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APPROXIMATION OF WEAKLY SINGULAR NON-LINEAR VOLTERRA-URYSOHN INTEGRAL EQUATIONS BY PIECEWISE POLYNOMIAL PROJECTION METHODS BASED ON GRADED MESH

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Abstract In this article, we address the approximation solution of Volterra-Urysohn integral equations which involves weakly singular kernels. In order to get better convergence rates, projection methods namely Galerkin and multi Galerkin methods, along with their iterated versions are used in the space of piecewise polynomials subspaces based on the graded mesh. In addition, we compute the superconvergence results for the proposed integral equation and show that iterated Galerkin method outperforms Galerkin method in terms of order of convergence. Further, we demonstrate numerical examples to verify the proposed theoretical framework.

Keywords Superconvergence results, Volterra integral equations with weakly singular kernel, Galerkin method, Multi-Galerkin method, piecewise polynomials.

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1. Introduction

The well-known Volterra-Urysohn integral equation of the second kind with a weakly singular kernel is given as follows:

$$u(\tau) = g(\tau) + \int_0^\tau (\tau - s)^{-\gamma} k(\tau, s, u(s))\, ds, \quad \tau \in [0, 1], \qquad 0 < \gamma < 1, \qquad (1.1)$$

where g is an inhomogeneous function in the Banach space $\mathbb{X} = L^\infty[0, 1]$, which is sufficiently smooth, the unknown function u in $\mathbb{X}$, is to be determined, and the kernel function $k(., ., .)$ is sufficiently smooth in the triangular area $C(\sigma)$, where $\sigma = \{(\tau, s) : 0 \le s \le \tau \le 1\}$ and $C(\sigma)$ signifies the set of all continuous functions in σ. We say the preceding Eq. (1.1) weakly singular because $(\tau - s)^{-\gamma}$ makes our

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kernel $k(.,.,.)$ unbounded at some $s = \tau$, although its integral is finite. This type of integral equation arises in the area of physics, chemistry, and biology [6,7,9,21,24]. With the use of the Laplace transformation, W. R. Mann and F. Wolf [21] explained a nonlinear BVP of heat transfer between solids and gases by converting it into a nonlinear integral equation. R.K. Miller discussed the regularity of solution of second kind nonlinear Volterra integral equations (VIE's) with convolution kernels in [23]. Regularity of the solution u of the aforementioned second kind weakly singular VIE (1.1) is discussed in [5].

The proposed Eq. (1.1) has an exact solution u that is either sufficiently smooth or its derivative $u'(\tau)$ contains a singularity around the origin $\tau = 0$ which implies $u'(\tau) \sim \tau^{-\gamma}$ [9]. In general, finding an exact solution to the aforesaid nonlinear VIE is difficult. The non-smoothness attribute of an exact solution causes a reduction in the global order of convergence of numerical methods based on uniform meshes. To retrieve the optimal order of convergence some classical approaches can be used. Instead of employing uniform mesh, graded mesh can be used to tackle this difficulty [8].

Several authors have studied and thoroughly documented many numerical approaches for finding numerical approximate solutions to non-linear VIE's in [6–8, 13, 27, 32]. The Galerkin, collocation, and product integration methods [6–10, 14, 20, 31, 34], are some of the most well-known numerical approaches to solve integral equations. Due to the usage of projection operators, these methods are referred to as projection methods. Brunner [8] developed the collocation method for nonlinear VIE's, and compared the convergence results in quasi-uniform and graded meshes successfully. Under certain assumptions, D O'regan [25] examined the existence of a solution for VIE's of Urysohn, and Hammerstein types. Following that, Brunner et al. [9] proposed a piecewise collocation approach based on the graded mesh for solving the preceding Eq. (1.1) as well as discussed the solution's smoothness. To avoid the situation of unbounded derivative of exact solution u at the left endpoint of the integration domain [0, 1], Orsi [26] introduced a new numerical approach to solve second kind weakly singular nonlinear VIE's with the product integration method. An extrapolation technique for solving proposed equation is described in [32], based on Gronwall inequality and Navot's quadrature rule with the endpoint singularity. In order to retrieve the optimal order, Rebelo et al. [28] used the Hybrid Collocation method and also obtained convergence rates by applying non-polynomial spline approximations and collocation on the graded mesh. Baratella [3] suggested a Nyström type interpolant of the solution for some weakly singular nonlinear VIE's based on Gauss Radau nodes and improved convergence results by employing a smoothing transformation technique.

