_n)-J_\lambda'(\theta_n x_n,\theta_n y_n)(\theta_n x_n,\theta_n y_n)\\
=&(\frac{q}{p}-1)\theta_n^p\parallel x_n\parallel^p_{(\alpha,p)}+(\frac{qa_1}{pb_1}-\frac{a_1}{b_1})\theta_n^p|x_n(0)|^p\\
&+(\frac{qa_1'}{pb_1'}-\frac{a_1'}{b_1'})\theta_n^p|x_n(T)|^p+\lambda\int_0^T D_xf(t,\theta_n x_n,\theta_n y_n)\cdot\theta_n x_n\end{aligned}$$

$$
\begin{aligned}
&+ D_y f(t,\theta_n x_n,\theta_n y_n)\cdot \theta_n y_n - q f(t,\theta_n x_n,\theta_n y_n)dt\\
\le&(\frac{q}{p}-1)\parallel x_n \parallel^p_{(\alpha,p)} +(\frac{qa_1}{pb_1}-\frac{a_1}{b_1})|x_n(0)|^p+(\frac{qa'_1}{pb'_1}-\frac{a'_1}{b'_1})|x_n(T)|^p\\
&+\lambda\int_0^T D_x f(t,x_n,y_n)\cdot x_n + D_y f(t,x_n,y_n)\cdot y_n\\
&- q f(t,x_n,y_n)dt+\lambda T d\\
=&qJ_\lambda(x_n,y_n)-J'_\lambda(x_n,y_n)(x_n,y_n)+\lambda T d\\
=&qK+qo(1)-o(1)+\lambda T d,
\end{aligned}
$$

which contradicts with (3.5), i.e., $\{(x_n,y_n)\}$ is bounded in $X_{(p,q)}$. Since $X_{(p,q)}$ is reflexive, then $(x_n,y_n)\rightharpoonup(x^*,y^*)$ in $X_{(p,q)}$ (up to subsequences). From Lemma 2.3, we have $(x_n(t),y_n(t))\to(x^*(t),y^*(t))$ uniformly on $t\in[0,T]$. Then

$$
\begin{cases}
(J'_\lambda(x_n,y_n)-J'_\lambda(x^*,y^*))((x_n,y_n)-(x^*,y^*))\to 0,\ n\to\infty,\\
\displaystyle\int_0^T[D_x f(t,x_n(t),y_n(t))-D_x f(t,x^*(t),y^*(t))](x_n(t)-x^*(t))dt\to 0,\ n\to\infty,\\
\displaystyle\int_0^T[D_y f(t,x_n(t),y_n(t))-D_y f(t,x^*(t),y^*(t))](y_n(t)-y^*(t))dt\to 0,\ n\to\infty,\\
x_n(0)-x^*(0)\to 0,\ x_n(T)-x^*(T)\to 0,\ n\to\infty,\\
y_n(0)-y^*(0)\to 0,\ y_n(T)-y^*(T)\to 0,\ n\to\infty.
\end{cases}
$$

Denote

$$
\begin{aligned}
J(\alpha,p)&=\int_0^T\Big(\Phi_p({}_0^C D_t^\alpha x_n)-\Phi_p({}_0^C D_t^\alpha x^*)\Big)({}_0^C D_t^\alpha x_n-{}_0^C D_t^\alpha x^*)dt,\\
J(\beta,q)&=\int_0^T\Big(\Phi_q({}_0^C D_t^\beta y_n)-\Phi_q({}_0^C D_t^\beta x^*)\Big)({}_0^C D_t^\beta y_n-{}_0^C D_t^\beta y^*)dt,\\
J(p)&=\int_0^T\Big(\Phi_p(x_n)-\Phi_p(x^*)\Big)(x_n-x^*)dt,\\
J(q)&=\int_0^T\Big(\Phi_q(y_n)-\Phi_q(y^*)\Big)(y_n-y^*)dt,\\
J_1&=\frac{a_1}{b_1}\Big(\Phi_p(x_n(0))-\Phi_p(x^*(0))\Big)(x_n(0)-x^*(0))\\
&\quad+\frac{a'_1}{b'_1}\Big(\Phi_p(x_n(T))-\Phi_p(x^*(T))\Big)(x_n(T)-x^*(T))\\
&\quad+\frac{a_2}{b_2}\Big(\Phi_q(y_n(0))-\Phi_q(y^*(0))\Big)(y_n(0)-y^*(0))\\
&\quad+\frac{a'_2}{b'_2}\Big(\Phi_q(y_n(T))-\Phi_q(y^*(T))\Big)(y_n(T)-y^*(T)),\\
J_2&=\lambda\int_0^T(D_x f(t,x_n,y_n)-D_x f(t,x^*,y^*))(x_n-x^*)\\
&\quad+(D_y f(t,x_n,y_n)-D_y f(t,x^*,y^*))(y_n-y^*)dt,
\end{aligned}
$$

by means of (2.9), we have $(J'_\lambda(x_n,y_n)-J'_\lambda(x^*,y^*))((x_n,y_n)-(x^*,y^*))=J(\alpha,p)+J(\beta,q)+J(p)+J(q)+J_1-J_2$. Thus, $J(\alpha,p)+J(\beta,q)+J(p)+J(q)\to 0$ as $n\to\infty$.

