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DYNAMICS OF A STOCHASTIC VECTOR-HOST EPIDEMIC MODEL WITH AGE-DEPENDENT OF VACCINATION AND DISEASE RELAPSE*

Zhen Cao[1] and Lin-Fei Nie[1,†]

Abstract Due to the ubiquitous stochastic interference in nature, the uncertainty of the disease relapse and the duration of immunity, we present a stochastic vector-host epidemic model with age-dependent of vaccination and disease relapse, where two general incidences are also introduced to depict the transmission of virus between vectors and hosts. By constructing a suitable Lyapunov function, the existence and uniqueness of the global positive solution of our model are proved. Further, the stochastic extinction of disease, the existence of stationary distribution are also discussed. Moreover, the stochastic extinction of disease and the existence of stationary distribution for special incidence are obtained as an application, where the general incidence degenerates into the billinear incidence. Finally, numerical simulations are given to illuminate the main results, which also suggest that the behaviors of vectors and the self-protection of hosts are the key factors to eliminate the disease relative to the quantity of vector population during the transmission of vector-host infectious diseases.

Keywords Vector-host disease, stochastic perturbation, age-dependent, general incidence rate, extinction and stationary.

MSC(2010) 37H10, 60H30, 92D30, 93E15.

1. Introduction

Vector-host infectious diseases, such as Malaria, Dengue fever, Japanese encephalitis and West Nile fever, etc, are transmitted to humans through the bites of insects or animals that carry certain pathogens or parasites. At present, vector-host infectious diseases have been involved in more than 100 countries and regions around the world, accounting for more than 17% of all infectious diseases [34]. Thus, it is very urgent to study the control and prevention of vector-host infectious diseases.

[†]The corresponding author.
Emails: 1364216327@qq.com(Z. Cao),
lfnie@163.com/nielinfei@xju.edu.cn(L. Nie)
[1]College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, China
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In order to explain the propagative law of vector-host diseases and the effectiveness of control strategies, many scholars have proposed various mathematical models to simulate the transmission of pathogen in vectors and hosts. For example, Esteva et al. [9] proposed an ordinary differential equation model for the spread of Dengue virus between mosquitoes and human population, and proved the existence and stability of equilibria. Bowman et al. [4] proposed a mathematical model to describe the spread of West Nile virus among mosquitoes, birds and humans, obtained the basic reproduction number and demonstrated the existence and stability of the disease-free equilibrium for the threshold value is less than 1. See Refs. [5, 15, 18, 22, 39, 40, 42] for more studies, and the topic is still continuous.

Whether in the past or now, vaccinations have always been considered one of the most effective strategies to curb the spread of diseases, such as Measles, Polio, Pertussis, Tuberculosis (TB) and Hepatitis B virus (HBV), etc [35]. A notable success story was the eradication of Small-pox in 1997 [36]. However, the effectiveness of vaccines is a problem that cannot be ignored, which have become the cause of recurrence and frequent of some diseases (such as, Rubella, Measles, Pertussis and Chickenpox). To this end, some researchers introduced imperfect immunity and reduced immunity into mathematical models of infectious diseases [6, 10, 12, 24, 28]. In particular, Nkamba et al. [23] proposed a SELI (susceptible, earyly latent, late latent, TB-infected) compartment model with immunization to assess the effect of vaccination rate on TB transmission. On the effectiveness and timeliness of vaccines, a reasonable assumption is that immunity depends on the duration of vaccination for susceptible individuals being vaccinated, but the time period is not always fixed. Based on this, Yang et al. [38] established an SIVS model with the age of vaccination and nonlinear incidence, and discussed th global dynamics of this model. Duan et al. [7] introduced a SVIR model with the age of inoculation, and discussed the global stability of equilibria. More related studies are available in Refs. [1, 8, 14, 20, 27, 31].

On the other hand, the spread of infectious diseases is constantly affected by uncertainties or stochastic perturbation in the environment, especially in vector-host diseases. These factors, such as temperature changes, seasonal changes, weather conditions and media coverage inevitably affect the quantity and behaviors of vectors or hosts, which in turn influence the infectious diseases that spread between them. Therefore, it is more reasonable to introduce stochastic differential equations into the modeling of vector-borne infectious diseases [11, 29, 30]. In particular, Jovanovi et al. [17] proposed a vector-host disease model with stochastic perturbation and direct transmission, and obtained some sufficient conditions for stochastic stability. Ran et al. [25] introduced uncertainty into a vector-host disease model with class-age structured and discussed global existence of positive solutions, the stochastic extinction of disease, and the existence of stationary distribution.

This paper proposes a stochastic vector-host disease model with the age of vaccination and relapse to study the impact of uncertainties. This work is organized as follows: the model is formulated and the global existence and boundedness of positive solutions are proved in Section 2 and Section 3, respectively. The main result on the stochastic extinction of disease is derived in Section 4. The existence and uniqueness of ergodic stationary distribution are analyzed in Section 5, and investigate the extinction and permanence in the mean and the existence and uniqueness of stationary distribution for special incidence in Section 6. We illustrate the main results, and a brief concluding remark in Section 7 and Section 8, respectively.

2. Model formulation

The quantity of hosts at time t (that is, $N_h(t)$) is split into susceptible class $S_h(t)$, vaccinated class $V_h(t,a)$ with the age of vaccination a (that is, the time-since-vaccination), infected class $I_h(t)$ and recovered class $R_h(t,b)$ with the age of recover b (that is, the time-since-recover). The vector population $N_v(t)$ at time t includes susceptible class $S_v(t)$ and infected class $I_v(t)$. Therefore, the total quantities of vaccinated class and recovered class at time t are $\int_0^\infty V_h(t,a)\mathrm{d}a$ and $\int_0^\infty R_h(t,b)\mathrm{d}b$, respectively. Assume that the age-dependent of immune loss rate and disease relapse rate are denoted by $\omega(a)$ and $r(b)$, respectively, and $0 \leqslant \omega(a),\ r(b) \leqslant 1$. Since the nonlinear incidence rate is very important in the mathematical modeling of infectious diseases [26, 33], the generalized incidence is used in this paper. More specifically, the rate of susceptible host population $S_h(t)$ get infectious by infected vectors $I_v(t)$ is governed by $\beta_1 f(S_h, I_v)$, where, β_1 is the probability of exposure of a susceptible person to an infected vector. Similarly, the rate of transmission from infected class $I_h(t)$ to susceptible class $S_v(t)$ is $\beta_2 g(S_v, I_h)$.

Based on the transmission pattern of pathogens between hosts and vectors, a model with age structured and general incidence reads

$$\begin{cases} \dfrac{\mathrm{d}S_h(t)}{\mathrm{d}t} = \Lambda_h - (\mu_h + \psi_h)S_h(t) - \beta_1 f(S_h(t), I_v(t)) + \displaystyle\int_0^\infty \omega(a)V_h(t,a)\mathrm{d}a, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) V_h(t,a) = -(\mu_h + \omega(a))V_h(t,a), \quad V_h(t,0) = \psi_h S_h(t), \\ \dfrac{\mathrm{d}I_h(t)}{\mathrm{d}t} = \beta_1 f(S_h(t), I_v(t)) - (\mu_h + k + \nu)I_h(t) + \displaystyle\int_0^\infty r(b)R_h(t,b)\mathrm{d}b, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) R_h(t,b) = -(\mu_h + r(b))R_h(t,b), \quad R_h(t,0) = kI_h(t), \\ \dfrac{\mathrm{d}S_v(t)}{\mathrm{d}t} = \Lambda_v - \beta_2 g(S_v(t), I_h(t)) - \mu_v S_v(t), \\ \dfrac{\mathrm{d}I_v(t)}{\mathrm{d}t} = \beta_2 g(S_v(t), I_h(t)) - \mu_v I_v(t) \end{cases} \tag{2.1}$$

with the initial conditions $S_h(0) = S_{h0}$, $V_h(0,a) = V_{h0}(a)$, $I_h(0) = I_{h0}$, $R_h(0,b) = R_{h0}(b)$, $S_v(0) = S_{v0}$, $I_v(0) = I_{v0}$ for $a \geqslant 0$, $b \geqslant 0$, where S_{h0}, I_{h0}, S_{v0} and I_{v0} are non-negative constants and $V_{h0}(a)$ and $R_{h0}(b)$ are non-negative essentialy bounded functions. In addition, let $S_{h0}+I_{h0}+S_{v0}+I_{v0}+\int_0^\infty V_{h0}(a)\mathrm{d}a+\int_0^\infty R_{h0}(b)\mathrm{d}b = N_{h0}$, where N_{h0} is the total size of hosts which is a positive constant.

Restricting the second and fourth equations of model (2.1) to the partial differential equations and solving then along the characteristic curve $t - a = const$ and $t - b = const$ (see Refs. [16, 32] for more details), one have

$$V_h(t,a) = \begin{cases} \psi_h S_h(t-a)\Gamma_0(a), & t > a \geqslant 0, \\ V_{h0}(a-t)\dfrac{\Gamma_0(a)}{\Gamma_0(a-t)}, & a \geqslant t > 0, \end{cases}$$

$$R_h(t,b) = \begin{cases} kI_h(t-b)\pi_0(b), & t > b \geqslant 0, \\ R_{h0}(b-t)\dfrac{\pi_0(b)}{\pi_0(b-t)}, & b \geqslant t > 0, \end{cases}$$

Table 1. The biological significance and value range of parameters for model (2.1), where, take the transmission of Malaria between mosquitoes and humans as an example.

Param.	Interpretation	Range	Source
Λ_h/Λ_v	Replenishment rate of hosts/vectors	Assumed	–
ν	Disease-related death rate of infected hosts	$0 \sim 1$	–
$1/\mu_h$	Average lifespan of hosts (*years*)	$68 \sim 79$	[5, 29]
$1/\mu_v$	Average life span of vectors (*days*)	$4 \sim 35$	[2]
ψ_h	Vaccination coverage rate	$0 \sim 1$	–
k	Proportional coefficient of recovered from infected	$0 \sim 1$	–
$\omega(a)$	Rate of vaccine shedding at age a	Assumed	–
$r(b)$	Relapse rate at relapse age b	Assumed	–
β_1	Probability of transmission from infectious vectors to susceptible hosts	$1.35e^{-6} \sim 2.09e^{-4}$	[29]
β_2	Probability of transmission from infectious hosts to susceptible vectors	$2.82e^{-6} \sim 3.65e^{-4}$	[29]

where $\Gamma_0(a) = \exp\{-\int_0^a (\mu_h + \omega(\tau))\mathrm{d}\tau\}$ and $\pi_0(b) = \exp\{-\int_0^b (\mu_h + r(\tau))\mathrm{d}\tau\}$. For $t \geqslant 0$, it follows that

$$\begin{aligned}\int_0^\infty \omega(a)V_h(t,a)\mathrm{d}a = &\int_0^t \omega(a)\psi_h S_h(t-a)\Gamma_0(a)\mathrm{d}a \\ &+ \int_t^\infty \omega(a)V_{h0}(a-t)\frac{\Gamma_0(a)}{\Gamma_0(a-t)}\mathrm{d}a. \end{aligned} \tag{2.2}$$

Due to

$$\int_t^\infty \omega(a)V_{h0}(a-t)\frac{\Gamma_0(a)}{\Gamma_0(a-t)}\mathrm{d}a \leqslant e^{-\mu_h t}\int_0^\infty V_{h0}(\hat{a})\mathrm{d}\hat{a} < e^{-\mu_h t}N_{h0},$$

where $\hat{a} = a - t$. Since $e^{-\mu_h t}N_{h0} \to 0$ as $t \to \infty$, equation (2.2) can be rewritten

$$\int_0^\infty \omega(a)V_h(t,a)\mathrm{d}a = \int_0^\infty \omega(a)\psi_h S_h(t-a)\Gamma_0(a)\mathrm{d}a = \int_0^\infty S_h(t-a)\Gamma(a)\mathrm{d}a,$$

where $\Gamma(a) = \omega(a)\psi_h\Gamma_0(a)$. Similarly, we can get

$$\int_0^\infty r(b)R_h(t,b)\mathrm{d}b = \int_0^\infty I_h(t-b)\pi(b)\mathrm{d}b,\ \ \pi(b) = r(b)k\pi_0(b).$$

Considering the effect of uncertainties (or, stochastic perturbation) on the behaviors of vectors and hosts, we assume that perturbation has an effect on the incidence of infected vectors/hosts to susceptible hosts/vectors, and the intensities of perturbations are denoted by σ_1^2 and σ_2^2, respectively. In addition, the quantity of vectors is also affected by uncertainties, such as temperature, rainfall, humidity, and so on, and the impact of these factors are denoted as σ_3^2 and σ_4^2, respectively. As pointed out in Ref. [21], this stochastic perturbation is known as white noise. Based on the above assumptions and model (2.1), an age-dependent vector-host epidemic model with stochastic perturbation reads

$$\begin{cases} \mathrm{d}S_h(t) = \Big(\Lambda_h - (\mu_h + \psi_h)S_h(t) - \beta_1 f(S_h(t), I_v(t)) \\ \qquad\qquad + \displaystyle\int_0^\infty \Gamma(a)S_h(t-a)\mathrm{d}a\Big)\mathrm{d}t - \sigma_1 f(S_h(t), I_v(t))\mathrm{dB}_1(t), \\ \mathrm{d}I_h(t) = \Big(\beta_1 f(S_h(t), I_v(t)) - (\mu_h + k + \nu)I_h(t) \\ \qquad\qquad + \displaystyle\int_0^\infty \pi(b)I_h(t-b)\mathrm{d}b\Big)\mathrm{d}t + \sigma_1 f(S_h(t), I_v(t))\mathrm{dB}_1(t), \\ \mathrm{d}S_v(t) = \big(\Lambda_v - \beta_2 g(S_v, I_h) - \mu_v S_v\big)\mathrm{d}t - \sigma_2 g(S_v, I_h)\mathrm{dB}_2(t) + \sigma_3 S_v \mathrm{dB}_3(t), \\ \mathrm{d}I_v(t) = \Big(\beta_2 g(S_v, I_h) - \mu_v I_v\Big)\mathrm{d}t + \sigma_2 g(S_v, I_h)\mathrm{dB}_2(t) + \sigma_4 I_v \mathrm{dB}_4(t), \end{cases} \tag{2.3}$$

where, $\mathrm{B}_i(t)$ is the standard Brownian motion and defined on $(\Omega, \mathcal{F}, \mathbb{P})$ $(i = 1, \cdots, 4)$. Here, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t>0}$ which is increasing and right continuous with $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) : x_i > 0,\ i = 1, 2, \cdots, n\}$, $(a_1 \wedge a_2) = \min\{a_1, a_2\}$ and $(a_1 \vee a_2) = \max\{a_1, a_2\}$, for all a_1, $a_2 \in \mathbb{R} = (-\infty, +\infty)$.

(H_1) Function $f(S_h, I_v)$ has two-order continuous derivative for any $S_h \geqslant 0$, $I_v \geqslant 0$, and $S_h + I_v > 0$. For each fixed $I_v \geqslant 0$, $f(S_h, I_v)$ is increasing for $S_h > 0$, and $f(S_h, I_v)/I_v$ is decreasing for $I_v > 0$ and each fixed $S_h \geqslant 0$, and $f(S_h, 0) = f(0, I_v) = 0$ for any $S_h > 0$ or $I_v > 0$.

(H_2) Function $g(S_v, I_h)$ has two-order continuously derivative for any $S_v \geqslant 0$, $I_h \geqslant 0$, and $S_v + I_h > 0$. For each fixed $I_h \geqslant 0$, $g(S_v, I_h)$ is increasing for $S_v > 0$, and $g(S_v, I_h)/I_h$ is decreasing for $I_h > 0$ and each fixed $S_v \geqslant 0$, and $g(S_v, 0) = g(0, I_h) = 0$ for any $S_v > 0$ or $I_h > 0$.

Remark 2.1. If $f(S_h, I_v) = S_h I_v/N$ and $g(S_v, I_h) = S_v I_h/N$ (standard incidence), where $N = S_h + I_h + S_v + I_v$, $f(S_h, I_v) = S_h I_v/(1 + \alpha_1 I_v + \alpha_2 S_h)$ and $g(S_v, I_h) = S_v I_h/(1 + \alpha_3 I_h + \alpha_4 S_v)$ (Beddington-DeAngelis incidence) with constants α_i $(i = 1, \cdots, 4)$, and $f(S_h, I_v) = S_h I_v/(1 + \alpha_5 I_v^2)$ and $g(S_v, I_h) = S_v I_h/(1 + \alpha_6 I_h^2)$ with constants $\alpha_5, \alpha_6 \geqslant 0$, then ($\mathrm{H}_1$) and ($\mathrm{H}_2$) are satisfied.

3. Global existence and uniqueness of solution

To prove the global existence and uniqueness of positive solutions for model (2.3), we show, firstly, the following result, whose proof is similar to the proof of Lemma 3 in Ref. [26].

