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AN EFFECTIVE SUSTAINABLE COLLOCATION METHOD FOR SOLVING REGULAR/SINGULAR SYSTEMS OF CONFORMABLE DIFFERENTIAL EQUATIONS SUBJECT TO INITIAL CONSTRAINT CONDITIONS

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Abstract The main objective of the present article is to provide an overview of the B-splines collocation methods (BSCM) to achieve practical analytical-numerical solutions for a family of regular/singular systems of initial constraints condition (ICC). Herein, the fractional derivatives are described by the conformable one, and an abundance of its basic theory is utilized. The useful properties of the cubic B-splines and collocation methods are employed to reduce the computations of both regular/singular systems of fractional order to a combination of linear/nonlinear algebraic equations. Numerical tests are treated quantitatively to demonstrate the technical statements and to exhibit the ability, perfection, and applicability of the suggested procedure for solving such conformable systems models. The outcomes confirm the reliability and efficacy of the technique improved. At the end of the manuscript, some notes were presented with some characteristics of the scheme and some possible future work.

Keywords B-spline collocation method, Lane-Emden type model, conformable fractional derivative, system of initial value problem.

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1. Introduction

The topic of fractional calculus has earned significant popularity and attention over the past three decades or so, owing essentially to its illustrated applications in various seemingly distinct and widespread domains of science and engineering [13, 23, 25, 30, 33]. It does surely give several probably helpful tools for determining the solutions of system of initial value problems (SIVPs) of conformable fractional derivative (CFD), and numerous other problems including special functions of mathematical physics as well as their expansions and generalizations in one

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and more variables. Real-world numerical data is usually difficult to analyze and any function which would effectively correlate the data would be difficult to obtain and highly unwieldy. In recent years, there has been a growing interest in the numerical treatment of fractional SIVPs, since they are usually difficult to solve analytically, so it is required to obtain efficient approximate solutions [1,7,10,12,14,15,28].

To this end, one of the most dominant universal mechanisms for forming approximate solutions of a wide range of linear/nonlinear and regular/singular SIVPs is the method of B-spline collocation based on the B-spline functions. The invasion of the splines in the approximation theory was due to the interpolation problems and gratitude for their excellent computational and approximation characteristics. Unquestionably, the splines own extremely excellent approximation properties and versatility, to see that we have the following papers [6,8,16,17,21,22,24,26]. They provide a smooth implementation of the computational algorithms created on their basis. At the identical time algorithms for forming spline functions coincide with the finite element method, which is the principal industrial method of structural analysis in the computer-aided plan. In computational mathematics, B-spline is a spline function owning the shortest interval of definition for a given degree of order of smoothness and region decomposition. The basic hypothesis declares that any spline function to a given degree, smoothness, and interval of definition can be expressed as a linear mixture of B-splines at the same degree and the same smoothness at the same interval of definition. The term B-spline was presented by Schoenberg and it is an abbreviation of the phrase "basis spline".

In 2014, Khalil and his colleagues presented a new definition of fractional derivatives to the world of mathematics as CFD [18]. This fractional derivative seems to be extra natural, and it agrees with the usual description of the original derivative. Further results, generalizations, developments, and applications of the CFD in science and engineering were investigated in [2,19,20,32,34]. Additionally, several numerical schemes are adaptive and formulated in the sense of the CFD as described in [3–5,9,11,27,29,31]. Although, very numerous techniques have been introduced, however, there are a vast number of all types of systems that can not find the exact solutions for them, consequently, the numerical method for simulations is submitted, and the outcomes are proved to be so perfect.

The attention here is to receive typical approximate collocation methods solutions for SIVPs of the next forms: Regular SIVPs of CFD order:

1. Regular SIVPs of CFD order:

$$\begin{cases} T^{\beta_1}\psi(\rho)+T^{\gamma_1}\phi(\rho)+q_1(\rho)T^{\alpha_1}\psi(\rho)+p_1(\rho)T^{\delta_1}\phi(\rho)=F_1(\rho,\phi(\rho),\psi(\rho)),\\ T^{\beta_2}\psi(\rho)+T^{\gamma_2}\phi(\rho)+q_2(\rho)T^{\alpha_2}\psi(\rho)+p_2(\rho)T^{\delta_2}\phi(\rho)=F_2(\rho,\phi(\rho),\psi(\rho)),\end{cases} \tag{1.1}$$

   subject to the attached ICCs:

$$\begin{cases} \phi(c)=\eta_1, & \phi'(c)=\theta_1,\\ \psi(c)=\eta_2, & \psi'(c)=\theta_2,\end{cases} \tag{1.2}$$

   where $T^{(.)}$ is the CFD of order (.), $0<\alpha_1,\delta_1,\alpha_2,\delta_2\leqslant 1$, $1<\beta_1,\gamma_1,\beta_2,\gamma_2\leqslant 2$, $a\leqslant\rho\leqslant b$.

2. Singular SIVPs of CFD order:

$$\begin{cases} T^{\beta_1}\psi(\rho) + T^{\gamma_1}\phi(\rho) + q_1(\rho)T^{\alpha_1}\psi(\rho) + p_1(\rho)T^{\delta_1}\phi(\rho) = F_1(\rho,\phi(\rho),\psi(\rho)), \\ T^{\beta_2}\psi(\rho) + T^{\gamma_2}\phi(\rho) + q_2(\rho)T^{\alpha_2}\psi(\rho) + p_2(\rho)T^{\delta_2}\phi(\rho) = F_2(\rho,\phi(\rho),\psi(\rho)), \end{cases} \tag{1.3}$$

subject to the attached ICCs:

$$\begin{cases} \phi(c) = \eta_1, & \phi'(c) = \theta_1, \\ \psi(c) = \eta_2, & \psi'(c) = \theta_2, \end{cases} \tag{1.4}$$

where $T^{(.)}$ is the CFD of order (.), $0 < \alpha_1, \delta_1, \alpha_2, \delta_2 \leqslant 1$, $1 < \beta_1, \gamma_1, \beta_2, \gamma_2 \leqslant 2$, $c \leqslant \rho \leqslant d$. Herein, $q_s(\rho) = \frac{k}{\rho}$ for $s = 1,2$ and/or $p_s(\rho) = \frac{k}{\rho}$ for $s = 1,2$ have a singularity at $\rho = 0$.

The numerical solutions of (1.1) and (1.3) had received lots of attention and it is still receiving such because many physical (Engineering, Medical, financial, population dynamics, and Biological Sciences) problems can be modeled and formulated into mathematical SIVPs with a singular matrix and such systems are so-called singular or differential-algebraic [13, 23, 25, 30, 33]. Anyhow, for computing ϕ and ψ we use the B-spline approximation in the formation of the BSCM that we will present its details in the next sections.

The study is organized in the following order: in Sections 2 and 3, we give fundamental data, characters, formulation, and introductory results linked to the CFD and the BSCM for linear/nonlinear regular/singular SIVPs. In Sections 4 and 5, we elaborate on our numerical procedure for finding solutions for linear/nonlinear SIVPs. In Section 6, we will apply the algorithms that are formed in the MATHEMATICA 11 programming environment and list our numerical findings, and then illustrate the efficiency of the suggested scheme by examining several numerical models. Finally, in Section 7 some concluding comments are exhibited.

2. Preliminaries: CFD and BSCM

This portion demonstrates some necessary definitions scores and the findings used in our scheme. Firstly, we recall the definition of the formation of the conformable operator approach. Secondly, the main conversion property of the CFD is given with other characterizations. Thirdly, the CBSM formulation and peculiar are utilized.

Fundamentally, for $\psi \in C([0,c] \longrightarrow \mathbb{R})$, then

$$T^{\alpha}(\psi)(\rho) = \lim_{\varepsilon\to 0} \frac{\psi(\rho + \varepsilon\rho^{1-\alpha}) - \psi(\rho)}{\varepsilon} \tag{2.1}$$

for all $\rho > 0, \alpha \in (0,1]$. If ψ is α-differentiable in some $(0,c)$, $c > 0$, and $\lim_{\rho\to 0^+} \psi^{(\alpha)}(\rho)$ exists, then $\psi^{\alpha}(0) = \lim_{\rho\to 0^+} \psi^{(\alpha)}(\rho)$. Indeed, when $\alpha \in (n, n+1]$, then

$$T^{\alpha}(\psi)(\rho) = \lim_{\varepsilon\to 0} \frac{\psi^{([\alpha]-1)}(\rho + \varepsilon\rho^{([\alpha]-\alpha)}) - \psi^{([\alpha]-1)}(\rho)}{\varepsilon} \tag{2.2}$$

Theorem 2.1 ([2]). *When $\alpha \in (n, n+1]$ with $n \in \mathbb{N} \cup \{0\}$ and ϕ, ψ are α-differentiable at $\rho > 0$. Then*

1. *$T^{\alpha}(c\psi + d\phi) = cT^{\alpha}(\psi) + dT^{\alpha}(\phi)$ for all $c, d \in \mathbb{R}$.*
2. *$T^{\alpha}(\phi \circ \psi)(\rho) = \psi'(\phi(\rho))T^{\alpha}(\phi(\rho))$.*
3. *If ψ is differentiable, then $T^{\alpha}\psi(\rho) = \rho^{1-\alpha}\frac{d\psi(\rho)}{d\rho}$.*
4. *If ψ is $([\alpha]+1)$ differentiable, then $T^{\alpha}\psi(\rho) = \rho^{[\alpha]-\alpha}\psi^{[\alpha]}(\rho)$, where $[\alpha] = n$.*