Similar with the discussion of $\Theta(\alpha,p),\Theta(\beta,q),\Theta(p),\Theta(q)$ in [12], we can get that

$$J(\alpha,p)\geq\begin{cases}k_1\|x_n-x^*\|^p_{(\alpha,p)}, & p\geq 2,\\ k_2\|x_n-x^*\|^2_{(\alpha,p)}(\|x_n\|^p_{L^p}+\|x^*\|^p_{L^p})^{\frac{p-2}{p}}, & 1<p<2,\end{cases}$$

$$J(\beta,q)\geq\begin{cases}k_3\|y_n-y^*\|^q_{(\beta,q)}, & q\geq 2,\\ k_4\|y_n-y^*\|^2_{(\beta,q)}(\|y_n\|^q_{L^q}+\|y^*\|^q_{L^q})^{\frac{q-2}{q}}, & 1<q<2,\end{cases}$$

$$J(p)\geq\begin{cases}k'_1\|x_n-x^*\|^p_{L^p}, & p\geq 2,\\ k'_2\|x_n-x^*\|^2_{L^p}(\|x_n\|^p_{L^p}+\|y^*\|^p_{L^p})^{\frac{p-2}{p}}, & 1<p<2,\end{cases}$$

$$J(q)\geq\begin{cases}k'_3\|y_n-y^*\|^q_{L^q}, & q\geq 2,\\ k'_4\|y_n-y^*\|^2_{L^q}(\|y_n\|^q_{L^q}+\|y^*\|^q_{L^q})^{\frac{q-2}{q}}, & 1<q<2.\end{cases}$$

where $k_i,k'_i,\ i=1,2,3,4$, are constants. At this point, we assert that $\|x_n-x^*\|_{(\alpha,p)}\to 0$, $\|y_n-y^*\|_{(\beta,q)}\to 0$, as $n\to\infty$. Hence, the Cerami condition holds.

Step 2: J_λ satisfies the geometry conditions of the Mountain pass theorem. Indeed, consider (A_2), there exists $\varepsilon>0$ such that

$$f(t,x,y)\leq\varepsilon L(|\ x\ |^p+|\ y\ |^q). \tag{3.7}$$

Based on (2.8), (3.7) and (2.3), we have

$$\begin{aligned}J_\lambda(x,y)\geq&\frac{1}{p}\|x\|^p_{(\alpha,p)}+\frac{1}{q}\|y\|^q_{(\beta,q)}-\lambda\int_0^T\varepsilon L(|\ x\ |^p+|\ y\ |^q)dt\\ \geq&\frac{1}{p}\|x\|^p_{(\alpha,p)}+\frac{1}{q}\|y\|^q_{(\beta,q)}-\lambda\varepsilon LT(\|x\|^p_\infty+\|y\|^q_\infty)\\ \geq&(\frac{1}{p}-\lambda\varepsilon LT(C_{(\alpha,p)})^p)\|x\|^p_{(\alpha,p)}+(\frac{1}{q}-\lambda\varepsilon LT(C_{(\beta,q)})^q)\|y\|^q_{(\beta,q)}.\end{aligned}\tag{3.8}$$

Let $\|x\|_{(\alpha,p)}=\rho_1$, $\|y\|_{(\beta,q)}=\rho_2$, $\|(x,y)\|_{X_{(p,q)}}=\rho_1+\rho_2=\rho$. Then, choosing $\varepsilon>0$ small enough, there exists $\sigma>0$ such that

$$\begin{aligned}J_\lambda(x,y)\geq&(\frac{1}{p}-\lambda\varepsilon LT(C_{(\alpha,p)})^p)\rho_1^p+(\frac{1}{q}-\lambda\varepsilon LT(C_{(\beta,q)})^q)\rho_2^q\\ :=&\sigma>0,\quad\forall\ (x,y)\in X_{(p,q)},\quad\|(x,y)\|_{X_{(p,q)}}=\rho.\end{aligned}$$

Additionally, from (A_1), we get that for any constant $K_0>0$, there exists a corresponding constant $C_{K_0}>0$ such that

$$f(t,x,y)\geq K_0(|\ x\ |^q+|\ y\ |^q)-C_{K_0}. \tag{3.9}$$

For any $(\omega_1(t),\omega_2(t))\in X_{(p,q)}$ with $\|(\omega_1,\omega_2)\|_{X_{(p,q)}}=1$, $r>0$, according to (2.8), (3.9) and (2.3) yields

$$\begin{aligned}J_\lambda(r\omega_1,r\omega_2)\leq&\frac{r^p}{p}+\frac{r^q}{q}+\left(\frac{a_1r^p}{pb_1}+\frac{a'_1r^p}{pb'_1}\right)(C_{(\alpha,p)})^p+\left(\frac{a_2r^q}{qb_2}+\frac{a'_2r^q}{qb'_2}\right)(C_{(\beta,q)})^q\\ &-\lambda K_0r^q(\|\omega_1\|^q_{L^q}+\|\omega_2\|^q_{L^q})+\lambda C_{K_0}T,\end{aligned}$$