Lemma 3.1. *For any positive constants $p > q$ and $D = \{(S_h, I_h, S_v, I_v) : S_h > 0, I_h > 0, S_v > 0, I_v > 0, q \leqslant S_h + I_h + S_v + I_v \leqslant p\}$, then,*

$$\max_{(S_h, I_v)\in D}\left\{\frac{f(S_h, I_v)}{S_h}, \frac{f(S_h, I_v)}{I_v}\right\} < \infty, \quad \max_{(S_v, I_h)\in D}\left\{\frac{g(S_v, I_h)}{S_v}, \frac{g(S_v, I_h)}{I_h}\right\} < \infty.$$

From Lemma 3.1, we introduce the following notations,

$$\max_{(S_h, I_v)\in\Omega}\left\{\frac{f(S_h, I_v)}{S_h}\right\} = K_1, \quad \max_{(S_h, I_v)\in\Omega}\left\{\frac{f(S_h, I_v)}{I_v}\right\} = K_2,$$

$$\max_{(S_v,I_h)\in\Omega}\left\{\frac{g(S_v,I_h)}{S_v}\right\}=K_3,\qquad \max_{(S_v,I_h)\in\Omega}\left\{\frac{g(S_v,I_h)}{I_h}\right\}=K_4.$$

Theorem 3.1. *For $(S_{h0},I_{h0},S_{v0},I_{v0})\in\mathbb{R}^4_+$, model (2.3) exists a unique solution $(S_h(t),I_h(t),\,S_v(t),I_v(t))$ on $[0,\infty)$, which remains in $\mathbb{R}^4_+$ with probability one. That is, $(S_h(t),I_h(t),\,S_v(t),I_v(t))\in\mathbb{R}^4_+$ for $t\in\mathbb{R}_+$ a.s. (almost surely).*

Proof. Since the local Lipschitz condition is valid for model (2.3), which has a unique positive solution $X(t)=(S_h(t),I_h(t),S_v(t),I_v(t))$ on $t\in[0,\tau_e)$ satisfies $X(0)=(S_h(0),I_h(0),S_v(0),I_v(0))=(S_{h0},I_{h0},S_{v0},I_{v0})\in\mathbb{R}^4_+$, where τ_e is the explosion time. This only needs to be proved $\tau_e=\infty$ a.s.

Choose m_0 to be a large integer so that S_{h0}, I_{h0}, S_{v0} and I_{v0} belongs to $[m_0^{-1},m_0]$. For any integer $m\geqslant m_0$, define stopping time by $\tau_m=\inf\{t\in[0,\tau_e):S_h(t)\notin(m^{-1},m),\text{ or } I_h(t)\notin(m^{-1},m),\text{or } S_v(t)\notin(m^{-1},m),\text{ or } I_v(t)\notin(m^{-1},m)\}$, where, $\inf\emptyset=\infty$. Due to the nondecreasing of the sequence $\{\tau_m\}$, limit $\tau_\infty=\lim_{t\to\infty}\tau_m$ exists, and $\tau_\infty\leqslant\tau_e$ a.s.

Next, we show $\tau_\infty=\infty$ a.s. If it is invalid, then there is a $T>0$ and $\epsilon\in(0,1)$ such that $\mathbb{P}\{\tau_\infty\leqslant T\}\geqslant\epsilon$. Thus, there is an integer $m_1\geqslant m_0$ such that

$$\mathbb{P}\{\tau_m\leqslant T\}\geqslant\epsilon\qquad\text{for all } m\geqslant m_1. \tag{3.1}$$

Let $N_h(t)=S_h(t)+I_h(t)$, it is obvious that $\limsup_{t\to\infty}N_h(t)\leqslant\Lambda_h/\mu_h$. Define a function V by

$$V(S_h,I_h,S_v,I_v)=(S_h+I_h)-1-\ln(S_h+I_h)+(S_v+I_v)-1-\ln(S_v+I_v).$$

Note that $V(S_h,I_h,S_v,I_v)$ is non-negative for $X(t)\in\mathbb{R}^4_+$. This yields from Itô's formula that

$$\mathrm{d}V=\mathcal{L}V\mathrm{d}t+\left(1-\frac{1}{S_v(t)+I_v(t)}\right)\left[\sigma_3S_v(t)\mathrm{dB}_3(t)+\sigma_4I_v(t)\mathrm{dB}_4(t)\right], \tag{3.2}$$

where

$$\begin{aligned}\mathcal{L}V=&\Lambda_h-(\mu_h+\psi_h)S_h(t)-(\mu_h+k+\nu)I_h(t)+\int_0^\infty\Gamma(a)S_h(t-a)\mathrm{d}a\\&+\frac{(\mu_h+\psi_h)S_h(t)}{S_h(t)+I_h(t)}-\frac{\int_0^\infty\Gamma(a)S_h(t-a)\mathrm{d}a+\int_0^\infty\pi(b)I_h(t-b)\mathrm{d}b}{S_h(t)+I_h(t)}\\&-\frac{\Lambda_h}{S_h(t)+I_h(t)}+\frac{(\mu_h+k+\nu)I_h(t)}{S_h(t)+I_h(t)}+\Lambda_v-\mu_vS_v(t)-\mu_vI_v(t)\\&-\frac{\Lambda_v}{S_v(t)+I_v(t)}+\frac{\mu_v(S_v(t)+I_v(t))}{S_v(t)+I_v(t)}+\frac{\sigma_1^2f^2(S_h(t),I_v(t))}{(S_h(t)+I_h(t))^2}\\&+\frac{\sigma_2^2g^2(S_v(t),I_h(t))}{(S_v(t)+I_v(t))^2}+\frac{\sigma_3^2S_v^2(t)+\sigma_4^2I_v^2(t)}{2(S_v(t)+I_v(t))^2}+\int_0^\infty\pi(b)I_h(t-b)\mathrm{d}b\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}V\leqslant&\Lambda_h+\Lambda_v+\mu_h+\psi_h+k+\nu+\mu_v+(\Gamma+\pi)\frac{\Lambda_h}{\mu_h}\\&+\sigma_1^2K_1^2+\sigma_2^2K_3^2+\frac{\max\{\sigma_3^2,\sigma_4^2\}}{2}.\end{aligned}$$

Here $\Gamma = \int_0^\infty \Gamma(a)\mathrm{d}a = \int_0^\infty \omega(a)\psi_h\Gamma_0(a)\mathrm{d}a \leqslant \int_0^\infty \psi_h \exp\{-\mu_h a\}\mathrm{d}a = \psi_h/\mu_h$ and $\pi = \int_0^\infty \pi(b)\mathrm{d}b = \int_0^\infty r(b)k\pi_0(b)\mathrm{d}b \leqslant \int_0^\infty k\exp\{-\mu_h b\}\mathrm{d}b = k/\mu_h$. Then it turns to

$$\begin{aligned}\mathcal{L}V \leqslant& \Lambda_h + \Lambda_v + \mu_h + \mu_v + \psi_h + k + \nu + \frac{\psi_h + k}{\mu_h}\frac{\Lambda_h}{\mu_h}\\ &+ \sigma_1^2 K_1^2 + \sigma_2^2 K_3^2 + \frac{\max\{\sigma_3^2, \sigma_4^2\}}{2} := K.\end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), and integrating both ends of (3.2) from 0 to $\tau_m \wedge T$ yields

$$\begin{aligned}&\int_0^{\tau_m\wedge T} \mathrm{d}V(S_h(t), I_h(t), S_v(t), I_v(t))\\ \leqslant& \int_0^{\tau_m\wedge T} K\mathrm{d}t + \int_0^{\tau_m\wedge T}\left(1 - \frac{1}{S_v(t)+I_v(t)}\right)[\sigma_3 S_v(t)\mathrm{dB}_3(t) + \sigma_4 I_v(t)\mathrm{dB}_4(t)].\end{aligned}$$

By taking expectations we have

$$\mathbb{E}[V(S_h(\tau_m\wedge T), \cdots, I_v(\tau_m \wedge T))] \leqslant V(S_h(0), I_h(0), S_v(0), I_v(0)) + KT.$$

Let $\Omega_m = \{\tau_m \leqslant T\}$ for $m \geqslant m_1$. From (3.1), it follows that $\mathbb{P}(\Omega_m) \geqslant \epsilon$ for $m > m_1$. For any $\omega \in \Omega_m$, at least one component of solution $(S_h(\tau_m,\omega)$, $I_h(\tau_m,\omega)$, $S_v(\tau_m,\omega)$, $I_v(\tau_m,\omega))$ equals m or $1/m$. Therefore, one get

$$\begin{aligned}\infty >& V(S_h(0), I_h(0), S_v(0), I_v(0)) + KT\\ \geqslant& \mathbb{P}\{\tau_m \leqslant T\}\min\{2m - 1 - \ln 2m, 1/2m - 1 + \ln 2m\}\\ \geqslant& \epsilon\min\big\{2m - 1 - \ln 2m, 1/2m - 1 + \ln 2m\big\}.\end{aligned}$$

This derives a contradiction $\infty > V(S_h(0), I_h(0), S_v(0), I_v(0)) + KT \geqslant \infty$ as $m \to \infty$. Therefore, one have $\tau_e = \infty$ a.s., which completes the proof. □

Theorem 3.2. *Assume* $(S_h(t), I_h(t), S_v(t), I_v(t))$ *is the solution of model* (2.3) *satisfies the initial value* $(S_{h0}, I_{h0}, S_{v0}, I_{v0}) \in \mathbb{R}_+^4$, *then* $\limsup_{t\to\infty}(S_v(t) + I_v(t)) < \infty$ *a.s. Moreover,* $\limsup_{t\to\infty}\langle S_v(t)+I_v(t)\rangle_t \leqslant \Lambda_v/\mu_v$ *a.s., where,* $\langle S_v(t)+I_v(t)\rangle_t = t^{-1}\int_0^t (S_v(s) + I_v(s))\mathrm{d}s$.

Proof. From the third and fourth equations of model (2.3), we have

$$\mathrm{d}[S_v(t) + I_v(t)] = \Lambda_v - \mu_v(S_v(t) + I_v(t)) + \sigma_3 S_v(t)\mathrm{dB}_3(t) + \sigma_4 I_v(t)\mathrm{dB}_4(t).$$

From the above equation and the principle of comparison of stochastic differential equations, it yields that

$$S_v(t) + I_v(t) \leqslant \frac{\Lambda_v}{\mu_v} + \left(S_v(0) + I_v(0) - \frac{\Lambda_v}{\mu_v}\right)e^{-\mu_v t} + M(t) := N_v(t)\ a.s., \tag{3.4}$$

where, $M(t) = \sigma_3\int_0^t e^{-\mu_v(t-s)}S_v(s)\mathrm{dB}_3(s) + \sigma_4\int_0^t e^{-\mu_v(t-s)}I_v(s)\mathrm{dB}_4(s)$. From the Definition 1.5.23 in Ref. [21], $M(t)$ is a continuous local martingale with $M(0) = 0$. Therefore, by Theorem 1.3.9 in Ref. [21], it follows that $\lim_{t\to\infty} M(t)$ exists and is finite almost surely. Thus, $\lim_{t\to\infty} N_v(t) < \infty$.

Now, we turn to the second conclusion. Define $\rho_1(t) = \int_0^t S_v(s)\mathrm{dB}_3(s)$, $\rho_2(t) = \int_0^t e^{-\mu_v(t-s)} S_v(s)\mathrm{dB}_3(s)$, $\rho_3(t) = \int_0^t I_v(s)\mathrm{dB}_4(s)$ and $\rho_4(t) = \int_0^t e^{-\mu_v(t-s)} I_v(s)\mathrm{dB}_4(s)$, it is easy to calculate the quadratic variations that

$$\begin{aligned}
\langle \rho_1(t), \rho_1(t) \rangle &= \int_0^t S_v^2(s)\mathrm{d}s \leqslant t \sup_{t\geqslant 0} S_v^2(t), \\
\langle \rho_2(t), \rho_2(t) \rangle &= \int_0^t e^{-2\mu_v(t-s)} S_v^2(s)\mathrm{d}s \leqslant t \sup_{t\geqslant 0} S_v^2(t), \\
\langle \rho_3(t), \rho_3(t) \rangle &= \int_0^t I_v^2(s)\mathrm{d}s \leqslant t \sup_{t\geqslant 0} I_v^2(t), \\
\langle \rho_4(t), \rho_4(t) \rangle &= \int_0^t e^{-2\mu_v(t-s)} I_v^2(s)\mathrm{d}s \leqslant t \sup_{t\geqslant 0} I_v^2(t).
\end{aligned}$$

By using the strong law of large number for martingales (see Theorem 1.3.4 in Ref. [21] for more detail), it can get that

$$\lim_{t\to\infty} \frac{\rho_i(t)}{t} = 0, \quad i = 1, 2, 3, 4. \tag{3.5}$$

It can be obtained by changing the order of integration that

$$\begin{aligned}
\langle M(t) \rangle_t =& \frac{\sigma_3}{t} \int_0^t \int_0^u e^{\mu_v(s-u)} S_v(s)\mathrm{dB}_3(s)\mathrm{d}u + \frac{\sigma_4}{t} \int_0^t \int_0^u e^{\mu_v(s-u)} I_v(s)\mathrm{dB}_4(s)\mathrm{d}u \\
=& \frac{\sigma_3}{\mu_v t} \left(\int_0^t S_v(s)\mathrm{dB}_3(s) - \int_0^t e^{-\mu_v(t-s)} S_v(s)\mathrm{dB}_3(s) \right) \\
&+ \frac{\sigma_4}{\mu_v t} \left(\int_0^t I_v(s)\mathrm{dB}_4(s) - \int_0^t e^{-\mu_v(t-s)} I_v(s)\mathrm{dB}_4(s) \right).
\end{aligned}$$

This yields from (3.5) that $\lim_{t\to\infty} \langle M(t) \rangle = 0$. In addition,

$$\lim_{t\to\infty} \frac{\int_0^t (S_v(0) + I_v(0) - \frac{\Lambda_v}{\mu_v}) e^{-\mu_v s}\mathrm{d}s}{t} = \lim_{t\to\infty} \frac{S_v(0) + I_v(0) - \frac{\Lambda_v}{\mu_v}}{\mu_v t} (1 - e^{-\mu_v t}) = 0.$$

Thus, from the above discussion and (3.4), we have

$$\limsup_{t\to\infty} \langle S_v(t) + I_v(t) \rangle_t \leqslant \limsup_{t\to\infty} \frac{1}{t} \int_0^t \frac{\Lambda_v}{\mu_v}\mathrm{d}s = \frac{\Lambda_v}{\mu_v}.$$

This completes the proof. □

4. The extinction of disease without relapse

The stochastic extinction of disease and the asymptotic behavior of solutions for model (2.3) without relapse will be discussed in this section. From Theorem 3.2, there is a constant $\check{N}_v$ such that $\limsup_{t\to\infty} N_v(t) \leqslant \check{N}_v$.