The derivation approach of the BSCM will be described completely in [13-21]. Anyhow, choose $R = [c, d]$ and fixed the partition $\chi: c = \rho_0 < \rho_s < \ldots < \rho_m = d$ on $[c, d]$ with mesh points $\rho_s = c + sh$, $s = 0, 1, \cdots, m$ with $h = \frac{d-c}{m}$. It is important in our case to add four extra points outside R. This procedure allows us to give a new partition as:

$$\chi: \rho_{-2} < \rho_{-1} < c = \rho_0 < \ldots < \rho_s < \ldots < \rho_m = d < \rho_{m+1} < \rho_{m+2}. \tag{2.3}$$

Let $\Psi_{3,\chi} = \left\{\varphi_{s,3}(\rho) | \varphi_{s,3}(\rho) \in C^2[c, d], s = -1, 0, 1, \cdots, m+1\right\}$ is a collection of all cubic B-splines on R. Anyhow, on $[\rho_s, \rho_{s+1}]$, the formation of $\varphi_{s,3}(\rho)$ takes the form

$$\varphi_{s,3}(\rho) = \frac{1}{6h^3}\begin{cases} (\rho - \rho_{s-2})^3, & \rho \in [\rho_{s-2}, \rho_{s-1}], \\ h^3 + 3h^2(\rho - \rho_{s-1}) + 3h(\rho - \rho_{s-1})^2 - 3(\rho - \rho_{s-1})^3, & \rho \in [\rho_{s-1}, \rho_s], \\ h^3 + 3h^2(\rho_{s+1} - \rho) + 3h(\rho_{s+1} - \rho)^2 - 3(\rho_{s+1} - \rho)^3, & \rho \in [\rho_s, \rho_{s+1}], \\ (\rho_{s+2} - \rho)^3, & \rho \in [\rho_{s+1}, \rho_{s+2}], \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Define $\Psi^*(\chi) = \text{span}\{\varphi_{-1,3}(\rho), \varphi_{0,3}(\rho), \cdots, \varphi_{m,3}(\rho), \varphi_{m+1,3}(\rho)\}$. At $\alpha \in (1, 2]$ and $\beta \in (0, 1]$ the CFD of $\varphi_{s,3}(\rho)$ and its nodal values are

$$T^{\beta}\varphi_{s,3}(\rho) = \frac{1}{6h^3}\begin{cases} 3\rho^{1-\beta}(\rho - \rho_{s-2})^2, & \rho \in [\rho_{s-2}, \rho_{s-1}], \\ \rho^{1-\beta}[3h^2 + 6h(\rho - \rho_{s-1}) - 9(\rho - \rho_{s-1})^2], & \rho \in [\rho_{s-1}, \rho_s], \\ \rho^{1-\beta}[-3h^2 - 6h(\rho_{s+1} - \rho) + 9(\rho_{s+1} - \rho)^2], & \rho \in [\rho_s, \rho_{s+1}], \\ \rho^{1-\beta}[-3(\rho_{s+2} - \rho)^2], & \rho \in [\rho_{s+1}, \rho_{s+2}], \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

$$T^{\alpha}\varphi_{s,3}(\rho) = \frac{1}{6h^3}\begin{cases} 6\rho^{2-\alpha}(\rho - \rho_{s-2}), & \rho \in [\rho_{s-2}, \rho_{s-1}], \\ \rho^{2-\alpha}(6h - 18(\rho - \rho_{s-1})), & \rho \in [\rho_{s-1}, \rho_s], \\ \rho^{2-\alpha}(6h - 18(\rho_{s+1} - \rho)), & \rho \in [\rho_s, \rho_{s+1}], \\ 6\rho^{2-\alpha}(\rho_{s+2} - \rho), & \rho \in [\rho_{s+1}, \rho_{s+2}], \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Table 1. Related results nodal values concerning $\varphi_{s,3}(\rho)$.

	ρ_{s-2}	ρ_{s-1}	ρ_s	ρ_{s+1}	ρ_{s+2}
$\varphi_{s,3}(\rho)$	0	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	0
$T^{\beta}\varphi_{s,3}(\rho)$	0	$\frac{\rho_{s-1}^{1-\beta}}{2h}$	0	$\frac{-\rho_{s+1}^{1-\beta}}{2h}$	0
$T^{\alpha}\varphi_{s,3}(\rho)$	0	$\frac{\rho_{s-1}^{2-\alpha}}{h^2}$	$\frac{-2\rho_s^{2-\alpha}}{h^2}$	$\frac{\rho_{s+1}^{2-\beta}}{h^2}$	0

3. Formulation of the singular SIVPs of CFDs

Usually, we are not able to determine the solution to a singular problem in a simple way like a regular one, as a result, we need to follow some rule that leads us to find the solution to the problem. Usually, the resolution process is divided into many steps which are performed.

A singular system is just a collection of the singular equation that has a singular point for $\rho= 0$. The function $q_s(\rho) =\frac{k}{\rho}$ for $s= 1, 2$ and/or $p_s(\rho) =\frac{k}{\rho}$ for $s= 1, 2$ have a singularity which needs at first a special treatment based on a sequence of steps that help us to get rid of the singularity.

The same following steps are applied wherever we find a singularity problem, the conventional procedures intend at getting a sufficient scheme that satisfies the claims of the problem. Herein, in these steps $s= 1, 2$.

(1) Multiply (1.3) by ρ yields

$$\rho T^{\beta_s}\psi(\rho)+\rho T^{\gamma_s}\phi(\rho)+kT^{\alpha_s}\psi(\rho)+p_s(\rho)T^{\delta_s}\phi(\rho) =\rho F_s(\rho, \phi(\rho),\psi(\rho)), \tag{3.1}$$

And \ Or

$$\rho T^{\beta_s}\psi(\rho)+\rho T^{\gamma_s}\phi(\rho)+q_s(\rho)T^{\alpha_s}\psi(\rho)+kT^{\delta_s}\phi(\rho) =\rho F_s(\rho, \phi(\rho),\psi(\rho)), \tag{3.2}$$

(2) Taking the CFD of order α_s or δ_s from both sides of step (1)

$$T^{\alpha_s}\left[\rho T^{\beta_s}\psi\left(\rho\right) + \rho T^{\gamma_s}\phi\left(\rho\right) + kT^{\alpha_s}\psi\left(\rho\right) + p_s\left(\rho\right)T^{\delta_s}\phi\left(\rho\right) =\rho F_s\left(\rho, \phi\left(\rho\right), \psi\left(\rho\right)\right)\right]. \tag{3.3}$$

And \ Or

$$T^{\delta_s}\left[\rho T^{\beta_s}\psi\left(\rho\right) +\rho T^{\gamma_s}\phi\left(\rho\right) +q_s\left(\rho\right)T^{\alpha_s}\psi\left(\rho\right) +kT^{\delta_s}\phi\left(\rho\right) =\rho F_s\left(\rho, \phi\left(\rho\right), \psi\left(\rho\right)\right)\right]. \tag{3.4}$$

(3) Using the Leibniz rule and linearity, we obtain

$$\begin{aligned} &T^{\beta_s}\psi(\rho) + \rho T^{\alpha_s}T^{\beta_s}\psi(\rho)+T^{\gamma_s}\phi(\rho)+\rho T^{\alpha_s}T^{\gamma_s}\phi(\rho)+kT^{\alpha_s}T^{\alpha_s}\psi(\rho)\\ &+ T^{\alpha_s}p_s\left(\rho\right)T^{\delta_s}\phi\left(\rho\right) + p_s(\rho)T^{\alpha_s}T^{\delta_s}\phi(\rho)\\ &=F(\rho, \phi(\rho),\psi(\rho))+\rho T^{\alpha_s}F_s(\rho, \phi(\rho),\psi(\rho)). \end{aligned} \tag{3.5}$$

And \ Or

$$\begin{aligned} &T^{\beta_s}\psi(\rho) + \rho T^{\delta_s}T^{\beta_s}\psi(\rho)+T^{\gamma_s}\phi(\rho)+\rho T^{\delta_s}T^{\gamma_s}\phi(\rho)+kT^{\delta_s}T^{\delta_s}\phi(\rho)\\ &+ T^{\delta_s}q_s(\rho)T^{\alpha_s}\psi(\rho) + q_s(\rho)T^{\delta_s}T^{\alpha_s}\psi(\rho)\\ &=F(\rho, \phi(\rho),\psi(\rho))+\rho T^{\delta_s}F_s(\rho, \phi(\rho),\psi(\rho). \end{aligned} \tag{3.6}$$

(4) Replacing ρ= 0 in step (3), we get

$$\begin{aligned}&T^{\beta_s}\psi\left(0\right)+T^{\gamma_s}\phi\left(0\right)+kT^{\alpha_s}T^{\alpha_s}\psi\left(0\right)+T^{\alpha_s}p_s\left(0\right)T^{\delta_s}\phi\left(0\right)+p_s\left(0\right)T^{\alpha_s}T^{\delta_s}\phi\left(0\right)\\&=F_s(0,\phi(0),\psi(0)),\end{aligned}\tag{3.7}$$

And \ Or

$$\begin{aligned}&T^{\beta_i}\psi(0)+T^{\gamma_i}\phi(0)+kT^{\delta_i}T^{\delta_i}\phi(0)+T^{\delta_i}q_i(0)T^{\alpha_i}\psi(0)+q_i\left(0\right)T^{\delta_i}T^{\alpha_i}\psi\left(0\right)\\&=F_i(0,\phi(0),\psi(0)),\end{aligned}\tag{3.8}$$

