Journal of Applied Analysis and Computation
Volume 13, Number 3, June 2023, 1255–1273
Website:http://www.jaac-online.com
DOI:10.11948/20220096

SUMUDU TRANSFORM FOR TIME FRACTIONAL PHYSICAL MODELS AN ANALYTICAL ASPECT

Mamta Kapoor[1,†]

Abstract Present paper deals with the development of a novel and reliable algorithm to solve various time-fractional differential prototypes regarding engineering and physics. The developed algorithm is named as Sumudu Iterative Transform Regime. In present work, proposed regime is applied to tackle different models of importance. The fetched results have shown the efficiency, efficacy and reliability of the developed scheme. In most of the cases, closed form of the solutions is provided. Moreover, profiles of solutions are provided to show the behavior of the fetched results. Error analysis of the results is already notified as well as convergence aspect is also mentioned. On the basis of the discussed aspects, it can be claimed the Sumudu Iterative Transform Regime is a robust technique to deal with the complex natured PDEs. Present scheme will surely add importance in the literature. With the aid of the present regime numerous fractional PDEs and partial-integro differential equations can be tackled.

Keywords Time-fractional fractional differential models, Sumudu Transform, closed form of solution, error analysis, convergence analysis, 2D and 3D plots.

MSC(2010) 41-XX, 42A10, 42A15, 65Dxx.

1. Introduction

1.1. Fractional Calculus

The calculus notion begun in the 17th century, gradually, it began a reliable and efficient tool to tackle various phenomena of importance. Moreover, due to demand of research in this area, researchers and investigators explored some limitations regarding calculus of integer order related to the phenomena like non-Markovian processes, memory-based processes, random walk, Brownian motion and many others. Soon after the traditional calculus, the calculus of integer order got progress. Many Pioneers have taken into account the importance of Fractional Calculus and therefore provided diversified definitions and properties to deal with differential operator and integral operators regarding fractional calculus. There exist numerous theories and applications of fractional calculus, which can relate to many real-world problems and many noticeable physical problems. Many researchers have worked upon different aspects of fractional order and have enriched the literature with the

[†]The corresponding author. Email: mamtakapoor.78@yahoo.com(M. Kapoor)
[1]Department of Mathematics, Lovely Professional University, Phagwara, Punjab, India-144411

updated knowledge. Atagane and Baleanu [4] provided the solution of fractional order Jaulent-Miodek and Whitham-Broer-Kaup equations using Sumudu transform. Ray [17] provided the analytical approximate solution of Whitham-Broer-Kaup (WBK) equation, fractional modified Boussinesq equation and fractional-approximate long wave equation using a novel method. Prakasha et al. [15] gave the novel approach to deal with Schrodinger-Boussinesq equation using the Mittag-Leffler kernel. In their paper, q-HATM was implemented to deal with the mentioned equation. Veeresha et al. [22] provided the analytical-approximated solution to deal with Lakshmanan-Porsezian-Danial model to deal with the analytical solution of the mentioned equation using q-HATM. q-HATM is treated as a fusion of Laplace transform and q-Homotopy analysis method. Loyinmi and Akinfe [8] provided the analytical solution to the Fisher's reaction-diffusion equation using the EHTPM (Elzaki Homotopy transform perturbation method). In their paper, a fusion of Elzaki transform and HPM was implemented to solve the prescribed equation. Cetinkaya et al. [5] provided the solution of space-time fractional equation using Shehu-Variational iteration method. Where a hybrid scheme using the Shehu transform and Variational iteration method was implemented. Akinyemi and Iyioly [3] provided the analytical solution of (3+1)-D fractional RD equation tri molecular models. Shah and Chung [18] gave the analytical solution of fractional Whitham-Broer-Kaup equation using Elzaki Decomposition method, where a hybrid scheme was implemented using Elzaki transform and Adomian Decomposition Method. Some more noteworthy work in this regard is provided in [12,14,16,20,21].

1.2. Sumudu Transform

Definition of Sumudu Transform. Sumudu transform of a function $f(t)$ is defined as follows [23]:

$$f(t) = \int_0^u \frac{1}{u} exp[-\frac{t}{u}] f(t) dt. \tag{1.1}$$

Riemann-Liouville Fractional Integral operator. Riemann-Liouville Fractional Integral operator is as follows [9–11,13]:

$$I^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha+1)} \int_0^t f(\tau)(d\tau)^{\alpha} \tag{1.2}$$

where, Γ is considered as the Gamma function.

Caputo Fractional Derivative. Caputo Fractional Derivative is defined as follows [9–11,13]:

$$D^{\alpha}(f(t)) = I^{m-\alpha} D^m[f(t)] = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\alpha)^{m-\alpha-1} f^m(x) dx \tag{1.3}$$

where, $m-1 < \alpha \leq m$.

Sumudu Transform of Caputo derivative. Sumudu transform of Caputo derivative is defined as follows [13]:

$$S[D_t^{\mu}\zeta(x,t)] = u^{-\mu} S[\zeta(x,t)] - \sum_{r=0}^{m-1} u^{-\mu+r} \zeta^r(0). \tag{1.4}$$

In Table 1, basic properties of Sumudu transform are provided.

Table 1. Basic properties of Sumudu transform.

$f(t)$	$S[f(t)]$
1	1
t	θ
$t^{n-1}/(n-1)!$	θ^{n-1}
e^{at}	$1/(1-a\theta)$
$\sin at/a$	$\theta/(1+a^2\theta^2)$
$\cos(at)$	$1/(1+a^2\theta^2)$
$\sinh(at)/a$	$\theta/(1-a^2\theta^2)$
$\cosh(at)$	$1/(1-a^2\theta^2)$

Linearity property of Sumudu Transform. Let $S[u_1] = V_1$ and $S[u_2] = V_2$, where S is considered as the Sumudu transform.

$$S[\alpha_1 u_1 + \alpha_2 u_2] = \alpha_1 S[u_1] + \alpha_2 S[u_2] = \alpha_1 V_1 + \alpha_2 V_2 = \alpha_1 S[u_1] + \alpha_2 S[u_2]. \quad (1.5)$$

Linearity property of Inverse Sumudu Transform. Let $u_1 = S^{-1}[V_1]$ and $u_2 = S^{-1}[V_2]$

$$S^{-1}[\alpha_1 V_1 + \alpha_2 V_2] = \alpha_1 S^{-1}[V_1] + \alpha_2 S^{-1}[V_2] = \alpha_1 u_1 + \alpha_2 u_2. \quad (1.6)$$

Mittag- Leffler function. Mittag- Leffler function considered for two parameters was given in [6, 7, 19].

$$E_{\mu,\nu}(n) = \sum_{k=0}^{\infty} \frac{n^k}{\Gamma(k\mu + \nu)} \quad (1.7)$$

where, $E_{1,1}(n) = \exp(n)$ and $E_{2,1}(n^2) = \cos(n)$.

Originality of the work. Originality of this paper lies in the development and implementation of efficient regime to solve complex-natured differential equations in different dimensions.

Merits of the proposed method. An iterative scheme is developed in the present research regarding the solution of various fractional equations. The present scheme is easy to implement and needs no complex programming regarding numerical discretization. Developing the numerical programs for the fractional PDEs is not an easy task; therefore, developing such iterative schemes is the need of time to find the approximated-analytical solutions. There are several transforms provided in the literature but from the calculation aspect some transforms are easy to implement and some are not. Sumudu transform is notified as one of the easiest methods to implement integral transform among all existing integral transforms. Due to importance of fractional equations, in this research, concentration is focused upon the solution for the same, which retains the novelty of the study. Moreover, error and convergence analysis are also incorporated in the article.

Limitations of the proposed method. Although, the developed regime is self-efficient to deal with most of the differential equations, but there exist some models which demands a lot of calculation, which is time taking by this approach. This is the only limitation of the proposed scheme.

Outline of the work. In Section 2, the Iterative Sumudu Transform Method is implemented upon the non-linear time fractional PDE. Where the generalized formulae are developed to deal with different examples.