Theorem 4.1. *Let $(S_h(t), I_h(t), S_v(t), I_v(t))$ be the solution of model (2.3) with $r(b) = 0$ satisfies the initial value $(S_h(0),\ I_h(0),\ S_v(0),\ I_v(0)) \in \mathbb{R}_+^4$. If one of the following conditions holds:*

(i) $\widetilde{\mathcal{R}}_0 < 1$ *and* $\sigma_1^2 \leqslant \beta_1/\big(\partial f(\Lambda_h/\mu_h, 0)/\partial I_v\big)$, $\sigma_2^2 \leqslant \beta_2/\big(\partial g(\check{N}_v, 0)/\partial I_h\big)$, *where*

$$\widetilde{\mathcal{R}}_0 = \frac{\beta_1 \frac{\partial f(\Lambda_h/\mu_h, 0)}{\partial I_v} + \beta_2 \frac{\partial g(\check{N}_v, 0)}{\partial I_h}}{\min\{\mu_h + k + \nu, \mu_v\} + \frac{\sigma_1^2}{2}\left(\frac{\partial f(\Lambda_h/\mu_h, 0)}{\partial I_v}\right)^2 + \frac{\sigma_2^2}{2}\left(\frac{\partial g(\check{N}_v, 0)}{\partial I_h}\right)^2}; \tag{4.1}$$

(ii) $\beta_1^2/2\sigma_1^2 + \beta_2^2/2\sigma_2^2 < \min\{\mu_h + k + \nu, \mu_v\}$;

then $\limsup_{t\to\infty} \ln(I_h(t) + I_v(t))/t < 0$ *a.s.*

Proof. For any $\eta > 0$, there exists a large enough $T_0 > 0$ such that $S_h(t) < \Lambda_h/\mu_h + \eta$ and $S_v(t) < \check{N}_v + \eta$ for $t \in [T_0, \infty)$. Further, by using the L'Hospital principle, it is easy to prove

$$\frac{f(S_h, I_v)}{I_v} \in \left(0, \frac{\partial f(\Lambda_h/\mu_h + \eta, 0)}{\partial I_v}\right], \quad \frac{g(S_v, I_h)}{I_h} \in \left(0, \frac{\partial g(\check{N}_v + \eta, 0)}{\partial I_h}\right]. \tag{4.2}$$

When $r(b) = 0$, let $V(t) = \ln(I_h(t) + I_v(t))$, from the Itô's formula, we obtain

$$\begin{aligned}
\mathrm{d}V =& \Bigg\{ \frac{1}{I_h(t) + I_v(t)}\Big(\beta_1 f(S_h(t), I_v(t)) - (\mu_h + k + \nu) I_h(t) + \beta_2 g(S_v(t), I_h(t)) \\
&- \mu_v I_v(t)\Big) - \frac{1}{2(I_h(t) + I_v(t))^2}\Big(\sigma_1^2 f^2(S_h(t), I_v(t)) + \sigma_2^2 g^2(S_v(t), I_h(t)) \\
&+ \sigma_4^2 I_v^2(t)\Big)\Bigg\}\mathrm{d}t + \frac{1}{I_h(t) + I_v(t)}\Big(\sigma_1 f(S_h(t), I_v(t))\mathrm{dB}_1(t) \\
&+ \sigma_2 g(S_v(t), I_h(t))\mathrm{dB}_2(t) + \sigma_4 I_v(t)\mathrm{dB}_4(t)\Big).
\end{aligned}$$

Directly integrating the above expression form 0 to t and dividing t, we can get

$$\begin{aligned}
&\frac{\ln(I_h(t) + I_v(t))}{t} \\
\leqslant& \frac{\ln(I_h(0) + I_v(0))}{t} + \frac{\beta_1 + \epsilon}{t}\int_0^t \frac{f(S_h(s), I_v(s))}{I_h(s) + I_v(s)}\mathrm{d}s - \frac{\mu_h + k + \nu}{t} \\
&\times \int_0^t \frac{I_h(s)\mathrm{d}s}{I_h(s) + I_v(s)} + \frac{\beta_2 + \epsilon}{t}\int_0^t \frac{g(S_v(s), I_h(s))}{I_h(s) + I_v(s)}\mathrm{d}s - \frac{\mu_v}{t}\int_0^t \frac{I_v(s)\mathrm{d}s}{I_h(s) + I_v(s)} \\
&- \frac{\sigma_1^2}{2t}\int_0^t \frac{f^2(S_h(s), I_v(s))}{(I_h(s) + I_v(s))^2}\mathrm{d}s - \frac{\sigma_2^2}{2t}\int_0^t \frac{g^2(S_v(s), I_h(s))}{(I_h(s) + I_v(s))^2}\mathrm{d}s \\
&- \frac{\sigma_4^2}{2t}\int_0^t \frac{I_v^2(s)}{(I_h(s) + I_v(s))^2}\mathrm{d}s + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{M_4(t)}{t} \\
\leqslant& \frac{\ln(I_h(0) + I_v(0))}{t} + \frac{\beta_1 + \epsilon}{t}\int_0^t \frac{f(S_h(s), I_v(s))}{I_h(s) + I_v(s)}\mathrm{d}s - \min\{\mu_h + k + \nu, \mu_v\} \\
&+ \frac{\beta_2 + \epsilon}{t}\int_0^t \frac{g(S_v(s), I_h(s))}{I_h(s) + I_v(s)}\mathrm{d}s - \frac{\sigma_1^2}{2t}\int_0^t \frac{f^2(S_h(s), I_v(s))}{(I_h(s) + I_v(s))^2}\mathrm{d}s \\
&- \frac{\sigma_2^2}{2t}\int_0^t \frac{g^2(S_v(s), I_h(s))}{(I_h(s) + I_v(s))^2}\mathrm{d}s + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{M_4(t)}{t},
\end{aligned}$$

where, $\epsilon > 0$ and

$$M_1(t) = \sigma_1 \int_0^t \frac{f(S_h(s), I_v(s))\mathrm{dB}_1(s)}{I_h(s) + I_v(s)}, \quad M_2(t) = \sigma_2 \int_0^t \frac{g(S_v(s), I_h(s))\mathrm{dB}_2(s)}{I_h(s) + I_v(s)},$$

$$M_4(t) = \sigma_4 \int_0^t \frac{I_v(s)\mathrm{dB}_4(s)}{I_h(s)+I_v(s)}$$

are local martingales that satisfy

$$\begin{aligned}
\langle M_1, M_1\rangle &= \sigma_1^2 \int_0^t \left(\frac{f(S_h(s), I_v(s))\mathrm{dB}_1(s)}{I_h(s)+I_v(s)}\right)^2 \leqslant \sigma_1^2 K_2^2 t,\\
\langle M_2, M_2\rangle &= \sigma_2^2 \int_0^t \left(\frac{g(S_v(s), I_h(s))\mathrm{dB}_2(s)}{I_h(s)+I_v(s)}\right)^2 \leqslant \sigma_2^2 K_4^2 t,\\
\langle M_4, M_4\rangle &= \sigma_4^2 \int_0^t \left(\frac{I_v(s)\mathrm{dB}_4(s)}{I_h(s)+I_v(s)}\right)^2 \leqslant \sigma_4^2 t.
\end{aligned}$$

By the strong law of large numbers of martingale, one have $\limsup_{t\to\infty} M_i(t)/t = 0$, $i = 1, 2, 4$.

Next, we define two functions $G_1(x) = -\sigma_1^2 x^2/2 + (\beta_1+\epsilon)x - \min\{\mu_h+k+\nu, \mu_v\}$, $G_2(x) = -\sigma_2^2 x^2/2 + (\beta_2+\epsilon)x$, where $G_i(x)$ is monotonically increasing for $x \in [0, (\beta_i+\epsilon)/\sigma_i^2)$ $(i = 1,\ 2)$. Since $\partial f(\Lambda_h/\mu_h, 0)/\partial I_v \leqslant \beta_1/\sigma_1^2$ and $\partial g(\check{N}_v, 0)/\partial I_h \leqslant \beta_2/\sigma_2^2$, we can choose $\eta > 0$ and $\eta \leqslant \epsilon$ such that $\partial f(\Lambda_h/\mu_h+\eta, 0)/\partial I_v \leqslant (\beta_1+\epsilon)/\sigma_1^2$ and $\partial g(\check{N}_v+\eta, 0)/\partial I_h \leqslant (\beta_2+\epsilon)/\sigma_2^2$. Obviously, from (4.2) we have, for $t \geqslant T_0$,

$$\begin{aligned}
G_1\left(\frac{f(S_h(t), I_v(t))}{I_v(t)}\right) &\leqslant G_1\left(\frac{\partial f(\Lambda_h/\mu_h+\eta, 0)}{\partial I_v}\right),\\
G_2\left(\frac{g(S_v(t), I_h(t))}{I_h(t)}\right) &\leqslant G_2\left(\frac{\partial g(\check{N}_v+\eta, 0)}{\partial I_h}\right).
\end{aligned}$$

Therefore, it yields from the above discussion that

$$\begin{aligned}
&\frac{\ln(I_h(t)+I_v(t))}{t}\\
\leqslant& \frac{\ln(I_h(0)+I_v(0))}{t} + \frac{1}{t}\int_0^t G_1\left(\frac{f(S_h(s), I_v(s))}{I_h(s)+I_v(s)}\right)\mathrm{d}s\\
&+ \frac{1}{t}\int_0^t G_2\left(\frac{g(S_v(s), I_h(s))}{I_h(s)+I_v(s)}\right)\mathrm{d}s + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{M_4(t)}{t}\\
\leqslant& \frac{\ln(I_h(0)+I_v(0))}{t} + \frac{1}{t}\int_0^t G_1\left(\frac{f(S_h(s), I_v(s))}{I_v(s)}\right)\mathrm{d}s\\
&+ \frac{1}{t}\int_0^t G_2\left(\frac{g(S_v(s), I_h(s))}{I_h(s)}\right)\mathrm{d}s + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{M_4(t)}{t}\\
\leqslant& \frac{\ln(I_h(0)+I_v(0))}{t} + \frac{1}{t}\int_0^{T_0} G_1\left(\frac{f(S_h(s), I_v(s))}{I_v(s)}\right)\mathrm{d}s\\
&+ G_1\left(\frac{\partial f(\Lambda_h/\mu_h+\eta, 0)}{\partial I_v}\right)\frac{t-T_0}{t} + \frac{1}{t}\int_0^{T_0} G_2\left(\frac{g(S_v(s), I_h(s))}{I_h(s)}\right)\mathrm{d}s\\
&+ G_2\left(\frac{\partial g(\check{N}_v+\eta, 0)}{\partial I_h}\right)\frac{t-T_0}{t} + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{M_4(t)}{t}
\end{aligned}$$

for $t \geqslant T_0$. Thus,

$$\limsup_{t\to\infty} \frac{\ln(I_h(t)+I_v(t))}{t}$$

$$
\begin{aligned}
\leqslant & G_1\left(\frac{\partial f(\Lambda_h/\mu_h,0)}{\partial I_v}\right)+G_2\left(\frac{\partial g(\check{N}_v,0)}{\partial I_h}\right)\\
= & (\beta_1+\epsilon)\frac{\partial f(\Lambda_h/\mu_h+\eta,0)}{\partial I_v}+(\beta_2+\epsilon)\frac{\partial g(\check{N}_v+\eta,0)}{\partial I_h}\\
& -\frac{\sigma_1^2}{2}\left(\frac{\partial f(\Lambda_h/\mu_h+\eta,0)}{\partial I_v}\right)^2-\frac{\sigma_2^2}{2}\left(\frac{\partial g(\check{N}_v+\eta,0)}{\partial I_h}\right)^2-\min\{\mu_h+k+\nu,\mu_v\}.
\end{aligned}
$$

From the arbitrariness of ϵ and η, and (4.1), one can get $\limsup_{t\to\infty}\ln\ (I_h(t)+I_v(t))/t<0$. This proves (i).

Finally, we consider (ii). Since $G_1(x)$ and $G_2(x)$ get their maximum values

$$
G_1^{\max}(x)=\frac{(\beta_1+\epsilon)^2}{2\sigma_1^2}-\min\{\mu_h+k+\nu,\mu_v\},\ G_2^{\max}(x)=\frac{(\beta_2+\epsilon)^2}{2\sigma_2^2}
$$

at $x=(\beta_1+\epsilon)/\sigma_1^2$ and $x=(\beta_2+\epsilon)/\sigma_2^2$, respectively, $G_1(x)$ and $G_2(x)$ satisfy

$$
\begin{aligned}
G_1\left(\frac{f(S_h,I_v)}{I_v(t)}\right)&\leqslant\frac{(\beta_1+\epsilon)^2}{2\sigma_1^2}-\min\{\mu_h+k+\nu,\mu_v\},\\
G_2\left(\frac{g(S_v,I_h)}{I_h(t)}\right)&\leqslant\frac{(\beta_2+\epsilon)^2}{2\sigma_2^2}.
\end{aligned}
$$

Discussions similar to the case (i), it follows that

$$
\limsup_{t\to\infty}\frac{\ln(I_h(t)+I_v(t))}{t}\leqslant\frac{\beta_1^2}{2\sigma_1^2}+\frac{\beta_2^2}{2\sigma_2^2}-\min\{\mu_h+k+\nu,\mu_v\}<0.
$$

This is (ii). The proof is completed. □

It is obvious that model (2.3) exists a disease-free equilibrium $E_0(S_h^0,0,S_v^0,0)$ for $\sigma_3=\sigma_4=0$, where

$$
S_h^0=\frac{\Lambda_h}{\mu_h+\psi_h-\int_0^\infty\Gamma(a)\mathrm{d}a},\qquad S_v^0=\frac{\Lambda_v}{\mu_v}.
$$

For $I_h(t)\equiv0$ and $I_v(t)\equiv0$, model (2.3) reduce as

$$
\begin{cases}
\mathrm{d}S_h(t)=\left[\Lambda_h-(\mu_h+\psi_h)S_h(t)+\displaystyle\int_0^\infty\Gamma(a)S_h(t-a)\mathrm{d}a\right]\mathrm{d}t,\\
\mathrm{d}S_v(t)=\left[\Lambda_v-\mu_vS_v(t)\right]\mathrm{d}t.
\end{cases}\tag{4.3}
$$

On the stability of equilibrium (S_h^0,S_v^0) of model (4.3), similar to the proof of Theorem 3.1 in Ref. [37], the following result is obvious.

Theorem 4.2. *The equilibrium (S_h^0,S_v^0) of (4.3) is globaly asymptotically stable .*

From Theorems 4.1 and 4.2, for $\sigma_3=\sigma_4=0$, we have the follow result.

Theorem 4.3. *Assume that $r(b)=\sigma_3=\sigma_4=0$ and $(S_h(t),I_h(t),S_v(t),I_v(t))$ is the solution of model (2.3) satisfies the initial value $(S_h(0),\ I_h(0),\ S_v(0),\ I_v(0))\in\mathbb{R}_+^4$. Assume that one of the following conditions is met,*

(i) $\widetilde{\mathcal{R}}_0<1$, $\sigma_1^2\leqslant\beta_1/\big(\partial f(S_h^0,0)/\partial I_v\big)$, $\sigma_2^2\leqslant\beta_2/\big(\partial g(S_v^0,0)/\partial I_h\big)$, *where*

$$
\widetilde{\mathcal{R}}_0=\frac{\beta_1\frac{\partial f(S_h^0,0)}{\partial I_v}+\beta_2\frac{\partial g(S_v^0,0)}{\partial I_h}}{\min\{\mu_h+k+\nu,\mu_v\}+\frac{\sigma_1^2}{2}\left(\frac{\partial f(S_h^0,0)}{\partial I_v}\right)^2+\frac{\sigma_2^2}{2}\left(\frac{\partial g(S_v^0,0)}{\partial I_h}\right)^2};
$$

(ii) $\beta_1^2/2\sigma_1^2 + \beta_2^2/2\sigma_2^2 < \min\{\mu_h + k + \nu, \mu_v\}$*;*

then $\lim_{t\to\infty} S_h(t) = S_h^0$ *a.s.,* $\lim_{t\to\infty} S_v(t) = S_v^0$ *a.s.*

5. The existence of stationary distribution

Since there is no equilibrium point in model (2.3), it is very necessary to consider the existence of stationary distribution for model (2.3), which means the stochastic persistence of disease. Let $X(t) \in \mathbb{R}^d$ is a homogeneous Markov process and satisfies

$$\mathrm{d}X(t) = b(X)\mathrm{d}t + \sum_{r=1}^{k} g_r(X)\mathrm{dB}_r(t), \tag{5.1}$$

and the diffusion matrix of $X(t)$ is defined as follows

$$\widetilde{A}(X) = (a_{ij}(X)),\ a_{ij}(X) = \sum_{r=1}^{k} g_r^i(X) g_r^j(X).$$

Lemma 5.1 is a criterion on the existence of stationary distribution of (5.1).

Lemma 5.1 ([19,41]). *The model* (5.1) *is positive recurrent if there is a bounded open subset* $\mathbb{D}$ *of* $\mathbb{R}^d$ *with a regular boundary (i.e., smooth), and* (*i*) *there is a constant* $\kappa > 0$ *such that* $a_{\iota\iota}(x) \geqslant \kappa$ *for* $x \in \mathbb{D}$*;* (*ii*) *there is a function* $V : \mathbb{D}^c \to \mathbb{R}_+$ *with second-order continuous derivative such that* $\mathcal{L}V(x) \leqslant -\theta$ *for* $x \in \mathbb{D}^c$, $\theta > 0$. *Then, stochastic process* $X(t)$ *admits a unique stationary distribution* $\mu(\cdot)$ *satisfies* $\lim_{t\to+\infty} \mathbb{P}(t, x, \mathbb{B}) = \mu(\mathbb{B})$ *for any Borel set* $\mathbb{B} \in \mathbb{R}^d$*, and*

$$\mathbb{P}_x\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))\mathrm{d}t = \int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x)\right\} = 1$$

for all $x \in \mathbb{R}^d$*, where* $f(\cdot)$ *is an integrable function with respect to* $\mu(\cdot)$*.*

Next, we turn to the existence of unique stationary distribution for model (2.3).