4. The BSCM for linear SIVPs of CFDs

The basic idea of cubic BSCM for solving linear/nonlinear SIVPs of CFD order is to extend the ϕ and ψ as a finite series of extremely smooth support functions. Anyhow, $\phi(\rho)$ and $\psi(\rho)$ can be expanded as

$$\begin{cases}\phi\left(\rho\right)=\displaystyle\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}\left(\rho\right),\\ \psi\left(\rho\right)=\displaystyle\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}\left(\rho\right).\end{cases}\tag{4.1}$$

Fundamentally, we define

$$\begin{cases}F_1\left(\rho,\phi\left(\rho\right),\psi\left(\rho\right)\right)=\phi(\rho)u_1\left(\rho\right)+\psi(\rho)v_1\left(\rho\right)+h_1\left(\rho\right),\\ F_2\left(\rho,\phi(\rho),\psi\left(\rho\right)\right)=\phi(\rho)u_2\left(\rho\right)+\psi(\rho)v_2\left(\rho\right)+h_2\left(\rho\right).\end{cases}\tag{4.2}$$

Thus (1.1) becomes as

$$\begin{cases}\quad T^{\beta_1}\psi\left(\rho\right)+T^{\gamma_1}\phi\left(\rho\right)+q_1\left(\rho\right)T^{\alpha_1}\psi(\rho)+p_1\left(\rho\right)T^{\delta_1}\phi(\rho)\\ =\phi\left(\rho\right)u_1(\rho)+\psi\left(\rho\right)v_1\left(\rho\right)+h_1\left(\rho\right),\\ \quad T^{\beta_2}\psi\left(\rho\right)+T^{\gamma_2}\phi\left(\rho\right)+q_2\left(\rho\right)T^{\alpha_2}\psi(\rho)+p_2\left(\rho\right)T^{\delta_2}\phi(\rho)\\ =\phi\left(\rho\right)u_2(\rho)+\psi\left(\rho\right)v_2\left(\rho\right)+h_2\left(\rho\right).\end{cases}\tag{4.3}$$

- **Regular case**: we define $q_1(\rho)$, $p_1\left(\rho\right)$, $q_2\left(\rho\right)$, and $p_2(\rho)$ in such a way non of those functions has a singular term. Thus, (1.1) become as

$$\begin{cases}\quad T^{\beta_1}\psi\left(\rho\right)+T^{\gamma_1}\phi\left(\rho\right)+q_1\left(\rho\right)T^{\alpha_1}\psi\left(\rho\right)+p_1\left(\rho\right)T^{\delta_1}\phi\left(\rho\right)\\ =\phi\left(\rho\right)u_1\left(\rho\right)+\psi\left(\rho\right)v_1\left(\rho\right)+h_1\left(\rho\right),\\ \quad T^{\beta_2}\psi\left(\rho\right)+T^{\gamma_2}\phi\left(\rho\right)+q_2\left(\rho\right)T^{\alpha_2}\psi\left(\rho\right)+p_2\left(\rho\right)T^{\delta_2}\phi\left(\rho\right)\\ =\phi\left(\rho\right)u_2\left(\rho\right)+\psi\left(\rho\right)v_2\left(\rho\right)+h_2\left(\rho\right).\end{cases}\tag{4.4}$$

Substituting (4.1) and their CFDs in (4.4) and applying interpolation conditions for the grid points $\rho{=}\rho_t$, we get when $t{=}\,1,2,\cdots,m$, the following:

$$\begin{aligned} & T^{\beta_1}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}\left(\rho_t\right)+T^{\gamma_1}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}\left(\rho_t\right)+q_1(\rho_t)T^{\alpha_1}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t) \\ & +p_1(\rho_t)T^{\delta_1}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t) \\ = & \sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)u_1(\rho_t)+\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)v_1(\rho_t)+h_1(\rho_t), \end{aligned} \tag{4.5}$$

$$\begin{aligned} & T^{\beta_2}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)+T^{\gamma_2}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)+q_2(\rho_t)T^{\alpha_2}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t) \\ & +p_2(\rho_t)T^{\delta_2}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t) \\ = & \sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)u_2(\rho_t)+\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)v_2(\rho_t)+h_2(\rho_t), \end{aligned} \tag{4.6}$$

So, when $t{=}\,1,2,\cdots,m$, one has

$$\begin{aligned} & \sum_{s=-1}^{m+1}\mu_s[T^{\gamma_1}\varphi_{s,3}(\rho_t)+p_1(\rho_t)T^{\delta_1}\varphi_{s,3}(\rho_t)-\varphi_{s,3}(\rho_t)u_1(\rho_t)]+\sum_{s=-1}^{m+1}\nu_s[T^{\beta_1}\varphi_{s,3}(\rho_t) \\ & +q_1(\rho_t)T^{\alpha_1}\varphi_{s,3}(\rho_t)-\varphi_{s,3}(\rho_t)v_1(\rho_t)]=h_1(\rho_t). \end{aligned} \tag{4.7}$$

$$\begin{aligned} & \sum_{s=-1}^{m+1}\mu_s[T^{\gamma_2}\varphi_{s,3}(\rho_t)+p_2(\rho_t)T^{\delta_2}\varphi_{s,3}(\rho_t)-\varphi_{s,3}(\rho_t)u_2(\rho_t)]+\sum_{s=-1}^{m+1}\nu_s[T^{\beta_2}\varphi_{s,3}(\rho_t) \\ & +q_2(\rho_t)T^{\alpha_2}\varphi_{s,3}(\rho_t)-\varphi_{s,3}(\rho_t)v_2(\rho_t)]\;=h_2(\rho_t). \end{aligned} \tag{4.8}$$

System in (4.7) and (4.8) consists $2m$ linear equalization in $(2m+6)$ unknowns as $\{\mu_{-1},\mu_0,\cdots,\mu_m,\mu_{m+1}\}$ and $\{\nu_{-1},\nu_0,\cdots,\nu_m,\nu_{m+1}\}$. To obtain its unique solution, six extra constraints are needed. Anyhow, at $\rho{=}c$, one has

$$\begin{cases} \sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}\left(\rho\right)=\eta_1, \ \sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}\left(\rho\right)=\eta_2, \\ \sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}^{'}\left(\rho\right)=\theta_1, \ \sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}^{'}\left(\rho\right)=\theta_2. \end{cases} \tag{4.9}$$

Thus, we have got $2m+4$ equalization for $2m+6$ unknowns. Indeed, consider (4.3) and set $\beta_1{=}\beta_2{=}\gamma_1{=}\gamma_2{=}\,2$ and $\alpha_1{=}\alpha_2{=}\delta_1{=}\delta_2{=}\,1$, one gained

$$\psi''(\rho)+\phi''(\rho)=-(q_1(\rho)\psi'(\rho)+p_1(\rho)\phi'(\rho))+\phi(\rho)u_1(\rho)+\psi(\rho)v_1(\rho)+h_1(\rho), \tag{4.10}$$

$$\psi''(\rho)+\phi''(\rho)=-(q_2(\rho)\psi'(\rho)+p_2(\rho)\phi'(\rho))+\phi(\rho)u_2(\rho)+\psi(\rho)v_2(\rho)+h_2(\rho), \tag{4.11}$$

Substituting $\rho{=}c$ and replacing the value of $\phi'(c)$, $\phi(c)$, $\psi'(c)$, and $\psi(c)$, from the ICCs (1.4), we obtain

$$\psi''(c)+\phi''(c)=-\left(q_1(c)\psi'(c)+p_1(c)\phi'(c)\right)+\phi(c)u_1(c)+\psi(\xi)v_1(c)+h_1(c)=\Lambda_1, \tag{4.12}$$

$$\psi''(c)+\phi''(c)=-\left(q_2(c)\psi'(c)+p_2(c)\phi'(c)\right)+\phi(c)u_2(c)+\psi(c)v_2(c)+h_2(c)=\Lambda_2, \tag{4.13}$$

Then, using the procedure above, we get the attached at $\rho{=}c$:

$$\begin{cases} \sum_{s=-1}^{m+1} \mu_s \varphi_{s,3}^{''}(\rho) + \sum_{s=-1}^{m+1} \nu_s \varphi_{s,3}^{''}(\rho) = \Lambda_1, \\ \sum_{s=-1}^{m+1} \mu_s \varphi_{s,3}^{''}(\rho) + \sum_{s=-1}^{m+1} \nu_s \varphi_{s,3}^{''}(\rho) = \Lambda_2. \end{cases} \tag{4.14}$$

To determine the unknown coefficients μ_s and ν_s, we write this system by practicing the notations in a matrix-vector scheme as

$$A\mu{=}H, \tag{4.15}$$

$$\mu{=}\left[\mu_{-1},\mu_0,\cdots,\mu_m,\mu_{m+1},\nu_{-1},\nu_0,\cdots,\nu_m,\nu_{m+1}\right]^t, \tag{4.16}$$

$$H{=}\left[\eta_1,\theta_1,\Lambda_1,h_1(\rho_1),\cdots,h_1(\rho_m),\eta_2,\theta_2,\Lambda_2,h_2(\rho_1),\cdots,h_2(\rho_m)\right]^t, \tag{4.17}$$

$$A=\begin{bmatrix} R_1 \mid R_2 \\ - \quad - \\ R_3 \mid R_4 \end{bmatrix}, \tag{4.18}$$

$$R_1{=}\begin{bmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & \cdots & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & \frac{1}{h} & \cdots & 0 & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & \cdots & 0 & 0 & 0 \\ c_1(\rho_1) & d_1(\rho_1) & e_1(\rho_1) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1(\rho_m) & d_1(\rho_m) & e_1(\rho_m) \end{bmatrix}, \tag{4.19}$$

$$R_2{=}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & \cdots & 0 & 0 & 0 \\ c_2(\rho_1) & d_2(\rho_1) & e_2(\rho_1) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_2(\rho_m) & d_2(\rho_m) & e_2(\rho_m) \end{bmatrix}, \tag{4.20}$$

$$R_3 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & \cdots & 0 & 0 & 0 \\ c_3(\rho_1) & d_3(\rho_1) & e_3(\rho_1) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_4(\rho_m) & d_4(\rho_m) & e_4(\rho_m) \end{bmatrix}, \tag{4.21}$$

$$R_4 = \begin{bmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & \cdots & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & \frac{1}{h} & \cdots & 0 & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & \cdots & 0 & 0 & 0 \\ c_4(\rho_1) & d_4(\rho_1) & e_4(\rho_1) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_3(\rho_m) & d_3(\rho_m) & e_3(\rho_m) \end{bmatrix}, \tag{4.22}$$

where c_s, d_s, and e_s are defined in Table 2.