Z. Xie et al. in [34] discussed convergence analysis of spectral Galerkin method for second kind Volterra type integral equations. S.Sohrabi et al. [30] proposed spectral-collocation method for a class of nonlinear weakly singular VIE's and obtained the convergence results. Kant et al. [18] studied the order of convergence of Galerkin and multi-Galerkin methods for Volterra Hammerstein integral equations. Micula [22] described numerical iterative method for the approximate solutions of weakly singular nonlinear VIE's of the second kind. On the basis of multivariate Jacobi approximation, Zaky et al. [35] recently discussed the spectral collocation method for nonlinear weakly singular VIE's.

This article discusses how to solve non-linear VIE's with weakly singular kernels using piecewise polynomials subspaces based on graded mesh to produce superconvergence results.

The significant contributions of this paper is given as follows:

- First, projection methods are discussed for solving the Volterra integral equations of type (1.1) with graded mesh in piecewise polynomials subspaces.
- In order to obtain the superconvergence rates, we transform weakly singular VIE to Fredholm integral equation.
- We show that the Galerkin and iterated Galerkin solutions converge with the order $\mathcal{O}(n^{-r})$ and $\mathcal{O}(n^{-2r})$, respectively, where r is the order of piecewise polynomials employed in the approximation and n signifies the number of partition points.
- In addition, we use an iterated multi-Galerkin approach to improve the convergence results. Indeed, we show that the iterated multi Galerkin approach converges with $\mathcal{O}(n^{-3r})$ order.

The remainder of the paper is depicted as follows: In Section 2, Galerkin and iterated Galerkin methods for the proposed VIE of the type (1.1) are developed. Preliminaries and auxiliary lemmas are discussed in Section 3. Section 4 discusses the convergence results and the improved superconvergence rate using the iterated multi-Galerkin method is presented in Section 5. Finally, numerical examples and conclusions are striated in Section 6.

2. Non-linear Volterra-Urysohn Integral Equation

We covered the Galerkin and iterated Galerkin method for VIE of the urysohn type with graded mesh in this section. To implement the enhanced solvability theory of the Fredholm integral equation, we begin by transforming the Urysohn VIE to the Fredholm integral equation.

Consider the second kind Volterra-Urysohn integral equation with a weakly singular kernel in a Banach space $\mathbb{X} = L^{\infty}[0,1]$

$$u(\tau) = g(\tau) + \int_0^{\tau} (\tau - s)^{-\gamma} k(\tau, s, u(s))\, ds, \quad \tau \in [0,\, 1],\ \gamma \in (0,\, 1). \tag{2.1}$$

In the Eq. (2.1), u is the unknown function in the Banach space $\mathbb{X}$, while k and g are known and sufficiently smooth functions which represent the kernel and source functions, respectively.

In order to obtain the superconvergence results, the domain of integration is transformed from $[0,\tau]$ to the interval $[0,1]$ by the transformation, $s(.,.) : ([0,1] \times [0,1]) \to [0,1]$

$$s = \tau\lambda, \qquad (\tau, \lambda) \in ([0,1] \times [0,1]). \tag{2.2}$$

It reduces the problem (2.1) to the following integral equation

$$u(\tau) = g(\tau) + \int_0^1 (1-\lambda)^{-\gamma} \tilde{k}\left(\tau, s(\tau,\lambda), u(s(\tau,\lambda))\right)\, d\lambda. \tag{2.3}$$

Let us define an operator $\mathcal{K} : L^{\infty}[0,1] \to L^{\infty}[0,1]$ for the integral portion of above Eq. (2.3)

$$\mathcal{K}(u)(\tau) := \int_0^1 (1-\lambda)^{-\gamma} \tilde{k}\left(\tau, s(\tau,\lambda), u(s(\tau,\lambda))\right)\, d\lambda, \tag{2.4}$$

with

$$\tilde{k}\left(\tau, s(\tau,\lambda), u(s(\tau,\lambda))\right) := \tau^{1-\gamma} k\left(\tau, s(\tau,\lambda), u(s(\tau,\lambda))\right). \quad (2.5)$$

The Fréchet derivative at u of integral operator (2.4) is given by

$$\mathcal{K}'(u)y(\tau) = \int_0^1 (1-\lambda)^{-\gamma} \tilde{k}_u\left(\tau, s(\tau,\lambda), u(s(\tau,\lambda))\right) y(s(\tau,\lambda))\, d\lambda, \ \ y \in L^\infty[0,1], \quad (2.6)$$

where $\tilde{k}_u\left(\tau, s(\tau,\lambda), u(s(\tau,\lambda))\right) = \frac{\partial}{\partial u}\tilde{k}\left(\tau, s(\tau,\lambda), u(s(\tau,\lambda))\right)$.