In Section 3, four examples are taken into account to find the analytical-approximate solution of the various fractional PDEs of importance. 2D and 3D plots are also provided for the comparison. In Section 4, concluding remarks are provided as a crux of this research.

2. Implementation of Iterative Sumudu Transform Method [ISTM]

Non-linear Fractional PDE is defined as follows:

$$D_t^\mu[\zeta(x,t)] = [R[\zeta(x,t)] + N[\zeta(x,t)] + \phi(x,t)] \tag{2.1}$$

where, D_t^μ is the derivative in Caputo sense. R is considered as the linear operator and N is considered as the non-linear operator. Applying Sumudu transform:

$$\Rightarrow S[D_t^\mu[\zeta(x,t)]] = S[R[\zeta(x,t)] + N[\zeta(x,t)] + \phi(x,t)] \tag{2.2}$$

$$\Rightarrow u^{-\mu}S[\zeta(x,t)] - \sum_{r=0}^{m-1} u^{-\mu+r}\zeta^r(0) = S[R[\zeta(x,t)] + N[\zeta(x,t)] + \phi(x,t)] \tag{2.3}$$

$$\Rightarrow u^{-\mu}S[\zeta(x,t)] = \sum_{r=0}^{m-1} u^{-\mu+r}\zeta^r(0) + S[R[\zeta(x,t)] + N[\zeta(x,t)] + \phi(x,t)] \tag{2.4}$$

$$\Rightarrow S[\zeta(x,t)] = u^\mu \sum_{r=0}^{m-1} u^{-\mu+r}\zeta^r(0) + u^\mu[S[R[\zeta(x,t)] + N[\zeta(x,t)] + \phi(x,t)]] \tag{2.5}$$

$$\Rightarrow \zeta(x,t) = S^{-1}[u^\mu \sum_{r=0}^{m-1} u^{-\mu+r}\zeta^r(0) + u^\mu[S[R[\zeta(x,t)] + N[\zeta(x,t)] + \phi(x,t)]]] \tag{2.6}$$

where,

$$\begin{aligned} N[\zeta(x,t)] &= N[\sum_{r-0}^{\infty}\zeta_r(x,t)] \\ \Rightarrow N[\zeta(x,t)] &= N[\zeta_0(x,t) + \sum_{r=1}^{\infty}\zeta_r(x,t)] \\ \Rightarrow N[\zeta(x,t)] &= N[\zeta_0(x,t)] + N[\sum_{r=1}^{\infty}\zeta_r(x,t)] \\ \Rightarrow N[\zeta(x,t)] &= N[\zeta_0(x,t)] + \sum_{r=1}^{\infty}[N(\sum_{j=0}^{r}\zeta_j(x,t) - \sum_{j=0}^{r-1}\zeta_j(x,t))] \end{aligned} \tag{2.7}$$

and

$$R[\zeta(x,t)] = \sum_{r=0}^{\infty}\zeta_r(x,t)$$

$$\Rightarrow R[\zeta(x,t)] = R[\zeta_0(x,t) + \sum_{r=1}^{\infty} \zeta_r(x,t)]$$

$$\Rightarrow R[\zeta(x,t)] = R[\zeta_0(x,t)] + R[\sum_{r=1}^{\infty} \zeta_r(x,t)]$$

$$\Rightarrow R[\zeta(x,t)] = R[\zeta_0(x,t)] + \sum_{r=1}^{\infty}[R(\sum_{j=0}^{r} \zeta_j(x,t) - \sum_{j=0}^{r-1} \zeta_j(x,t))]. \tag{2.8}$$

Using Equation (2.7) and Equation (2.8) in Equation (2.6):

$$\begin{aligned}\sum_{k=0}^{\infty} \zeta_k(x,t) = S^{-1}[u^{\mu} \sum_{r=0}^{m-1} u^{r-\mu} \zeta^r(0) + S[\phi(x,t)]] \\ + S^{-1}[u^{\mu} SR[\zeta_0(x,t)] + \sum_{r=1}^{\infty} R[\zeta_r(x,t)] + N[\zeta_0(x,t)] \\ + \sum_{r=1}^{\infty} N(\sum_{j=0}^{r} \phi_j(x,t)) - N(\sum_{j=0}^{r-1} \phi_j(x,t))]\end{aligned} \tag{2.9}$$

$$\begin{aligned}\Rightarrow \sum_{k=0}^{\infty} \zeta_k(x,t) = S^{-1}[u^{\mu} \sum_{r=0}^{m-1} u^{r-\mu} \zeta^r(0) + S(\phi(x,t))] \\ + S^{-1}[u^{\mu} SR[\zeta_0(x,t)] + N[\zeta_0(x,t)] \\ + \sum_{r=1}^{\infty} R(\zeta_r(x,t) + N(\sum_{j=0}^{r} \zeta_j(x,t)) - N(\sum_{j=0}^{r-1} \zeta_j(x,t))).\end{aligned} \tag{2.10}$$

Comparing terms in Equation (2.10):

$$\zeta_0 = S^{-1}[u^{\mu} \sum_{r=0}^{m-1} u^{r-\mu} \zeta^r(0) + u^{\mu} S[\phi(x,t)]], \tag{2.11}$$

$$\zeta_1(x,t) = S^{-1}[u^{\mu} S\{R[\zeta_0(x,t)] + N[\zeta_0(x,t)]\}] \tag{2.12}$$

where, $r = 1, 2, 3, \ldots$

$$\zeta_{r+1}(x,t) = S^{-1}[u^{\mu} S\{R[\zeta_r(x,t)] + N(\sum_{j=0}^{r} \zeta_j(x,t)) - N(\sum_{j=0}^{r-1} \zeta_j(x,t))\}]. \tag{2.13}$$

If $\phi(x,t) = 0$: then

$$\zeta_0 = S^{-1}[u^{\mu} \sum_{r=0}^{m-1} u^{r-\mu} \zeta^r(0)] \tag{2.14}$$

$$\zeta_1(x,t) = S^{-1}[u^{\mu} S\{R[\zeta_0(x,t)] + N[\zeta_0(x,t)]\}], \tag{2.15}$$

where, $r = 1, 2, 3, \ldots$

$$\zeta_{r+1}(x,t) = S^{-1}[u^{\mu} S\{R[\zeta_r(x,t)] + N(\sum_{j=0}^{r} \zeta_j(x,t)) - N(\sum_{j=0}^{r-1} \zeta_j(x,t))\}]. \tag{2.16}$$

3. Applications to Fractional equations

In present section, four examples are elaborated to test the proposed scheme. Closed form of the solution is tried to be fetched. 2D and 3D plots are also provided in some cases. In Plot 1, approx. and exact results are provided at $t = 0.1, 0.2, 0.3$ and 0.4 for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.1$, $\mu = 1$, $N = 50$ regarding Example 3.1. In Table 2, error analysis and convergence aspects are provided at various time levels regarding Example 3.1. In Plot 2, approx. and exact results are provided at $t = 1, 2, 3$ and 4 for ϵ_1=ϵ_2=$\epsilon_3 = 0.1$,$\mu = 1$, $N = 50$ regarding Example 3.1. In Plot 3, 2D plot for approx. and exact results are discussed at $t = 1, 2$, where $\mu = 1$, $\gamma = 0.1$ and $N = 30$ regarding Example 3.4. In Plot 4, 3D plot for approx. and exact results are discussed at $t = 1, 2$, where $\mu = 1$, $\gamma = 0.1$ and $N = 30$ regarding Example 3.4. Via Plot 5 and Plot 6 approx. and exact compatibility of solution profiles is validated at $t = 3$ and $t = 4$ regarding Example 3.4, where mesh, surface and contour plots are provided. In Table 3, error analysis and convergence aspects are validated at various time levels for Example 3.4.