Theorem 5.1. *Assume that* $r(b) = 0$*. If* $\mathcal{R}_0^s > 1$ *and*

$$(\mu_h \wedge \mu_v) > \left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \sigma_3^2/2 \vee \sigma_4^2/2\right),$$

where,

$$\mathcal{R}_0^s = 4\sqrt{\frac{\Lambda_h \Lambda_v \mu_v (\mu_h + \psi_h)}{\left(\Lambda_h + \Lambda_v + \beta_1 M_1 + \frac{\sigma_1^2}{2}K_1^2 + \frac{\psi_h \Lambda_h}{\mu_h^2}\right)^3 \left(\mu_v + \frac{\sigma_3^2}{2} + \beta_2 K_3 + \frac{\sigma_2^2}{2}K_3^2\right)}}$$

and $\rho_h = \psi_h + \mu_h$*, then solution* $(S_h(t), I_h(t), S_v(t), I_v(t))$ *of model* (2.3) *with the initial value* $(S_h(0),\ I_h(0),\ S_v(0),\ I_v(0)) \in \Gamma = \{(S_h, I_h, S_v, I_v) : (S_h, I_h, S_v, I_v) \in \mathbb{R}_+^4, S_h + I_h + S_v + I_v \leqslant \Lambda_h/\mu_h + \check{N}_v\}$ *is positive recurrent and exists a stationary distribution in* Γ *which is unique and ergodic.*

Proof. Define a bounded open subset of Γ as $\mathbb{D}_\varepsilon = \{(S_h, I_h, S_v, I_v) \in \Gamma : \varepsilon \leqslant S_h, I_h, S_v, I_v \leqslant 1/\varepsilon\}$, where ε is a small enough positive constant. The diffusion matrix associated with model (2.3) is given by

$$\tilde{A} = (a_{ij})_{4\times 4} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \sigma_1^2 f^2(S_h, I_v) & -\sigma_1^2 f^2(S_h, I_v) \\ -\sigma_1^2 f^2(S_h, I_v) & \sigma_1^2 f^2(S_h, I_v) \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} \sigma_2^2 g^2(S_v, I_h) + \sigma_3^2 S_v^2 & -\sigma_2^2 g^2(S_v, I_h) \\ -\sigma_2^2 g^2(S_v, I_h) & \sigma_2^2 g^2(S_v, I_h) + \sigma_4^2 I_v^2 \end{pmatrix}.$$

Since $\overline{\mathbb{D}}_\varepsilon \subset \mathbb{R}_+^4$, then

$$\sigma_1^2 f^2(S_h, I_v) = \sigma_1^2 \frac{f^2(S_h, I_v)}{I_v^2} I_v^2 \geqslant \sigma_1^2 \frac{f^2(\varepsilon, I_v)}{I_v^2} I_v^2 \geqslant \varepsilon^4 \sigma_1^2 f^2\left(\varepsilon, \frac{1}{\varepsilon}\right),$$
$$\sigma_2^2 g^2(S_v, I_h) = \sigma_2^2 \frac{g^2(S_v, I_h)}{I_h^2} I_h^2 \geqslant \sigma_2^2 \frac{g^2(\varepsilon, I_h)}{I_h^2} I_h^2 \geqslant \varepsilon^4 \sigma_2^2 g^2\left(\varepsilon, \frac{1}{\varepsilon}\right).$$

Therefore, one verify the condition (i) of Lemma 5.1.

Next, we turn to condition (ii) of Lemma 5.1. Define a function $\widetilde{V} : \mathbb{R}_+^4 \to \mathbb{R}$ as $\widetilde{V}(S_h, I_h, S_v, I_v) = MV_1 + NV_2 + V_3 + V_4$, where $V_1 = -\ln S_h - c\ln S_v + S_h + S_v - I_h$, $V_2 = -\ln S_h - c\ln S_v + S_h + S_v - I_v$, $V_3 = -\ln S_h - \ln S_v$ and

$$V_4 = \frac{1}{\theta+2}(S_h + I_h + S_v + I_v)^{\theta+2},$$
$$c = \frac{\mu_h + \psi_h + \beta_1 K_1 + \Lambda_h + \Lambda_v + \frac{\psi_h \Lambda_h}{\mu_h^2} + \frac{\sigma_1^2}{2} K_1^2}{\mu_v + \beta_2 K_3 + \frac{\sigma_3^2}{2} + \frac{\sigma_2^2}{2} K_3^2}.$$

Here, M and N are sufficiently large positive constants which satisfy

$$-M\lambda + F_2 \leqslant -2, \qquad -N\lambda + F_3 \leqslant -2, \tag{5.2}$$

and

$$\lambda = 2\left(\mu_h + \psi_h + \beta_1 K_1 + \Lambda_h + \frac{\psi_h \Lambda_h}{\mu_h^2} + \Lambda_v + \frac{\sigma_1^2}{2} K_1^2\right)\left(\sqrt{\mathcal{R}_0^s} - 1\right), \tag{5.3}$$

$$\begin{aligned} F_1 = \sup_{(S_h, I_h, S_v, I_v) \in \mathbb{R}_+^4} & \Bigg\{ M(\mu_h + k + \nu) I_h + N\mu_v I_v - \frac{1}{2}\Big[(\mu_h \wedge \mu_v) - (\theta+1) \\ & \times \left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \frac{1}{2}\sigma_3^2 \vee \frac{1}{2}\sigma_4^2\right)\Big] \left(S_h^{\theta+2} + I_h^{\theta+2} + S_v^{\theta+2} + I_v^{\theta+2}\right) \\ & + \mu_h + \psi_h + \beta_1 K_1 + \frac{\sigma_1^2}{2} K_1^2 + \beta_2 K_3 + \mu_v + \frac{\sigma_2^2}{2} K_3^2 + \frac{\sigma_3^2}{2} + B \Bigg\}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} F_2 = \sup_{(S_h, I_h, S_v, I_v) \in \mathbb{R}_+^4} & \Bigg\{ N\mu_v I_v - \frac{1}{2}\Big[(\mu_h \wedge \mu_v) - (\theta+1) \\ & \times \left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \frac{1}{2}\sigma_3^2 \vee \frac{1}{2}\sigma_4^2\right)\Big] \left(S_h^{\theta+2} + I_h^{\theta+2} + S_v^{\theta+2} + I_v^{\theta+2}\right) \end{aligned} \tag{5.5}$$

$$+\mu_h+\psi_h+\beta_1K_1+\frac{\sigma_1^2}{2}K_1^2+\beta_2K_3+\mu_v+\frac{\sigma_2^2}{2}K_3^2+\frac{\sigma_3^2}{2}+B\bigg\},$$

$$\begin{aligned}F_3=&\sup_{(S_h,I_h,S_v,I_v)\in\mathbb{R}_+^4}\bigg\{M(\mu_h+k+\nu)I_h-\frac{1}{2}\bigg[(\mu_h\wedge\mu_v)-(\theta+1)\\&\times\left(\sigma_1^2K_1^2\vee\sigma_2^2K_4^2\vee\frac{1}{2}\sigma_3^2\vee\frac{1}{2}\sigma_4^2\right)\bigg]\left(S_h^{\theta+2}+I_h^{\theta+2}+S_v^{\theta+2}+I_v^{\theta+2}\right)\\&+\mu_h+\psi_h+\beta_1K_1+\frac{\sigma_1^2}{2}K_1^2+\beta_2K_3+\mu_v+\frac{\sigma_2^2}{2}K_3^2+\frac{\sigma_3^2}{2}+B\bigg\},\end{aligned}\tag{5.6}$$

$$\begin{aligned}F_4=&\sup_{(S_h,I_h,S_v,I_v)\in\mathbb{R}_+^4}\bigg\{M(\mu_h+k+\nu)I_h+N\mu_vI_v-\frac{1}{4}\bigg[(\mu_h\wedge\mu_v)-(\theta+1)\\&\times\left(\sigma_1^2K_1^2\vee\sigma_2^2K_4^2\vee\frac{1}{2}\sigma_3^2\vee\frac{1}{2}\sigma_4^2\right)\bigg]\left(S_h^{\theta+2}+I_h^{\theta+2}+S_v^{\theta+2}+I_v^{\theta+2}\right)\\&+\mu_h+\psi_h+\beta_1K_1+\frac{\sigma_1^2}{2}K_1^2+\beta_2K_3+\mu_v+\frac{\sigma_2^2}{2}K_3^2+\frac{\sigma_3^2}{2}+B\bigg\},\end{aligned}\tag{5.7}$$

$$\begin{aligned}B=&\sup_{(S_h,I_h,S_v,I_v)\in\mathbb{R}_+^4}\bigg\{(S_h+I_h+S_v+I_v)^{\theta+1}\left(\Lambda_h+\Lambda_v+\frac{\Lambda_h(\psi_h+k)}{\mu_h^2}\right)\\&-\frac{1}{2}\bigg[(\mu_h\wedge\mu_v)-(\theta+1)\left(\sigma_1^2K_1^2\vee\sigma_2^2K_4^2\vee\frac{1}{2}\sigma_3^2\vee\frac{1}{2}\sigma_4^2\right)\bigg]\\&\times(S_h+I_h+S_v+I_v)^{\theta+2}\bigg\},\end{aligned}\tag{5.8}$$

and $\theta>0$ satisfying $(\mu_h\wedge\mu_v)>(\theta+1)(\sigma_1^2K_1^2\vee\sigma_2^2K_4^2\vee\sigma_3^2/2\vee\sigma_4^2/2)$. Due to

$$\liminf_{\substack{k\to\infty\\(S_h,I_h,S_v,I_v)\in\mathbb{R}_+^4\setminus\mathbb{D}_k}}\widetilde{V}(S_h,I_h,S_v,I_v)=+\infty,$$

then $\widetilde{V}(S_h,I_h,S_v,I_v)$ has a minimum point $(\bar{S}_h,\bar{I}_h,\bar{S}_v,\bar{I}_v)\in\mathbb{R}_+^4$. So, one can construct a function $V:\mathbb{R}_+^4\to\mathbb{R}_+$ as $V(S_h,I_h,S_v,I_v)=MV_1+NV_2+V_3+V_4-\widetilde{V}(\bar{S}_h,\bar{I}_h,\bar{S}_v,\bar{I}_v)$. It can be obtained by direct calculation that

$$\begin{aligned}\mathcal{L}V_1=&-\frac{1}{S_h}\left[\Lambda_h-\rho_hS_h-\beta_1f(S_h,I_v)+\int_0^\infty\Gamma(a)S_h(t-a)\mathrm{d}a\right]\\&-\frac{c}{S_v}\left[\Lambda_v-\beta_2g(S_v,I_h)-\mu_vS_v\right]+\frac{c\sigma_2^2}{2}\frac{g^2(S_v,I_h)}{S_v^2}+\frac{c\sigma_3^2}{2}+\Lambda_h-\rho_hS_h\\&-\beta_1f(S_h,I_v)+\int_0^\infty\Gamma(a)S_h(t-a)\mathrm{d}a+\Lambda_v-\beta_2g(S_v,I_h)-\mu_vS_v\\&-\beta_1f(S_h,I_v)+(\mu_h+k+\nu)I_h-\int_0^\infty\pi(b)I_h(t-b)\mathrm{d}b+\frac{\sigma_1^2}{2}\frac{f^2(S_h,I_v)}{S_h^2}\\\leqslant&-\frac{\Lambda_h}{S_h}+\rho_h+\beta_1K_1-\frac{c\Lambda_v}{S_v}+c\beta_2K_3+c\mu_v+\Lambda_h-\rho_hS_h\\&+\frac{\psi_h\Lambda_h}{\mu_h^2}+\Lambda_v-\mu_vS_v+(\mu_h+k+\nu)I_h+\frac{\sigma_1^2}{2}K_1^2+\frac{c\sigma_2^2}{2}K_3^2+\frac{c\sigma_3^2}{2}\\\leqslant&-4\sqrt[4]{c\Lambda_h\Lambda_v\mu_v\rho_h}+\rho_h+\beta_1K_1+\Lambda_h+\frac{\psi_h\Lambda_h}{\mu_h^2}+\Lambda_v\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma_1^2}{2}K_1^2+c\left(\beta_2K_3+\mu_v+\frac{\sigma_2^2}{2}K_3^2+\frac{\sigma_3^2}{2}\right)+(\mu_h+k+\nu)I_h\\
=&-2\big(\rho_h+\beta_1K_1+\Lambda_h+\frac{\psi_h\Lambda_h}{\mu_h^2}+\Lambda_v+\frac{\sigma_1^2}{2}K_1^2\big)\big(\sqrt{\mathcal{R}_0^s}-1\big)\\
&+(\mu_h+k+\nu)I_h\\
:=&-\lambda+(\mu_h+k+\nu)I_h,
\end{aligned}
$$

where λ is given by equation (5.3). Similarly

$$
\begin{aligned}
\mathcal{L}V_2=&-\frac{1}{S_h}\left[\Lambda_h-\rho_hS_h-\beta_1f(S_h,I_v)+\int_0^\infty\Gamma(a)S_h(t-a)\mathrm{d}a\right]\\
&-\frac{c}{S_v}\left[\Lambda_v-\beta_2g(S_v,I_h)-\mu_vS_v\right]+\frac{c\sigma_2^2}{2}\frac{g^2(S_v,I_h)}{S_v^2}+\frac{c\sigma_3^2}{2}+\Lambda_h\\
&-\rho_hS_h+\int_0^\infty\Gamma(a)S_h(t-a)\mathrm{d}a+\Lambda_v-\beta_2g(S_v,I_h)\\
&-\mu_vS_v-\beta_2g(S_v,I_h)+\mu_vI_v+\frac{\sigma_1^2}{2}\frac{f^2(S_h,I_v)}{S_h^2}-\beta_1f(S_h,I_v)\\
\leqslant&-\frac{\Lambda_h}{S_h}+\rho_h+\beta_1K_1-\frac{c\Lambda_v}{S_v}+c\beta_2K_3+c\mu_v+\Lambda_h-\rho_hS_h+\frac{\psi_h\Lambda_h}{\mu_h^2}\\
&+\Lambda_v-\mu_vS_v+\mu_vI_v+\frac{\sigma_1^2}{2}K_1^2+\frac{c\sigma_2^2}{2}K_3^2+\frac{c\sigma_3^2}{2}\\
\leqslant&-4\sqrt[4]{c\Lambda_h\Lambda_v\mu_v\rho_h}+\rho_h+\beta_1K_1+\Lambda_h+\frac{\psi_h\Lambda_h}{\mu_h^2}\\
&+\Lambda_v+\frac{\sigma_1^2}{2}K_1^2+c\left(\beta_2K_3+\mu_v+\frac{\sigma_2^2}{2}K_3^2+\frac{\sigma_3^2}{2}\right)+\mu_vI_v\\
=&-2\left(\rho_h+\beta_1K_1+\Lambda_h+\frac{\psi_h\Lambda_h}{\mu_h^2}+\Lambda_v+\frac{\sigma_1^2}{2}K_1^2\right)\left(\sqrt{\mathcal{R}_0^s}-1\right)+\mu_vI_v\\
=&-\lambda+\mu_vI_v,\\
\mathcal{L}V_3=&-\frac{1}{S_h}\left[\Lambda_h-\rho_hS_h-\beta_1f(S_h,I_v)+\int_0^\infty\Gamma(a)S_h(t-a)\mathrm{d}a\right]+\frac{\sigma_1^2}{2}\\
&\times\frac{f^2(S_h,I_v)}{S_h^2}-\frac{1}{S_v}\left[\Lambda_v-\beta_2g(S_v,I_h)-\mu_vS_v\right]+\frac{\sigma_2^2}{2}\frac{g^2(S_v,I_h)}{S_v^2}+\frac{\sigma_3^2}{2}\\
\leqslant&-\frac{\Lambda_h}{S_h}-\frac{\Lambda_v}{S_v}+\rho_h+\beta_1K_1+\frac{\sigma_1^2}{2}K_1^2+\beta_2K_3+\mu_v+\frac{\sigma_2^2}{2}K_3^2+\frac{\sigma_3^2}{2},\\
\mathcal{L}V_4=&(S_h+I_h+S_v+I_v)^{\theta+1}\bigg[\Lambda_h-\rho_hS_h+\int_0^\infty\Gamma(a)S_h(t-a)\mathrm{d}a\\
&-(\mu_h+k+\nu)I_h+\int_0^\infty\pi(b)I_h(t-b)\mathrm{d}b+\Lambda_v-\mu_v(S_v+I_v)\bigg]+(\theta+1)\\
&\times(S_h+I_h+S_v+I_v)^\theta\left[\sigma_1^2f^2(S_h,I_v)+\sigma_2^2g^2(S_v,I_h)+\frac{\sigma_3^2}{2}S_v^2+\frac{\sigma_4^2}{2}I_v^2\right]\\
\leqslant&(S_h+I_h+S_v+I_v)^{\theta+1}\left(\Lambda_h+\Lambda_v+\frac{\Lambda_h(\psi_h+k)}{\mu_h^2}\right)\\
&+(\theta+1)(S_h+I_h+S_v+I_v)^\theta\left(\sigma_1^2K_1^2\vee\sigma_2^2K_4^2\vee\frac{1}{2}\sigma_3^2\vee\frac{1}{2}\sigma_4^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times (S_h+I_h+S_v+I_v)^2-(S_h+I_h+S_v+I_v)^{\theta+2}(\mu_h\wedge\mu_v)\\
\leqslant &B-\frac{1}{2}\left[(\mu_h\wedge\mu_v)-(\theta+1)\left(\sigma_1^2K_1^2\vee\sigma_2^2K_4^2\vee\frac{1}{2}\sigma_3^2\vee\frac{1}{2}\sigma_4^2\right)\right]\\
&\times\left(S_h^{\theta+2}+I_h^{\theta+2}+S_v^{\theta+2}+I_v^{\theta+2}\right),
\end{aligned}
$$

where B is given by (5.8). From the above calculation, one can get that

$$
\begin{aligned}
\mathcal{L}V\leqslant &-M\lambda+M(\mu_h+k+\nu)I_h-N\lambda+N\mu_vI_v-\frac{\Lambda_h}{S_h}-\frac{\Lambda_v}{S_v}\\
&-\frac{1}{2}\left[(\mu_h\wedge\mu_v)-(\theta+1)\left(\sigma_1^2K_1^2\vee\sigma_2^2K_4^2\vee\frac{1}{2}\sigma_3^2\vee\frac{1}{2}\sigma_4^2\right)\right]\\
&\times\left(S_h^{\theta+2}+I_h^{\theta+2}+S_v^{\theta+2}+I_v^{\theta+2}\right)+\mu_h+\psi_h+\beta_1K_1\\
&+\frac{\sigma_1^2}{2}K_1^2+\beta_2K_3+\mu_v+\frac{\sigma_2^2}{2}K_3^2+\frac{\sigma_3^2}{2}+B.
\end{aligned}
$$

