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THE NUMBER OF LIMIT CYCLES FROM ELLIPTIC HAMILTONIAN VECTOR FIELDS BY HIGHER ORDER MELNIKOV FUNCTIONS

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Abstract In this paper, the perturbed Hamiltonian system $dH = \epsilon F_4 + \epsilon^2 F_3 + \epsilon^3 F_2 + \epsilon^4 F_1$, with F_i the vector valued homogeneous polynomials of degree i. The Hamiltonian function is $H = y^2/2 + U(X)$, where U is a univariate polynomial of degree four without symmetry. By computing higher order Melnikov functions, the upper bounds for the number of limit cycles that bifurcate from $dH = 0$ are deserved.

Keywords Melnikov functions, bifurcation, limit cycles, generators.

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1. Introduction

On the number of limit cycles bifurcated from the double homoclinic loop there are many results, see [10, 11, 18, 20, 21] for example. Some new results on upper bound of the number of limit cycles bifurcated from the period annuluses and Hopf bifurcation and Poincare bifurcation are given, one can see [12, 16, 23].

By using the method of computing the higher order Melnikov functions of some perturbed systems developed in [4, 13], the number of limit cycles bifurcated from the period orbits is considered , see [1–3, 5–8, 15, 17, 19, 22]. In general, the system takes the form

$$\begin{cases} \dot{x} = H_y + \epsilon f(x, y, \epsilon), \\ \dot{y} = -H_x + \epsilon g(x, y, \epsilon). \end{cases}$$

Gavrilov and Iliev [7] studied the perturbed Hamiltonian planar vector field X_ϵ,

$$X_\epsilon : \begin{cases} \dot{x} = H_y + \epsilon f(x, y), \\ \dot{y} = -H_x + \epsilon g(x, y), \end{cases} \tag{1.1}$$

with

$$H = \frac{1}{2}y^2 + U(x),\ U(x) = \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4, \quad \left(a \neq 0, \frac{8}{9}\right) \tag{1.2}$$

is a univariate polynomial of degree four without symmetry and arbitrary cubic polynomial perturbations f and g.

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Then, Asheghi and Babavi [1] considered the perturbed system

$$Y_{1\epsilon}:\begin{cases}\dot{x}=H_y+\sum_{i=1}^{3}\epsilon^i f_i(x,y),\\ \dot{y}=-H_x+\sum_{i=1}^{3}\epsilon^i g_i(x,y),\end{cases} \tag{1.3}$$

and

$$Y_{2\epsilon}:\begin{cases}\dot{x}=H_y+\sum_{i=1}^{3}\epsilon^i f_{4-i}(x,y),\\ \dot{y}=-H_x+\sum_{i=1}^{3}\epsilon^i g_{4-i}(x,y),\end{cases} \tag{1.4}$$

where H is the same as (1.2),

$$\begin{aligned}f_1=&f_{10}x+f_{01}y,\quad g_1=g_{10}x+g_{01}y,\\ f_2=&f_{20}x^2+f_{11}xy+f_{02}y^2,\quad g_2=g_{20}x^2+g_{11}xy+g_{02}y^2,\\ f_3=&f_{30}x^3+f_{21}x^2y+f_{12}xy^2+f_{03}y^3,\quad g_3=g_{30}x^3+g_{21}x^2y+g_{12}xy^2+g_{03}y^3.\end{aligned}$$

Through computing higher order Melnikov functions until the presentation of reversible perturbations. The upper bounds for the number of limit cycles bifurcated from the periodic orbits of $dH=0$ are found.

Motivated by the above references, we consider the following perturbed system

$$Z_{\epsilon}:\begin{cases}\dot{x}=H_y+\sum_{i=1}^{4}\epsilon^i f_{5-i}(x,y),\\ \dot{y}=-H_x+\sum_{i=1}^{4}\epsilon^i g_{5-i}(x,y),\end{cases} \tag{1.5}$$

where

$$\begin{aligned}f_4&=f_{40}x^4+f_{31}x^3y+f_{22}x^2y^2+f_{13}xy^3+f_{04}y^4,\\ g_4&=g_{40}x^4+g_{31}x^3y+g_{22}x^2y^2+g_{13}xy^3+g_{04}y^4\end{aligned}$$

and f_i, $g_i(i=1,2,3)$ are the same as that in system (1.3), H is the same as (1.2). Parametrizing the displacement map $d(h,\epsilon)$ by the energy level $H=h$ and the small parameter ϵ, one can obtain

$$d(h,\epsilon)=\epsilon M_1(h)+\epsilon^2M_2(h)+\cdots\epsilon^kM_k(h)+\cdots, \tag{1.6}$$

where $M_k(h)$ is called the k-th order Melnikov function.

The system Z_ϵ can be written as the Pffafian form

$$dH=\epsilon\omega_1+\epsilon^2\omega_2+\epsilon^3\omega_3+\epsilon^4\omega_4+\cdots,$$

with

$$\begin{aligned}\omega_i=&g_i(x,y)dx-f_i(x,y)dy, i=1,2,3,4,\\ \omega_i=&0, i=5,6,7,\cdots.\end{aligned}$$

We will use the algorithm of Iliev [14] to calculate higher order Melnikov funtions.

Theorem 1.1 ([14]). *For $k \geq 2$ and if $M_1(h) = M_2(h) = \cdots = M_{k-1}(h) = 0$, then there exist polynomials q_1, q_2, $\cdots$, q_{k-1} and Q_1, Q_2, $\cdots$, Q_{k-1} such that $\Omega_1 = dQ_1 + q_1 dH$, $\cdots$, $\Omega_{k-1} = dQ_{k-1} + q_{k-1} dH$ and*

$$M_k(h) = \oint_{\delta(h)} \Omega_k, \tag{1.7}$$

where $\Omega_1 = \omega_1$, $\Omega_l = \omega_l + \sum_{i+j=l} q_i \omega_j$, $2 \leq l \leq k$.

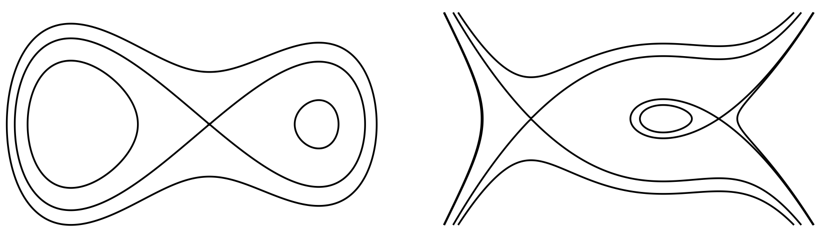
Define

$$J_k(h) = \oint_{\delta(h)} x^k y dx, J_{k,j}(h) = \oint_{\delta(h)} x^k y^j dx, k \geq 0, j \geq 2.$$

Corollary 1.1 ([1]). *Let $\alpha(h)$, $\beta(h)$ and $\gamma(h)$ be real or complex polynomials in h. The Abelian integral*

$$J(h) = \alpha(h) J_0(h) + \beta(h) J_1(h) + \gamma(h) J_2(h),$$

is identically zero if and only if $\alpha(h)$, $\beta(h)$ and $\gamma(h)$ are identically zero.