Example 3.1. The generalized time-fractional Burgers-Fisher equation is as follows [2]:

$$D_t^{\mu}[\zeta(x,t)] = \epsilon_1(\zeta(x,t))_{xx} - \epsilon_2(\zeta(x,t))^{\beta}(\zeta(x,t))_x + \epsilon_3\zeta(x,t) - \epsilon_3(\zeta(x,t))^{\beta+1}. \tag{3.1}$$

I.C.:

$$\begin{aligned}
&\zeta(x,0) = e^{((-\epsilon_3/\epsilon_2)x)},\\
&\phi(x,t) = 0,\\
&R[\zeta(x,t)] = \epsilon_1(\zeta(x,t))_{xx} + \epsilon_3(\zeta(x,t)),\\
&N[\zeta(x,t)] = -\epsilon_2(\zeta(x,t))^{\beta}(\zeta(x,t))_x - \epsilon_3(\zeta(x,t))^{\beta+1},\\
&\zeta_0(x,t) = S^{-1}[u^{\mu}\sum_{r=0}^{m-1} u^{r-\mu}(\zeta(x,t))^r(0)].
\end{aligned} \tag{3.2}$$

Considering $m = 1$ in Equation (3.2):

$$\zeta_0(x,t) = S^{-1}[u^{\mu}u^{0-\mu}\zeta(x,0)] \Rightarrow \zeta_0(x,t) = S^{-1}[u^0\zeta(x,0)]$$
$$\Rightarrow \zeta_0(x,t) = S^{-1}[u^0 e^{(-\epsilon_3/\epsilon_2)x}] \Rightarrow \zeta_0(x,t) = e^{(-\epsilon_3/\epsilon_2)x}S^{-1}[u^0]$$
$$\Rightarrow \zeta_0(x,t) = e^{(-\epsilon_3/\epsilon_2)x}S^{-1}[1] \Rightarrow \zeta_0(x,t) = e^{(-\epsilon_3/\epsilon_2)x}, \tag{3.3}$$
$$\zeta_1(x,t) = S^{-1}[u^{\mu}SR[\zeta_0(x,t)] + N[\zeta_0(x,t)]] \tag{3.4}$$

where,

$$R[\zeta_0(x,t)] = \epsilon_1(\zeta_0(x,t))_{xx} + \epsilon_3\zeta_0(x,t)$$

and

$$\begin{aligned}
&N[\zeta_0(x,t)] = -\epsilon_2(\zeta_0)^{\beta}(\zeta_0(x,t))_x - \epsilon_3(\zeta_0(x,t))^{\beta+1},\\
&(\zeta_0(x,t))_x = e^{(-\epsilon_3/\epsilon_2)x}(-\epsilon_3/\epsilon_2),\\
&(\zeta_0(x,t))_{xx} = e^{((-\epsilon_3/\epsilon_2)x)}(\epsilon_3/\epsilon_2)^2 = (\epsilon_3/\epsilon_2)^2\zeta_0(x,t).
\end{aligned}$$

From Equation (3.4):

$$\zeta_1(x,t) = S^{-1}[u^{\mu}S\{\epsilon_1(\zeta_0(x,t))_{xx} + \epsilon_3\zeta_0(x,t)$$

$$
\begin{aligned}
&\quad - \epsilon_2(\zeta_0(x,t))^{\beta}(\zeta_0(x,t))_x - \epsilon_3(\zeta_0(x,t))^{(\beta+1)}\}] && (3.5)\\
&\Rightarrow \zeta_1(x,t) = S^{-1}[u^{\mu} S\{\epsilon_1(\epsilon_3/\epsilon_2)^2 \zeta_0(x,t) + \epsilon_3\zeta_0(x,t)\\
&\quad - \epsilon_2(\zeta_0(x,t))^{\beta}(-\epsilon_3/\epsilon_2)\zeta_0(x,t) - \epsilon_3(\zeta_0(x,t))^{(\beta+1)}\}] && (3.6)\\
&\Rightarrow \zeta_1(x,t) = S^{-1}[u^{\mu} S\{\epsilon_1(\epsilon_3/\epsilon_2)^2 \zeta_0(x,t) + \epsilon_3\zeta_0(x,t) + \epsilon_3(\zeta_0(x,t))^{(\beta+1)}\\
&\quad - \epsilon_3(\zeta_0(x,t))^{(\beta+1)}\}] && (3.7)\\
&\Rightarrow \zeta_1(x,t) = S^{-1}[u^{\mu} S\{\epsilon_1(\epsilon_3/\epsilon_2)^2 \zeta_0(x,t) + \epsilon_3\zeta_0(x,t)\}]\\
&\Rightarrow \zeta_1(x,t) = S^{-1}[u^{\mu} S\{(\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)\zeta_0(x,t)\}]\\
&\Rightarrow \zeta_1(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)S^{-1}[u^{\mu} S\{\zeta_0(x,t)\}]\\
&\Rightarrow \zeta_1(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)S^{-1}[u^{\mu} S[e^{(-\epsilon_3/\epsilon_2)x}]]\\
&\Rightarrow \zeta_1(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)e^{(-\epsilon_3/\epsilon_2)x}S^{-1}[u^{\mu} S[1]]\\
&\Rightarrow \zeta_1(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)e^{(-\epsilon_3/\epsilon_2)x}S^{-1}[u^{\mu}]\\
&\Rightarrow \zeta_1(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)e^{(-\epsilon_3/\epsilon_2)x}\frac{t^{\mu}}{(\Gamma(\mu+1))}\\
&\Rightarrow \zeta_2(x,t) = S^{-1}[u^{\mu} S\{R[\zeta_1(x,t)] + N[\zeta_0(x,t)+\zeta_1(x,t)] - N[\zeta_0(x,t)]\}] && (3.8)
\end{aligned}
$$

where,

$$
\begin{aligned}
R[\zeta_1(x,t)] &= \epsilon_1(\zeta_1(x,t))_{xx} + \epsilon_3\zeta_1(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)\zeta_1(x,t),\\
N[\zeta_0(x,t)+\zeta_1(x,t)] &= -\epsilon_2[\zeta_0(x,t)+\zeta_1(x,t)]^{\beta}[\zeta_0(x,t)+\zeta_1(x,t)]_x\\
&\quad - \epsilon_3[\zeta_0(x,t)+\zeta_1(x,t)]^{\beta+1}\\
&= -(\epsilon_2+\epsilon_3)[\zeta_0(x,t)+\zeta_1(x,t)]^{\beta+1},\\
N[\zeta_0(x,t)] &= -\epsilon_2[\zeta_0(x,t)]^{\beta}[\zeta_0(x,t)]_x - \epsilon_3[\zeta_0](x,t)^{\beta+1} = 0.
\end{aligned}
$$

From Equation (3.8):

$$
\begin{aligned}
&\zeta_2(x,t) = S^{-1}[u^{\mu} S\{R[\zeta_1(x,t)] + N[\zeta_0(x,t)+\zeta_1(x,t)]\}] && (3.9)\\
&\Rightarrow \zeta_2(x,t) = S^{-1}[u^{\mu} S\{(\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)\zeta_1(x,t) - (\epsilon_2+\epsilon_3)(\zeta_0(x,t)+\zeta_1(x,t))^{\beta+1}\}]\\
&\Rightarrow \zeta_2(x,t) = S^{-1}[u^{\mu} S\{(\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)\zeta_1(x,t)\}]\\
&\Rightarrow \zeta_2(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)S^{-1}[u^{\mu} S\{\zeta_1(x,t)\}]\\
&\Rightarrow \zeta_2(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)S^{-1}[u^{\mu} S\{(\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)e^{(-\epsilon_3/\epsilon_2)x} \quad t^{\mu}/(\Gamma(\mu+1))\}]\\
&\Rightarrow \zeta_2(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)(\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)e^{(-\epsilon_3/\epsilon_2)x}S^{-1}[u^{\mu} S\{t^{\mu}/(\Gamma(\mu+1))\}]\\
&\Rightarrow \zeta_2(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^2 e^{(-\epsilon_3/\epsilon_2)x}S^{-1}[u^{\mu} S\{t^{\mu}/(\Gamma(\mu+1))\}]\\
&\Rightarrow \zeta_2(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^2 e^{(-\epsilon_3/\epsilon_2)x}S^{-1}[u^{\mu}u^{\mu}]\\
&\Rightarrow \zeta_2(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^2 e^{(-\epsilon_3/\epsilon_2)x}S^{-1}[u^{2\mu}]\\
&\Rightarrow \zeta_2(x,t) = (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^2 e^{(-\epsilon_3/\epsilon_2)}t^{2\mu}/(\Gamma(2\mu+1)). && (3.10)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\zeta_3(x,t) &= (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^3 e^{(-\epsilon_3/\epsilon_2)}t^{3\mu}/(\Gamma(3\mu+1)) && (3.11)\\
\zeta_m(x,t) &= (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^m e^{(-\epsilon_3/\epsilon_2)}t^{m\mu}/(\Gamma(m\mu+1)) && (3.12)\\
\zeta^{(m)}(x,t) &= \zeta_0(x,t)+\zeta_1(x,t)+\zeta_2(x,t)+\zeta_3(x,t)+\ldots
\end{aligned}
$$