choose K_0 large enough, we have $J_\lambda(r\omega_1, r\omega_2) \to -\infty$ as $r \to +\infty$, i.e., there exists $r_0 > 0$ such that $\inf_{(x,y)\in\partial\Omega_\rho} J_\lambda(x,y) > J_\lambda(r_0\omega_1, r_0\omega_2)$ with $\|(r_0\omega_1, r_0\omega_2)\|_{X_{(p,q)}} > \rho$. Using Theorem 2.2, the problem (P_λ) exists at least one nontrivial weak solution $(x_0, y_0) \in X_{(p,q)}$ such that

$$J'_\lambda(x_0, y_0) = 0 \text{ with } \ J_\lambda(x_0, y_0) > \max\{J_\lambda(0,0), J_\lambda(r_0\omega_1, r_0\omega_2)\} \geq J_\lambda(0,0). \quad (3.10)$$

Step 3: We prove that the problem (P_λ) has at least two nontrivial weak solutions in $X_{(p,q)}$. Since $\overline{\Omega}_\rho$ is a closed-convex set, from the Mazur Theorem [14], it is easy to get that $\overline{\Omega}_\rho$ is bounded and weakly sequentially closed. Furthermore, J_λ is sequentially weakly lower semi-continuous and $X_{(p,q)}$ is a reflexive Banach space. Then, from Theorem 2.3, there exists a local minimum $(x_1, y_1) \in \overline{\Omega}_\rho$ such that $J_\lambda(x_1, y_1) = \min\{J_\lambda(x,y) : (x,y) \in \overline{\Omega}_\rho\}$.

In what follows, (x_0, y_0) and (x_1, y_1) are two different bounded solutions of the problem (P_λ) is shown. Obviously, $J_\lambda(x_0, y_0) > J_\lambda(0,0) \geq J_\lambda(x_1, y_1)$, which implies that (x_0, y_0) and (x_1, y_1) are different. Define

$$\begin{aligned} G(s) =& \frac{r_0^p}{p} s^p + \frac{r_0^q}{q} s^q + \left(\frac{a_1 r_0^p}{p b_1} + \frac{a'_1 r_0^p}{p b'_1} \right) (C_{(\alpha,p)})^p s^p + \left(\frac{a_2 r_0^q}{q b_2} + \frac{a'_2 r_0^q}{q b'_2} \right) (C_{(\beta,q)})^q s^q \\ & - \lambda K_0 r_0^q (\|\omega_1\|_{L^q}^q + \|\omega_2\|_{L^q}^q) s^q + \lambda C_{K_0} T, \end{aligned}$$

where r_0 and K_0 are introduced in (3.10) and (3.9). Obviously, $G(s)$ is continuous on $[0,1]$. Combining the inf-max characterization of (x_0, y_0) in Theorem 2.2 with (3.10), we derive

$$J_\lambda(x_0, y_0) = \inf_{g\in\Upsilon} \max_{s\in[0,1]} J(g_x(s), g_y(s)) \leq \max_{s\in([0,1])} J_\lambda(r_0\omega_1 s, r_0\omega_2 s) \leq G(s),$$

where $g(s) = (g_x(s), g_y(s))$. Thus, $J_\lambda(x_0, y_0)$ is bounded as well as $J_\lambda(x_1, y_1)$.

In a nutshell, the problem (P_λ) has two different nontrivial weak solutions (x_0, y_0) and (x_1, y_1) in $X_{(p,q)}$. $\square$

3.3. Proof of Theorem 1.3.

Proof. Let (u, v) be a nontrivial solution of the problem (P_λ), from (2.9) and (A_4), we can get

$$\begin{aligned} 0 = J'_\lambda(u,v)(u,v) \geq& \|u\|_{(\alpha,p)}^p + \|v\|_{(\beta,q)}^q - \lambda \int_0^T D_u f(t, u(t), v(t)) u(t) \\ & + D_v f(t, u(t), v(t)) v(t) dt \\ \geq& \|u\|_{(\alpha,p)}^p + \|v\|_{(\beta,q)}^q - \lambda \int_0^T \mu f(t, u(t), v(t)) dt. \end{aligned} \quad (3.11)$$

Since (A_2) implies (3.7), then (3.11) can be further processed by

$$J'_\lambda(u,v)(u,v) \geq (1 - \lambda\varepsilon L\mu (C_{(\alpha,p)})^p) \|u\|_{(\alpha,p)}^p + (1 - \lambda\varepsilon L\mu (C_{(\beta,q)})^q) \|v\|_{(\beta,q)}^q.$$

Choose ε small enough such that $J'_\lambda(u,v)(u,v) > 0$, we get a contradiction. Hence, the problem (P_λ) does not exist any nontrivial weak solution. $\square$

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