(a) (b)

Figure 1. (a) $\frac{8}{9} < a < 1$, eight-loop case, (b) $a < 0$, saddle-loop case.

Theorem 1.2 ([7]). *Let the coefficients $\alpha_k(h)$, $\beta_k(h)$, $\gamma_k(h)$ in the expression of $M_k(h)$ be polynomials of degree n with real coefficients. Then $M_k(h)$ has in the respective interval $\sum$ at most $3n + 2$ zeros in the interior eight-loop case, at most $4n + 4$ in the exterior eight-loop case and at most $4n + 3$ zeros in the saddle-loop case, see Fig. 1.*

The main results in [1] showed that $\alpha_k(h)$, $\beta_k(h)$, $\gamma_k(h)$ are polynomials in h of degree at most one. However, For system (1.5), we deduce that $\alpha_k(h)$, $\beta_k(h)$, $\gamma_k(h)$ are polynomials in h of degree at most three, and there are more limit cycles for this system.

The organization of the paper is as follows: In section 2, some useful results are given as preliminaries; In section 3, the higher order Melnikov functions are computed; In section 4, the main results are given; In section 5, a brief discussion is shown.

2. Preliminaries

Lemma 2.1 ([1]). *By using the Hamiltonnian function, it is easy to obtain*

$$\begin{aligned} &\oint_{\delta(h)} P(x)y^{2k}dx = 0, k = 0,1,2,3,\cdots, \\ &\oint_{\delta(h)} P(x)y^{2k+1}dy = 0, k = 0,1,2,3,\cdots, \\ &\frac{k+6}{6}aJ_{k+3} = \frac{4k+18}{9}J_{k+2} - \frac{k+3}{3}J_{k+1} + \frac{2k}{3}hJ_{k-1}, k = 0,1,2,\cdots. \end{aligned} \tag{2.1}$$

Define

$$\delta_{kj} = x^k y^j dy, \quad \omega_{kj} = x^k y^j dx.$$

Lemma 2.2. *For j is even, $x^k y^j dx$ can be written as $dQ + qdH$, $x^k y^j dx$ as $dQ + qdx + \bar{q}dH$ for j is odd, where Q, q and $\bar{q}$ are some polynomials of x and y. Concretely, we have the following formulas,*

$$\begin{aligned} x^k y^2 dx =& d\left(\frac{2Hx^{k+1}}{k+1} - \frac{x^{k+3}}{k+3} + \frac{4x^{k+4}}{3(k+4)} - \frac{ax^{k+5}}{2(k+5)}\right) - \frac{2x^{k+1}dH}{k+1}, \\ x^k y^4 dx =& d\left(\frac{a^2x^{k+9}}{4(k+9)} - \frac{4ax^{k+8}}{3(k+8)} + \frac{x^{k+7}}{k+7}\left(a+\frac{16}{9}\right) - \frac{8x^{k+6}}{3(k+6)} + \frac{x^{k+5}}{k+5}\right. \\ &\left. + \left(-\frac{2ax^{k+5}}{k+5} + \frac{16x^{k+4}}{3(k+4)} - \frac{4x^{k+3}}{k+3}\right)H + \frac{4x^{k+1}H^2}{k+1}\right) \\ &+ \left(\frac{2ax^{k+5}}{k+5} - \frac{16x^{k+4}}{3(k+4)} + \frac{4x^{k+3}}{k+3} - \frac{8Hx^{k+1}}{k+1}\right)dH, \\ x^k y^6 dx =& \left[-\frac{24H^2x^{k+1}}{k+1} + \frac{12aHx^{k+5}}{k+5} - \frac{32Hx^{k+4}}{k+4} + \frac{24Hx^{k+3}}{k+3} - \frac{3a^2x^{k+9}}{2(k+9)}\right. \\ &\left. + \frac{8ax^{k+8}}{k+8} - \frac{x^{k+7}}{k+7}\left(6a+\frac{32}{3}\right) + 16\frac{x^{k+6}}{k+6} - 6\frac{x^{k+5}}{k+5}\right]dH + dQ_{k,6}(x,H), \\ x^k y^8 dx =& \left[-64\,\frac{x^{k+1}H^3}{k+1} - 3\left(\frac{128\,x^{k+4}}{3\,k+12} - \frac{32x^{k+3}}{k+3} - \frac{16x^{k+5}a}{k+5}\right)H^2 - 2\left(\frac{6a^2x^{k+9}}{k+9}\right.\right. \\ &\left. - \frac{32ax^{k+8}}{k+8} + \frac{8\,(9\,a+16)\,x^{k+7}}{3(k+7)} - \frac{64x^{k+6}}{k+6} + \frac{24x^{k+5}}{k+5}\right)H - \frac{8x^{k+12}a^2}{k+12} \\ &+ \frac{x^{k+13}a^3}{k+13} + \frac{2x^{k+11}a\,(9\,a+32)}{3(k+11)} - \frac{32x^{k+10}\,(27a+16)}{27\,k+270} + \frac{4x^{k+9}\,(9a+32)}{3(k+9)} \\ &\left. - \frac{32x^{k+8}}{k+8} + \frac{8x^{k+7}}{k+7}\right]dH + dQ_{k,8}(x,H), \\ x^k y^j dy =& d\left(\frac{y^{j+1}x^k}{j+1}\right) - \frac{k}{j+1}x^{k-1}y^{j+1}dx. \end{aligned}$$

For j odd and $k \geq 3$,

$$x^k y^j dx = \frac{1}{a}\left[\left(\frac{(k-3)\,y^{j+2}x^{k-4}}{j+2} + x^{k-2}y^j\,(2x-1)\right)dx + x^{k-3}y^j\,dH - d\left(\frac{x^{k-3}y^{j+2}}{j+2}\right)\right]$$

$$\Longrightarrow \omega_{k-4,j+2}=\frac{j+2}{k-3}\left[y^j\left(ax^k-2x^{k-1}+x^{k-2}\right)dx-x^{k-3}y^j\,dH+d\left(\frac{x^{k-3}y^{j+2}}{j+2}\right)\right].$$

Moreover,

$$J_{i-4,j+2}=\frac{\left(aJ_{i,j}+J_{i-2,j}-2J_{i-1,j}\right)(j+2)}{i-3}. \tag{2.2}$$

Proof. For j even, by substituting $y^2=2H-x^2+\frac{4}{3}x^3-\frac{1}{2}ax^4$ into $x^ky^j\,dx$, and using the integral by parts, we can compute the expressions of $x^ky^j\,dx$.

$$\begin{aligned}
&x^ky^j\,dy=x^kd\left(\frac{y^{j+1}}{j+1}\right)=d\left(\frac{x^ky^{j+1}}{j+1}\right)-\frac{k}{j+1}x^{k-1}y^{j+1}dx,\\
&\quad x^ky^j\,dx\\
&=x^{k-3}y^j\,d\left(\frac{x^4}{4}\right)=\frac{1}{a}x^{k-3}y^j\,d\left(H-\frac{1}{2}y^2-\frac{1}{2}x^2+\frac{2}{3}x^3\right)\\
&=\frac{1}{a}x^{k-3}y^j\left(dH-ydy+x(2x-1)dx\right)\\
&=\frac{1}{a}\left[x^{k-3}y^j\,dH-x^{k-3}d\left(\frac{y^{j+2}}{j+2}\right)+x^{k-2}y^j(2x-1)dx\right]\\
&=\frac{1}{a}\left[x^{k-3}y^j\,dH-d\left(\frac{x^{k-3}y^{j+2}}{j+2}\right)+\frac{y^{j+2}}{j+2}(k-3)x^{k-4}dx+x^{k-2}y^j(2x-1)dx\right].
\end{aligned}$$

Solving $x^ky^j\,dx$ about $x^{k-4}y^{j+2}dx$, we can calculate the formula of $\omega_{k-4,j+2}$. By integrating it on the interval $\delta(h)$, (2.2) is obtained. The proof is completed. □

Furthermore, by the third equation of (2.1) and (2.2), we have the following recursion formulas.