Table 2. The entries element in the matrices R_1, R_2, R_3, and R_4:

$c_1(\rho_t)$	$c_2(\rho_t)$	$c_3(\rho_t)$	$c_4(\rho_t)$
$\frac{\rho_{t-1}^{2-\alpha_1}}{h^2}+p_1(\rho_t)\frac{\rho_{t-1}^{1-\beta_1}}{2h}+\frac{1}{6}u_1(\rho_t)$	$\frac{\rho_{t-1}^{2-\alpha_1}}{h^2}+q_1(\rho_t)\frac{\rho_{t-1}^{1-\beta_1}}{2h}+\frac{1}{6}v_1(\rho_t)$	$\frac{\rho_{t-1}^{2-\gamma_2}}{h^2}+p_2(\rho_t)\frac{\rho_{t-1}^{1-\delta_2}}{2h}+\frac{1}{6}u_2(\rho_t)$	$\frac{\rho_{t-1}^{2-\alpha_2}}{h^2}+q_2(\rho_t)\frac{\rho_{t-1}^{1-\beta_2}}{2h}+v_2(\rho_t)\frac{1}{6}$
$d_1(\rho_t)$	$d_2(\rho_t)$	$d_3(\rho_t)$	$d_4(\rho_t)$
$-\frac{2\rho_t^{2-\gamma_1}}{h^2}+\frac{4}{6}u_1(\rho_t)$	$-\frac{2\rho_t^{2-\alpha_1}}{h^2}+\frac{4}{6}v_1(\rho_t)$	$-\frac{2\rho_t^{2-\gamma_2}}{h^2}+\frac{4}{6}u_2(\rho_t)$	$-\frac{2\rho_t^{2-\alpha_2}}{h^2}+v_2(\rho_t)\frac{4}{6}$
$e_1(\rho_t)$	$e_2(\rho_t)$	$e_3(\rho_t)$	$e_4(\rho_t)$
$\frac{\rho_{t+1}^{2-\gamma_1}}{h^2}-p_1(\rho_t)\frac{\rho_{t+1}^{1-\delta_1}}{2h}+\frac{1}{6}u_1(\rho_t)$	$\frac{\rho_{t+1}^{2-\alpha_1}}{h^2}-q_1(\rho_t)\frac{\rho_{t+1}^{1-\beta_1}}{2h}+\frac{1}{6}v_1(\rho_t)$	$\frac{\rho_{t+1}^{2-\gamma_2}}{h^2}-p_2(\rho_t)\frac{\rho_{t+1}^{1-\delta_2}}{2h}+\frac{1}{6}u_2(\rho_t)$	$\frac{\rho_{t+1}^{2-\alpha_1}}{h^2}-q_2(\rho_t)\frac{\rho_{t+1}^{1-\beta_1}}{2h}+v_2(\rho_t)\frac{1}{6}$

We have a linear system of $2m+6$ equalizations in $2m+6$ unknown coefficients given by (4.15). When it is solved using Mathematica 11, one can obtain the unknown coefficients that are necessary for the approximations ϕ and ψ.

- **Singular case**: In section 3, we utilized details about the singularity problem and some tricks to eliminate these issues. Anyhow, we will be using these results to solve the attached linear SIVPs of fractional order using CBSM:

$$\begin{cases} T^{\beta_1}\psi(\rho) + T^{\gamma_1}\phi(\rho) + q_1(\rho)T^{\alpha_1}\psi(\rho) + p_1(\rho)T^{\delta_1}\phi(\rho) \\ =\phi(\rho)\,u_1(\rho) + \psi(\rho)v_1(\rho) + h_1(\rho), \\ T^{\beta_2}\psi(\rho) + T^{\gamma_2}\phi(\rho) + q_2(\rho)T^{\alpha_2}\psi(\rho) + p_2(\rho)T^{\delta_2}\phi(\rho) \\ =\phi(\rho)u_2(\rho) + \psi(\rho)v_2(\rho) + h_2(\rho). \end{cases} \tag{4.23}$$

Because we are in the linear case; for $s=1,2$ (3.7) and (3.8) becomes as

$$T^{\beta_s}\psi(0) + T^{\gamma_s}\phi(0) + kT^{\alpha_s}T^{\alpha_s}\psi(0) + T^{\alpha_s}p_s(0)\,T^{\delta_s}\phi(0) + p_s(0)\,T^{\alpha_s}T^{\delta_s}\phi(0)$$
$$=\phi(0)u_s(0) + \psi(0)\,v_s(0) + h_s(0)\,, \tag{4.24}$$

and/or

$$T^{\beta_s}\psi(0) + T^{\gamma_s}\phi(0) + kT^{\delta_s}T^{\delta_s}\phi(0) + T^{\delta_s}q_s(0)T^{\alpha_s}\psi(0) + q_s(0)T^{\delta_s}T^{\alpha_s}\psi(0)$$
$$=\phi(0)u_s(0) + \psi(0)v_s(0) + h_s(0). \tag{4.25}$$

To get a new constraint, we give the following values for $\beta_s=\gamma_s=2$ and $\alpha_s=\delta_s=1$ for $s=1,2$ in (4.24) and (4.25) to get

$$(1+k)\psi''(0) + p_s'(0)\,\phi'(0) + (p_s(0)+1)\phi''(0) = \phi(0)\,u_s(0) + \psi(0)\,v_s(0) + h_s(0)\,, \tag{4.26}$$

and/or

$$(1+k)\,\phi''(0) + q_s'(0)\,\psi'(0) + (q_s(0)+1)\psi''(0) = \phi(0)\,u_s(0) + \psi(0)\,v_s(0) + h_s(0)\,. \tag{4.27}$$

It is required that the approximate solutions in (4.1) satisfy the SIVPs at $\rho=\rho_t$ for $t=1,2,\cdots,m$ as

$$T^{\beta_1}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t) + T^{\gamma_1}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t) + q_1(\rho_t)\,T^{\alpha_1}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)$$
$$+\,p_1(\rho_t)\,T^{\delta_1}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t) \tag{4.28}$$
$$=u_1(\rho_t)\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t) + v_1(\rho_t)\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t) + h_1(\rho_t)\,,$$

$$T^{\beta_2}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t) + T^{\gamma_2}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t) + q_2(\rho_t)\,T^{\alpha_2}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)$$
$$+\,p_2(\rho_t)\,T^{\delta_2}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t) \tag{4.29}$$
$$=u_2(\rho_t)\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t) + v_2(\rho_t)\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t) + h_2(\rho_t)\,.$$

Together with the attached ICCs at $\rho=c$

$$\begin{cases} \sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho) = \eta_1, \sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho) = \eta_2, \\ \sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}'(\rho) = \theta_1, \sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}'(\rho) = \theta_2. \end{cases} \tag{4.30}$$

Together also with the attached singularity conditions for $s= 1,2$ and $\rho= 0$

$$\begin{aligned}&(1+k)\sum_{s=-1}^{m+1}\nu_s\varphi''_{s,3}(\rho)+p'_s(\rho)\sum_{s=-1}^{m+1}\mu_s\varphi'_{s,3}(\rho)+(p_s(\rho)+1)\sum_{s=-1}^{m+1}\mu_s\varphi''_{s,3}(\rho)\\&=\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho)\,u_s(\rho)+\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho)v_s(\rho)+h_s(\rho),\end{aligned}\tag{4.31}$$

and/or

$$\begin{aligned}&(1+k)\sum_{s=-1}^{m+1}\mu_s\varphi''_{s,3}(\rho)+q'_s(\rho)\sum_{s=-1}^{m+1}\nu_s\varphi'_{s,3}(\rho)+(q_s(\rho)+1)\sum_{s=-1}^{m+1}\nu_s\varphi''_{s,3}(\rho)\\&=\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho)\,u_s(\rho)+\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho)\,v_s(\rho)+h_s(\rho).\end{aligned}\tag{4.32}$$