We claim $\mathcal{L}V(S_h,I_h,S_v,I_v)\leqslant -1$ on $\mathbb{R}_+^4\setminus\mathbb{D}_\varepsilon$, this is equivalent to prove it on the following eight regions

$$
\begin{aligned}
&\mathbb{D}_1=\{(S_h,I_h,S_v,I_v)\mathbb{R}_+^4:\ S_h<\varepsilon\}, &&\mathbb{D}_2=\{(S_h,I_h,S_v,I_v)\mathbb{R}_+^4:\ I_h<\varepsilon\},\\
&\mathbb{D}_3=\{(S_h,I_h,S_v,I_v)\mathbb{R}_+^4:\ S_v<\varepsilon\}, &&\mathbb{D}_4=\{(S_h,I_h,S_v,I_v)\mathbb{R}_+^4:\ I_v<\varepsilon\},\\
&\mathbb{D}_5=\{(S_h,I_h,S_v,I_v)\mathbb{R}_+^4:\ S_h>1/\varepsilon\}, &&\mathbb{D}_6=\{(S_h,I_h,S_v,I_v)\mathbb{R}_+^4:\ I_h>1/\varepsilon\},\\
&\mathbb{D}_7=\{(S_h,I_h,S_v,I_v)\mathbb{R}_+^4:\ S_v>1/\varepsilon\}, &&\mathbb{D}_8=\{(S_h,I_h,S_v,I_v)\mathbb{R}_+^4:\ I_v>1/\varepsilon\}.
\end{aligned}
$$

Case 1. If $(S_h,I_h,S_v,I_v)\in\mathbb{D}_1$, it can be easily shown that $\mathcal{L}V\leqslant-\Lambda_h/S_h+F_1\leqslant-\Lambda_h/\varepsilon+F_1$, where F_1 is given by (5.4). Therefore, we could pick a very small constant $\varepsilon>0$ such that $-\Lambda_h/\varepsilon+F_1\leqslant-1$. Then, it yields

$$
\mathcal{L}V\leqslant-1,\ \text{for any}\ \ (S_h,I_h,S_v,I_v)\in\mathbb{D}_1. \tag{5.9}
$$

Case 2. If $(S_h,I_h,S_v,I_v)\in\mathbb{D}_2$, it follows that $\mathcal{L}V\leqslant-M\lambda+M(\mu_h+k+\nu)I_h+F_2$, where F_2 is given by (5.5). Therefore, it is possible to select a $\varepsilon>0$ so that $M(\mu_h+k+\nu)\varepsilon\leqslant1$. Hence, from (5.2),

$$
\mathcal{L}V\leqslant-1,\ \text{for any}\ \ (S_h,I_h,S_v,I_v)\in\mathbb{D}_2. \tag{5.10}
$$

Case 3. For $(S_h,I_h,S_v,I_v)\in\mathbb{D}_3$, one have $\mathcal{L}V\leqslant-\Lambda_v/S_v+F_1\leqslant-\Lambda_v/\varepsilon+F_1$. That is, there exists a $\varepsilon>0$ so that $-\Lambda_v/\varepsilon+F_1\leqslant-1$. Therefore

$$
\mathcal{L}V\leqslant-1,\ \text{for any}\ \ (S_h,I_h,S_v,I_v)\in\mathbb{D}_3. \tag{5.11}
$$

Case 4. If $(S_h,I_h,S_v,I_v)\in\mathbb{D}_4$, it yields that $\mathcal{L}V\leqslant-N\lambda+N\mu_vI_v+F_3$, where F_3 is given by (5.6).Therefore, we can select a $\varepsilon>0$ such that $N\mu_v\varepsilon\leqslant1$. Hence, from (5.2),

$$
\mathcal{L}V\leqslant-1,\ \text{for any}\ \ (S_h,I_h,S_v,I_v)\in\mathbb{D}_4. \tag{5.12}
$$

Case 5. If $(S_h,I_h,S_v,I_v)\in\mathbb{D}_5$, we can get

$$
\begin{aligned}
\mathcal{L}V\leqslant&-\frac{1}{4}\left[(\mu_h\wedge\mu_v)-(\theta+1)\left(\sigma_1^2K_1^2\vee\sigma_2^2K_4^2\vee\frac{1}{2}\sigma_3^2\vee\frac{1}{2}\sigma_4^2\right)\right]\\
&\times\left(S_h^{\theta+2}+I_h^{\theta+2}+S_v^{\theta+2}+I_v^{\theta+2}\right)+F_4
\end{aligned}
$$

$$\leqslant -\frac{1}{4}\left[(\mu_h \wedge \mu_v) - (\theta+1)\left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \frac{1}{2}\sigma_3^2 \vee \frac{1}{2}\sigma_4^2\right)\right]\frac{1}{\varepsilon^{\theta+2}} + F_4,$$

where F_4 is given by (5.7). So, one have

$$-\frac{1}{4}\left[(\mu_h \wedge \mu_v) - (\theta+1)\left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \frac{1}{2}\sigma_3^2 \vee \frac{1}{2}\sigma_4^2\right)\right]\frac{1}{\varepsilon^{\theta+2}} + F_4 \leqslant -1$$

for some $\varepsilon > 0$. Then it follows that

$$\mathcal{L}V \leqslant -1 \quad \text{for any} \quad (S_h, I_h, S_v, I_v) \in \mathbb{D}_5. \tag{5.13}$$

Case 6. If $(S_h, I_h, S_v, I_v) \in \mathbb{D}_6$, for ε is small enough, one have

$$\begin{aligned}\mathcal{L}V \leqslant& -\frac{1}{4}\left[(\mu_h \wedge \mu_v) - (\theta+1)\left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \frac{1}{2}\sigma_3^2 \vee \frac{1}{2}\sigma_4^2\right)\right] I_h^{\theta+2} + F_4\\ \leqslant& -\frac{1}{4}\left[(\mu_h \wedge \mu_v) - (\theta+1)\left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \frac{1}{2}\sigma_3^2 \vee \frac{1}{2}\sigma_4^2\right)\right]\frac{1}{\varepsilon^{\theta+2}} + F_4\\ \leqslant& -1.\end{aligned} \tag{5.14}$$

Case 7. If $(S_h, I_h, S_v, I_v) \in \mathbb{D}_7$, for small enough ε, we can get

$$\begin{aligned}\mathcal{L}V \leqslant& -\frac{1}{4}\left[(\mu_h \wedge \mu_v) - (\theta+1)\left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \frac{1}{2}\sigma_3^2 \vee \frac{1}{2}\sigma_4^2\right)\right] S_v^{\theta+2} + F_4\\ \leqslant& -\frac{1}{4}\left[(\mu_h \wedge \mu_v) - (\theta+1)\left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \frac{1}{2}\sigma_3^2 \vee \frac{1}{2}\sigma_4^2\right)\right]\frac{1}{\varepsilon^{\theta+2}} + F_4\\ \leqslant& -1.\end{aligned} \tag{5.15}$$

Case 8. If $(S_h, I_h, S_v, I_v) \in \mathbb{D}_8$, for small enough ε, we have

$$\begin{aligned}\mathcal{L}V \leqslant& -\frac{1}{4}\left[(\mu_h \wedge \mu_v) - (\theta+1)\left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \frac{\sigma_3^2}{2} \vee \frac{\sigma_4^2}{2}\right)\right] I_v^{\theta+2} + F_4\\ \leqslant& -\frac{1}{4}\left[(\mu_h \wedge \mu_v) - (\theta+1)\left(\sigma_1^2 K_1^2 \vee \sigma_2^2 K_4^2 \vee \frac{\sigma_3^2}{2} \vee \frac{\sigma_4^2}{2}\right)\right]\frac{1}{\varepsilon^{\theta+2}} + F_4\\ \leqslant& -1.\end{aligned} \tag{5.16}$$

Therefore, from (5.9)-(5.16), one finally drive $\mathcal{L}V(S_h, I_h, S_v, I_v) \leqslant -1$ for all $(S_h, I_h, S_v, I_v) \in \mathbb{R}_+^4 \setminus \mathbb{D}_\varepsilon$. This is, all conditions of Lemma 5.1 hold. Thus, model (2.3) admits a unique stationary distribution which has ergodicity. This finishes the proof. □

6. A special case

Note that the conditions of Theorems 4.1 and 5.1 are not easy to test due to the general incidences $f(S_h, I_v)$ and $g(S_v, I_h)$. Therefore, we discuss, in this section, the stochastic dynamics of model (2.3) with a special case of general incidences

$f(S_h, I_v) = S_h I_v$ and $g(S_v, I_h) = S_v I_h$. As a result, the model is stated as follows

$$\begin{cases} \mathrm{d}S_h(t) = &\Big(\Lambda_h - (\mu_h + \psi_h)S_h(t) - \beta_1 S_h(t)I_v(t) \\ & + \displaystyle\int_0^\infty \Gamma(a)S_h(t-a)\mathrm{d}a\Big)\mathrm{d}t - \sigma_1 S_h(t)I_v(t)\mathrm{dB}_1(t), \\ \mathrm{d}I_h(t) = &\Big(\beta_1 S_h(t)I_v(t) - (\mu_h + k + \nu)I_h(t) + \displaystyle\int_0^\infty r(b)I_h(t-b)\mathrm{d}b\Big)\mathrm{d}t \\ & + \sigma_1 S_h(t)I_v(t)\mathrm{dB}_1(t), \\ \mathrm{d}S_v(t) = &\big(\Lambda_v - \beta_2 S_v(t)I_h(t) - \mu_v S_v(t)\big)\mathrm{d}t \\ & - \sigma_2 S_v(t)I_h(t)\mathrm{dB}_2(t) + \sigma_3 S_v(t)\mathrm{dB}_3(t), \\ \mathrm{d}I_v(t) = &\big(\beta_2 S_v(t)I_h(t) - \mu_v I_v(t)\big)\mathrm{d}t + \sigma_2 S_v(t)I_h(t)\mathrm{dB}_2(t) \\ & + \sigma_4 I_v(t)\mathrm{dB}_4(t). \end{cases} \tag{6.1}$$

According to Theorem 4.1, we have the following corollary.

Corollary 6.1. *Let $(S_h(t), I_h(t), S_v(t), I_v(t))$ be the solution of model (6.1) satisfies the initial value $(S_h(0),\ I_h(0),\ S_v(0),\ I_v(0)) \in \mathbb{R}^4_+$. Assume that $r(b) = 0$ and one of the following conditions is met:*

(*i*) $\widetilde{\mathcal{R}}_0^* < 1$ *and* $\Lambda_h/\mu_h \leqslant \beta_1/\sigma_1^2$, $\check{N}_v \leqslant \beta_2/\sigma_2^2$, *where,*

$$\widetilde{\mathcal{R}}_0^* = \frac{\beta_1 \frac{\Lambda_h}{\mu_h} + \beta_2 \check{N}_v}{\min\{\mu_h + k + \nu, \mu_v\} + \frac{\sigma_1^2}{2}\left(\frac{\Lambda_h}{\mu_h}\right)^2 + \frac{\sigma_2^2}{2}\check{N}_v^2};$$

(*ii*) $\beta_1^2/2\sigma_1^2 + \beta_2^2/2\sigma_2^2 < \min\{\mu_h + k + \nu, \mu_v\}$.

Then disease is extinct with probability one, that is, $\limsup_{t\to\infty} \ln\,(I_h(t)+I_v(t))/t < 0$ *a.s.*

Lemma 6.1. *Assume that $(S_h(t), I_h(t), S_v(t), I_v(t))$ is the solution of model (6.1) satisfies the initial value $(S_{h0}, I_{h0}, S_{v0}, I_{v0}) \in \mathbb{R}^4_+$, then*

$$\langle S_h \rangle_t \geqslant \frac{\Lambda_h}{\mu_h + \psi_h} - \frac{\mu_h + k + \nu}{\mu_h + \psi_h}\langle I_h \rangle_t + \varphi_h(t),\ \ \langle S_v \rangle_t = \frac{\Lambda_v}{\mu_v} - \langle I_v \rangle_t + \varphi_v(t), \tag{6.2}$$

where,

$$\begin{aligned} \varphi_h(t) = &- \frac{1}{\mu_h + \psi_h}\left[\frac{S_h(t) - S_h(0)}{t} + \frac{I_h(t) - I_h(0)}{t}\right], \\ \varphi_v(t) = &- \frac{1}{\mu_v}\bigg[\frac{S_v(t) - S_v(0)}{t} + \frac{I_v(t) - I_v(0)}{t} \\ & - \frac{\sigma_3}{t}\int_0^t S_v(s)\mathrm{dB}_3(s) - \frac{\sigma_4}{t}\int_0^t I_v(s)\mathrm{dB}_4(s)\bigg]. \end{aligned}$$

Proof. It can be obtained by calculating the sum of the first and second equations of model (6.1)

$$\mathrm{d}(S_h(t) + I_h(t)) = \Big[\Lambda_h - (\mu_h + \psi_h)S_h(t) + \int_0^\infty \Gamma(a)S_h(t-a)\mathrm{d}a$$

$$
\begin{aligned}
&- (\mu_h + k + \nu) I_h(t) + \int_0^\infty r(b) I_h(t-b)\mathrm{d}b\Big]\mathrm{d}t \\
\geqslant &[\Lambda_h - (\mu_h + \psi_h) S_h(t) - (\mu_h + k + \nu) I_h(t)]\mathrm{d}t.
\end{aligned}
$$

Directly integrating the above expression form 0 to t and dividing t gives

$$
\frac{S_h(t) - S_h(0)}{t} + \frac{I_h(t) - I_h(0)}{t} \geqslant \Lambda_h - (\mu_h + \psi_h)\langle S_h\rangle_t - (\mu_h + k + \nu)\langle I_h\rangle_t.
$$

Hence

$$
\langle S_h\rangle_t \geqslant \frac{\Lambda_h}{\mu_h + \psi_h} - \frac{\mu_h + k + \nu}{\mu_h + \psi_h}\langle I_h\rangle_t + \varphi_h(t).
$$

Similarly, $\mathrm{d}(S_v(t)+I_v(t)) = [\Lambda_v - \mu_v S_v(t) - \mu_v I_v(t)]\mathrm{d}t + \sigma_3 S_v(t)\mathrm{dB}_3(t) + \sigma_4 I_v(t)\mathrm{dB}_4(t)$. Thus, $\langle S_v\rangle_t = \Lambda_v/\mu_v - \langle I_v\rangle_t + \varphi_v(t)$. This completes the proof. □

Now, we explain the stochastic persistence of disease of model (6.1).