Set

$$W(\tau, s(\tau,\lambda), u(s(\tau,\lambda))) := (1-\lambda)^{-\gamma} \tilde{k}_u\left(\tau, s(\tau,\lambda), u(s(\tau,\lambda))\right). \quad (2.7)$$

The integral equation (2.3) has the following operator equation form:

$$u(\tau) = g(\tau) + \mathcal{K}(u)(\tau), \qquad \tau \in [0,1]. \quad (2.8)$$

2.1. Assumptions

Throughout this paper following assumptions are made on g, kernel $k(.,.,.)$, and transformed kernel $\tilde{k}(.,.,)$:

(i) The source function g is r times continuously differentiable over an interval $[0,1]$, where r is greater than or equal to 1 and it also satisfies Lipschitz continuity with order θ. That is, for any $\tau_1, \tau_2 \in [0,1]$ there exists a positive number a such that

$$|g(s(\tau_1,\lambda)) - g(s(\tau_2,\lambda))| \le a|s(\tau_1,\lambda) - s(\tau_2,\lambda)|^\theta, \quad (2.9)$$

where $\theta \ge 1-\gamma$.

(ii) The kernel $k(.,.,.)$ satisfies the Lipschitz continuity with respect to u and also it is bounded over $[0,1]\times[0,1]\times\mathbb{R}$. That is, for any $u_1, u_2 \in \mathbb{R}$, there exists a positive number a_1 such that

$$|k\left(\tau, s, u_1\right) - k\left(\tau, s, u_2\right)| \le a_1|u_1 - u_2|. \quad (2.10)$$

(iii) The partial derivative with respect to u of the transformed kernel $\tilde{k}(.,.,.)$ exists and it also satisfies Lipschitz continuity with respect to u. That is, for any $u_1, u_2 \in \mathbb{R}$, there exists a positive number a_3 such that

$$|\tilde{k}_u\left(\tau, s, u_1\right) - \tilde{k}_u\left(\tau, s, u_2\right)| \le a_3|u_1 - u_2|. \quad (2.11)$$

(iv) Further, the partial derivative $\tilde{k}_u(.,.,.)$ satisfies the Lipschitz continuity with respect to s. Then, for any $s_1, s_2 \in [0,1]$ there exists a positive number a_4 such that

$$|\tilde{k}_u\left(\tau, s_1, u\right) - \tilde{k}_u\left(\tau, s_2, u\right)| \le a_4|s_1 - s_2|. \quad (2.12)$$

Remark 2.1. (i) By Eq. (2.5) and assumption (ii), we have

$$|\tilde{k}\left(\tau, s, u_1\right) - \tilde{k}\left(\tau, s, u_2\right)| = |\tau^{1-\gamma} k\left(\tau, s, u_1\right) - \tau^{1-\gamma} k\left(\tau, s, u_2\right)|$$

$$= |\tau^{1-\gamma}||k(\tau, s, u_1) - k(\tau, s, u_2)|$$
$$\leq |\tau^{1-\gamma}|a_1|u_1 - u_2|$$
$$\leq a_2|u_1 - u_2|, \quad (2.13)$$

where $a_2 = \sup_{\tau \in [0,1]} |\tau^{1-\gamma}|a_1$.

Hereby, the transformed kernel $\tilde{k}(.,.,.)$ also satisfies Lipschitz continuity with respect to u.

(ii) As the kernel $k(.,.,.)$ is sufficiently smooth, then the partial derivative $\tilde{k}_u(\tau, s(\tau,\lambda), u(s(\tau,\lambda)))$ of transformed kernel $\tilde{k}(\tau, s(\tau,\lambda), u(s(\tau,\lambda)))$ is sufficiently smooth for any $u \in \mathcal{C}^r(0,1]$, where $r \geq 1$, the space of r-times continuously differentiable functions.

Let us introduce an operator T on $\mathbb{X}$ such that:

$$T(u) = g + \mathcal{K}(u), \quad u \in \mathbb{X}. \quad (2.14)$$

The Fréchet derivative of T at any $u \in \mathbb{X}$ is as follows:

$$T^{'}(u) = \mathcal{K}^{'}(u).$$

From Eq. (2.14), the Eq. (2.8) can be written as:

$$u = T(u). \quad (2.15)$$

For $\tilde{a}a_2 < 1$, $\tilde{a} = \frac{2^{1-\gamma}}{1-\gamma}$, $T : \mathbb{X} \to \mathbb{X}$ is a contraction mapping (by Eq. (2.13)) then $\exists$ unique solution u_0 in a complete metric space $\mathbb{X}$, for the Eq. (2.15).

Suppose that 1 is not an eigenvalue of linear operator $T^{'}(u_0)$ then the inverse of integral operator $\left(I - T^{'}(u_0)\right)$ exists and uniformly bounded in infinity norm.