This will gain an algebraic linear system of rank $m+3$ with $s= 1,2$, $t= 1,2,\cdots,m$, and $\rho= 0$ as

$$\begin{aligned}&\sum_{s=-1}^{m+1}\mu_s\left[T^{\gamma_1}\varphi_{s,3}(\rho_t)+p_1(\rho_t)T^{\delta_1}\varphi_{s,3}(\rho_t)-\varphi_{s,3}(\rho_t)u_1(\rho_t)\right]\\&+\sum_{s=-1}^{m+1}\nu_s\left[T^{\beta_1}\varphi_{s,3}(\rho_t)+q_1(\rho_t)T^{\alpha_1}\varphi_{s,3}(\rho_t)-\varphi_{s,3}(\rho_t)v_1(\rho_t)\right]=h_1(\rho_t),\end{aligned}\tag{4.33}$$

$$\begin{aligned}&\sum_{s=-1}^{m+1}\mu_s\left[T^{\gamma_2}\varphi_{s,3}(\rho_t)+p_2(\rho_t)T^{\delta_2}\varphi_{s,3}(\rho_t)-\varphi_{s,3}(\rho_t)u_2(\rho_t)\right]\\&+\sum_{s=-1}^{m+1}\nu_s\left[T^{\beta_2}\varphi_{s,3}(\rho_t)+q_2(\rho_t)T^{\alpha_2}\varphi_{s,3}(\rho_t)-\varphi_{s,3}(\rho_t)v_2(\rho_t)\right]=h_2(\rho_t),\end{aligned}\tag{4.34}$$

and/or at $\boldsymbol{\rho = c}$

$$\begin{aligned}&\sum_{s=-1}^{m+1}\nu_s\left[(1+k)\varphi''_{s,3}(\rho)-\varphi_{s,3}(\rho)\,v_s(\rho)\right]\\&+\sum_{s=-1}^{m+1}\mu_s\left[p'_s(\rho)\,\varphi'_{s,3}(\rho)+(p_s(\rho)+1)\varphi''_{s,3}(\rho)-\varphi_{s,3}(\rho)\,u_s(\rho)\right]=h_s(\rho),\end{aligned}\tag{4.35}$$

$$\begin{aligned}&\sum_{s=-1}^{m+1}\mu_s\left[(1+k)\varphi''_{s,3}(\rho)-\varphi_{s,3}(\rho)\,u_s(\rho)\right]\\&+\sum_{s=-1}^{m+1}\nu_s\left[q'_i(\rho)\,\varphi'_{s,3}(\rho)+(q_s(\rho)+1)\varphi''_{s,3}(\rho)-\varphi_{s,3}(\rho)\,v_s(\rho)\right]=h_s(\rho),\end{aligned}\tag{4.36}$$

$$\begin{cases}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho)=\eta_1, & \sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho)=\eta_2,\\ \sum_{s=-1}^{m+1}\mu_s\varphi'_{s,3}(\rho)=\theta_1, & \sum_{s=-1}^{m+1}\nu_s\varphi'_{s,3}(\rho)=\theta_2.\end{cases}\tag{4.37}$$

5. The BSCM for nonlinear SIVPs of CFD

For the nonlinear case, we also approximate $\phi(\rho)$ and $\psi(\rho)$ using B-spline basis functions. Thereafter, we collocate our nonlinear system at fitting collocation points $2m$. Therefore we possess a nonlinear system of equations including the same number of unknowns which can be done by Mathematica 11 to obtain the vector. Anyhow, we assume that F_1 and F_2 are nonlinear in (1.1) and (1.3).

- **Regular case**: in (1.1) define $q_1(\rho)$, $p_1(\rho)$, $q_2(\rho)$, and $p_2(\rho)$ in such a way non of those functions has a singular term for the singularity point. Substituting the approximate solutions in (4.1) with their CFDs and utilizing interpolation at $\rho{=}\rho_t$ with $t{=}\,1, 2, \cdots, m$, we get

$$T^{\beta_1}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)+T^{\gamma_1}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)+q_1(\rho_t)\,T^{\alpha_1}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)$$
$$+\,p_1(\rho_t)\,T^{\delta_1}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)=F_1\left(\rho_t,\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t),\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)\right), \tag{5.1}$$

$$T^{\beta_2}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)+T^{\gamma_2}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)+q_2(\rho_t)\,T^{\alpha_2}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)$$
$$+\,p_2(\rho_t)\,T^{\delta_2}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)=F_2\left(\rho_t,\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t),\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)\right). \tag{5.2}$$

Systems (5.1) and (5.2) include $2m$ equalization in $2m+6$ unknowns $\{\mu_{-1},\mu_0,\cdots,\mu_m,\mu_{m+1}\}$ and $\{\nu_{-1},\nu_0,\cdots,\nu_m,\nu_{m+1}\}$. To proceed, six extra constraints are needed. Anyhow, at $\rho{=}c$, one has

$$\begin{cases}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho)=\eta_1, & \sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho)=\eta_2,\\ \sum_{s=-1}^{m+1}\mu_s\varphi'_{s,3}(\rho)=\theta_1, & \sum_{s=-1}^{m+1}\nu_s\varphi'_{s,3}(\rho)=\theta_2.\end{cases} \tag{5.3}$$

We have got $2m+4$ equalization with $2m+6$ unknowns and required two more equalizations. Anyhow, set $\beta_1{=}\beta_2{=}\gamma_1{=}\gamma_2{=}\,2$ and $\alpha_1{=}\delta_1{=}\alpha_2{=}\delta_2{=}\,1$, we get

$$\psi''(\rho)+\phi''(\rho)=-(q_1(\rho)\psi'(\rho)+p_1(\rho)\phi'(\rho))+F_1(\rho,\phi(\rho),\psi(\rho)), \tag{5.4}$$
$$\psi''(\rho)+\phi''(\rho)=-\,(q_2(\rho)\psi'(\rho)+p_2(\rho)\phi'(\rho))+F_2(\rho,\phi(\rho),\psi(\rho)). \tag{5.5}$$

Substituting $\rho{=}c$ and replacing the value of $\phi'(c)$, $\phi(c)$, $\psi'(c)$, and $\psi(c)$ from the ICCs, we obtain

$$\psi''(c)+\phi''(c)=-\,(q_1(c)\psi'(c)+p_1(c)\phi'(c))+F_1(c,\phi(c),\psi(c))=\Lambda_1, \tag{5.6}$$
$$\psi''(c)+\phi''(c)=-\,(q_2(c)\psi'(c)+p_2(c)\phi'(c))+F_2(c,\phi(c),\psi(c))=\Lambda_2. \tag{5.7}$$

Then, for $\rho{=}c$, one has

$$\begin{cases}\sum_{s=-1}^{m+1}\mu_s\varphi''_{s,3}(\rho)+\sum_{s=-1}^{m+1}\nu_s\varphi''_{s,3}(\rho)=\Lambda_1,\\ \sum_{s=-1}^{m+1}\mu_s\varphi''_{s,3}(\rho)+\sum_{s=-1}^{m+1}\nu_s\varphi''_{s,3}(\rho)=\Lambda_2.\end{cases} \tag{5.8}$$

Formulas in (5.8) may be formulated as $A\mu{=}H$ with A is the matrix with dimensions $(2m{+}3){\times}(2m{+}3)$ and

$$\begin{cases} \mu{=} \left[\mu_{-1},\mu_0,\cdots,\mu_m,\mu_{m+1},\nu_{-1},\nu_0,\cdots,\nu_m,\nu_{m+1}\right]^t, \\ H{=} \left[\eta_1,\theta_1,\Lambda_1,h_1(\rho_1),\cdots,h_1(\rho_m),\eta_2,\theta_2,\Lambda_2,h_2(\rho_1),\cdots,h_2(\rho_m)\right]^t. \end{cases} \tag{5.9}$$

- **Singular case**: as in the previous section procedure, let us consider (1.3) and we will use the result of the singularity point which we discussed early. Because we are in the nonlinear case, (3.7) and (3.8) for $s{=}\,1,2$ becomes as

$$\begin{aligned} &T^{\beta_s}\psi\left(0\right)+T^{\gamma_s}\phi\left(0\right)+kT^{\alpha_s}T^{\alpha_s}\psi\left(0\right)+T^{\alpha_s}p_s(0)T^{\delta_s}\phi\left(0\right)+p_s\left(0\right)T^{\alpha_s}T^{\delta_s}\phi(0) \\ &{=}F_s\left(0,\phi(0),\psi\left(0\right)\right), \end{aligned} \tag{5.10}$$

and/or

$$\begin{aligned} &T^{\beta_s}\psi\left(0\right)+T^{\gamma_s}\phi\left(0\right)+kT^{\delta_s}T^{\delta_s}\psi\left(0\right)+T^{\delta_s}q_s\left(0\right)T^{\alpha_s}\psi\left(0\right)+q_s\left(0\right)T^{\delta_s}T^{\alpha_s}\psi(0) \\ &{=}F_s\left(0,\phi\left(0\right),\psi\left(0\right)\right). \end{aligned} \tag{5.11}$$

To get new constraints, we give $\beta_s{=}\gamma_s{=}\,2$ for $s{=}\,1,2$ and $\alpha_s{=}\delta_s{=}\,1$ for $s{=}\,1,2$ in (5.10) and (5.11) as

$$(1{+}k)\psi''\left(0\right)+p_s'\left(0\right)\phi'\left(0\right)+(p_s\left(0\right)+1)\phi''\left(0\right)=F_s\left(0,\phi\left(0\right),\psi\left(0\right)\right), \tag{5.12}$$

and/or

$$(1{+}k)\phi''\left(0\right)+q_s'\left(0\right)\psi'\left(0\right)+(q_s\left(0\right)+1)\psi''\left(0\right)=F_s\left(0,\phi\left(0\right),\psi\left(0\right)\right). \tag{5.13}$$

Following the same previous procedure, it demanded that (4.1) satisfies (1.3) at the $\rho{=}\rho_t$ with $t{=}\,1,2,\cdots,m$ as

$$\begin{aligned} &T^{\beta_1}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}\left(\rho_t\right)+T^{\gamma_1}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}\left(\rho_t\right)+q_1\left(\rho_t\right)T^{\alpha_1}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}\left(\rho_t\right) \\ &+p_1\left(\rho_t\right)T^{\delta_1}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}\left(\rho_t\right)=F_1\left(\rho_t,\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}\left(\rho_t\right),\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}\left(\rho_t\right)\right), \end{aligned} \tag{5.14}$$

$$\begin{aligned} &T^{\beta_2}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}\left(\rho_t\right)+T^{\gamma_2}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}\left(\rho_t\right)+q_2\left(\rho_t\right)T^{\alpha_2}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}\left(\rho_t\right) \\ &+p_2\left(\rho_t\right)T^{\delta_2}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}\left(\rho_t\right)\;=F_2\left(\rho_t,\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}\left(\rho_t\right),\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}\left(\rho_t\right)\right). \end{aligned} \tag{5.15}$$