$$\Rightarrow \zeta^m(x,t) = e^{(-\epsilon_3/\epsilon_2)x} + (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)e^{(-\epsilon_3/\epsilon_2)x}t^{\mu}/(\Gamma(\mu+1))$$
$$+ (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^3 e^{(-\epsilon_3/\epsilon_2)} t^{3\mu}/(\Gamma(3\mu+1)) + \dots$$
$$+ (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^m e^{(-\epsilon_3/\epsilon_2)} t^{m\mu}/(\Gamma(m\mu+1))$$
$$\Rightarrow \zeta^m(x,t) = e^{(-\epsilon_3/\epsilon_2)x} \sum_{i=0}^{m} (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^i \quad t^{i\mu}/(\Gamma(i\mu+1))$$

where, $\zeta(x,t) = \lim_{x\to\infty} \zeta^m(x,t)$

$$\Rightarrow \zeta(x,t) = \lim_{n\to\infty} e^{(-\epsilon_3/\epsilon_2)x} \sum_{i=0}^{m} (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^i \quad t^{i\mu}/(\Gamma(i\mu+1))$$
$$\Rightarrow \zeta(x,t) = e^{(-\epsilon_3/\epsilon_2)x} \lim_{n\to\infty} \sum_{i=0}^{m} (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^i \quad t^{i\mu}/(\Gamma(i\mu+1))$$
$$\Rightarrow \zeta(x,t) = e^{(-\epsilon_3/\epsilon_2)x} \sum_{i=0}^{\infty} (\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)^i \quad t^{i\mu}/(\Gamma(i\mu+1))$$
$$\Rightarrow \zeta(x,t) = e^{(-\epsilon_3/\epsilon_2)x} E_{\mu}[(\epsilon_1(\epsilon_3/\epsilon_2)^2 + \epsilon_3)t^{\mu}]. \tag{3.13}$$

Considering $\mu = 1$ in Equation (3.13),

$$\Rightarrow \zeta(x,t) = e^{(-\epsilon_3/\epsilon_2)x} E_1[(\epsilon_1(\epsilon_3/\epsilon_1)^2 + \epsilon_3)t]$$
$$\Rightarrow \zeta(x,t) = e^{((-\epsilon_3/\epsilon_2)x} + ((\epsilon_1\epsilon_3^2)/(\epsilon_2^2) + \epsilon_3)t). \tag{3.14}$$

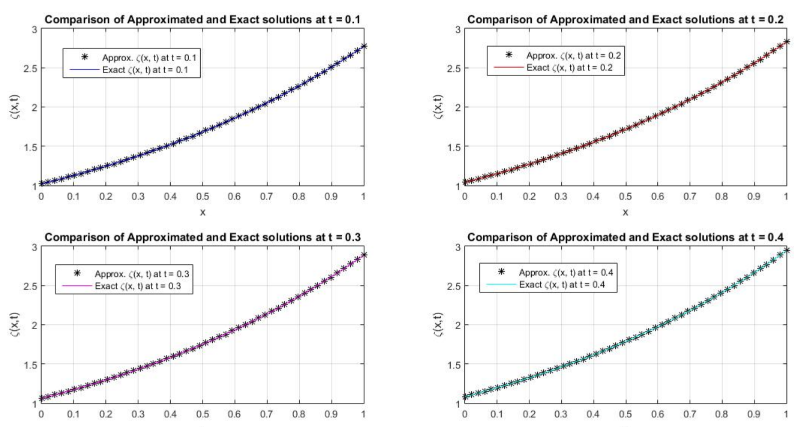

Figure 1. Approx. and exact results at $t = 0.1, 0.2, 0.3$ and 0.4 for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.1$, $\mu = 1, N = 50$ regarding Example 3.1.

Example 3.2. Considered non-homogenous time-fractional backward Klomogorov equation as follows [2]:

$$D_t^{\mu}\zeta(x,t) = -x^2 e^t \zeta_{xx} + (x+1)\zeta_x + xt \tag{3.15}$$

where,

$$\zeta(x,0) = x+1,$$

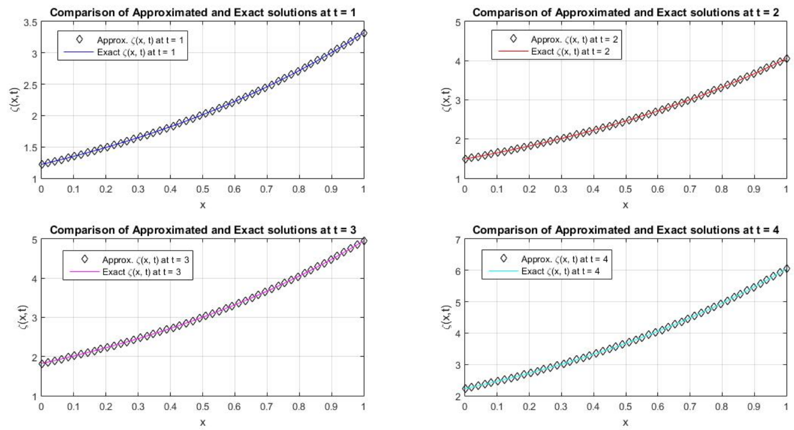

Figure 2. Approx. and exact results at $t = 1, 2, 3$ and 4 for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.1$, $\mu = 1$,$N = 50$ regarding Example 3.1.

Table 2. Error analysis regarding Example 3.1

N	$t = \mathbf{1}$	$t = \mathbf{2}$	$t = \mathbf{3}$
	L_∞	L_∞	L_∞
10	7.86E-14	8.15E-11	4.79E-09
20	1.33E-15	8.88E-16	1.78E-15
30	1.33E-15	1.33E-15	1.78E-15
	Converging up to 10^{-15}	Converging up to 10^{-15}	Converging up to 10^{-15}

$$\phi(x,t) = xt,$$
$$R[\zeta(x,t)] = -x^2 e^t (\zeta(x,t))_{xx} + (x+1)(\zeta(x,t))_x + xt,$$
$$N[\zeta(x,t)] = 0.$$

Considering Sumudu transform in Equation (3.15):

$$S[D_t^\mu \zeta(x,t)] = S[-x^2 e^t (\zeta(x,t))_{xx} + (x+1)(\zeta(x,t))_x + xt],$$
$$\zeta_0 = S^{-1}[u^\mu \sum_{r=0}^{m-1} u^{r-\mu} \zeta^r(x,0) + u^\mu S[\phi(x,t)]]. \tag{3.16}$$