Theorem 6.1. *Assume that* $(S_h(t), I_h(t), S_v(t), I_v(t))$ *is the solution of model* (6.1) *satisfies the initial value* $(S_{h0}, I_{h0}, S_{v0}, I_{v0}) \in \mathbb{R}_+^4$. *If* $\mathcal{R}_0^m > 1$, *then*

$$
\lim_{t\to\infty}\langle I_h\rangle_t + \lim_{t\to\infty}\langle I_v\rangle_t \geqslant \frac{D}{C}(\mathcal{R}_0^m - 1) > 0 \ a.s., \tag{6.3}
$$

where

$$
\begin{aligned}
&\mathcal{R}_0^m = \left(\frac{\beta_1\Lambda_h}{\mu_h + \psi_h} + \frac{\beta_2\Lambda_v}{\mu_v}\right)\frac{1}{D}, \qquad C = \frac{\beta_1(\mu_h + k + \nu)}{\mu_h + \psi_h} \vee \beta_2, \\
&D = \left(\frac{\beta_1\Lambda_h}{\mu_h} + \mu_h + k + \nu\right) \vee \left(\frac{\beta_2\Lambda_v}{\mu_v} + \mu_v\right) + \frac{\sigma_1^2}{2}\frac{\Lambda_h^2}{\mu_h^2} + \frac{\sigma_2^2}{2}\frac{\Lambda_v^2}{\mu_v^2} + \frac{\sigma_4^2}{2}.
\end{aligned}
$$

Proof. Let $V = \ln(I_h + I_v)$, from Itô's formula, the following inequality can be obtained by direct calculation

$$
\begin{aligned}
\mathrm{d}V = &\Bigg\{\frac{1}{I_h(t) + I_v(t)}\Big(\beta_1 S_h(t) I_v(t) - (\mu_h + k + \nu) I_h(t) + \int_0^\infty r(b) I_h(t-b)\mathrm{d}b \\
&+ \beta_2 S_v(t) I_h(t) - \mu_v I_v(t)\Big) - \frac{1}{2(I_h(t) + I_v(t))^2}\Big(\sigma_1^2 S_h^2(t) I_v^2(t) \\
&+ \sigma_2^2 S_v^2(t) I_h^2(t) + \sigma_4^2 I_v^2(t)\Big)\Bigg\}\mathrm{d}t + \frac{1}{I_h(t) + I_v(t)} \\
&\times \Big(\sigma_1 S_h(t) I_v(t)\mathrm{dB}_1(t) + \sigma_2 S_v(t) I_h(t)\mathrm{dB}_2(t) + \sigma_4 I_v(t)\mathrm{dB}_4(t)\Big) \\
\geqslant &\Bigg\{\frac{1}{I_h(t) + I_v(t)}\Big(\beta_1 S_h(t)(I_v(t) + I_h(t)) - \beta_1 S_h(t) I_h(t) - (\mu_h + k + \nu) I_h(t) \\
&+ \beta_2 S_v(t)(I_h(t) + I_v(t)) - \beta_2 S_v(t) I_v(t) - \mu_v I_v(t)\Big) \\
&- \frac{\sigma_1^2 S_h^2(t) I_v^2(t) + \sigma_2^2 S_v^2(t) I_h^2(t) + \sigma_4^2 I_v^2(t)}{2(I_h(t) + I_v(t))^2}\Bigg\}\mathrm{d}t + \frac{1}{I_h(t) + I_v(t)} \\
&\times \Big(\sigma_1 S_h(t) I_v(t)\mathrm{dB}_1(t) + \sigma_2 S_v(t) I_h(t)\mathrm{dB}_2(t) + \sigma_4 I_v(t)\mathrm{dB}_4(t)\Big) \\
\geqslant &\Bigg\{\beta_1 S_h(t) + \beta_2 S_v(t) - \big(\frac{\beta_1\Lambda_h}{\mu_h} + \mu_h + k + \nu\big) \vee \big(\frac{\beta_2\Lambda_v}{\mu_v} + \mu_v\big)
\end{aligned}
$$

$$- \frac{\sigma_1^2\Lambda_h^2}{2\mu_h^2} - \frac{\sigma_2^2\Lambda_v^2}{2\mu_v^2} - \frac{\sigma_4^2}{2}\bigg\}\mathrm{d}t + \frac{1}{I_h(t)+I_v(t)}\Big(\sigma_1 S_h(t)I_v(t)\mathrm{dB}_1(t) \tag{6.4}$$
$$+ \sigma_2 S_v(t)I_h(t)\mathrm{dB}_2(t) + \sigma_4 I_v(t)\mathrm{dB}_4(t)\Big).$$

Integrating the both ends of (6.4) from 0 to t and then dividing t on both ends

$$\begin{aligned}\frac{V(t)}{t} \geqslant & \beta_1\langle S_h\rangle_t + \beta_2\langle S_v\rangle_t - \left(\frac{\beta_1\Lambda_h}{\mu_h} + \mu_h + k + \nu\right) \vee \left(\frac{\beta_2\Lambda_v}{\mu_v} + \mu_v\right) \\ & - \frac{\sigma_1^2}{2}\frac{\Lambda_h^2}{\mu_h^2} - \frac{\sigma_2^2}{2}\frac{\Lambda_v^2}{\mu_v^2} - \frac{\sigma_4^2}{2} + \frac{V(0)}{t} + \frac{M_5(t)}{t} + \frac{M_6(t)}{t} + \frac{M_7(t)}{t},\end{aligned} \tag{6.5}$$

where,

$$M_5(t) = \int_0^t \frac{\sigma_1 S_h(s)I_v(s)}{I_h(s)+I_v(s)}\mathrm{dB}_1(s),\ M_6(t) = \int_0^t \frac{\sigma_2 S_v(s)I_h(s)}{I_h(s)+I_v(s)}\mathrm{dB}_2(s),$$
$$M_7(t) = \int_0^t \frac{\sigma_4 I_v(s)\mathrm{dB}_4(s)}{I_h(s)+I_v(s)}.$$

Substituting (6.2) into (6.5)

$$\begin{aligned}\frac{V(t)}{t} \geqslant & \beta_1\left(\frac{\Lambda_h}{\mu_h+\psi_h} - \frac{\mu_h+k+\nu}{\mu_h+\psi_h}\langle I_h\rangle_t + \varphi_h(t)\right) + \beta_2\left(\frac{\Lambda_v}{\mu_v} - \langle I_v\rangle_t + \varphi_v(t)\right) \\ & - \left(\frac{\beta_1\Lambda_h}{\mu_h} + \mu_h + k + \nu\right) \vee \left(\frac{\beta_2\Lambda_v}{\mu_v} + \mu_v\right) \\ & - \frac{\sigma_1^2}{2}\frac{\Lambda_h^2}{\mu_h^2} - \frac{\sigma_2^2}{2}\frac{\Lambda_v^2}{\mu_v^2} - \frac{\sigma_4^2}{2} + \frac{V(0)}{t} + \sum_{i=5}^{7}\frac{M_i(t)}{t} \\ = & - \frac{\beta_1(\mu_h+k+\nu)}{\mu_h+\psi_h}\langle I_h\rangle_t - \beta_2\langle I_v\rangle_t + \beta_1\left(\frac{\Lambda_h}{\mu_h+\psi_h} + \varphi_h(t)\right) \\ & + \beta_2\left(\frac{\Lambda_v}{\mu_v} + \varphi_v(t)\right) - \left(\frac{\beta_1\Lambda_h}{\mu_h} + \mu_h + k + \nu\right) \vee \left(\frac{\beta_2\Lambda_v}{\mu_v} + \mu_v\right) \\ & - \frac{\sigma_2^2\Lambda_v^2}{2\mu_v^2} - \frac{\sigma_4^2}{2} + \frac{V(0)}{t} + \sum_{i=5}^{7}\frac{M_i(t)}{t} - \frac{\sigma_1^2\Lambda_h^2}{2\mu_h^2}.\end{aligned} \tag{6.6}$$

By taking the limit of (6.6) we have

$$\begin{aligned}&\frac{\beta_1(\mu_h+k+\nu)}{\mu_h+\psi_h}\lim_{t\to\infty}\langle I_h\rangle_t + \beta_2\lim_{t\to\infty}\langle I_v\rangle_t \\ &\geqslant \lim_{t\to\infty}\left[\beta_1\left(\frac{\Lambda_h}{\mu_h+\psi_h} + \varphi_h(t)\right) + \beta_2\left(\frac{\Lambda_v}{\mu_v} + \varphi_v(t)\right) - \left(\frac{\beta_1\Lambda_h}{\mu_h} + \mu_h + k + \nu\right)\right. \\ &\left.\vee\left(\frac{\beta_2\Lambda_v}{\mu_v} + \mu_v\right) - \frac{\sigma_1^2}{2}\frac{\Lambda_h^2}{\mu_h^2} - \frac{\sigma_2^2}{2}\frac{\Lambda_v^2}{\mu_v^2} - \frac{\sigma_4^2}{2} + \frac{V(0)-V(t)}{t} + \sum_{i=5}^{7}\frac{M_i(t)}{t}\right].\end{aligned} \tag{6.7}$$

From the large number theorem of martingales and Theorem 3.2, this yields

$$\lim_{t\to\infty}\varphi_k(t) = 0,\ k = h, v;\ \lim_{t\to\infty}\frac{V(0)-V(t)}{t} = 0,\ \lim_{t\to\infty}\frac{M_i(t)}{t} = 0,\ i = 5, 6, 7.$$

Therefore, from (6.7) we finally obtain (6.3). The proof is completed. □

Finally, we consider the stationary distribution of (6.1) for $\sigma_1 = \sigma_2 = r(b) = 0$.

Corollary 6.2. *Assume that $\sigma_1 = \sigma_2 = r(b) = 0$. If $\mathcal{R}_*^s > 1$ and $(\mu_h \wedge \mu_v) > (\sigma_3^2 \vee \sigma_4^2)/2$, then the solution $(S_h(t), I_h(t), S_v(t), I_v(t))$ of model* (6.1) *satisfies the initial value $(S_h(0), I_h(0), S_v(0), I_v(0)) \in \Gamma$ is positive recurrent and has a ergodic unique stationary distribution in Γ, where*

$$\mathcal{R}_*^s = \sqrt{\frac{\Lambda_h \Lambda_v \beta_1 \beta_2}{(\mu_v + \frac{\sigma_3^2}{2})(\mu_v + \frac{\sigma_4^2}{2})(\mu_h + \psi_h)(\mu_h + k + \nu)}}$$

and $\Gamma = \{(S_h, I_h, S_v, I_v) \in \mathbb{R}_+^4 : \ S_h + I_h + S_v + I_v \leqslant \Lambda_h/\mu_h + \check{N}_v\}$.

Proof. According to the proof of Theorem 5.1, it can easily get that condition (i) of Lemma 5.1 holds. Now, one verify condition (ii). By the condition $(\mu_h \wedge \mu_v) > (\sigma_3^2 \vee \sigma_4^2)/2$, it can choose a sufficiently small positive constant θ which satisfies $\rho := (\mu_h \wedge \mu_v) - (\theta + 1)(\sigma_3^2 \vee \sigma_4^2)/2 > 0$. Define a function $\overline{V} : \mathbb{R}_+^4 \to \mathbb{R}$ as $\overline{V}(S_h, I_h, S_v, I_v) = V_5 + \overline{N} V_6 + V_7$, where $V_7 = (\theta + 2)^{-1}(S_h + I_h + S_v + I_v)^{\theta+2}$,

$$\begin{aligned}
V_5 &= -\ln S_h - \ln S_v - \ln I_h + \frac{\beta_2 I_h}{\mu_h + k + \nu}, \\
V_6 &= -\ln S_v - c_1 \ln I_v - c_2 \ln S_h - c_3 \ln I_h + \frac{\beta_2 I_h}{\mu_h + k + \nu}, \\
c_1 &= \frac{\mu_v + \sigma_3^2/2}{\mu_v + \sigma_4^2/2}, \ c_2 = \frac{\mu_v + \sigma_3^2/2}{\mu_h + \psi_h}, \ c_3 = \frac{\mu_v + \sigma_3^2/2}{\mu_h + k + \nu},
\end{aligned}$$

and $\overline{N} > 0$ satisfies the following condition

$$-4\overline{N}\left(\mu_v + \frac{\sigma_3^2}{2}\right)\left(\sqrt{\mathcal{R}_*^s} - 1\right) + J \leqslant -2, \tag{6.8}$$

where

$$\begin{aligned}
J = &\sup_{(S_h, I_h, S_v, I_v) \in \mathbb{R}_+^4} \bigg\{ \beta_1 I_v + \frac{I_v^2}{4} + \frac{2\beta_1^2 \beta_2^2 S_h^2}{(\mu_h + k + \nu)^2} - \frac{\rho}{2}\Big(S_h^{\theta+2} + I_h^{\theta+2} \\
&+ S_v^{\theta+2} + I_v^{\theta+2} \Big) + 2\mu_h + k + \nu + \psi_h + \mu_v + \frac{\sigma_3^2}{2} + Q \bigg\}, \\
Q = &\sup_{(S_h, I_h, S_v, I_v) \in \mathbb{R}_+^4} \bigg\{ (S_h + I_h + S_v + I_v)^{\theta+1} \left(\Lambda_h + \Lambda_v + \frac{\Lambda_h \psi_h}{\mu_h^2} \right) \\
&- \frac{\rho}{2}\big(S_h + I_h + S_v + I_v \big)^{\theta+2} \bigg\}.
\end{aligned}$$

Note that

$$\liminf_{\substack{k \to \infty \\ (S_h, I_h, S_v, I_v) \in \mathbb{R}_+^4 \setminus \mathbb{D}_k}} \overline{V}(S_h, I_h, S_v, I_v) = +\infty$$

and $\overline{V}(S_h, I_h, S_v, I_v)$ is a continuous function, $\overline{V}(S_h, I_h, S_v, I_v)$ has a unique minimum value at point $(\bar{S}_h, \bar{I}_h, \bar{S}_v, \bar{I}_v) \in \mathbb{R}_+^4$. Therefore, one introduce a non-negative function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$ as $V(S_h, I_h, S_v, I_v) = V_5 + \overline{N} V_6 + V_7 - \overline{V}(\bar{S}_h, \bar{I}_h, \bar{S}_v, \bar{I}_v)$.

Directly applying Itô's formula, one can separately obtain

$$\mathcal{L}V_5 \leqslant -\frac{\Lambda_h}{S_h} - \frac{\Lambda_v}{S_v} - \frac{\beta_1 S_h I_v}{I_h} + \beta_1 I_v + 2\mu_h + k + \nu$$

$$
\begin{aligned}
&+\psi_h+\mu_v+\frac{\sigma_3^2}{2}+\frac{\beta_1\beta_2 S_h I_v}{\mu_h+k+\nu}\\
\leqslant&-\frac{\Lambda_h}{S_h}-\frac{\Lambda_v}{S_v}-\frac{\beta_1 S_h I_v}{I_h}+\beta_1 I_v+2\mu_h+k+\nu+\psi_h\\
&+\mu_v+\frac{\sigma_3^2}{2}+\frac{\beta_1^2\beta_2^2 S_h^2}{(\mu_h+k+\nu)^2}+\frac{I_v^2}{4},\\
\mathcal{L}V_6\leqslant&-\frac{\Lambda_v}{S_v}-\frac{c_2\Lambda_h}{S_h}-\frac{c_1\beta_2 S_v I_h}{I_v}-\frac{c_3\beta_1 S_h I_v}{I_h}+\mu_v+\frac{\sigma_3^2}{2}+c_1\Big(\mu_v+\frac{\sigma_4^2}{2}\Big)\\
&+c_2(\mu_h+\psi_h)+c_3(\mu_h+k+\nu)+\frac{\beta_1\beta_2}{\mu_h+k+\nu}S_h I_v+c_2\beta_1 I_v\\
\leqslant&-4\left(\mu_v+\frac{\sigma_3^2}{2}\right)\left(\sqrt{\mathcal{R}_*^s}-1\right)+\frac{\beta_1\beta_2}{\mu_h+k+\nu}S_h I_v+c_2\beta_1 I_v,\\
\mathcal{L}V_7\leqslant&(S_h+I_h+S_v+I_v)^{\theta+1}\left(\Lambda_h+\Lambda_v+\frac{\Lambda_h\psi_h}{\mu_h^2}\right)-(S_h+I_h+S_v+I_v)^{\theta+2}\\
&\times(\mu_h\wedge\mu_v)+\frac{\theta+1}{2}(S_h+I_h+S_v+I_v)^{\theta}(\sigma_3^2\vee\sigma_4^2)(S_h+I_h+S_v+I_v)^2\\
\leqslant&Q-\frac{\rho}{2}\left(S_h^{\theta+2}+I_h^{\theta+2}+S_v^{\theta+2}+I_v^{\theta+2}\right).
\end{aligned}
$$

Moreover, from the above discussion, we can get that

$$
\begin{aligned}
\mathcal{L}V\leqslant&-\frac{\Lambda_h}{S_h}-\frac{\Lambda_v}{S_v}-\frac{\beta_1 S_h I_v}{I_h}-\frac{\rho}{2}(S_h^{\theta+2}+I_h^{\theta+2}+S_v^{\theta+2}+I_v^{\theta+2})+\beta_1 I_v+2\mu_h\\
&+\frac{\sigma_3^2}{2}+Q-4\overline{N}\left(\mu_v+\frac{\sigma_3^2}{2}\right)\left(\sqrt{\mathcal{R}_*^s}-1\right)+\frac{2\beta_1^2\beta_2^2 S_h^2}{(\mu_h+k+\nu)^2}\\
&+\frac{I_v^2}{4}+k+\nu+\psi_h+\mu_v+\frac{\overline{N}}{4}I_v^2+c_2\overline{N}\beta_1 I_v.
\end{aligned}\tag{6.9}
$$

Choose a bounded closed set

$$
\mathbb{D}_\epsilon=\left\{(S_h,\cdots,I_v)\in\mathbb{R}_+^4:\epsilon\leqslant S_h\leqslant\frac{1}{\epsilon},\epsilon^3\leqslant I_h\leqslant\frac{1}{\epsilon^3},\epsilon\leqslant S_v\leqslant\frac{1}{\epsilon},\epsilon\leqslant I_v\leqslant\frac{1}{\epsilon}\right\},
$$

where, $0<\epsilon<1$ is a sufficiently small constant such that

$$
(1+F)\max\left\{\frac{1}{\Lambda_h},\frac{1}{\Lambda_v},\ \frac{1}{\beta_1}\right\}\leqslant\frac{1}{\epsilon},\ \ \frac{\overline{N}\epsilon^2}{4}+c_2\overline{N}\beta_1\epsilon\leqslant 1,\ \ -\frac{\rho}{4\epsilon^{\theta+2}}+F\leqslant -1,\tag{6.10}
$$

where,

$$
\begin{aligned}
F=\sup_{(S_h,I_h,S_v,I_v)\in\mathbb{R}_+^4}&\Big\{-\frac{\rho}{4}(S_h^{\theta+2}+I_h^{\theta+2}+S_v^{\theta+2}+I_v^{\theta+2})+\beta_1 I_v+2\mu_h+k\\
&+\nu+\psi_h+\mu_v+\frac{\sigma_3^2}{2}+Q+\frac{2\beta_1^2\beta_2^2 S_h^2}{(\mu_h+k+\nu)^2}+\frac{I_v^2}{4}+\frac{\overline{N}}{4}I_v^2+c_2\overline{N}\beta_1 I_v\Big\}.
\end{aligned}
$$

For convenience, we divide $\mathbb{R}_+^4\setminus\mathbb{D}_\epsilon$ into the following eight regions

$$
\begin{aligned}
&\mathbb{D}_1=\left\{(S_h,\cdots,I_v)\in\mathbb{R}_+^4:S_h<\epsilon\right\}, &&\mathbb{D}_3=\left\{(S_h,\cdots,I_v)\in\mathbb{R}_+^4:S_v<\epsilon\right\},\\
&\mathbb{D}_4=\left\{(S_h,\cdots,I_v)\in\mathbb{R}_+^4:I_v<\epsilon\right\}, &&\mathbb{D}_5=\left\{(S_h,\cdots,I_v)\in\mathbb{R}_+^4:S_h>1/\epsilon\right\},
\end{aligned}
$$

$$\mathbb{D}_6 = \left\{(S_h, \cdots, I_v) \in \mathbb{R}^4_+ : I_h > 1/\epsilon^3\right\}, \quad \mathbb{D}_7 = \left\{(S_h, \cdots, I_v) \in \mathbb{R}^4_+ : S_v > 1/\epsilon\right\},$$

and $\mathbb{D}_2 = \left\{(S_h, \cdots, I_v) \in \mathbb{R}^4_+ : I_h < \epsilon^3, S_h \geqslant \epsilon, I_v \geqslant \epsilon\right\}$, $\mathbb{D}_8 = \{(S_h, \cdots, I_v) \in \mathbb{R}^4_+ : I_v > 1/\epsilon\}$.