Together with the ICCs for $\rho{=}c$ as

$$\begin{cases} \sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho)=\eta_1,\ \sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho)=\eta_2, \\ \sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}^{'}(\rho)=\theta_1,\ \sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}^{'}(\rho)=\theta_2. \end{cases} \tag{5.16}$$

Indeed, with the attached singularity conditions for $s= 1,2$ and $\rho= 0$:

$$(1+k)\sum_{s=-1}^{m+1}\nu_s\varphi''_{s,3}(\rho)+p'_s(\rho)\sum_{s=-1}^{m+1}\mu_s\varphi'_{s,3}(\rho)+(p_s(\rho)+1)\sum_{s=-1}^{m+1}\mu_s\varphi''_{s,3}(\rho)$$
$$=F_s\left(\rho,\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho),\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho)\right), \tag{5.17}$$

$$(1+k)\sum_{s=-1}^{m+1}\mu_s\varphi''_{s,3}(\rho)+q'_s(\rho)\sum_{s=-1}^{m+1}\mu_s\varphi'_{s,3}(\rho)+(q_s(\rho)+1)\sum_{s=-1}^{m+1}\mu_s\varphi''_{s,3}(\rho)$$
$$=F_s\left(\rho,\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho),\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho)\right) \tag{5.18}$$

This gained algebraic linear system of rank $(2+6)$ with $t= 1,2,\cdots,m$, $s= 1,2$, and $\rho= 0$ given by

$$T^{\beta_1}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)+T^{\gamma_1}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)+q_1(\rho_t)T^{\alpha_1}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)$$
$$+p_1(\rho_t)T^{\delta_1}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)=F_1\left(\rho_t,\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t),\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)\right), \tag{5.19}$$

$$T^{\beta_2}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)+T^{\gamma_2}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)+q_2(\rho_t)T^{\alpha_2}\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t)$$
$$+p_2(\rho_t)T^{\delta_2}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)=F_2\left(\rho_t,\sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho_t),\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho_t)\right), \tag{5.20}$$

$$\sum_{s=-1}^{m+1}\nu_s\left[(1+k)\varphi''_{s,3}(\rho)\right]+\sum_{s=-1}^{m+1}\mu_s\left[p'_s(\rho)\varphi'_{s,3}(\rho)+(p_s(\rho)+1)\varphi''_{s,3}(\rho)\right]$$
$$=F_s\left(\rho,\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho),\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho)\right), \tag{5.21}$$

and/or at $\rho=c$

$$\sum_{s=-1}^{m+1}\mu_s\left[(1+k)\varphi''{}_{s,3}(\rho)\right]+\sum_{s=-1}^{m+1}\nu_s\left[q'_s(\rho)\varphi'{}_{s,3}(\rho)+(q_s(\rho)+1)\varphi''_{s,3}(\rho)\right]$$
$$=F_s\left(\rho,\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho),\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho)\right), \tag{5.22}$$

$$\begin{cases}\sum_{s=-1}^{m+1}\mu_s\varphi_{s,3}(\rho)=\eta_1,\ \sum_{s=-1}^{m+1}\nu_s\varphi_{s,3}(\rho)=\eta_2,\\ \sum_{s=-1}^{m+1}\mu_s\varphi'_{s,3}(\rho)=\theta_1,\ \sum_{s=-1}^{m+1}\nu_s\varphi'_{s,3}(\rho)=\theta_2.\end{cases} \tag{5.23}$$

6. Numerical outcomes and simulations

This section's purpose is to show the behavior, properties, efficiency, and applicability of the BSCM in the solutions of the linear/nonlinear SIVPs of CFD order. Via exercising the introduced method, the calculation here is reduced to a set of algebraic linear/nonlinear equalizations that can be more simply answered.

Herein, we propose a few numerical simulations of specific examples, we offer the absolute errors as $E_{\psi}(\rho)=\left|\psi(\rho)-\widetilde{\psi}(\rho)\right|$ and $E_{\phi}(\rho)=\left|\phi(\rho)-\widetilde{\phi}(\rho)\right|$, where $a \leq \rho \leq d$. Indeed, $\phi(\rho), \psi(\rho)$ and $\widetilde{\phi}(\rho), \widetilde{\psi}(\rho)$ denote the exact and approximate solutions, respectively.

Problem 1. As a primary model for illustration, consider the linear SIVP of CFD order

$$\begin{cases} T^{\frac{5}{4}}\psi(\rho)+T^{\frac{3}{2}}\phi(\rho)+\sin(\rho)\, T^{\frac{1}{2}}\psi(\rho)+\cos(\pi\rho)\,\phi(\rho)=F_1(\rho), \\ T^{\frac{5}{4}}\phi(\rho)+\rho\exp\left(\rho^3+1\right)\, T^{\frac{1}{4}}\phi(\rho)+\left(\rho^2+2\right)\psi(\rho)=F_2(\rho), \end{cases} \tag{6.1}$$

$$\begin{cases} F_1(\rho)=\rho^{\frac{1}{2}}\left(-\pi^2\rho^{\frac{1}{4}}\sin(\pi\rho)+\left((\rho-1)\rho^{\frac{1}{2}}+\pi\sin(\rho)\right)\cos(\pi\rho)+2\right), \\ F_2(\rho)=\left(\rho^2+2\right)\sin(\pi\rho)+\left(\exp\left(\rho^3+1\right)(2\rho-1)+2\right)\rho^{\frac{3}{4}}, \end{cases} \tag{6.2}$$

subject to the attached ICCs:

$$\begin{cases} \psi(0)=0,\ \psi'(0)=\pi, \\ \phi(0)=0,\ \phi'(0)=-1. \end{cases} \tag{6.3}$$

Herein, $0\leqslant\rho\leqslant1$ and the exact solutions to Problem 1 are $\psi(\rho)=\sin(\pi\rho)$ and $\phi(\rho)=\rho^2-\rho$.

We employ the BSCM to solve Problem 1 by taking different values for m. In Table 3, we examine E_{ψ} of ψ and E_{ϕ} of ϕ at different points when $m=50$. In Table 4, we examine the maximum E_{ψ} and E_{ϕ} of ψ and ϕ, respectively, at different values of m. In Figure 1 $\widetilde{\psi}(\rho_t)$ and $\widetilde{\phi}(\rho_t)$ against of $\psi(\rho_t)$ and $\phi(\rho_t)$ when $m=50$ are plotted. Indeed, $E_{\psi}(\rho_t)$ and $E_{\phi}(\rho_t)$ when $m=40$ are plotted in Figure 2.

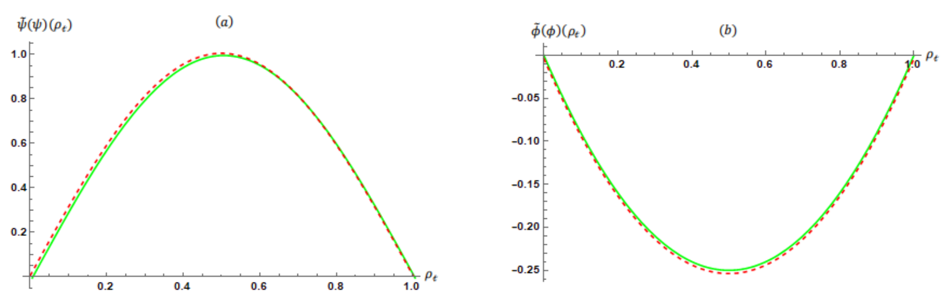

Figure 1. Results in Problem 1 when $m=50$: (a) $\psi(\rho_t)$ (dashed line) and (b) $\widetilde{\psi}(\rho_t)$ (solid line).