Considering m=1 in Equation (3.16):

$$\zeta_0(x,t) = S^{-1}[u^\mu u^{0-\mu} \zeta(x,0) + u^\mu S[xt]]$$
$$\Rightarrow \zeta_0(x,t) = S^{-1}[Q(x,0) + u^\mu S[xt]]$$
$$\Rightarrow \zeta_0(x,t) = S^{-1}[(x+1) + u^\mu S[xt]]$$
$$\Rightarrow \zeta_0(x,t) = S^{-1}[(x+1) + xu^\mu S[t]]$$
$$\Rightarrow \zeta_0(x,t) = S^{-1}[(x+1) + xu^\mu u]$$
$$\Rightarrow \zeta_0(x,t) = S^{-1}[(x+1) + xu^{\mu+1}]$$

$$\Rightarrow \zeta_0(x,t) = (x+1)S^{-1}[1] + xS^{-1}[u^(\mu+1)]$$
$$\Rightarrow \zeta_0(x,t) = (x+1) + xS^{-1}[u^{\mu+1}]$$
$$\Rightarrow \zeta_0(x,t) = (x+1) + x \quad t^{\mu+1}/(\Gamma(\mu+1)), \tag{3.17}$$
$$\zeta_1(x,t) = S^{-1}[u^\mu SR[\zeta_0(x,t)] + N[\zeta_0(x,t)]] \tag{3.18}$$

where,

$$R[\zeta_0(x,t)] = -x^2e^t(\zeta_0(x,t))_{xx} + (x+1)(\zeta_0(x,t))_{xx} + (x+1)(\zeta_0(x,t))_x + xt$$
$$\Rightarrow R[\zeta_0(x,t)] = (x+1)[1 + t^{\mu+1}/\Gamma(\mu+2)]$$

and $N[\zeta_0(x,t)] = 0$ From Equation (3.18):

$$\Rightarrow \zeta_1(x,t) = S^{-1}[u^\mu S(x+1)1 + t^{\mu+1}/\Gamma(\mu+2)]$$
$$\Rightarrow \zeta_1(x,t) = (x+1)S^{-1}[u^\mu S1 + t^{\mu+1}/\Gamma(\mu+2)]$$
$$\Rightarrow \zeta_1(x,t) = (x+1)S^{-1}[u^\mu 1 + S(t^{\mu+1}/\Gamma(\mu+2))]$$
$$\Rightarrow \zeta_1(x,t) = (x+1)S^{-1}[u^\mu 1 + u^{\mu+1}]$$
$$\Rightarrow \zeta_1(x,t) = (x+1)S^{-1}[u^\mu + u^{2\mu+1}]$$
$$\Rightarrow \zeta_1(x,t) = (x+1)[S^{-1}[u^\mu] + S^{-1}[u^{2\mu+1}]]$$
$$\Rightarrow \zeta_1(x,t) = (x+1)[t^\mu/(\Gamma(\mu+1)) + t^{2\mu+1}/(\Gamma(2\mu+2))], \tag{3.19}$$
$$\zeta_2(x,t) = S^{-1}[u^\mu R[\zeta_1(x,t)] + N[\zeta_0(x,t) + \zeta_1(x,t)] - N[\zeta_0(x,t)]],$$
$$\zeta_2(x,t) = S^{-1}[u^\mu SR[\zeta_1]] \tag{3.20}$$

where,

$$N[\zeta_0(x,t) + \zeta_1(x,t)] = 0,$$
$$R[\zeta_1] = -x^2e^t(\zeta_1(x,t))_{xx} + (x+1)(\zeta_1(x,t))_x$$

where,

$$\zeta_1(x,t) = (x+1)[t^\mu/\Gamma(\mu+1) + t^{2\mu+1}/\Gamma(2\mu+2)],$$
$$(\zeta_1(x,t))_x = [t^\mu/(\Gamma(\mu+1)) + t^{2\mu+1}/(\Gamma(2\mu+2))],$$
$$(\zeta_1(x,t))_{xx} = 0,$$
$$R[\zeta_1(x,t)] = (x+1)[t^\mu/(\Gamma(\mu+1)) + t^{2\mu+1}/(\Gamma(2\mu+2))].$$

From Equation (3.20):

$$\zeta_2(x,t) = S^{-1}[u^\mu S(x+1)[t^\mu/\Gamma(\mu+1) + t^{2\mu+1}/\Gamma(2\mu+2)]]$$
$$\Rightarrow \zeta_2(x,t) = (x+1)S^{-1}[u^\mu S[t^\mu/\Gamma(\mu+1) + t^{2\mu+1}/\Gamma(2\mu+2)]]$$
$$\Rightarrow \zeta_2(x,t) = (x+1)S^{-1}[u^\mu u^\mu + u^{2\mu+1}]$$
$$\Rightarrow \zeta_2(x,t) = (x+1)S^{-1}[u^{2\mu} + u^{3\mu+1}]$$
$$\Rightarrow \zeta_2(x,t) = (x+1)[S^{-1}(u^{2\mu}) + S^{-1}(u^{3\mu+1})]$$
$$\Rightarrow \zeta_2(x,t) = (x+1)[t^{2\mu}/\Gamma(2\mu+1) + t^{3\mu+1}/\Gamma(3\mu+2)]. \tag{3.21}$$

Similarly,

$$\zeta_3(x,t) = (x+1)[t^{3\mu}/\Gamma(3\mu+1) + t^{4\mu+1}/\Gamma(4\mu+2)], \tag{3.22}$$

....

....

....

$$\zeta_m(x,t) = (x+1)[t^{m\mu}/\Gamma(m\mu+1) + t^{m\mu+1}/\Gamma(m\mu+2)], \tag{3.23}$$

$$\zeta^{(m)}(x,t) = \sum_{i=0}^{m} \zeta_i(x,t),$$

$$\zeta^{(m)}(x,t) = \zeta_0(x,t) + \zeta_1(x,t) + \zeta_2(x,t) + \zeta_3(x,t) + \cdots + \zeta_m(x,t),$$

$$\begin{aligned}\Rightarrow \zeta^{(m)}(x,t) = {} & [(x+1) + xt^{\mu+1}/\Gamma(\mu+1)] + [(x+1)[t^{\mu}/\Gamma(\mu+1) \\ & + t^{2\mu+1}/\Gamma(2\mu+2)]] + [(x+1)[t^{3\mu}/\Gamma(3\mu+1) + t^{4\mu+1}/\Gamma(4\mu+2)]] \\ & + \cdots + [(x+1)[t^{m\mu}/\Gamma(m\mu+1) + t^{m\mu+1}/\Gamma(m\mu+2)]]\end{aligned}$$

$$\begin{aligned}\Rightarrow \zeta^{(m)}(x,t) = {} & (x+1)[\sum_{i=0}^{m} t^{i\mu}/\Gamma(i\mu+1) + \sum_{i=0}^{m} t^{(i+2)\mu+1}/\Gamma((i+2)\mu+2)] \\ & + xt^{\mu}/(\Gamma(\mu+2))\end{aligned}$$

$$\begin{aligned}\Rightarrow \zeta^{(m)}(x,t) = {} & (x+1)[\sum_{i=0}^{m} t^{i\mu}/\Gamma(i\mu+1) + \sum_{i=0}^{m} t^{(i+2)\mu+1}/\Gamma((i+2)\mu+2)] \\ & + [(x+1)-1]t^{\mu+1}/(\Gamma(\mu+2))\end{aligned}$$

$$\begin{aligned}\Rightarrow \zeta^{(m)}(x,t) = {} & (x+1)[\sum_{i=0}^{m} t^{i\mu}/\Gamma(i\mu+1) + \sum_{i=0}^{m} t^{(i+2)\mu+1}/\Gamma((i+2)\mu+2)] \\ & + (x+1)t^{\mu+1}/\Gamma(\mu+2) - t^{\mu+1}/\Gamma(\mu+2),\end{aligned}$$

$$\begin{aligned}\zeta^{(m)}(x,t) = {} & -t^{\mu+1}/\Gamma(\mu+2) + (x+1)[\sum_{i=0}^{m} t^{i\mu}/\Gamma(i\mu+1) \\ & + \sum_{i=0}^{m} t^{(i+1)\mu+1}/(\Gamma((i+1)\mu+2))],\end{aligned}$$

$$\zeta(x,t) = \lim_{n\to\infty} \zeta^{(m)}(x,t),$$

$$\begin{aligned}\zeta(x,t) = {} & (x+1)[\lim_{n\to\infty} \sum_{i=0}^{m} t^{i\mu}/(\Gamma(i\mu+1)) + \lim_{n\to\infty} \sum_{i=0}^{\infty} t^{(i+1)\mu+1}/(\Gamma((i+1)\mu+2))] \\ & - t^{\mu+1}/\Gamma(\mu+2),\end{aligned}$$

$$\zeta(x,t) = (x+1)[E_\mu(t^\mu) + \lim_{n\to\infty} \sum_{i=0}^{\infty} t^{(i+1)\mu+1}/\Gamma((i+1)\mu+2)] - t^{\mu+1}/\Gamma(\mu+2). \tag{3.24}$$