Lemma 2.3.

$$\begin{aligned}
J_3=&\frac{2J_2-J_1}{a},\ J_4=-\frac{(24a-88)\,J_2}{21a^2}-\frac{44J_1}{21a^2}+\frac{4hJ_0}{7a},\\
J_5=&\frac{26hJ_0}{21a^2}+\frac{\left(252ha^2+315a-1144\right)J_1}{252a^3}-\frac{(627a-1144)\,J_2}{126a^3},\\
J_6=&\left(\frac{4h}{3a}+\frac{32}{21a^2}-\frac{3146}{189a^3}+\frac{11440}{567a^4}\right)J_2+\left(\frac{20h}{9a^2}+\frac{39}{7a^3}-\frac{5720}{567a^4}\right)J_1\\
&+\left(\frac{520}{189a^3}-\frac{16}{21a^2}\right)hJ_0,\\
J_7=&\left(\frac{56h}{9a^2}+\frac{1313}{126a^3}-\frac{28600}{567a^4}+\frac{77792}{1701a^5}\right)J_2+\left(\left(\frac{136}{27a^3}-\frac{3}{a^2}\right)h-\frac{7}{4a^3}+\frac{1196}{63a^4}\right.\\
&\left.-\frac{38896}{1701a^5}\right)J_1+\left(\frac{3536}{567a^4}-\frac{218}{63a^3}\right)hJ_0,\\
J_8=&\left(\left(\frac{45632}{2079a^3}-\frac{928}{231a^2}\right)h-\frac{512}{231a^3}+\frac{33410}{693a^4}-\frac{35360}{243a^5}+\frac{537472}{5103a^6}\right)J_2\\
&+\left(\left(\frac{10336}{891a^4}-\frac{9668}{693a^3}\right)h-\frac{2803}{231a^4}+\frac{364208}{6237a^5}-\frac{268736}{5103a^6}\right)J_1\\
&+\left(\frac{80h}{77a^2}+\frac{256}{231a^3}-\frac{8296}{693a^4}+\frac{268736}{18711a^5}\right)hJ_0,
\end{aligned}$$

$$
\begin{aligned}
J_9 =& \left(\left(\frac{432680}{6237a^4} - \frac{19861}{693a^3} \right) h - \frac{57665}{2772a^4} + \frac{130390}{693a^5} - \frac{2082704}{5103a^6} + \frac{537472}{2187a^7} \right) J_2 \\
& + \left(\frac{2h^2}{a^2} + \left(\frac{7}{a^3} - \frac{102260}{2079a^4} + \frac{72352}{2673a^5} \right) h + \frac{21}{8a^4} - \frac{39355}{693a^5} + \frac{3191240}{18711a^6} \right. \\
& \left. - \frac{268736}{2187a^7} \right) J_1 + \left(\frac{1132h}{231a^3} + \frac{5389}{693a^4} - \frac{25840}{693a^5} + \frac{268736}{8019a^6} \right) hJ_0, \\
J_{10} =& \left(\frac{112h^2}{39a^2} + \left(\frac{9472}{1001a^3} - \frac{3708820}{27027a^4} + \frac{50377120}{243243a^5} \right) h + \frac{10240}{3003a^4} - \frac{3330895}{27027a^5} \right. \\
& \left. + \frac{12493640}{18711a^6} - \frac{2459968}{2187a^7} + \frac{3803648}{6561a^8} \right) J_2 + \left(\frac{1112h^2}{117a^3} \right. \\
& + \left(\frac{450232}{9009a^4} - \frac{12616720}{81081a^5} + \frac{6656384}{104247a^6} \right) h + \frac{149311}{6006a^5} - \frac{18146140}{81081a^6} + \frac{1292000}{2673a^7} \\
& \left. - \frac{1901824}{6561a^8} \right) J_1 + \left(\left(\frac{472592}{27027a^4} - \frac{9728}{3003a^3} \right) h - \frac{5120}{3003a^4} + \frac{993548}{27027a^5} - \frac{3824320}{34749a^6} \right. \\
& \left. + \frac{1901824}{24057a^7} \right) hJ_0, \\
J_{11} =& \left(\frac{2464h^2}{117a^3} + \left(\frac{823285}{9009a^4} - \frac{44678720}{81081a^5} + \frac{436117184}{729729a^6} \right) h + \frac{490195}{12012a^5} - \frac{47765240}{81081a^6} \right. \\
& \left. + \frac{5963872}{2673a^7} - \frac{140734976}{45927a^8} + \frac{190182400}{137781a^9} \right) J_2 + \left(\left(\frac{11984}{351a^4} - \frac{10}{a^3} \right) h^2 + \left(\frac{239916}{1001a^5} \right. \right. \\
& \left. - \frac{15}{a^4} - \frac{37726400}{81081a^6} + \frac{47545600}{312741a^7} \right) h - \frac{33}{8a^5} + \frac{148580}{1001a^6} - \frac{9276560}{11583a^7} + \frac{32331008}{24057a^8} \\
& \left. - \frac{95091200}{137781a^9} \right) J_1 + \left(\left(\frac{4531424}{81081a^5} - \frac{210116}{9009a^4} \right) h - \frac{48887}{3003a^5} + \frac{1692520}{11583a^6} - \frac{10935488}{34749a^7} \right. \\
& \left. + \frac{95091200}{505197a^8} \right) hJ_0,
\end{aligned}
$$

$$
\begin{aligned}
J_{0,3} =& 3(aJ_4 + J_2 - 2J_3), \quad J_{1,3} = \frac{3}{2}(aJ_5 + J_3 - 2J_4), \quad J_{2,3} = aJ_6 + J_4 - 2J_5, \\
J_{3,3} =& \frac{3}{4}(aJ_7 + J_5 - 2J_6), \; J_{4,3} = \frac{3}{5}(aJ_8 + J_6 - 2J_7), \; J_{5,3} = \frac{1}{2}(aJ_9 + J_7 - 2J_8), \\
J_{6,3} =& \frac{3}{7}(aJ_{10} + J_8 - 2J_9), \; J_{7,3} = \frac{3}{8}(aJ_{11} + J_9 - 2J_{10}), \; J_{8,3} = \frac{1}{3}(aJ_{12} + J_{10} - 2J_{11}), \\
J_{0,5} =& 5(aJ_{4,3} + J_{2,3} - 2J_{3,3}), \quad J_{1,5} = \frac{5}{2}(aJ_{5,3} + J_{3,3} - 2J_{4,3}), \\
J_{2,5} =& \frac{5}{3}(aJ_{6,3} + J_{4,3} - 2J_{5,3}), \\
J_{3,5} =& \frac{5}{4}(aJ_{7,3} + J_{5,3} - 2J_{6,3}), \quad J_{4,5} = aJ_{8,3} + J_{6,3} - 2J_{7,3}.
\end{aligned}
$$