For any $v \in \mathcal{C}^r[0,1]$, we can write

$$\|v\|_{r,\infty} = \max\left(\left\|\frac{\partial^k v}{\partial \tau^k}\right\|_\infty : 0 \leq k \leq r\right),$$

where $\dfrac{\partial^k v}{\partial \tau^k}$ denotes the k^{th} order derivative of v w.r.t. τ for any $r \in \mathbb{N}$.

Now we discuss the graded mesh for the given interval $[0,1]$. Since the kernel function has singularity at $\lambda = 1$ and the derivative of exact solution u_0 is unbounded at $\tau = 0$ in $[0,1]$, we consider the graded mesh $\Pi_n = \{\tau_0, \tau_1, \ldots, \tau_n\}$ of the following form:

$$\tau_l = \frac{1}{2}\left(\frac{2l}{n}\right)^q, \qquad 0 \leq l \leq \frac{n}{2}, \quad (2.16)$$
$$\tau_l = 1 - \tau_{n-l}, \qquad \frac{n}{2} \leq l \leq n, \quad (2.17)$$

wherein $q = \frac{r}{1-\gamma}, r \geq 1$.

Next, we set $\Lambda_l = [\tau_{l-1}, \tau_l]$, $\forall\ l$ be the sub-intervals of $[0,1]$ and $h = \max_l\{h_l = \tau_l - \tau_{l-1}\}$, $(l = 1, 2, \ldots, n)$ be the norm of partition, which tends to 0 when $n \to \infty$. The approximating subspace $\mathbb{X}_n = S^{\mu}_{r,n}(\Pi_n)$ over the interval $[0,1]$, the space of piecewise polynomials with degree less than and equals to $r - 1$, with continuous derivatives $-1 \leq \mu \leq r-2$ at the break points $\tau_l,\ l = 1, 2, \ldots (n-1)$. If $\mu = 0$, then approximating subspace $S^0_{r,n}(\Pi_n)$ becomes the case of continuous piecewise polynomials while for $\mu = -1$ approximating subspace $S^{-1}_{r,n}(\Pi_n)$ has no requirement of continuity at the breakpoints. In this case $u_n \in \mathbb{X}_n$ is arbitrarily taken to be left continuous at $\tau_l,\ \forall\ l$ and right continuous at τ_0. Then

$$h_1 = \tau_1 - \tau_0 = \frac{1}{2}\Big(\frac{2}{n}\Big)^q - 0 = 2^{q-1}\Big(\frac{1}{n}\Big)^q = \mathcal{O}(n^{-q}), \tag{2.18}$$

and

$$h_n = \tau_n - \tau_{n-1} = 1 - 1 + \frac{1}{2}\Big(\frac{2}{n}\Big)^q = 2^{q-1}\Big(\frac{1}{n}\Big)^q = \mathcal{O}(n^{-q}). \tag{2.19}$$

Hence

$$h_1 = h_n = \mathcal{O}\left(n^{\frac{-r}{1-\gamma}}\right),\ 0 < \gamma < 1. \tag{2.20}$$

After that, for rest of the sub-intervals Λ_l, h_l $(l = 2, 3, \ldots, (n-1))$ are computed by using Mean Value Theorem (MVT), which is given as follows:

$$\begin{aligned} h_l = \tau_l - \tau_{l-1} &= \frac{1}{2}\left(\frac{2l}{n}\right)^q - \frac{1}{2}\left(\frac{2l-2}{n}\right)^q \\ &= \frac{1}{2}\left(\frac{2}{n}\right)^q \{l^q - (l-1)^q\} \\ &\leq \frac{q}{2}\left(\frac{2}{n}\right)^q \zeta^{q-1}, \end{aligned} \tag{2.21}$$

where $\zeta \in (l-1,\ l)$ and $l \leq n-1$. Hence

$$\begin{aligned} h_l \leq \frac{q}{2}\left(\frac{2}{n}\right)^q l^{q-1} &\leq \frac{q}{2}\left(\frac{2}{n}\right)^q (n-1)^{q-1} \\ &\leq q\ 2^{(q-1)}\frac{1}{n}\left[1 - \frac{1}{n}\right]^{q-1} \\ &\leq q\ 2^{(q-1)}\frac{1}{n} = \mathcal{O}(n^{-1}). \end{aligned} \tag{2.22}$$

Orthogonal projection:

Consider that $\mathcal{P}_n : L^\infty[0,1] \to \mathbb{X}_n$ represents the orthogonal projection operator which is given as follows:

$$\langle \mathcal{P}_n v_1, v_2 \rangle = \langle v_1, v_2 \rangle, \quad \forall\ v_1 \in L^\infty[0,1], \quad \forall\ v_2 \in \mathbb{X}_n, \tag{2.23}$$

$$\text{where} \quad \langle v_1, v_2 \rangle = \int_0^1 v_1(\tau) v_2(\tau) d\tau.$$