It can be seen from the Tables and Figures, that our estimated numerical solutions are in totally great agreement with exact solutions. Moreover, we found from those results that the absolute errors are so small, which shows that the rate of

Table 3. Numerical results for Problem 1

ρ_t	$\widetilde{\psi}(\rho_t)$	$\widetilde{\phi}(\rho_t)$	$E_\psi(\rho_t)$	$E_\phi(\rho_t)$
0	0	0	0	0
0.1	0.309019	−0.090000	1.74349×10^{-6}	1.47579×10^{-8}
0.2	0.587799	−0.160000	1.36785×10^{-5}	2.48675×10^{-7}
0.3	0.809062	−0.210001	4.51107×10^{-5}	1.31242×10^{-6}
0.4	0.951160	−0.240004	1.03291×10^{-4}	4.23185×10^{-6}
0.5	1.000190	−0.250010	1.92503×10^{-4}	1.03632×10^{-5}
0.6	0.951370	−0.240021	3.13502×10^{-4}	2.12337×10^{-5}
0.7	0.809480	−0.210038	4.63293×10^{-4}	3.82799×10^{-5}
0.8	0.588420	−0.160062	6.35233×10^{-4}	6.24541×10^{-5}
0.9	0.309836	−0.090093	5.19393×10^{-4}	9.36878×10^{-5}
1	0.001003	−0.000130	1.00320×10^{-3}	1.30294×10^{-4}

Table 4. Maximum E_ψ and E_ϕ of ψ and ϕ for Problem 1

m	20	30	40	80	100
$\max(E_\psi(\rho_t))$	6.27×10^{-3}	2.78×10^{-3}	1.56×10^{-3}	3.91×10^{-4}	1.74×10^{-4}
$\max(E_\phi(\rho_t))$	8.17×10^{-4}	3.62×10^{-4}	2.03×10^{-4}	5.08×10^{-5}	2.26×10^{-5}

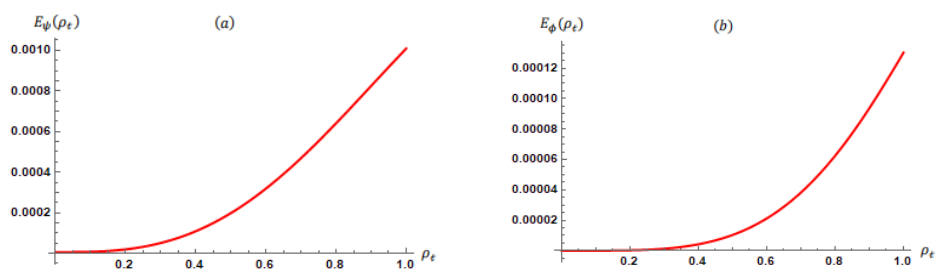

Figure 2. Results in Problem 1 when m= 40: (a) $E_\psi(\rho_t)$ and (b) $E_\phi(\rho_t)$.

convergence of the instant process is so active, and the overall errors can be made remarkably petite by calculating more extra terms in approaches.

Problem 2. As a basic example, consider the nonlinear SIVPs of CFD order

$$\begin{cases} T^{\frac{4}{3}}\psi(\rho) - \phi(\rho)\,T^{\frac{1}{2}}\psi(\rho) - \cos(\phi(\rho)) = F_1(\rho), \\ T^{\frac{4}{3}}\phi(\rho) - \exp(-\rho)\,T^{\frac{1}{2}}\psi(\rho) + \left(T^{\frac{1}{3}}\phi(\rho)\right)^2 + \rho^2\psi(\rho) = F_2(\rho), \end{cases} \tag{6.4}$$

$$\begin{cases} F_1(\rho) = \exp(\rho)\,\rho^{\frac{2}{3}}(2+\rho(4+\rho)) - \cos(1+4\rho\ln(1+\rho)) \\ \qquad\qquad - \exp(\rho)\,\rho^{\frac{2}{3}}\,(2+\rho)\,(1+4\rho\ln(1+\rho)), \\ F_2(\rho) = -(\rho+2)\rho^{\frac{3}{2}} + \frac{4(\rho+2)\rho^{\frac{2}{3}}}{(\rho+1)^2} + 16\rho^{\frac{4}{3}}\left(\frac{\rho}{\rho+1} + \ln(\rho+1)\right)^2 + \exp(\rho)\,\rho^4, \end{cases} \tag{6.5}$$

subject to the attached ICCs:

$$\begin{cases} \psi(0)=0,\ \psi'(0)=0, \\ \phi(0)=1,\ \phi'(0)=0. \end{cases} \tag{6.6}$$

Herein, $0\leqslant\rho\leqslant 1$ and the exact solutions to Problem 2 are $\psi(\rho)=\rho^2\exp(\rho)$ and $\phi(\rho)=4\rho\ln(\rho+1)+1$.

For Problem 2, the numerical results of E_ψ of ψ and E_ϕ of ϕ obtained by the BSCM for $m=40$ are given in Table 5 whilst the corresponding maximum E_ψ and E_ϕ of ψ and ϕ, respectively, are provided in Table 6. Here, the experimental rates of convergence for the different number of collocation points which is approximately equal to 2 are calculated and confirm that the process is a very strong tool to gain approximate solutions.

Table 5. Numerical results for Problem 2.

ρ_t	$\widetilde{\psi}(\rho_t)$	$\widetilde{\phi}(\rho_t)$	$E_\psi(\rho_t)$	$E_\phi(\rho_t)$
0	0	1	0	0
0.1	0.01106	1.03813	3.46044×10^{-6}	7.52781×10^{-6}
0.2	0.04887	1.14588	1.51681×10^{-5}	2.72322×10^{-5}
0.3	0.12153	1.31489	3.76092×10^{-5}	5.47939×10^{-5}
0.4	0.23877	1.53844	7.45048×10^{-5}	8.58982×10^{-5}
0.5	0.41231	1.81105	1.31695×10^{-4}	1.16719×10^{-4}
0.6	0.65618	2.12815	2.18450×10^{-4}	1.44579×10^{-4}
0.7	0.98709	2.48593	3.49526×10^{-4}	1.68264×10^{-4}
0.8	1.42489	2.88111	5.48569×10^{-4}	1.87821×10^{-4}
0.9	1.99313	3.31088	8.53993×10^{-4}	2.03991×10^{-4}
1	2.71918	3.77296	9.02480×10^{-3}	3.74582×10^{-4}

Table 6. Maximum E_ψ and E_ϕ of ψ and ϕ for Problem 2.

m	20	40	60	80	100
$\max(E_\psi(\rho_t))$	5.34×10^{-3}	1.32×10^{-3}	5.90×10^{-4}	3.31×10^{-4}	2.12×10^{-4}
$\max(E_\phi(\rho_t))$	8.69×10^{-4}	2.17×10^{-4}	9.67×10^{-5}	5.44×10^{-5}	3.48×10^{-5}

Problem 3. Consider the linear system of the Lane-Emden model of CFD order

$$\begin{cases} T^{\frac{6}{5}}\psi(\rho)+\frac{2}{\rho}T^{\frac{1}{5}}\psi(\rho)+\ln(\rho+1)\ \psi(\rho)+\cos(\pi\rho)\ T^{\frac{1}{2}}\phi(\rho)=F_1(\rho), \\ T^{\frac{3}{2}}\phi(\rho)+T^{\frac{6}{5}}\psi(\rho)+\frac{2}{\rho}T^{\frac{1}{2}}\phi(\rho)+\left(\rho^3+\rho^2+3\right)\psi(\rho)=F_2(\rho), \end{cases} \tag{6.7}$$

$$F_1(\rho)=\left(6\rho+\frac{9}{4}\exp\left(\frac{3}{2}\rho\right)\right)\rho^{\frac{4}{5}}+\left(\rho^3+\exp\left(\frac{3}{2}\rho\right)+1\right)\ln(\rho+1)$$
$$+\frac{2\left(3\rho^2+\frac{3}{2}\exp\left(\frac{3}{2}\rho\right)\right)}{\rho^{\frac{1}{5}}}+\left(1-3\rho^2\right)\rho^{\frac{1}{2}}\cos(\pi\rho)\ ,$$
$$F_2(\rho)=-\,6\rho^{\frac{3}{2}}+\left(6\rho+\frac{9}{4}\exp\left(\frac{3}{2}\rho\right)\right)\rho^{\frac{4}{5}}+\frac{1-3\rho^2}{\rho^{\frac{1}{2}}}$$
$$+\left(\rho^3+\exp\left(\frac{3}{2}\rho\right)+1\right)\left(\rho^3+\rho^2+3\right), \tag{6.8}$$

subject to the attached ICCs:

$$\begin{cases}\psi(0)=2, & \psi'(0)=1.5,\\ \phi(0)=-\,5, & \phi'(0)=1.\end{cases} \tag{6.9}$$

Herein, $0\leqslant\rho\leqslant1$ and the exact solutions to Problem 3 are $\psi(\rho)=\exp\left(\frac{3}{2}\rho\right)+\rho^3+1$ and $\phi(\rho)=-\,\rho^3+\rho-5$.

For Problem 3, the numerical results of E_ψ of ψ and E_ϕ of ϕ for $m=40$ are given in Table 6. The idea is to rewrite this conformable problem as an algebraic system of equalizations depending only on the value of cubic B-spline across used all nodes, and then we can apply any numerical tool available to solve it. Table 7 shows the numerical results by applying the proposed iterative method to different sizes of grids. Figure 3 utilizes plots of absolute errors and presents comparisons between $m=$ 100 and $m=$ 200.

Table 7. Numerical results for Problem 3.