3. Calculation of the coefficients $M_k(h)$

3.1. Calculation of $M_1(h)$

Lemma 3.1. (i) *The function $M_1(h)$ has the form*

$$
M_1(h) = \alpha_1 J_0 + \beta_1 J_1 + \gamma_1 J_2, \tag{3.1}
$$

where α_1 and β_1 are all polynomials of degree one of h, γ_1 is constant coefficient.
(ii) *If $M_1(h) = 0$, then there exists $\Omega_1 = q_1 dH + dQ_1$, where Q_1 and q_1 are polynomials.*

Proof.

$$\Omega_1 = \omega_1 = g_1 dx - f_1 dy = \sum_{i+j=4} (g_{ij} x^i y^j dx - f_{ij} x^i y^j dy).$$

By using the formulas (2.1) and Lemma 2.2, we have

$$\begin{aligned} M_1(h) = \oint_{\delta(h)} \Omega_1 =& a\left(f_{22} + \frac{3}{2} g_{13}\right) J_5 - (2f_{22} + 3g_{13}) J_4 \\ &+ \left(f_{22} + 4f_{40} + \frac{3}{2} g_{13} + g_{31}\right) J_3. \end{aligned}$$

Together with Lemma 2.3, $M_1(h)$ has the form (3.1), with

$$\begin{aligned} \alpha_1 &= \frac{h\,(2f_{22} + 3g_{13})}{21a}, \\ \beta_1 &= \frac{1}{2}(2f_{22} + 3g_{13})\,h + \frac{(63a - 88)(2f_{22} + 3g_{13})}{504a^2} - \frac{4f_{40} + g_{31}}{a}, \\ \gamma_1 &= \frac{2(4f_{40} + g_{31})}{a} - \frac{(87a - 88)(2f_{22} + 3g_{13})}{252a^2}. \end{aligned}$$

Then $M_1(h) \equiv 0 \iff$

$$g_{13} = -\frac{2}{3} f_{22}, g_{31} = -4f_{40}, \tag{3.2}$$

at the same time, $\Omega_1 = q_1 dH + dQ_1$ with

$$\begin{aligned} q_1 =& -\frac{1}{30} x\,(f_{13} + 4g_{04})\left(-3ax^4 + 10x^3 - 10x^2 + 60H\right) - \frac{1}{3} x^3 (2g_{22} + 3f_{31}), \\ dQ_1 =& \; x\,(f_{13} + 4g_{04})\,H^2 + x^3 \left(\frac{1}{3}(2g_{22} + 3f_{31}) - \frac{1}{30}(f_{13} + 4g_{04})\left(3ax^2 - 10x + 10\right)\right) H \\ &+ \frac{x^8 a\,(f_{13} + 4g_{04})(ax - 6)}{144} + \left(\frac{(f_{13} + 4g_{04})(9a + 16)}{252} - \frac{1}{28} a\,(2g_{22} + 3f_{31})\right) x^7 \\ &- \frac{1}{9}(f_{13} - 3f_{31} + 4g_{04} - 2g_{22})\,x^6 + \frac{1}{20}(4g_{40} + f_{13} - 6f_{31} + 4g_{04} - 4g_{22})\,x^5 \\ &- yx^4 f_{40} - \frac{1}{2} x^3 y^2 f_{31} - \frac{1}{3} x^2 y^3 f_{22} - \frac{1}{4} x y^4 f_{13} - \frac{1}{5} y^5 f_{04}. \end{aligned}$$

□

3.2. Calculation of $M_2(h)$

Lemma 3.2. (i) *If $M_1(h) \equiv 0$, then the function $M_2(h)$ has the form*

$$M_2(h) = \alpha_2 J_0 + \beta_2 J_1 + \gamma_2 J_2, \tag{3.3}$$

where β_2 and γ_2 are all polynomials of degree two of h, α_2 is polynomials of degree threee of h.
(ii) *If $M_1(h) = M_2(h) = 0$, then there exists $\Omega_2 = q_2 dH + dQ_2$ under the* **Case(a)** *or* **Case(b)**, *where Q_2 and q_2 are all polynomials forms.*

Proof. By Theorem 1.1 and following the formulas (2.1) and Lemma 2.2,

$$\begin{aligned}
M_2(h) &= \oint_{\delta(h)} \Omega_2 = \oint_{\delta(h)} (\omega_2 + q_1\omega_1) \\
&= \frac{(f_{13}+4g_{04})}{90}\left[(45aJ_8 - 180hJ_4 - 120J_7)\, f_{40} + (15aJ_{6,3} - 60hJ_{2,3} - 40J_{5,3})\, f_{22}\right. \\
&\quad \left. + (9aJ_{4,5} - 36hJ_{0,5} - 24J_{3,5})\, f_{04}\right] + (g_{21}+3f_{30})J_2 + \frac{1}{3}J_{0,3}\,(3g_{03}+f_{12}) \\
&\quad + \frac{1}{15}\,(f_{13}+4g_{04}-2g_{22}-3f_{31})\,(15J_6 f_{40} + 3J_{2,5} f_{04} + 5J_{4,3} f_{22})\,.
\end{aligned}$$