Considering $\mu = 1$ in Equation (3.24):

$$\zeta(x,t) = (x+1)[E_1(t^1) + \lim_{n\to\infty} \sum_{i=0}^{\infty} t^{(i+1)1+1}/\Gamma((i+1)1+2)] - t^{1+1}/\Gamma(1+2),$$

$$\zeta(x,t) = (x+1)[E_1(t) + \lim_{n\to\infty} \sum_{i=0}^{\infty} t^{i+2}/\Gamma(i+3)] - t^2/\Gamma(3). \tag{3.25}$$

Example 3.3. Considered the fractional time Klein–Gordon equation as follows [2]:

$$D_t^{\mu}\zeta = \zeta_{xx} - \zeta + 2\sin x \tag{3.26}$$

where,

$$\begin{aligned}
&\zeta(x,0) = \sin x, \zeta'(x,0) = 1,\\
&\phi(x,t) = 2\sin x,\\
&R[\zeta(x,t)] = (\zeta(x,t))_{xx} - \zeta(x,t),\\
&N[\zeta(x,t)] = 0,\\
&\zeta_0(x,t) = S^{-1}[u^{\mu}\sum_{r=0}^{m-1} u^{r-\mu}\zeta^{r}(0) + u^{\mu}S[\phi(x,t)]]. \qquad (3.27)
\end{aligned}$$

Considering $m = 2$ in Equation (3.27):

$$\begin{aligned}
&\zeta_0(x,t) = S^{-1}[u^{\mu}\sum_{r=0}^{1} u^{r-\mu}\zeta^{r}(0) + u^{\mu}S[\phi(x,t)]] \qquad (3.28)\\
&\Rightarrow \zeta_0(x,t) = S^{-1}[u^{\mu}u^{0-\mu}\zeta(0) + u^{1-\mu}\zeta'(0) + u^{\mu}S[\phi(x,t)]]\\
&\Rightarrow \zeta_0(x,t) = S^{-1}[u^{\mu}u^{-\mu}\zeta(0) + u^{1-\mu}\zeta'(0) + u^{\mu}S[\phi(x,t)]]\\
&\Rightarrow \zeta_0(x,t) = S^{-1}[\zeta(0) + u\zeta'(0) + u^{\mu}S\{\phi(x,t)\}]\\
&\Rightarrow \zeta_0(x,t) = S^{-1}[\sin x + u + u^{\mu}S\{\phi(x,t)\}]\\
&\Rightarrow \zeta_0(x,t) = S^{-1}[\sin x + u + u^{\mu}S\{2\sin x\}]\\
&\Rightarrow \zeta_0(x,t = S^{-1}[\sin x + u + 2\sin x\ u^{\mu}S\{1\}]\\
&\Rightarrow \zeta_0(x,t) = S^{-1}[\sin x + u + 2\sin x\ u^{\mu}]\\
&\Rightarrow \zeta_0(x,t) = \sin x + t + 2\sin x\ t^{\mu}/(\Gamma(\mu+1)),\\
&\zeta_1 = S^{-1}[u^{\mu}SR[\zeta_0] + N[\zeta_0]] \qquad (3.29)
\end{aligned}$$

where, $R[\zeta_0] = (\zeta_0)_{xx} - \zeta_0 - 2\sin x - 4\sin x\ t^{\mu}/\Gamma(\mu+1) - t$ and $N[\zeta_0] = 0$.

From Equation (3.29):

$$\begin{aligned}
&\zeta_1(x,t) = S^{-1}[u^{\mu}S{-2}\sin x - 4\sin x\ t^{\mu}/\Gamma(\mu+1) - t]\\
&\Rightarrow \zeta_1(x,t) = S^{-1}[u^{\mu}{-2}\sin x - 4\sin x\ S(t^{\mu}/\Gamma(\mu+1)) - S(t)]\\
&\Rightarrow \zeta_1(x,t) = S^{-1}[u^{\mu}{-2}\sin x - 4\sin x\ u^{\mu} - u]\\
&\Rightarrow \zeta_1(x,t) = S^{-1}[-2\sin x u^{\mu} - 4\sin x\ u^{2\mu} - u^{\mu+1}]\\
&\Rightarrow \zeta_1(x,t) = -2\sin x S^{-1}[u^{\mu}] - 4\sin x\ S^{-1}[u^{2\mu}] - S^{-1}[u^{\mu+1}]\\
&\Rightarrow \zeta_1(x,t) = -2\sin x t^{\mu}/\Gamma(\mu+1) - 4\sin x \quad t^{2\mu}/\Gamma(2\mu+1) - t^{\mu+1}/\Gamma(\mu+2)\\
&\zeta_2(x,t) = S^{-1}[u^{\mu}SR[\zeta_1] + N[\zeta_0+\zeta_1] - N[\zeta_0]],\\
&\zeta_2(x,t) = S^{-1}[u^{\mu}SR[\zeta_1] + N[\zeta_1]] \qquad (3.30)
\end{aligned}$$

where,

$$\begin{aligned}
&R[\zeta_1] = (\zeta_1)_{xx} - \zeta_1,\\
&R[\zeta_1] = [2\sin x\ t^{\mu}/\Gamma(\mu+1) + 4\sin x\ t^{2\mu}/\Gamma(2\mu+1)]
\end{aligned}$$

$$
\begin{aligned}
&\qquad - [-2\sin x\ t^{\mu}/\Gamma(\mu+1) - 4\sin x\ t^{2\mu}/\Gamma(2\mu+1) - t^{\mu+1}/\Gamma(\mu+2)]\\
&\Rightarrow R[\zeta_1] = 4\sin x\ t^{\mu}/\Gamma(\mu+1) + 8\sin x\ t^{2\mu}/\Gamma(2\mu+1) + t^{\mu+1}/\Gamma(\mu+2),\\
&N[\zeta_1] = 0.
\end{aligned}
$$

From Equation (3.30):

$$
\begin{aligned}
&\zeta_2(x,t) = S^{-1}[u^{\mu}S[R[\zeta_1]]]\\
\Rightarrow\ &\zeta_2(x,t) = S^{-1}[u^{\mu}S[4\sin xt^{\mu}/\Gamma(\mu+1) + 8\sin xt^{2\mu}/\Gamma(2\mu+1) + t^{\mu+1}/\Gamma(\mu+2)]]\\
\Rightarrow\ &\zeta_2(x,t) = S^{-1}[u^{\mu}[4\sin xS(t^{\mu}/\Gamma(\mu+1)) + 8\sin xS(t^{2\mu}/\Gamma(2\mu+1))\\
&\qquad + S(t^{\mu+1}/\Gamma(\mu+2))]]\\
\Rightarrow\ &\zeta_2(x,t) = S^{-1}[u^{\mu}[4\sin xu^{\mu} + 8\sin xu^{2\mu} + u^{\mu+1}]]\\
\Rightarrow\ &\zeta_2(x,t) = S^{-1}[4\sin xu^{2\mu} + 8\sin xu^{3\mu} + u^{2\mu+1}]\\
\Rightarrow\ &\zeta_2(x,t) = 4\sin xS^{-1}[u^{2\mu}] + 8\sin xS^{-1}[u^{3\mu}] + S^{-1}[u^{2\mu+1}]\\
\Rightarrow\ &\zeta_2(x,t) = 4\sin xt^{2\mu}/\Gamma(2\mu+1) + 8\sin xt^{3\mu}/\Gamma(3\mu+1) + t^{2\mu+1}/\Gamma(2\mu+2).
\end{aligned}
\tag{3.31}
$$