2.2. Galerkin method

Here, the Galerkin and iterated Galerkin methods are discussed for obtaining approximate solutions to the integral equation (2.8). The Galerkin approximate solution $u_n \in \mathbb{X}_n$ of the integral equation (2.8) is defined as follows:

$$u_n = \mathcal{P}_n \mathcal{K}(u_n) + \mathcal{P}_n g. \tag{2.24}$$

Let us define the operator T_n as follows:

$$T_n(u) = \mathcal{P}_n\mathcal{K}(u) + \mathcal{P}_n g. \tag{2.25}$$

So, we can write Eq. (2.24) as

$$u_n = T_n(u_n). \tag{2.26}$$

Next, we introduce the iterated Galerkin solution $\tilde{u}_n$ for the integral equation (2.8):

$$\tilde{u}_n = \mathcal{K}(u_n) + g. \tag{2.27}$$

By replacing $u_n = \mathcal{P}_n\tilde{u}_n$, the Eq. (2.27) becomes:

$$\tilde{u}_n = \mathcal{K}(\mathcal{P}_n\tilde{u}_n) + g. \tag{2.28}$$

Let us define an integral operator $S_n(u) = \mathcal{K}(\mathcal{P}_n u) + g$, then the above Eq. (2.28) becomes

$$\implies \tilde{u}_n = S_n(\tilde{u}_n). \tag{2.29}$$

3. Preliminaries and auxiliary lemmas

The purpose of this part is to state and prove some lemmas and theorems which are needed to get convergence results. Let b be a generic constant that is independent of n and can take different values depending on context. $BL(\mathbb{X})$ denotes the set of bounded linear operators on $\mathbb{X}$.

The following result stated by Ivan G. Graham [12] is significant in explaining convergence analysis of the proposed integral equation (2.1).

Lemma 3.1. *Let $\mathcal{P}_n$ be the orthogonal projection operator defined by* (2.23). *Then*

(i) there exist a positive number ρ, such that

$$\|\mathcal{P}_n\|_{L^\infty} \leq \rho < \infty. \tag{3.1}$$

(ii) in the sub-interval $\Lambda_i = [\tau_{i-1}, \tau_i]$, $i = 1, 2, \cdots, n$ for any $u \in \mathcal{C}^r[0,1]$ there exist a positive number C such that

$$\|(\mathcal{I} - \mathcal{P}_n)u\|_{L^\infty(\Lambda_i)} \leq C h_i^r \|u\|_{r, L^\infty(\Lambda_i)}. \tag{3.2}$$

Next, we state the following lemma concerning about exact solution u_0 as its derivative is unbounded at the starting point of the domain of integration.

Lemma 3.2. *Suppose the source function $g \in \mathcal{C}^r[0,1], r \geq 1$ satisfies Lipschitz continuity with order θ. The exact solution $u_0 \in \mathcal{C}^r(0,1]$ and $\mathcal{P}_n$ be the orthogonal projection operator, then it follows:*

(i) $$\|(\mathcal{I} - \mathcal{P}_n)u_0(s(\tau, .))\|_{L^\infty[0,1]} = \mathcal{O}(n^{-r}). \tag{3.3}$$

(ii) $$|u_0(s(\tau, \lambda + \delta)) - u_0(s(\tau, \lambda))| = \mathcal{O}(\delta^{1-\gamma}). \tag{3.4}$$

Proof. Similar to the Lemma 1 of [18], it can be proved. □

Now, we quote a [33] Theorem, which helps us to show that the solution of an equation $u = \widehat{\mathcal{H}}u$ can be obtained if the solution of other equation $u = \widetilde{\mathcal{H}}u$ is already known.

Theorem 3.1. *Let $\widehat{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$ be continuous operators over an open set Λ in a Banach space $\mathbb{X}$. Suppose $u_0 \in \Lambda$ be an isolated solution of equation $u = \widetilde{\mathcal{H}}u$ and $\widehat{\mathcal{H}'}$ is Fréchet derivative of $\widehat{\mathcal{H}}$ at u_0. If the inverse of operator $\mathcal{I} - \widehat{\mathcal{H}'}(u_0)$ exists in the given norm then for sufficiently small δ, there exists a positive constant q lying in the open interval $(0, 1)$ such that*

$$\sup_{u \in B(u_0,\delta)} \|(\mathcal{I} - \widehat{\mathcal{H}'}(u_0))^{-1}(\widehat{\mathcal{H}'}(u) - \widehat{\mathcal{H}'}(u_0))\| \le q, \tag{3.5}$$