By using the formulas in Lemma 2.3, $M_2(h)$ has the form (3.3), with

$$\begin{aligned}
\alpha_2 = &-\frac{64 f_{04}\,(f_{13}+4g_{04})}{55}h^3 + \Bigg[(3f_{31}+2g_{22})\left(\left(\frac{320}{3003a} - \frac{4244}{27027a^2}\right) f_{04} - \frac{16 f_{22}}{77a}\right) \\
&+ \left(\left(\frac{38}{231a} - \frac{5854}{27027a^2}\right) f_{22} + \left(-\frac{3578}{45045a} + \frac{20968}{81081a^2} - \frac{179912}{1216215a^3}\right) f_{04} - \frac{48 f_{40}}{77a}\right) \\
&\times (f_{13}+4g_{04})\Bigg] h^2 + \Bigg[(3f_{31}+2g_{22}) \\
&\times \left(\frac{(144a-520)\, f_{40}}{189a^3} + \left(\frac{16}{1001a^2} - \frac{131}{819a^3} + \frac{65552}{243243a^4} - \frac{20672}{168399a^5}\right) f_{04}\right. \\
&\left. + \left(-\frac{16}{231a^2} + \frac{956}{2079a^3} - \frac{7072}{18711a^4}\right) f_{22}\right) + \frac{4}{7} f_{12} + \frac{12 g_{03}}{7} + (f_{13}+4g_{04}) \\
&\times \left(\left(\frac{956}{693a^3} - \frac{16}{77a^2} - \frac{7072}{6237a^4}\right) f_{40} + \left(\frac{16}{693a^2} - \frac{2456}{11583a^3} + \frac{21562}{66339a^4} - \frac{880624}{6567561a^5}\right) f_{22}\right. \\
&\left. + \left(\frac{14579}{162162a^3} - \frac{304}{45045a^2} - \frac{23354}{104247a^4} + \frac{48008}{243243a^5} - \frac{5672384}{98513415a^6}\right) f_{04}\right)\Bigg] h, \\
\beta_2 = &\left[(f_{13}+4g_{04})\left(\left(\frac{1064}{1755a} - \frac{6724}{57915a^2}\right) f_{04} - \frac{215 f_{22}}{1287a}\right) - \frac{14 f_{04}\,(3f_{31}+2g_{22})}{117a}\right] h^2 \\
&+ \Bigg[(4g_{04}+f_{13})\left(\frac{(7146a-1904)\, f_{40}}{2079a^3} + \left(\frac{258121}{243243a^3} - \frac{106151}{108108a^2} - \frac{26344}{243243a^4}\right) f_{22}\right. \\
&\left. + \left(\frac{61753}{108108a^2} - \frac{1551428}{1216215a^3} + \frac{7702852}{10945935a^4} - \frac{305440}{6567561a^5}\right) f_{04}\right) \\
&+ \left(\left(\frac{794}{693a^2} - \frac{272}{891a^3}\right) f_{22} + \left(-\frac{668}{1001a^2} + \frac{64660}{81081a^3} - \frac{10336}{104247a^4}\right) f_{04} - \frac{20 f_{40}}{9a^2}\right) \\
&\times (3f_{31}+2g_{22})\Bigg] h + \left(\left(\frac{283}{154a^3} - \frac{12896}{2079a^4} + \frac{7072}{1701a^5}\right) f_{40}\right. \\
&+ \left(\frac{148465}{138996a^4} - \frac{2060}{9009a^3} - \frac{2905366}{2189187a^5} + \frac{9686840}{19702683a^6}\right) f_{22} \\
&\left. + \left(\frac{18569}{240240a^3} - \frac{2479033}{4864860a^4} + \frac{10996207}{10945935a^5} - \frac{25672928}{32837805a^6} + \frac{62396128}{295540245a^7}\right) f_{04}\right) \\
&\times (4g_{04}+f_{13}) + \left(\left(\frac{5720}{567a^4} - \frac{39}{7a^3}\right) f_{40} + \left(\frac{283}{462a^3} - \frac{12896}{6237a^4} + \frac{7072}{5103a^5}\right) f_{22}\right. \\
&\left. + \left(\frac{22277}{27027a^4} - \frac{3935}{24024a^3} - \frac{2312}{2079a^5} + \frac{20672}{45927a^6}\right) f_{04}\right) (2g_{22}+3f_{31}) - \frac{2(f_{12}+3g_{03})}{21a},
\end{aligned}$$

$$\gamma_2 = \left[\left(\left(\frac{4538}{6435} - \frac{18152}{19305a}\right) f_{04} - \frac{2}{3} f_{22}\right)(f_{13} + 4g_{04}) - \frac{16 f_{04}(3f_{31} + 2g_{22})}{39}\right] h^2$$
$$+ \left[\left(\left(\frac{24}{77a} - \frac{3952}{693a^2}\right) f_{40} + \left(\frac{28541}{12474a^2} - \frac{256}{693a} - \frac{418928}{243243a^3}\right) f_{22}\right.\right.$$
$$\left. + \left(\frac{7688}{45045a} - \frac{10240}{6237a^2} + \frac{9630122}{3648645a^3} - \frac{961792}{841995a^4}\right) f_{04}\right)(4g_{04} + f_{13})$$
$$+ \left(\left(\frac{124}{231a} - \frac{3952}{2079a^2}\right) f_{22} + \left(\frac{42649}{27027a^2} - \frac{724}{3003a} - \frac{309896}{243243a^3}\right) f_{04} - \frac{4f_{40}}{3a}\right)$$
$$\left.\times (3f_{31} + 2g_{22})\right] h + \left[\left(\frac{32}{77a^2} - \frac{637}{99a^3} + \frac{8320}{567a^4} - \frac{14144}{1701a^5}\right) f_{40}\right.$$
$$+ \left(\frac{71464}{81081a^3} - \frac{32}{693a^2} - \frac{4066493}{1459458a^4} + \frac{2741920}{938223a^5} - \frac{19373680}{19702683a^6}\right) f_{22}$$
$$+ \left(\frac{608}{45045a^2} - \frac{98593}{294840a^3} + \frac{10706659}{7297290a^4} - \frac{78939398}{32837805a^5} + \frac{11025488}{6567561a^6}\right.$$
$$\left.\left. - \frac{124792256}{295540245a^7}\right) f_{04}\right](4g_{04} + f_{13}) + \left[\left(\frac{3146}{189a^3} - \frac{32}{21a^2} - \frac{11440}{567a^4}\right) f_{40}\right.$$
$$+ \left(\frac{32}{231a^2} - \frac{637}{297a^3} + \frac{8320}{1701a^4} - \frac{14144}{5103a^5}\right) f_{22}$$
$$\left. + \left(\frac{23333}{36036a^3} - \frac{32}{1001a^2} - \frac{40930}{18711a^4} + \frac{37808}{15309a^5} - \frac{41344}{45927a^6}\right) f_{04}\right](2g_{22} + 3f_{31})$$
$$+ g_{21} + 3f_{30} - \frac{(3a-4)(3g_{03} + f_{12})}{21a}.$$

$M_2(h) \equiv 0 \iff Case(a)$ or $Case(b)$, with

Case(a) : $f_{04} = f_{22} = f_{40} = 0, g_{04} = -\frac{1}{4} f_{13}, g_{03} = -\frac{1}{3} f_{12}, g_{21} = -3f_{30};$

$$\Longrightarrow q_2 = -\frac{1}{4} f_{13}(3f_{31} + 2g_{22})\left(\frac{1}{4}ax^8 - \frac{16x^7}{21} + \frac{2}{3}x^6 - 2Hx^4\right)$$
$$+ \frac{1}{9}(3f_{31} + 2g_{22})(3f_{31} + g_{22})x^6 - (f_{21} + g_{12})x^2,$$

Case(b) : $f_{04} = f_{22} = 0, g_{04} = -\frac{1}{4} f_{13}, g_{22} = -\frac{3}{2} f_{31}, g_{03} = -\frac{1}{3} f_{12}, g_{21} = -3f_{30}$

$$\Longrightarrow q_2 = -x^2(f_{21} + g_{12}).$$

□

Note that by computation, we find that the conditions of higher order Melnikov functions of **Case (b)** are similar as them of **Case(a)**, we will not show them in this paper.