Similarly,

$$
\zeta_3(x,t) = -8\sin xt^{3\mu}/\Gamma(3\mu+1) - 16\sin xt^{4\mu}/\Gamma(4\mu+1) + t^{3\mu+1}/\Gamma(3\mu+2),
\tag{3.32}
$$

$$
\begin{aligned}
&\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\\
&\zeta_m(x,t) = (-1)^m(2)^m\sin xt^{m\mu}/\Gamma(m\mu+1)\\
&\qquad + (-1)^m 2^{m+1}\sin xt^{(m+1)\mu}/\Gamma((m+1)\mu+1) + t^{m\mu+1}/\Gamma(m\mu+2)
\end{aligned}
\tag{3.33}
$$

where, m = 2,3,4,5,...

$$
\begin{aligned}
&\zeta^{(m)}(x,t) = \sum_{i=0}^{m}\zeta_i\\
\Rightarrow\ &\zeta^{(m)}(x,t) = \zeta_0(x,t) + \zeta_1(x,t) + \zeta_2(x,t) + \zeta_3(x,t) + \cdots + \zeta_m(x,t)\\
\Rightarrow\ &\zeta^{(m)}(x,t) = \sin x + \sum_{i=0}^{m}(-1)^i t^{i\mu+1}/\Gamma(i\mu+2),\\
&\zeta(x,t) = \lim_{m\to\infty}\zeta^{(m)}(x,t)\\
\Rightarrow\ &\zeta(x,t) = \lim_{m\to\infty}[\sin x + \sum_{i=0}^{m}(-1)^i t^{i\mu+1}/\Gamma(i\mu+2)],\\
&\zeta(x,t) = \sin x + \sin t.
\end{aligned}
\tag{3.34}
$$

Example 3.4. Considered generalized 2D time-fractional biological population model as follows [2]:

$$
D_t^{\mu}\zeta = (\zeta^2(x,y,t))_{xx} + (\zeta^2(x,y,t))_{yy} + \zeta(x,y,t) - \gamma(\zeta(x,y,t))^2
\tag{3.35}
$$

where, $0 < \mu \leq 1$.

$$
\begin{aligned}
&\zeta(x,y,0) = e^{\sqrt{(\gamma/8)(x+y)}},\\
&\phi(x,y,t) = 0,\\
&R[\zeta(x,y,t)] = (\zeta^2(x,y,t))_{xx} + (\zeta^2(x,y,t))_{yy} + \zeta(x,y,t),\\
&N[\zeta(x,y,t)] = -\gamma(\zeta(x,y,t))^2,\\
&\zeta_0(x,y,t) = S^{-1}[u^{\mu}\sum_{r=0}^{m-1} u^{r-\mu}(\zeta(x,y,t))^{(r)}(0) + S\phi(x,y,t)]. \qquad (3.36)
\end{aligned}
$$

Considered $m = 1$ in Equation (3.36):

$$
\begin{aligned}
&\zeta_0(x,y,t) = S^{-1}[\zeta(x,y,0) + S\phi(x,y,t)]\\
\Rightarrow &\zeta_0(x,y,t) = S^{-1}[\zeta(x,y,0)] \Rightarrow \zeta_0(x,y,t) = S^{-1}[e^{\sqrt{(\gamma/8)(x+y)}}]\\
\Rightarrow &\zeta_0(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}}S^{-1}[1] \Rightarrow \zeta_0(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}} \qquad (3.37)\\
\Rightarrow &\zeta_1(x,y,t) = S^{-1}[u^{\mu}SR[\zeta_0] + N[\zeta_0]] \qquad (3.38)
\end{aligned}
$$

where,

$$
\begin{aligned}
&R[\zeta_0(x,y,t)] = (\zeta_0^2(x,y,t))_{xx} + (\zeta_0^2(x,y,t))_{yy} + \zeta_0(x,y,t),\\
&\zeta_0^2 = e^{2\sqrt{(\gamma/8)(x+y)}},\\
&(\zeta_0^2)_x = e^{2\sqrt{(\gamma/8)(x+y)}}[2\sqrt{\gamma/8}],\\
&(\zeta_0^2)_{xx} = e^{2\sqrt{(\gamma/8)(x+y)}}[2\sqrt{\gamma/8}]^2,\\
&R[\zeta_0(x,y,t)] = (\zeta_0^2(x,y,t))_{xx} + (\zeta_0^2(x,y,t))_{yy} + \zeta_0(x,y,t),\\
&R[\zeta_0(x,y,t)] = e^{2\sqrt{(\gamma/8)(x+y)}}[2\sqrt{\gamma/8}]^2 + e^{2\sqrt{(\gamma/8)(x+y)}}[2\sqrt{\gamma/8}]^2 + e^{\sqrt{(\gamma/8)(x+y)}},\\
&R[\zeta_0(x,y,t)] + N[\zeta_0(x,y,t)] = e^{\sqrt{(\gamma/8)(x+y)}}.
\end{aligned}
$$

From Equation (3.38):

$$
\begin{aligned}
&\zeta_1(x,y,t) = S^{-1}[u^{\mu}Se^{\sqrt{(\gamma/8)(x+y)}}]\\
\Rightarrow\ &\zeta_1(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}}S^{-1}[u^{\mu}S1] \Rightarrow \zeta_1(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}}S^{-1}[u^{\mu}]\\
\Rightarrow\ &\zeta_1(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}}t^{\mu}/\Gamma(\mu+1) \qquad (3.39)\\
&\zeta_2(x,y,t) = S^{-1}[u^{\mu}SR[\zeta_1(x,y,t)] + N[\zeta_0(x,y,t) + \zeta_1(x,y,t)] - N[\zeta_0(x,y,t)]]\\
&\zeta_2(x,y,t) = S^{-1}[u^{\mu}SR[\zeta_1(x,y,t)] + N[\zeta_1(x,y,t)]] \qquad (3.40)
\end{aligned}
$$

where,

$$
\begin{aligned}
R[\zeta_1(x,y,t)] &= (\zeta_1^2(x,y,t))_{xx} + (\zeta_1^2(x,y,t))_{yy} + \zeta_1(x,y,t)\\
R[\zeta_1(x,y,t)] &= [2\sqrt{\gamma/8}]^2 e^{2\sqrt{(\gamma/8)(x+y)}}t^{\mu}/\Gamma(\mu+1)\\
&\quad + [2\sqrt{\gamma/8}]^2 e^{2\sqrt{(\gamma/8)(x+y)}}t^{\mu}/\Gamma(\mu+1) + e^{\sqrt{(\gamma/8)(x+y)}}t^{\mu}/\Gamma(\mu+1)
\end{aligned}
$$

and

$$
N[\zeta_1(x,y,t)] = -\gamma e^{2\sqrt{(\gamma/8)(x+y)}}[t^{\mu}/\Gamma(\mu+1)]^2,
$$

$$R[\zeta_1(x,y,t)] + N[\zeta_1(x,y,t)] = e^{\sqrt{(\gamma/8)(x+y)}} t^{\mu}/\Gamma(\mu+1).$$

From Equation (3.40):

$$\begin{aligned}
&\zeta_2(x,y,t) = S^{-1}[u^{\mu} S e^{\sqrt{(\gamma/8)(x+y)}} t^{\mu}/\Gamma(\mu+1)]\\
&\Rightarrow \zeta_2(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}} S^{-1}[u^{\mu} S t^{\mu}/\Gamma(\mu+1)]\\
&\Rightarrow \zeta_2(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}} S^{-1}[u^{\mu} u^{\mu}] \Rightarrow \zeta_2(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}} S^{-1}[u^{2\mu}]\\
&\Rightarrow \zeta_2 = e^{\sqrt{(\gamma/8)(x+y)}} t^{2\mu}/\Gamma(2\mu+1). \qquad (3.41)
\end{aligned}$$

Similarly,

$$\zeta_3(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}} t^{3\mu}/\Gamma(3\mu+1), \qquad (3.42)$$

$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$

$$\zeta_m(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}} t^{m\mu}/\Gamma(m\mu+1), \qquad (3.43)$$

$$\zeta^{(m)}(x,y,t) = \sum_{i=0}^{m} \zeta_i(x,y,t),$$

$$\zeta^{(m)}(x,y,t) = \zeta_0(x,y,t) + \zeta_1(x,y,t) + \zeta_2(x,y,t) + \zeta_3(x,y,t) + \cdots + \zeta_m(x,y,t),$$

$$\zeta(x,y,t) = \lim_{m\to\infty} \sum_{i=0}^{m} e^{\sqrt{(\gamma/8)(x+y)}} t^{i\mu}/\Gamma(i\mu+1),$$

$$\zeta(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}} \lim_{m\to\infty} \sum_{i=0}^{m} t^{i\mu}/\Gamma(i\mu+1),$$

$$\zeta(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)}} e^{t\mu}. \qquad (3.44)$$

Considering μ=1 in Equation (3.44),

$$\zeta(x,y,t) = e^{\sqrt{(\gamma/8)(x+y)+t}}. \qquad (3.45)$$

Table 3. Error analysis regarding Example 3.4.