$$\chi = \|(\mathcal{I} - \widehat{\mathcal{H}'}(u_0))^{-1}(\widehat{\mathcal{H}}(u_0) - \widetilde{\mathcal{H}}(u_0))\| \le \delta(1 - q), \tag{3.6}$$

holds. Then a unique solution $\hat{u}_0$ of $u = \hat{\mathcal{H}}u$ lying in the open ball $B(u_0, \delta)$ with centre u_0 and radius $\delta > 0$ such that

$$\frac{\chi}{1+q} \le \|u_0 - \hat{u}_0\| \le \frac{\chi}{1-q}, \tag{3.7}$$

satisfies.

The following Lemma is useful to prove the existence and to determine the order of convergence of the approximate solution of the iterated Galerkin method.

Lemma 3.3. *Let the function $W(\tau, s(\tau,\lambda), u(s(\tau,\lambda))) = (1–\lambda)^{-\gamma}\tilde{k}_u\,(\tau, s(\tau,\lambda), u(s(\tau,\lambda)))$, which is defined by Eq. (2.7). Then for an exact solution $u_0 \in \mathcal{C}^r(0, 1]$ following result holds:*

$$\begin{aligned}
&\left|\int_0^1 W(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))(\mathcal{P}_n - I)u_0(s(\tau,\lambda))\,d\lambda\right| \\
&\le \|W(\tau, s(\tau,.), u_0(s(\tau,.))) - \nu_\tau\|_{L^1_{(\Lambda_1)}} \|(\mathcal{P}_n - I)u_0(s(\tau,.))\|_{L^\infty_{(\Lambda_1)}} \\
&\quad + \sum_{i=2}^{n-1} \|(\mathcal{P}_n - I)W(\tau, s(\tau,.), u_0(s(\tau,.)))\|_{L^2_{(\Lambda_i)}} \|(\mathcal{P}_n - I)u_0(s(\tau,.))\|_{L^2_{(\Lambda_i)}} \\
&\quad + \|W(\tau, s(\tau,.), u_0(s(\tau,.))) - \nu_\tau\|_{L^1_{(\Lambda_n)}} \|(\mathcal{P}_n - I)u_0(s(\tau,.))\|_{L^\infty_{(\Lambda_n)}}.
\end{aligned}$$

Proof. The derivative of exact solution u_0 is unbounded in the sub-interval $[0, \tau_1]$, but is sufficiently smooth in rest of the sub-intervals whereas the kernel function exhibits singularity at $\lambda = 1$ in the last sub-interval $[\tau_{n-1}, 1]$, although it is sufficiently smooth in rest of the sub-intervals.

As a result, $W(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))) = (1 - \lambda)^{-\gamma}\tilde{k}_u\,(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))$ is sufficiently smooth in the sub-intervals $[\tau_{i-1}, \tau_i]$ where i varies from 2 to $(n - 1)$, it is singular in the sub-intervals $[0,\ \tau_1]$ and $[\tau_{n-1}, 1]$. Thus to obtain the solution, the standard integration domain $[0, 1]$ can be divided into n sub-intervals such as $[0, \tau_1]$, $[\tau_{i-1}, \tau_i]$, and $[\tau_{n-1}, 1]$.

$$\begin{aligned}
&\left|\int_0^1 W(\tau, s(\tau,\ \lambda),\ u_0(s(\tau,\ \lambda)))(\mathcal{P}_n - I)u_0(s(\tau,\lambda))\,d\lambda\right| \\
&= \left|\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i} W(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))(\mathcal{P}_n - I)u_0(s(\tau,\lambda))\,d\lambda\right| \\
&\le \left|\int_0^{\tau_1} W(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))(\mathcal{P}_n - I)u_0(s(\tau,\lambda))\,d\lambda\right|
\end{aligned}$$

$$+\left|\sum_{i=2}^{n-1}\int_{\tau_{i-1}}^{\tau_i} W(\tau,s(\tau,\lambda),u_0(s(\tau,\lambda)))(\mathcal{P}_n-I)u_0(s(\tau,\lambda))\,d\lambda\right|$$
$$+\left|\int_{\tau_{n-1}}^{1} W(\tau,s(\tau,\lambda),u_0(s(\tau,\lambda)))(\mathcal{P}_n-I)u_0(s(\tau,\lambda))\,d\lambda\right|.$$