3.3. Calculation of $M_3(h)$

Lemma 3.3. (i) *If $M_1(h) = M_2(h) \equiv 0$, then the function $M_3(h)$ has the form*

$$M_3(h) = \alpha_3 J_0 + \beta_3 J_1 + \gamma_3 J_2, \tag{3.4}$$

where α_3, β_3 and γ_3 are all polynomials of degree one of h.
(ii) *If $M_1(h) = M_2(h) = M_3(h) \equiv 0$, then $\Omega_3 = q_3 dH + dQ_3$, where Q_3 and q_3 are polynomials.*

Proof. **Case (a):** It follows from (2.1) and Lemma 2.2,

$$\begin{aligned}M_3(h) &= \oint_{\delta(h)} \Omega_3 = \oint_{\delta(h)} \omega_3 + q_1\omega_2 + q_2\omega_1 \\ &= \oint_{\delta(h)} \left[-\frac{1}{3}\left(3f_{31}+2g_{22}\right) y^3 f_{12} x^3 + \left(-\left(3f_{31}+2g_{22}\right) f_{30} x^5 + (2f_{20}+g_{11})x\right) y\right] dx \\ &= -\frac{1}{3}\left(3f_{31}+2g_{22}\right)\left(3J_5 f_{30} + J_{3,3} f_{12}\right) + \left(2f_{20}+g_{11}\right) J_1.\end{aligned}$$

By Lemma 2.3, $M_3(h)$ has the form (3.4), where

$$\begin{aligned}\alpha_3 =& \frac{\left(3f_{31}+2g_{22}\right) h \left(99 f_{12} a - 702 f_{30} a - 104 f_{12}\right)}{567 a^3}, \\ \beta_3 =& \left[\frac{\left(3f_{31}+2g_{22}\right)\left(27 a f_{12} - 54 a f_{30} - 8 f_{12}\right)}{54 a^2}\right] h \\ &+ \left[\left(\frac{286}{63a^3} - \frac{5}{4a^2}\right) f_{30} + \left(\frac{1}{8a^2} - \frac{52}{63a^3} + \frac{1144}{1701a^4}\right) f_{12}\right] (2g_{22}+3f_{31}) + 2f_{20} + g_{11}, \\ \gamma_3 =& \left[\left(\frac{209}{42a^2} - \frac{572}{63a^3}\right) f_{30} + \left(\frac{1144}{567a^3} - \frac{151}{252a^2} - \frac{2288}{1701a^4}\right) f_{12} - \frac{8 f_{12} h}{9a}\right] \left(3f_{31}+2g_{22}\right).\end{aligned}$$

By $M_3(h)=0$, we can deduce that the following two cases:
Case (a1):

$$\begin{aligned}& g_{22} = -\frac{3}{2} f_{31}, g_{11} = -2 f_{20} \\ \Longrightarrow & q_3 = -\frac{1}{2} f_{13}\left(f_{21}+g_{12}\right)\left(\frac{2}{7} a x^7 - \frac{8x^6}{9} + \frac{4}{5} x^5 - \frac{8}{3} H x^3\right) + \frac{2}{5} f_{31}\left(f_{21}+g_{12}\right) x^5 \\ & \quad - 2\left(\frac{1}{2} f_{11} + g_{02}\right) x,\end{aligned}$$

Case (a2):

$$\begin{aligned}& f_{12} = f_{30} = 0, g_{11} = -2 f_{20} \\ \Longrightarrow & q_3 = -\frac{7 f_{13}^2 \left(3f_{31}+2g_{22}\right) x^5 y^4}{40} + \left[-\frac{1}{30} f_{13}^2 \left(3f_{31}+2g_{22}\right) x^8 (4ax-9)\right. \\ & - \frac{f_{13}\left(3f_{31}+2g_{22}\right)\left(225 f_{31} + 72 f_{13} + 100 g_{22}\right) x^7}{420} \\ & \left. + \left(\frac{2}{3}\left(f_{21}+g_{12}\right) f_{13} + \frac{1}{3} f_{03}\left(3f_{31}+2g_{22}\right)\right) x^3\right] y^2 - \frac{x^{12} f_{13}^2 a \left(3f_{31}+2g_{22}\right)}{2340} \\ & \times (48ax - 221) - \left[\left(\frac{64a}{1155} - \frac{6}{55}\right) f_{13} + \frac{197 a f_{31}}{1232} + \frac{101 a g_{22}}{1848}\right] f_{13}(3f_{31}+2g_{22}) x^{11} \\ & + \frac{f_{13}(3f_{31}+2g_{22})(75 f_{31} + 27 f_{13} + 26 g_{22}) x^{10}}{210} - \left[\frac{4 g_{22}^3}{81} + \left(\frac{4}{9} f_{31} + \frac{1}{7} f_{13}\right) g_{22}^2\right. \\ & - \left(\frac{2}{3} f_{21} - \frac{11 f_{31}^2}{9} - \frac{13 f_{31} f_{13}}{21} - \frac{1}{9} f_{13} g_{40} - \frac{8 f_{13}^2}{105}\right) g_{22} + f_{31}^3 + \frac{17 f_{13} f_{31}^2}{28} \\ & \left. + \left(\frac{1}{6} f_{13} g_{40} + \frac{4 {f_{13}}^2}{35} - 2 f_{21} - g_{12}\right) f_{31}\right] x^9 + \frac{2}{21} a \left(2 f_{13} f_{21} + 2 f_{13} g_{12}\right.\end{aligned}$$

$$+3f_{03}f_{31}+2f_{03}g_{22})\,x^7-\left(\left(\frac{2}{3}f_{31}+\frac{4}{9}g_{22}\right)f_{03}+\frac{4}{9}\left(f_{21}+g_{12}\right)f_{13}\right)x^6$$
$$+\left(\frac{2}{5}(f_{03}+g_{12})\,f_{31}+\left(\frac{4f_{03}}{15}+\frac{2}{5}f_{221}+\frac{2}{3}g_{12}\right)g_{122}+\left(\frac{4f_{21}}{15}+\frac{4g_{12}}{15}\right)f_{13}\right)x^5$$
$$-\left(f_{11}+2g_{02}\right)x$$

by using Lemma 2.2 and substituting $H=\frac{1}{2}y^2+\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}ax^4$ into q_2 of **Case(a)**. □

3.4. Calculation of $M_4(h)$

Lemma 3.4. (i) *If $M_1(h)=M_2(h)=M_3(h)\equiv 0$, then the function $M_4(h)$ has the form*

$$M_4(h)=\alpha_4J_0+\beta_4J_1+\gamma_4J_2, \tag{3.5}$$

where α_4, β_4 and γ_4 are all polynomials of degree one of h.
(ii) *If $M_1(h)=M_2(h)=M_3(h)=M_4(h)\equiv 0$, then $\Omega_4=q_4dH+dQ_4$, where Q_4 and q_4 are polynomials.*

Proof. Case (a1):

$$\begin{aligned}M_4(h)&=\oint_{\delta(h)}\Omega_4\\&=\oint_{\delta(h)}\left[-\frac{2}{3}f_{12}\left(f_{21}+g_{12}\right)x^2y^3+\left(f_{10}+g_{01}-2f_{30}\left(f_{21}+g_{12}\right)x^4\right)y\right]dx\\&=-\frac{2}{3}f_{12}\left(f_{21}+g_{12}\right)J_{2,3}+(f_{10}+g_{01})J_0-2f_{30}\left(f_{21}+g_{12}\right)J_4.\end{aligned}$$