Next, we verify that $\mathcal{L}V(S_h, I_h, S_v, I_v) \leqslant -1$ on $\mathbb{R}^4_+ \setminus \mathbb{D}_\epsilon$. This is equivalent to prove the inequality on the above domains.

Case 1. If $(S_h, I_h, S_v, I_v) \in \mathbb{D}_1$, one can see from (6.9) that $\mathcal{L}V \leqslant -\Lambda_h/S_h + F \leqslant -\Lambda_h/\epsilon + F$. Thus, one can conclude that $\mathcal{L}V \leqslant -1$ on $\mathbb{D}_1$ for small enough ϵ.

Case 2. For $(S_h, I_h, S_v, I_v) \in \mathbb{D}_2$, we have from (6.9) that $\mathcal{L}V \leqslant -\beta_1 S_h I_v/I_h + F \leqslant -\beta_1/\epsilon + F$. According to (6.10), one get $\mathcal{L}V \leqslant -1$ on $\mathbb{D}_2$ for small enough ϵ.

Case 3. If $(S_h, I_h, S_v, I_v) \in \mathbb{D}_3$, it yields from (6.9) that $\mathcal{L}V \leqslant -\Lambda_v/S_v + F \leqslant -\Lambda_v/\epsilon + F$. In view of (6.10), it follows that $\mathcal{L}V \leqslant -1$ on $\mathbb{D}_3$ for small enough ϵ.

Case 4. If $(S_h, I_h, S_v, I_v) \in \mathbb{D}_4$, it yields that

$$\begin{aligned}
\mathcal{L}V \leqslant & -4\overline{N}\left(\mu_v + \frac{\sigma_3^2}{2}\right)\left(\sqrt{\mathcal{R}_*^s} - 1\right) + \frac{\overline{N}}{4} I_v^2 + c_2 \overline{N} \beta_1 I_v + J \\
\leqslant & -2 + \frac{\overline{N}\epsilon^2}{4} + c_2 \overline{N} \beta_1 \epsilon,
\end{aligned}$$

which follows from the condition (6.10) that $\mathcal{L}V \leqslant -1$ on $\mathbb{D}_4$ for small enough ϵ.

Case 5. If $(S_h, I_h, S_v, I_v) \in \mathbb{D}_5$, then $\mathcal{L}V \leqslant -\rho S_h^{\theta+2}/4 + F \leqslant -\rho/4\epsilon^{\theta+2} + F$. It follows from (6.10) that $\mathcal{L}V \leqslant -1$ on $\mathbb{D}_5$ for small enough ϵ.

Case 6. If $(S_h, I_h, S_v, I_v) \in \mathbb{D}_6$, one can get $\mathcal{L}V \leqslant -\rho I_h^{\theta+2}/4 + F \leqslant -\rho/4\epsilon^{3(\theta+2)} + F \leqslant -\rho/4\epsilon^{\theta+2} + F$. Combining with (6.10) that $\mathcal{L}V \leqslant -1$ on $\mathbb{D}_6$ for small ϵ.

Case 7. For $(S_h, I_h, S_v, I_v) \in \mathbb{D}_7$, it can get $\mathcal{L}V \leqslant -\rho S_v^{\theta+2}/4 + F \leqslant -\rho/4\epsilon^{\theta+2} + F$. This together with (6.10), one conclude that $\mathcal{L}V \leqslant -1$ on $\mathbb{D}_7$ for small enough ϵ.

Case 8. If $(S_h, I_h, S_v, I_v) \in \mathbb{D}_8$, we have $\mathcal{L}V \leqslant -\rho I_v^{\theta+2}/4 + F \leqslant -\rho/4\epsilon^{\theta+2} + F$. By virtue of (6.10), it obtained that $\mathcal{L}V \leqslant -1$ on $\mathbb{D}_8$ for small ϵ.

To sum up, we finally drive $\mathcal{L}V(S_h, I_h, S_v, I_v) \leqslant -1$ for $(S_h, I_h, S_v, I_v) \in \mathbb{R}^4_+ \setminus \mathbb{D}_\epsilon$. Therefore, from Lemma 3.1, model (6.1) has a unique stationary distribution which has the ergodic property. The proof of Corollary 6.2 is completed. □

Remark 6.1. The stationary distribution of model (6.1) is obtained under additional conditions. Of course, we can also get the existence and uniqueness of the stationary distribution of model (6.1) with $\sigma_1 \neq 0$, $\sigma_2 \neq 0$ and $r(b) \neq 0$, which makes the criteria more complicated. How to obtain a simple and easy-to-verify criteria for the stationary distribution will be an interesting question.

7. Numerical simulations and discussion

In this section, some numerical simulations are given to show the effect of stochastic perturbation. According to the possible values in Table 1, some basic model parameters are fixed as following: $\mu_h = 1/(72 \times 365)$, $k = 1/7$, $\nu = 0.0035$, $\Lambda_v = 5000$, $r(b) = 0.003(1 + \sin(b - 10)\pi/20)$ and

$$\omega(a) = \begin{cases} 0, & \text{if } \ 0 < a \leqslant 10; \\ 0.6667(a - 200)^2 e^{-0.6(a-200)}, & \text{if } \ 10 < a \leqslant 30; \\ 0.0185, & \text{if } \ 30 < a. \end{cases}$$

By using the discretization methods of class-age-structure model and stochastic differential equation which are proposed in Refs. [3, 13] , the discretization system of model (6.1) writes

$$\begin{aligned} S_{h_{i+1}} =& S_{h_i} + \left[\Lambda_h - (\mu_h + \psi_h) S_{h_i} - \beta_1 S_{h_i} I_{v_i} + D_1\right] \Delta t \\ & - \sigma_1 S_{h_i} I_{v_i} \sqrt{\Delta t} \xi_{1i} - \frac{\sigma_1^2}{2} S_{h_i} I_{v_i} (\xi_{1i}^2 - 1) \Delta t \\ I_{h_{i+1}} =& I_{h_i} + \left[\beta_1 S_{h_i} I_{v_i} - (\mu_h + k + \nu) I_{h_i} + D_2\right] \Delta t \\ & + \sigma_1 S_{h_i} I_{v_i} \sqrt{\Delta t} \xi_{1i} + \frac{\sigma_1^2}{2} S_{h_i} I_{v_i} (\xi_{1i}^2 - 1) \Delta t \\ S_{v_{i+1}} =& S_{v_i} + \left[\Lambda_v - \beta_2 S_{v_i} I_{h_i} - \mu_v S_{v_i}\right] \Delta t - \sigma_2 S_{v_i} I_{h_i} \sqrt{\Delta t} \xi_{2i} \\ & - \frac{\sigma_2^2}{2} S_{v_i} I_{h_i} (\xi_{2i}^2 - 1) \Delta t + \sigma_3 S_{v_i} \sqrt{\Delta t} \xi_{3i} + \frac{\sigma_3^2}{2} S_{v_i} (\xi_{3i}^2 - 1) \Delta t \\ I_{v_{i+1}} =& I_{v_i} + \left[\beta_2 S_{v_i} I_{h_i} - \mu_v I_{v_i}\right] \Delta t + \sigma_2 S_{v_i} I_{h_i} \sqrt{\Delta t} \xi_{2i} \\ & + \frac{\sigma_2^2}{2} S_{v_i} I_{h_i} (\xi_{2i}^2 - 1) \Delta t + \sigma_4 I_{v_i} \sqrt{\Delta t} \xi_{4i} + \frac{\sigma_4^2}{2} I_{v_i} (\xi_{4i}^2 - 1) \Delta t, \end{aligned}$$

where, $D_1 = \int_0^\infty \Gamma(a) S_h(t-a)\mathrm{d}a$, $\Gamma(a) = \psi_h \omega(a) \exp\left\{-\int_0^a (\mu_h + \omega(\tau))\mathrm{d}\tau\right\}$, $D_2 = \int_0^\infty \pi(b) I_h(t-b)\mathrm{d}b$ and $\pi(b) = kr(b)\exp\{-\int_0^b (\mu_h + r(\tau))\mathrm{d}\tau\}$, and ξ_{1i}, ξ_{2i}, ξ_{3i} and ξ_{4i} are independent stochastic variables which obey Gaussian distribution $N(0,1)$.

Firstly, one choose parameters as $\Lambda_h = 1$, $\mu_v = 5000$, $\psi_h = 0.375$, $\mu_v = 0.12$, $\beta_1 = 2.875 \times 10^{-6}$, $\beta_2 = 4.275 \times 10^{-6}$, $\sigma_1 = 1.0 \times 10^{-5}$, $\sigma_2 = 9.0 \times 10^{-6}$, $\sigma_3 = 3.2 \times 10^{-6}$ and $\sigma_4 = 2.3 \times 10^{-6}$. It can be obtained by direct calculation that $\widetilde{\mathcal{R}}_0^* \approx 0.9360 < 1$, $\sigma_1^2 - \beta_1/\check{N}_h \approx -1.0781 \times 10^{-6} < 0$, $\sigma_2^2 - \beta_2/\check{N}_v \approx -2.16 \times 10^{-11} < 0$. Thus, all conditions in (i) of Corollary 6.1 are satisfied. That is, the disease is stochastic extinction, which are shown that infected hosts $I_h(t)$ and infected vectors $I_v(t)$ tend to 0 a.s., as $t \to \infty$ in Figure 1(a)-1(c). At the same time, it is not difficult to find that infected hosts and vectors are fluctuating in the initial stage of disease outbreak because of the effects of stochastic perturbations. In addition, the Figure 1(d) also show that the quantity of susceptible vectors tend to the value Λ_v/μ_v as $t \to \infty$, this is the conclusion of Theorem 4.2. Both theoretical results and numerical simulations show that as long as $\widetilde{\mathcal{R}}_0^* < 1$, regardless of the initial state of model (6.1), the disease will eventually become extinct.

However, if we change $\Lambda_h = 10$, $\psi_h = 0.175$, $\mu_v = 1/30$, $\beta_1 = 2.875 \times 10^{-4}$, $\beta_2 = 4.275 \times 10^{-4}$, $\sigma_3 = 8.678 \times 10^{-38}$, $\sigma_4 = 8.678 \times 10^{-3}$ and other parameters are fixed as Figure 1. We can get $\mathcal{R}_*^s \approx 14.6748 > 1$ and $(\mu_h \wedge \mu_v) - (\sigma_3^2 \vee \sigma_4^2)/2 \approx 3.9791 \times 10^{-7} > 0$ by direct calculation. From Corollary 6.2, model (6.1) has a unique stationary distribution which has the ergodic property. The Figure 2(a) displays the trajectory of the element $I_v(t)$ of solution $(S_h(t), I_h(t), S_v(t), I_v(t))$ and the plot of Figure 2(b) shows the distribution of $I_v(t)$ after some initial transients. Further, the trajectories of Figures 2(c) and 2(d) show that the distributions of infected classes $I_h(t)$ and $I_v(t)$ of model (6.1), which is obtained by 5000 numerical simulations with the same initial condition and the same sufficiently large t, this implies that the disease is endemic.

Finally, we consider the influence of temperature, humidity and other factors on transmission of vector-borne infectious disease. To do so, the basic parameters of model are fixed as follows: $\Lambda_h = 10$, $\Lambda_v = 5000$, $\psi = 0.175$, $k = 1/7$, $\mu_h = 1/(72 \times 365)$, $\beta_1 = \beta_2 = 2.875 \times 10^{-5}$, $\mu_v = 1/20$. In Figure 3(a), we fix $\sigma_3 = \sigma_4 = 2.5 \times 10^{-6}$,

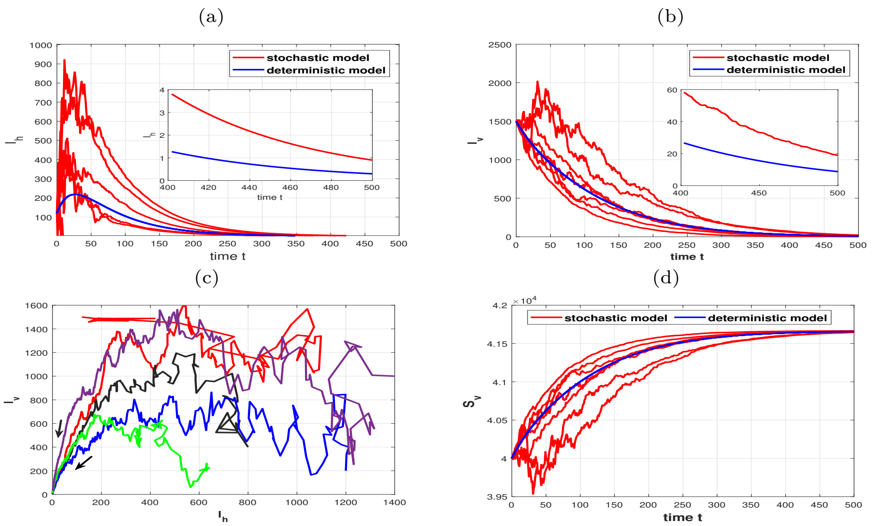

Figure 1. The stochastic extinction of disease and asymptotical stability of the disease-free steady state for model (6.1) with $\Lambda_h = 1$, $\mu_v = 5000$, $\psi_h = 0.375$, $\mu_v = 0.12$, $\beta_1 = 2.875 \times 10^{-6}$, $\beta_2 = 4.275 \times 10^{-6}$, $\sigma_1 = 1.0 \times 10^{-5}$, $\sigma_2 = 9.0 \times 10^{-6}$, $\sigma_3 = 3.2 \times 10^{-6}$ and $\sigma_4 = 2.3 \times 10^{-6}$. Here, $\widetilde{\mathcal{R}}_0^* \approx 0.9360 < 1$, $\sigma_1^2 - \beta_1/S_h^0 \approx -1.0781 \times 10^{-6} < 0$, $\sigma_2^2 - \beta_2/S_v^0 \approx -2.16 \times 10^{-11} < 0$.