For any $v_\tau \in \mathbb{X}_n$, $\langle v_\tau,(\mathcal{P}_n-I)u_0(s(\tau,.))\rangle = 0$, by applying orthogonality of $(\mathcal{P}_n-I)$, and Hölder's inequality, we get

$$\begin{aligned}
&\left|\int_0^1 W(\tau,s(\tau,\lambda),u_0(s(\tau,\lambda)))(\mathcal{P}_n-I)u_0(s(\tau,\lambda))\,d\lambda\right|\\
&=|\langle W(\tau,s(\tau,.),u_0(s(\tau,.)))-\nu_\tau,(\mathcal{P}_n-I)u_0(s(\tau,.))\rangle_{\Lambda_1}|\\
&\quad+\left|\sum_{i=2}^{n-1}\langle W(\tau,s(\tau,.),u_0(s(\tau,.))),(\mathcal{P}_n-I)u_0(s(\tau,.))\rangle_{\Lambda_i}\right|\\
&\quad+|\langle W(\tau,s(\tau,.),u_0(s(\tau,.)))-\nu_\tau,(\mathcal{P}_n-I)u_0(s(\tau,.))\rangle_{\Lambda_n}|\\
&=|\langle W(\tau,s(\tau,.),u_0(s(\tau,.)))-\nu_\tau,(\mathcal{P}_n-I)u_0(s(\tau,.))\rangle_{\Lambda_1}|\\
&\quad+\left|\sum_{i=2}^{n-1}\langle(\mathcal{P}_n-I)W(\tau,s(\tau,.),u_0(s(\tau,.))),(\mathcal{P}_n-I)u_0(s(\tau,.))\rangle_{\Lambda_i}\right|\\
&\quad+|\langle W(\tau,s(\tau,.),u_0(s(\tau,.)))-\nu_\tau,(\mathcal{P}_n-I)u_0(s(\tau,.))\rangle_{\Lambda_n}|\\
&\le\|W(\tau,s(\tau,.),u_0(s(\tau,.)))-\nu_\tau\|_{L^1_{(\Lambda_1)}}\|(\mathcal{P}_n-I)u_0(s(\tau,.))\|_{L^\infty_{(\Lambda_1)}}\\
&\quad+\sum_{i=2}^{n-1}\|(\mathcal{P}_n-I)W(\tau,s(\tau,.),u_0(s(\tau,.)))\|_{L^2_{(\Lambda_i)}}\|(\mathcal{P}_n-I)u_0(s(\tau,.))\|_{L^2_{(\Lambda_i)}}\\
&\quad+\|W(\tau,s(\tau,.),u_0(s(\tau,.)))-\nu_\tau\|_{L^1_{(\Lambda_n)}}\|(\mathcal{P}_n-I)u_0(s(\tau,.))\|_{L^\infty_{(\Lambda_n)}}.
\end{aligned}$$

Thus, this proves the anticipated result. □

Lemma 3.4. *For any $\delta > 0$ and $u_0 \in \mathcal{C}^r(0,1]$, the following hold:*

(i) $\sup\limits_{0\le\tau,\lambda\le1}\left|\tilde{k}_u(\tau,s(\tau,\lambda+\delta),u_0(s(\tau,\lambda+\delta)))-\tilde{k}_u(\tau,s(\tau,\lambda),u_0(s(\tau,\lambda)))\right|\le b\delta+b_1\delta^{1-\gamma}.$

(ii) $\displaystyle\int_0^{1-\delta}\left|\frac{1}{(1-(\lambda+\delta))^\gamma}-\frac{1}{(1-\lambda)^\gamma}\right|d\lambda\le 2\frac{1}{1-\gamma}\left[\delta^{1-\gamma}\right].$

Proof.

(i) Using assumptions (iii), (iv) and estimate (3.4), we obtain

$$\begin{aligned}
&\left|\tilde{k}_u(\tau,s(\tau,\lambda+\delta),u_0(s(\tau,\lambda+\delta)))-\tilde{k}_u(\tau,s(\tau,\lambda),u_0(s(\tau,\lambda)))\right|\\
&\le\left|\tilde{k}_u(\tau,s(\tau,\lambda+\delta),u_0(s(\tau,\lambda+\delta)))-\tilde{k}_u(\tau,s(\tau,\lambda),u_0(s(\tau,\lambda+\delta)))\right|\\
&\quad+\left|\tilde{k}_u(\tau,s(\tau,\lambda),u_0(s(\tau,\lambda+\delta)))-\tilde{k}_u(\tau,s(\tau,\lambda),u_0(s(\tau,\lambda)))\right|\\
&\le a_4|s(\tau,\lambda+\delta)-s(\tau,\lambda)|+a_3|u_0(s(\tau,\lambda+\delta))-u_0(s(\tau,\lambda))|\\
&\le a_4|\tau(\lambda+\delta)-\tau(\lambda)|+a_3b\ \delta^{1-\gamma}\\
&=a_4|\tau|\,|\delta|+b_1\delta^{1-\gamma},
\end{aligned}$$

where $b_1 = a_3 b$. Thus,

$$\sup_{0\le\tau,\lambda\le1}\left|\tilde{k}_u(\tau, s(\tau, \lambda+\delta), u_0(s(\tau, \lambda+\delta))) - \tilde{k}_u(\tau, s(\tau, \lambda), u_0(s(\tau, \lambda)))\right| \le b\delta + b_1\delta^{1-\gamma}, \tag{3.8}$$

where $\sup_{0\le\tau\le1} a_4 |\tau| = b$.