It follows from the formulas in Lemma 2.3, $M_4(h)$ has the form (3.5) with

$$\begin{aligned}\alpha_4&=\left(\left(\frac{8}{63a}-\frac{104}{567a^2}\right)f_{12}-\frac{8f_{30}}{7a}\right)\left(f_{21}+g_{12}\right)h+f_{10}+g_{01},\\\beta_4&=\left[-\frac{4hf_{12}}{27a}+\left(\frac{1144}{1701a^3}-\frac{41}{63a^2}\right)f_{12}+\frac{88f_{30}}{21a^2}\right]\left(f_{21}+g_{12}\right),\\\gamma_4&=\left[-\frac{8hf_{12}}{9}+\left(\frac{16}{7a}-\frac{176}{21a^2}\right)f_{30}+\left(\frac{946}{567a^2}-\frac{16}{63a}-\frac{2288}{1701a^3}\right)f_{12}\right]\left(f_{21}+g_{12}\right),\end{aligned}$$

$M_4(h)=0$ implies
Case(a11):

$$\begin{aligned}&g_{01}=-f_{10},g_{12}=-f_{21}\\\Longrightarrow\ &q_4=\left(f_{11}+2g_{02}\right)x^2\left[\left(H-\frac{1}{12}ax^4+\frac{4x^3}{15}-\frac{1}{4}x^2\right)f_{13}+\frac{1}{4}f_{31}x^2\right];\end{aligned}$$

Case(a12):

$$\begin{aligned}&g_{01}=-f_{10},f_{30}=f_{12}=0\\\Longrightarrow\ &q_4=-\frac{5f_{13}^2\left(g_{12}+f_{21}\right)x^4y^4}{12}\end{aligned}$$

$$
\begin{aligned}
&+\left[-\frac{x^6 f_{13}\left(g_{12}+f_{21}\right)\left(63\,a x^2 f_{13}-144\,x f_{13}+84\,f_{13}+56\,f_{31}\right)}{168}\right.\\
&\left.+\left(f_{03}\left(g_{12}+f_{21}\right)+\frac{1}{2} f_{113}\left(f_{311}+2\,g_{302}\right)\right) x^2\right] y^2\\
&-\frac{x^{11} f_{13}^2 a\left(g_{12}+f_{21}\right)(77 a x-360)}{1232}\\
&-\frac{f_{13}\left(g_{12}+f_{21}\right)\left(49 a f_{13}+56 a f_{31}+96 f_{13}\right) x^{10}}{280}\\
&+\frac{2 f_{13}\left(g_{12}+f_{21}\right)\left(13\, f_{13}+14 f_{31}\right) x^9}{63}\\
&-\frac{1}{24}\left(3 f_{13}^2+6 f_{13} f_{31}+4 f_{13} g_{40}+6 f_{31}^2\right)\left(g_{12}+f_{21}\right) x^8\\
&+\frac{1}{30} x^5\left(f_{13}\left(f_{11}+2 g_{02}\right)+2 f_{03}\left(g_{12}+f_{21}\right)\right)(5 a x-12)\\
&+\left(\frac{1}{4}\left(f_{11}+2 g_{02}\right)\left(f_{13}+f_{31}\right)+\frac{1}{2}\left(g_{12}+f_{21}\right)\left(2\, f_{21}+f_{03}+g_{12}\right)\right) x^4 .
\end{aligned}
$$

Case (a2):

$$
\begin{aligned}
M_4(h) &= \oint_{\delta(h)} \Omega_4\\
&= \oint_{\delta(h)} \left(-\frac{1}{3} f_{02}\left(3 f_{31}+2 g_{22}\right) x^2 y^3+\left(f_{10}+g_{01}-f_{20}\left(3 f_{31}+2 g_{22}\right) x^4\right) y\right) dx\\
&= -\frac{1}{3} f_{02}\left(3 f_{31}+2 g_{22}\right) J_{2,3}+\left(f_{10}+g_{01}\right) J_0-f_{20}\left(3 f_{31}+2 g_{22}\right) J_4,
\end{aligned}
$$

$M_4(h)$ has the form (3.5) with

$$
\begin{aligned}
\alpha_4 &= \left[\left(\frac{4}{63a}-\frac{52}{567a^2}\right) f_{02}-\frac{4 f_{20}}{7a}\right]\left(3 f_{31}+2 g_{22}\right) h+f_{10}+g_{01},\\
\beta_4 &= \left[-\frac{2 f_{02} h}{27a}+\left(\frac{572}{1701a^3}-\frac{41}{126a^2}\right) f_{02}+\frac{44 f_{20}}{21a^2}\right]\left(3 f_{31}+2 g_{22}\right),\\
\gamma_4 &= \left[-\frac{4}{9} f_{02} h+\left(\frac{8}{7a}-\frac{88}{21a^2}\right) f_{20}+\left(-\frac{8}{63a}+\frac{473}{567a^2}-\frac{1144}{1701a^3}\right) f_{02}\right]\left(3 f_{31}+2 g_{22}\right).
\end{aligned}
$$

Following $M_4(h)=0$, we have $g_{01}=-f_{10}, g_{22}=-\frac{3}{2} f_{31}$, we can compute that q_4 is the same as **Case(a12)** or $g_{01}=-f_{10}, f_{02}=f_{20}=0$, since the expression of q_4 is very long, we will not consider it in the following. □

3.5. Calculation of $M_5(h)$

Lemma 3.5. (i) *If $M_1(h)=M_2(h)=M_3(h)=M_4(h)\equiv 0$, then the function $M_5(h)$ has the form*

$$
M_5(h)=\alpha_5 J_0+\beta_5 J_1+\gamma_5 J_2, \tag{3.6}
$$

where α_5, β_5 are all polynomials of degree one of h, γ_5 is constant.
(ii) *If $M_1(h)=\cdots=M_5(h)\equiv 0$, then $\Omega_5=q_5 dH+dQ_5$, where Q_5 and q_5 are polynomials.*

Proof. **Case(a11):**

$$M_5(h) = \oint_{\delta(h)} \Omega_5 = \oint_{\delta(h)} -\frac{1}{3}\left(f_{11}+2g_{02}\right)\left(3yf_{30}x^3+y^3f_{12}x\right)dx$$
$$= -\frac{1}{3}\left(f_{11}+2g_{02}\right)\left(3f_{30}J_3+f_{12}J_{1,3}\right),$$

which implies $M_5(h)$ has the form (3.6), and

$$\alpha_5 = -\frac{\left(f_{11}+2g_{02}\right)hf_{12}}{21a},$$
$$\beta_5 = \left(-\frac{1}{2}f_{12}h+\left(\frac{11}{63a^2}-\frac{1}{8a}\right)f_{12}+\frac{f_{30}}{a}\right)\left(f_{11}+2g_{02}\right),$$
$$\gamma_5 = \left(\left(\frac{29}{84a}-\frac{22}{63a^2}\right)f_{12}-2\frac{f_{30}}{a}\right)\left(f_{11}+2g_{02}\right).$$