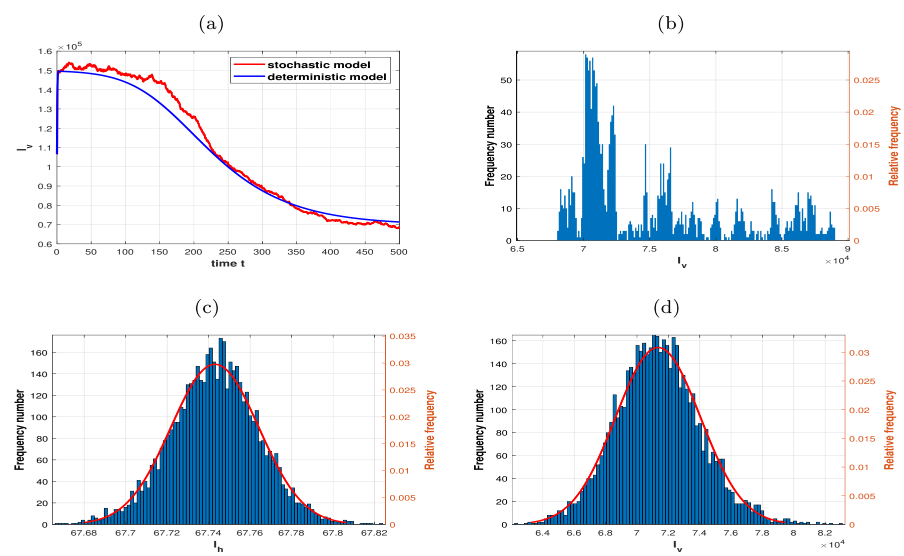

Figure 2. The persistence of disease of model (6.1) with $\Lambda_h = 10$, $\psi_h = 0.175$, $\mu_v = 1/30$, $\beta_1 = 2.875 \times 10^{-4}$, $\beta_2 = 4.275 \times 10^{-4}$, $\sigma_3 = 8.678 \times 10^{-38}$, $\sigma_4 = 8.678 \times 10^{-3}$. Here, $\mathcal{R}_*^s \approx 14.6748 > 1$ and $(\mu_h \wedge \mu_v) - (\sigma_3^2 \vee \sigma_4^2)/2 \approx 3.9791 \times 10^{-7} > 0$: (a) the trajectory of $I_v(t)$; (b) the distributions of $I_v(t)$ after some initial transients; (c) the histogram of $I_h(t)$; (d) the histogram of $I_v(t)$.

and choose $\sigma_1 = \sigma_2$ as 1.2×10^{-6}, 1.2×10^{-5} and 5.2×10^{-5} to discuss the impacts of strengths of σ_1^2 and σ_2^2 on the disease transmission. This can be understood as the impacts of stochastic perturbation on the behaviors and scope of activities of vectors, or the protective measures of humans. The plots in Figures 3(a) and 3(b) imply that σ_1 and σ_2 have an impact on the peak value of disease outbreak and the arrival time of the peak value. However, if we fix $\sigma_1 = \sigma_2 = 2.5 \times 10^{-6}$, and choose

$\sigma_3 = \sigma_4$ as 8.2×10^{-4}, 8.2×10^{-2} and 8.2×10^{-1} in Figures 3(c) and 3(d). The plots show that σ_3 and σ_4 have very little influence on disease transmission. This seems to imply that the size of vector population has a smaller impact on disease than the behavior of vectors. Further, we choose the following three sets of values in turn: (i) $\sigma_1 = \sigma_2 = 2.5 \times 10^{-6}$, $\sigma_3 = \sigma_4 = 1.2 \times 10^{-5}$; (ii) $\sigma_1 = \sigma_2 = 2.5 \times 10^{-5}$, $\sigma_3 = \sigma_4 = 1.2 \times 10^{-4}$; (iii) $\sigma_1 = \sigma_2 = 5.2 \times 10^{-5}$, $\sigma_3 = \sigma_4 = 1.2 \times 10^{-2}$. The plots in Figures 4(a) and 4(b) imply that the transmission of disease is unpredictable due to the change of white noise strength, this may be one of the reasons why it is difficult to control the vector-borne infectious diseases.

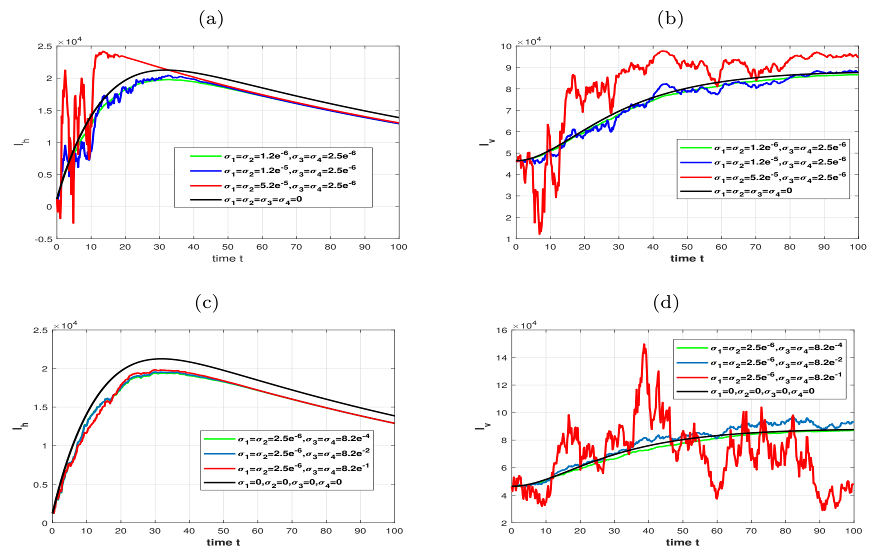

Figure 3. The effect of strength of $\sigma_i^2 (i = 1, 2, 3, 4)$ on the transmission of disease with $\Lambda_h = 10$, $\Lambda_v = 5000$, $\psi = 0.175$, $k = 1/7$, $\mu_h = 1/(72 \times 365)$, $\beta_1 = \beta_2 = 2.875 \times 10^{-5}$, $\mu_v = 1/20$. Here, (a) and (b): $\sigma_3 = \sigma_4 = 2.5 \times 10^{-6}$ and $\sigma_1 = \sigma_2 = 1.2 \times 10^{-6}$, 1.2×10^{-5}, 5.2×10^{-4}, respectively; (c) and (d): $\sigma_1 = \sigma_1 = 2.5 \times 10^{-6}$ and $\sigma_3 = \sigma_4 = 8.2 \times 10^{-4}$, 8.2×10^{-2}, 8.2×10^{-1}, respectively.

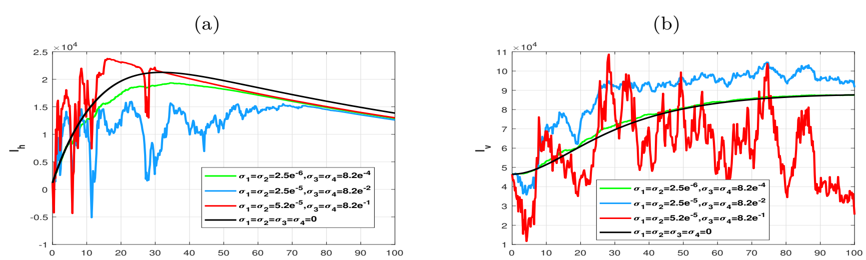

Figure 4. The effect of strength of $\sigma_i^2 (i = 1, 2, 3, 4)$ on the transmission of disease with $\Lambda_h = 10$, $\Lambda_v = 5000$, $\psi = 0.175$, $k = 1/7$, $\mu_h = 1/(72 \times 365)$, $\beta_1 = \beta_2 = 2.875 \times 10^{-5}$, $\mu_v = 1/20$. Here, (i) $\sigma_1 = \sigma_2 = 2.5 \times 10^{-6}$, $\sigma_3 = \sigma_4 = 1.2 \times 10^{-5}$; (ii) $\sigma_1 = \sigma_2 = 2.5 \times 10^{-5}$, $\sigma_3 = \sigma_4 = 1.2 \times 10^{-4}$; (iii) $\sigma_1 = \sigma_2 = 5.2 \times 10^{-5}$, $\sigma_3 = \sigma_4 = 1.2 \times 10^{-2}$, respectively.

8. Concluding remark

In view of stochastic factors (such as, weather changes, living habitats and the amount of medical resources, etc.) in the transmission of vector-host infectious diseases, as well as the immune loss rate and disease relapse rate are related to the vaccine effectiveness and recovery cycle, we propose a stochastic vector-host epidemic model with the age of vaccination and relapse, to study the impacts of random factors and compartment-age. Our model can be used to characterize the transmission of Tuberculosis, Malaria, and so on.

By using Itô's formula, and the techniques of some inequalities, the global existence and uniqueness of positive solution, some criteria of stochastic extinction of disease, the existence of stationary distribution are obtained for this stochastic model with general incidence. Further, when the general incidences are reduced to bilinear incidences, some sufficient conditions on extinction, persistence, and existence of ergodic stationary distribution are obtained. Numerical simulations illustrate the main theoretical results and the impacts of stochastic perturbation. Numerical simulations also imply that the transmission of vector-host disease become more frequent and unpredictable due to uncertain factors. In addition, during the spread of vector-host diseases, the behavior of vectors and the self-protection of humans are the key factors to control the disease relative to the number of vector population. Therefore, the application of limited medical resources to personal protection to avoid mosquito bites, rather than to reduce the number of vectors and destroy the natural environment, may be one of the keys to control vector-host infectious diseases in the future.

Vector-host infectious diseases are transmitted by infected vectors, and therefore, its transmission has a significant period. How seasonal fluctuations affect the dynamics of vector-host infectious disease models with seasonal and stochastic disturbances is a problem worthy of further study. In addition, we also pay close attention to the dynamic properties of the stochastic vector-host models with impulsive control. All these will be studied in the future.

References

[1] A. Alexanderian, M. Gobbert, K. R. Fister, H. Gaff, S. Lenhart and E. Schaefer, *An age-structured model for the spread of epidemic cholera: analysis and simulation*, Nonlinear Anal. Real., 2011, 12, 3483–3498.

[2] M. Andraud, N. Hens, C. Marais and P. Beutels, *Dynamic epidemiological models for dengue transmission: a systematic review of structural approaches*, PLoS One, 2012, 7(11), e49085.

[3] S. Anita, V. Arnautu and V. Capasso, *An Introduction to Optimal Control Problems in Life Sciences*, Springer Science, New York, 2011.

[4] C. Bowman, A. B. Gumel and P. V. D. Driessche, *A mathematical model for assessing control strategies against West Nile virus*, B. Math. Biol., 2005, 67(5), 1107–1133.

[5] N. Chitnis, J. Hyman and J. Cushing, *Determining important parameters in the spread of malaria through the sensitivity analysis of a mathematical model*, B. Math. Biol., 2008, 70(5), 1272–1296.

[6] K. W. Chung and R. Lui, *Dynamics of two-strain influenza model with cross-immunity and no quarantine class*, J. Math. Biol., 2016, 73, 1467–1489.

[7] X. Duan, S. Yuan and X. Li, *Global stability of an SVIR model with ages of vaccination*, Appl. Math. Comput., 2014, 226, 528–540.

[8] X. Duan, S. Yuan, Z. Qiu and J. Ma, *Stability of an SVEIR epidemic model with ages of vaccination and latency*, Comput. Math. Appl., 2014, 68, 288–308.

[9] L. Esteva and C. Vargas, *Analysis of a dengue disease transmission model*, J. Math. Biol., 1998, 150(2), 131–151.

[10] H. Gulbudak and M. Martcheva, *A structured avian influenza model with imperfect vaccination and vaccine-induced asymptomatic infection*, B. Math. Biol., 2014, 76, 2389–2425.

[11] M. Guo, L. Hu and L. Nie, *Stochastic dynamics of the transmission of Dengue fever virus between mosquitoes and humans*, Int. J. Biomath, 2021, 14(07), 2150062.

[12] H. Hethcote, *An immunization model for a heterogeneous population*, Theor. Popul. Biol., 1978, 14, 338–349.

[13] D. J. Higham, *An algorithmic introduction to numerical simulation of stochastic differential equations*, SIAM Rev., 2001, 43(3), 525–546.

[14] F. Hoppensteadt, *An age dependent epidemic model*, J. Franklin I., 1974, 297(5), 325–333.

[15] Z. Hu, S. Yin and H. Wang, *Stability and Hopf bifurcation of a vector-borne disease model with saturated infection rate and reinfection*, Comput. Math. Method. M., 2019, 2019, 1352698

[16] M. Iannelli, *Mathematical Theory of Age-structured Population Dynamics*, Giadini Editorie Stampatori, Pisa, 1994.

[17] M. Jovanovi and M. Krsti, *Stochastically perturbed vector-borne disease models with direct transmission*, Appl. Math. Model., 2012, 36(11), 5214–5228.

[18] S. A. Kumar and G. Mini, *Assessing the impact of treatment on the dynamics of dengue fever: A case study of India*, Appl. Math. Comput., 2019, 362(1), 124533.

[19] D. Li, J. Cui, M. Liu and S. Liu, *The evolutionary dynamics of stochastic epidemic model with nonlinear incidence rate*, B. Math. Biol., 2015, 77(9), 1705–1743.

[20] F. M. G. Magpantay, *Vaccine impact in homogeneous and age-structured models*, J. Math. Biol., 2017, 75, 1591–1617.

[21] X. Mao, *Stochastic Differential Equations and Applications (second ed.)*, Horwood publishing, Chichester, 2007.

[22] X. Meng and C. Yin, *Dynamics of a Dengue fever model with unreported cases and asymptomatic infected classes in Singapore, 2020*, J. Appl. Anal. Comput., 2022. DOI: 10.11948/20220111.

[23] L. N. Nkamba, T. T. Manga, F. Agouanet and M. L. Manyombe, *Mathematical model to assess vaccination and effective contact rate impact in the spread of tuberculosis*, J. Biol. Dynam., 2019, 13, 26–42.

[24] K. Nudee, S. Chinviriyasit and W. Chinviriyasit, *The effect of backward bifurcation in controlling measles transmission by vaccination*, Chaos Soliton. Fract., 2019, 123, 400–412.

[25] X. Ran, L. Hu, L. Nie and Z. Teng, *Effects of stochastic perturbation and vaccinated age on a vector-borne epidemic model with saturation incidence rate*, Appl. Math. Comput., 2021, 394, 125798.

[26] R. Rifhat, Q. Ge and Z. Teng, *The dynamical behaviors in a stochastic SIS epidemic model with nonlinear incidence*, Comput. Math. Method. M., 2016, 2016, 5218163.

[27] E. Shim, Z. Feng, M. Martcheva and C. Castillo-Chavez, *An age-structured epidemic model of rotavirus with vaccination*, J. Math. Biol., 2006, 53, 719–746.

[28] Z. Shuai, J. Tien and P. van den Driessche, *Cholera models with hyperinfectivity and temporary immunity*, B. Math. Biol., 2012, 74, 2423–2445.

[29] W. Sun, L. Xue and X. Yan, *Stability of a dengue epidemic model with independent stochastic perturbations*, J. Math. Anal. Appl., 2018, 468, 998–1017.

[30] L. Wang, Z. Teng, C. Ji, X. Feng and K. Wang, *Dynamical behaviors of a stochastic malaria model: A case study for Yunnan, China*, Physica A, 2019, 521, 435–454.

[31] S. Wang and L. Nie, *Global dynamics for a vector-borne disease model with class-age-dependent vaccination, latency and general incidence rate*, Qual. Theor. Dyn. Syst., 2020, 19, 72.

[32] G. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics*, Marcel Dekker, New York, 1985.

[33] B. Wen, R. Rifhat and Z. Teng, *The stationary distribution in a stochastic SIS epidemic model with general nonlinear incidence*, Physica A, 2019, 524, 258–271.

[34] World Health Organization, *Vector-borne diseases*, https://www.who.int/news-room/fact-sheets/detail/vector-borne-diseases, 2017 (accessed 31 October 2017).

[35] World Health Organization, *Poliomyelitis*, https://www.who.int/news-room/fact-sheets/detail/poliomyelitis, 2019 (accessed 22 July 2019).

[36] World Health Organization, *WHO commemorates the 40th anniversary of smallpox eradication*, https://www.who.int/news-room/detail/13-12-2019-who-commemorates-the-40th-anniversary-of-smallpox-eradication, 2019 (accessed 13 December 2019).

[37] J. Xu and Y. Zhou, *Global stability of a multi-group model with vaccination age, distributed delay and random perturbation*, Math. Biosci. Eng., 2015, 12, 1083–1106.

[38] J. Yang, M. Martcheva and L. Wang, *Global threshold dynamics of an SIVS model waning vaccine-induced immunity and nonlinear incidence*, Math. Biosci., 2015, 268, 1–8.

[39] T. Zhang and X. Zhao, *Mathematical modeling for Schistosomiasis with seasonal influence: a case study in Hubei, China*, SIAM J. Appl. Dyn. Syst., 2020, 19, 1438–1471.

[40] T. Zheng and L. Nie, *Modelling the transmission dynamics of two-strain Dengue in the presence awareness and vector control*, J. Theor. Biol., 2018, 443, 82–91.

[41] C. Zhu and G. Yin, *Asymptotic properties of hybrid diffusion systems*, SIAM J. Control. Optim., 2007, 46(4), 1155–1179.

[42] L. Zou, J. Chen, X. Feng and S. Ruan, *Analysis of a dengue model with vertical transmission and application to the 2014 dengue outbreak in Guangdong Province, China*, B. Math. Biol., 2018, 80, 2633–2651.