N	$t = \mathbf{1}$	$t = \mathbf{2}$	$t = \mathbf{3}$
	L_∞	L_∞	L_∞
10	6.11E-07	6.94E-04	4.47E-02
20	1.78E-15	9.59E-13	3.37E-09
30	2.66E-15	5.33E-15	2.13E-14
	Converging up to 10^{-15}	**Converging up to** 10^{-15}	**Converging up to** 10^{-14}

Convergence Analysis. Convergence of the proposed scheme can be affirmed via following two lemmas.

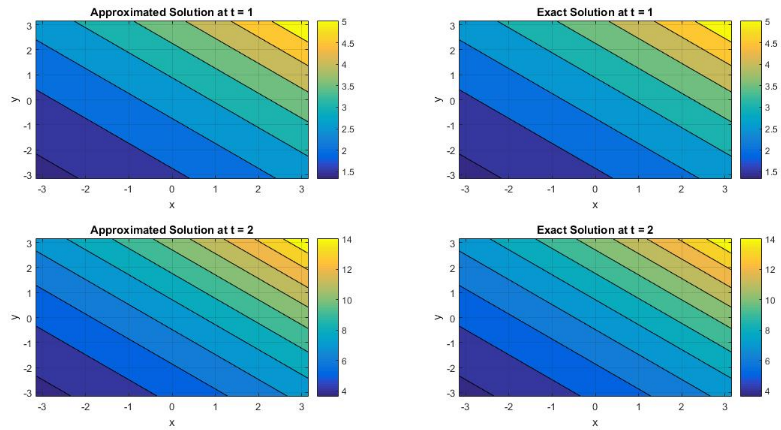

Figure 3. 2D plot for Approx. and Exact results at $t = 1, 2$, where $\mu = 1$, $\gamma = 0.1$ and $N = 30$ regarding Example 3.4.

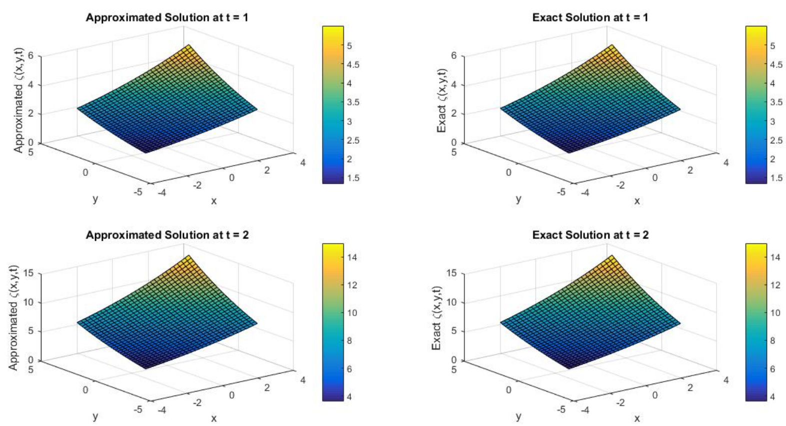

Figure 4. 3D plot for Approx. and Exact results at $t = 1, 2$, where $\mu = 1$, $\gamma = 0.1$ and $N = 30$ regarding Example 3.4.

Lemma 3.1 ([1]). *If the integral* $\frac{1}{s}\int_0^\infty exp[-\frac{t}{s}]f(t)dt$ *converges at* $s = s_0$ *then the integral converges for* $s < s_0$.

Lemma 3.2 ([1]). *If the integral* $h(x,u) = \frac{1}{u}\int_0^\infty exp[-\frac{t}{s}]f(t)dt$ *converges for* $s \leq s_0$ *and the integral* $\frac{1}{p}\int_0^\infty exp[-\frac{x}{p}]h(s)dx$ *converges at* $p = p_0$ *then the above-mentioned integral converges for* $p < p_0$.

4. Concluding Remarks

Present research is related to the implementation of a novel approach Sumudu Transform for the solution of time fractional PDEs. Sumudu transform is employed to fetch the analytical-approx. results of the various time fractional PDEs. Proposed scheme is easy to implement and also do not require a lengthy and cum-

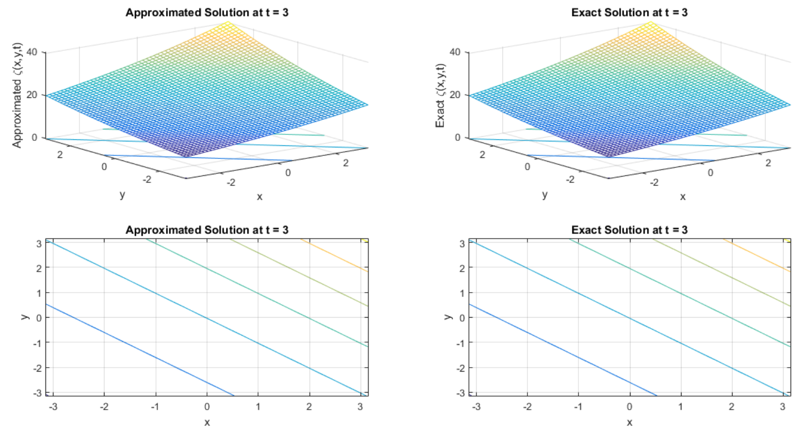

Figure 5. Mesh and contour plot for approx. and exact solution profiles at $t = 3$, $N = 50$ regarding Example 3.4.

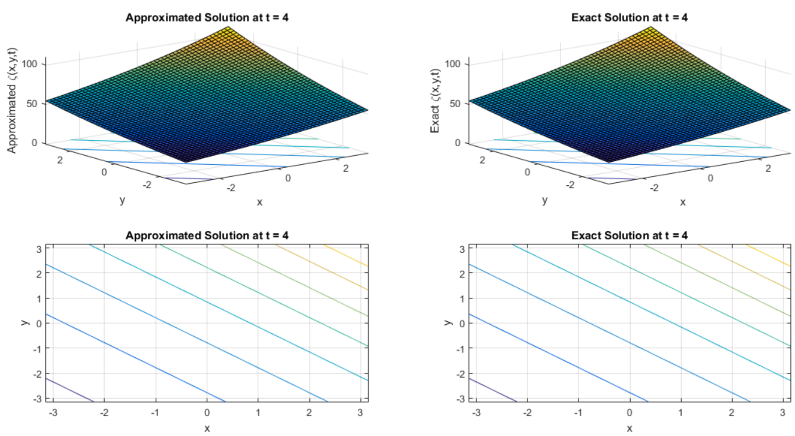

Figure 6. Surface and contour plot for approx. and exact solution profiles at $t = 4$, $N = 50$ regarding Example 3.4.

bersome numerical program. Graphical matching of the approx. and exact solution profile is matched with aid of 2D and 3D plots. Present scheme will surely be helpful to solve complex natured fractional PDEs where, developing the numerical scheme is not an easy task. As the present regime does not demand any discretization or complex numerical algorithm, it can be utilized to tackle the complex natured fractional ODEs, PDEs, fractional systems, and integro differential equations in an efficient way.

Conflict of Interest. Not applicable.

Data Availability Statement. All data is included within the manuscript.

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