$M_5 = 0 \Longrightarrow$
Case(a11-1): $g_{02} = -\frac{1}{2}f_{11} \Longrightarrow q_5 = 0$,
Case(a11-2):

$$f_{212} = f_{230} = 0 \Longrightarrow$$
$$q_5 = x\left(2g_{02}+f_{11}\right)\left[-\frac{1}{4}x^2y^4f_{13}^2+\left(\left(\frac{2}{3}x^5-\frac{2}{7}ax^6-\frac{2}{5}x^4\right)f_{13}^2-\frac{1}{10}x^4f_{13}f_{31}+f_{03}\right)y^2\right.$$
$$-\frac{2ax^9f_{13}^2\left(30ax-143\right)}{1155}-\left(\left(\frac{16a}{105}+\frac{16}{27}\right)f_{13}^2+\frac{2}{15}af_{13}f_{131}\right)x^8$$
$$+\frac{1}{30}f_{13}\left(11f_{13}+9f_{31}\right)x^7-\left(\frac{4f_{13}^2}{35}+\left(\frac{6f_{131}}{35}+\frac{1}{7}g_{40}\right)f_{13}+\frac{1}{7}f_{31}^2\right)x^6$$
$$\left.+\frac{1}{5}x^3f_{03}\left(2ax-5\right)+\frac{1}{3}\left(2f_{03}+f_{21}\right)x^2+\frac{8f_{13}^2}{27}\right].$$

Case(a12):

$$M_5(h) = \oint_{\delta(h)} \Omega_5 = \left(f_{21}+g_{12}\right)\oint_{\delta(h)} \left(x^4f_{20}+x^2y^2f_{02}\right)dy+2yf_{20}x^3\,dx$$
$$= -2(f_{21}+g_{12})\left(\frac{1}{3}J_{1,3}f_{02}+f_{20}J_3\right) = \alpha_5J_0+\beta_5J_1+\gamma_5J_2,$$

where

$$\alpha_5 = -\frac{2hf_{02}}{21a}(f_{21}+g_{12}), \quad \beta_5 = \left[-hf_{02}+\left(\frac{22}{63a^2}-\frac{1}{4a}\right)f_{02}+\frac{2f_{20}}{a}\right](f_{21}+g_{12}),$$
$$\gamma_5 = \left[\left(\frac{29}{42a}-\frac{44}{63a^2}\right)f_{02}-\frac{4f_{20}}{a}\right](f_{21}+g_{12}),$$

Clearly,

$$M_5(h) = 0 \Longrightarrow \mathbf{Case(a12-1)}: f_{02} = f_{20} = 0,$$

(since the expression of q_5 is too long, we will not show it here) or $g_{12} = -f_{21}$, this case is the same as **Case(a11 − 2)**. □

3.6. Calculation of $M_6(h)$ and $M_7(h)$

Case (a11-2):

$$M_6(h) = \oint_{\delta(h)} \Omega_6 = -\frac{1}{3}\left(f_{11} + 2\,g_{02}\right)\left(3\,f_{20}J_2 + f_{02}J_{0,3}\right) = \alpha_6 J_0 + \beta_6 J_1 + \gamma_6 J_2,$$

with

$$\alpha_6 = -\frac{4}{7}\left(f_{11} + 2\,g_{02}\right) h f_{02}, \quad \beta_6 = \frac{2\left(f_{11} + 2\,g_{02}\right) f_{02}}{21a},$$
$$\gamma_6 = \frac{\left(f_{11} + 2\,g_{02}\right)\left(3\,f_{02}a - 21\,f_{20}a - 4\,f_{02}\right)}{21a}.$$

$M_6 = 0$ implies $g_{02} = -\frac{1}{2}f_{11} \Longrightarrow q_6 = 0$, system (1.5) is integrable or $f_{02} = f_{20} = 0$, since q_6 is too long, we will not consider it in the rest part.

Case (a12-1):

$$M_6(h) = \oint \Omega_6 = -2\,f_{10}\left(f_{21} + g_{12}\right) J_2,$$

by $M_6(h) = 0$ we have or $f_{10} = 0$ (the expression of q_7 is too long, we do not consider this case) or $g_{12} = -f_{21} \Longrightarrow M_7(h) = -f_{10}\left(f_{11} + 2\,g_{02}\right) J_1$, if $g_{02} = -f_{11}$ we deduce that $q_7 = 0$ which implies that system (1.5) is integrable.

4. Main results

Theorem 4.1. *System* (1.5) *is integrable if* $g_{13} = g_{31} = f_{04} = f_{22} = f_{40} = 0$, $g_{04} = -\frac{1}{4}f_{13}$, $g_{03} = -\frac{1}{3}f_{12}$, $g_{21} = -3f_{30}$, $g_{22} = -\frac{3}{2}f_{31}$, $g_{11} = -2f_{20}$, $g_{01} = -f_{10}$, $g_{12} = -f_{21}$, $g_{02} = -\frac{1}{2}f_{11}$.

Proof. It follows from the expressions of $M_i(h)$, $i = 1, 2, \cdots, 7$ in Section 3.1-3.6, we achieve this conclusion. On the other hand, under these condition, system (1.5) becomes

$$Z_\epsilon : \begin{cases} \dot{x} = y + \epsilon^4\left(f_{10}x + f_{0,1}y\right) + \epsilon^3\left(f_{20}x^2 + f_{11}xy + f_{02}y^2\right) \\ \qquad + \epsilon^2\left(f_{30}x^3 + f_{21}x^2y + f_{12}xy^2 + f_{03}y^3\right) + \epsilon\left(f_{31}x^3y + f_{13}xy^3\right) \\ \quad := F_1, \\ \dot{y} = -x + 2x^2 - ax^3 + \epsilon^4\left(xg_{10} - yf_{10}\right) + \epsilon^3\left(g_{20}x^2 - 2xyf_{20} - \frac{1}{2}y^2f_{11}\right) \\ \qquad + \epsilon^2\left(g_{30}x^3 - 3x^2yf_{30} - xy^2f_{21} - \frac{1}{3}y^3f_{12}\right) + \epsilon\left(g_{40}x^4 - \frac{3}{2}x^2y^2f_{31} - \frac{1}{4}y^4f_{13}\right) \\ \quad := G_1, \end{cases}$$

since $\frac{\partial G_1}{\partial y} + \frac{\partial F_1}{\partial x} = 0$, we can obtain the results. □

Theorem 4.2. (i) *At most 11 limit cycles can bifurcate from each one of the annuli inside the loop in the interior eight-loop case.*
(ii) *At most 16 limit cycles can bifurcate from the annulus outside the loop in the exterior eight-loop case.*
(iii) *At most 15 limit cycles can bifurcate from the period annulus in the saddle-loop case.*

Proof. By Theorem 1.2 and the expressions of $M_i(h)$, $i = 1, 2, \cdots, 7$ in Section (3.1)-(3.6), one can see that under the condition (3.2),

$$M_2(h) = \alpha_2 J_0 + \beta_2 J_1 + \gamma_2 J_2,$$

where β_2 and γ_2 are all polynomials of degree two of h, α_2 is polynomials of degree threee of h. But for other $M_i(h)$, the coefficients $\alpha_i(h)$, $\beta_i(h)$, $\gamma_i(h)$ in the expression of $M_i(h)$ be polynomials of degree at most one, hence, we can obtain this conclusion. □

5. Disscussion

For system (1.5), we can see that $M_2(h)$ can bifurcate the maximum number limit cycles, by computating the higher order Melnikov functions, we can obtain the conditions that the system becomes integral. However, if f_i and g_i are homogeneous polynomials of degree i, $i = 1, 2, 3, 4$, there are less number limit cycles, we do not show them in this paper. For cubic perturbation, the authors in [1, 7] obtained the same number limit cycles.

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