ρ_t	$\widetilde{\psi}(\rho_t)$	$\widetilde{\phi}(\rho_t)$	$E_\psi(\rho_t)$	$E_\phi(\rho_t)$
0	2	−5	0	0
0.08	2.12801	−4.92051	2.04332×10^{-7}	7.23940×10^{-8}
0.16	2.27535	−4.84410	8.11001×10^{-7}	3.46140×10^{-7}
0.24	2.44716	−4.77382	1.90375×10^{-6}	8.91999×10^{-7}
0.32	2.64885	−4.71277	3.56758×10^{-6}	1.76467×10^{-6}
0.4	2.88612	−4.66400	5.90879×10^{-6}	3.00720×10^{-6}
0.48	3.16503	−4.63059	9.05877×10^{-6}	4.65023×10^{-6}
0.56	3.49200	−4.61561	1.31762×10^{-5}	6.70983×10^{-6}
0.64	3.87386	−4.62213	1.84470×10^{-5}	9.18531×10^{-6}
0.72	4.31795	−4.65324	2.50824×10^{-5}	1.20574×10^{-5}
0.80	4.83215	−4.71198	3.33158×10^{-5}	1.52865×10^{-5}
0.88	5.42494	−4.80145	4.33982×10^{-5}	1.88099×10^{-5}
0.96	6.10549	−4.92471	5.55950×10^{-5}	2.25364×10^{-5}

Problem 4. As an elementary example, consider the nonlinear singular SIVPs of

Table 8. Numerical results for Problem 3.

m	20	40	80	100	120
$\max\left(E_{\psi}\left(\rho_t\right)\right)$	3.91×10^{-4}	9.77×10^{-5}	2.44×10^{-5}	1.56×10^{-5}	1.08×10^{-5}
$\max\left(E_{\phi}\left(\rho_t\right)\right)$	1.52×10^{-4}	3.81×10^{-5}	9.54×10^{-6}	6.11×10^{-6}	4.24×10^{-6}

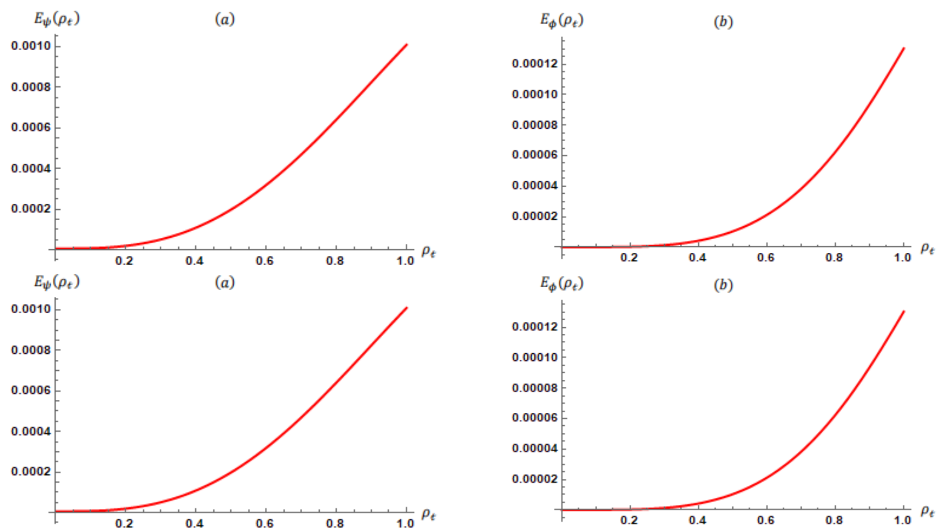

Figure 3. Results in Problem 3: (a) $E_{\psi}\left(\rho_t\right)$ when m= 100, (b) $E_{\phi}\left(\rho_t\right)$ when m= 100, (c) $E_{\psi}\left(\rho_t\right)$ when m= 200, (d) $E_{\phi}\left(\rho_t\right)$ when m= 200.

CFD order

$$\begin{cases} T^{\frac{3}{2}}\psi\left(\rho\right)-\frac{2}{\rho}T^{\frac{1}{4}}\phi\left(\rho\right)+\frac{T^{\frac{1}{4}}\psi(\rho)}{1+T^{\frac{1}{4}}\psi(\rho)}+\phi\left(\rho\right)-\cos(\phi\left(\rho\right))=F_1\left(\rho\right), \\ T^{\frac{4}{3}}\phi\left(\rho\right)+\frac{1}{\rho}T^{\frac{1}{2}}\psi\left(\rho\right)+T^{\frac{1}{3}}\phi\left(\rho\right)\phi\left(\rho\right)+\psi\left(\rho\right)=F_2\left(\rho\right), \end{cases} \tag{6.10}$$

$$\begin{aligned} F_1\left(\rho\right)=&\ln\left(\left(\rho-1\right)^2\rho^2+1\right)-\cos\left(\ln\left(\left(\rho-1\right)^2\rho^2+1\right)\right)+\frac{\left(6\rho^2-6\rho+1\right)\rho^{\frac{3}{4}}}{\left(6\rho^2-6\rho+1\right)\rho^{\frac{3}{4}}+1} \\ &+\frac{4\left(2\rho^6-6\rho^5+7\rho^4-4\rho^3-5\rho^2+6\rho-1\right)\rho^{\frac{3}{4}}}{\left(\rho^4-2\rho^3+\rho^2+1\right)^2} \\ &-\frac{2\left(2\rho^6-6\rho^5+7\rho^4-4\rho^3-5\rho^2+6\rho-1\right)\rho^{\frac{7}{6}}}{\left(\rho^4-2\rho^3+\rho^2+1\right)^2}, \end{aligned} \tag{6.11}$$

$$\begin{aligned} F_2\left(\rho\right)=&6\left(2\rho-1\right)\rho^{\frac{2}{3}}+\rho^2\left(2\rho-3\right)+\frac{6\rho^2-6\rho+1}{\rho^{\frac{1}{2}}} \\ &+\frac{2\rho\left(\rho-1\right)\left(2\rho-1\right)\rho^{\frac{2}{3}}\ln\left(\left(\rho-1\right)^2\rho^2+1\right)}{\left(\rho-1\right)^2\rho^2+1}+\rho \end{aligned} \tag{6.12}$$

subject to the attached ICCs:

$$\begin{cases} \psi(0) = 0, \quad \psi'(0) = 0, \\ \phi(0) = 0, \quad \phi'(0) = 1. \end{cases} \tag{6.13}$$

Herein, $0 \leqslant \rho \leqslant 1$ and the exact solutions to Problem 4 are $\psi(\rho) = 2\rho^3 - 3\rho^2 + \rho$ and $\phi(\rho) = \ln\left((\rho-1)^2\rho^2 + 1\right)$.

In Table 8, the approximation results from the BSCM are tabulated. Maximum E_ψ and E_ϕ of ψ and ϕ at varying scale levels are presented in Table 9 when $m = 40$. In Figure 4, we plotted $\widetilde{\psi}(\rho_t)$ and $\widetilde{\phi}(\rho_t)$ against of $\psi(\rho_t)$ and $\phi(\rho_t)$ for $m = 20$. It can be noticed from this table that increasing m produces better approximate solutions in the neighborhood of a singular point. Encore with the use of the BSCM to calculate numerical solutions, we observe that E_ψ and E_ϕ at various scale levels are extremely smaller which confirms the efficiency and usability of the studied technique.

Table 9. Numerical results for Problem 4.

ρ_t	$\widetilde{\psi}(\rho_t)$	$\widetilde{\phi}(\rho_t)$	$E_\psi(\rho_t)$	$E_\phi(\rho_t)$
0	0	0	0	0
0.1	0.07200	0.00807	2.03087×10^{-10}	3.10106×10^{-6}
0.2	0.09600	0.02529	6.67799×10^{-9}	1.49296×10^{-5}
0.3	0.08400	0.04319	4.67665×10^{-8}	3.65078×10^{-5}
0.4	0.04799	0.05607	1.70058×10^{-7}	6.70792×10^{-5}
0.5	0.06073	0.06073	4.30030×10^{-7}	1.05828×10^{-4}
0.6	-0.04800	0.05615	8.62834×10^{-7}	1.52673×10^{-4}
0.7	-0.08400	0.04336	1.47172×10^{-6}	2.08502×10^{-4}
0.8	-0.09600	0.02555	2.21750×10^{-6}	2.74990×10^{-4}
0.9	-0.07200	0.00842	3.01795×10^{-6}	3.53813×10^{-4}
1	0.00049	0.00044	3.77244×10^{-6}	4.44886×10^{-4}

Table 10. Maximum E_ψ and E_ϕ of ψ and ϕ for Problem 4.

m	20	40	60	80	100
$\max(E_\psi(\rho_t))$	2.40×10^{-5}	5.90×10^{-6}	2.61×10^{-6}	2.61×10^{-6}	9.40×10^{-7}
$\max(E_\phi(\rho_t))$	2.78×10^{-3}	6.95×10^{-4}	3.08×10^{-4}	3.08×10^{-4}	1.11×10^{-4}

7. Conclusion

The goal of the present study was to provide the approximate solutions system of singular and nonsingular initial value problems of fractional order differential equations for both linear/nonlinear cases in terms of piecewise polynomials using

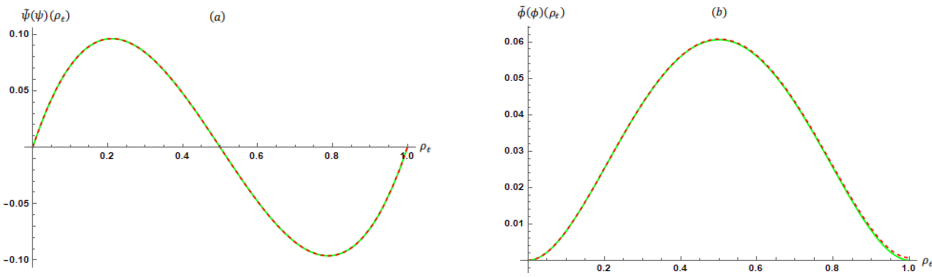

Figure 4. Results in Problem 4 when m= 20: (a) $\psi\left(\rho_t\right)$ (dashed line) and (b) $\widetilde{\psi}\left(\rho_t\right)$ (solid line).

an efficient and accurate method. The work emphasized our belief that the useful properties of the cubic B-spline and collocation method are reliable to handle linear and nonlinear SIVPs of fractional order differential equations. The introduced method is practiced in some particular examples to demonstrate its applicability and efficiency. The results of the approximate solutions are in excellent agreement with the exact solutions. Our future suggested work will focus on the SIVPs of CFD order concerning nonclassical constraints.

Declarations

Conflicts of Interest. The authors declare that they have no conflicts of interest.
Ethical Approval. This article does not contain any studies with human participants or animals performed by any of the authors.
Data Availability Statement. No datasets are associated with this manuscript. The datasets used for generating the plots and results during the current study can be directly obtained from the numerical simulation of the related mathematical equations in the manuscript.

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