(ii) Consider,

$$\int_0^{1-\delta}\left|\frac{1}{(1-(\lambda+\delta))^\gamma} - \frac{1}{(1-\lambda)^\gamma}\right| d\lambda$$
$$= \frac{1}{1-\gamma}\left[\delta^{1-\gamma} + (1-\delta)^{1-\gamma} - 1^{1-\gamma}\right] \le 2\frac{1}{1-\gamma}\left[\delta^{1-\gamma}\right]. \tag{3.9}$$

□

We now state and prove a Theorem which is useful in our convergence analysis, with the help of a result from ([29], Schumaker, p.92.)

Theorem 3.2. *Let the function*

$$W(\tau, s(\tau, \lambda), u_0(s(\tau, \lambda))) = (1-\lambda)^{-\gamma}\tilde{k}_u(\tau, s(\tau, \lambda), u_0(s(\tau, \lambda))),$$

is defined by Eq. (2.7)*. Then for every* $\tau \in [0, 1]$, $\exists$ *a polynomial* $v_\tau \in \mathbb{P}_r$ *such that*

$$\|W(\tau, s(\tau, .), u_0(s(\tau, .))) - v_\tau\|_{L^1[0,1]} = \mathcal{O}(h^{1-\gamma}), \tag{3.10}$$

where h *is the norm of the partition and* $\mathbb{P}_r$ *is the set of polynomials of degree* $\le r-1$.

Proof. $W(\tau, s(\tau, .), u_0(s(\tau, .))) \in L^1[0, 1]$ and $\omega_{1,1}$ is the modulus of smoothness of order 1 (cf. [29] Schumaker, p. 92), then for every $\tau \in [0, 1]$ $\exists$ a polynomial $v_\tau \in \mathbb{P}_r$ such that

$$\|W(\tau, s(\tau, .), u_0(s(\tau, .))) - v_\tau\|_{L^1[0,1]} \le C_1\omega_{1,1}(W(\tau, s(\tau, .), u_0(s(\tau, .))), h),$$

where C_1 is a constant free from n.

Let $I_\delta = [0, 1-\delta]$, for very small $\delta > 0$. Thus,

$$\begin{aligned}
&\|W(\tau, s(\tau, \lambda), u_0(s(\tau, \lambda))) - v_\tau\|_{L^1[0,1]}\\
&\le C_1\omega_{1,1}(W(\tau, s(\tau, .), u_0(s(\tau, .))), h)\\
&= C_1 \sup_{0\le\delta\le h}\|\Delta_\delta W(\tau, s(\tau, .), u_0(s(\tau, .)))\|_{L^1(I_\delta)}\\
&= C_1 \sup_{0\le\delta\le h}\int_0^{1-\delta}|W(\tau, s(\tau, \lambda+\delta), u_0(s(\tau, \lambda+\delta)))\\
&\quad - W(\tau, s(\tau, \lambda), u_0(s(\tau, \lambda)))|\, d\lambda.
\end{aligned} \tag{3.11}$$

Now consider,

$$\|W(\tau, s(\tau, \lambda+\delta), u_0(s(\tau, \lambda+\delta))) - W(\tau, s(\tau, \lambda), u_0(s(\tau, \lambda)))\|_{L^1(I_\delta)}$$
$$= \int_0^{1-\delta}\left|\frac{\tilde{k}_u(\tau, s(\tau, \lambda+\delta), u_0(s(\tau, \lambda+\delta)))}{(1-(\lambda+\delta))^\gamma} - \frac{\tilde{k}_u(\tau, s(\tau, \lambda), u_0(s(\tau, \lambda)))}{(1-\lambda)^\gamma}\right| d\lambda$$

$$\leq \int_0^{1-\delta} \left| \frac{\tilde{k}_u\left(\tau, s(\tau,\lambda+\delta), u_0(s(\tau,\lambda+\delta))\right)}{(1-(\lambda+\delta))^{\gamma}} - \frac{\tilde{k}_u\left(\tau, s(\tau,\lambda+\delta), u_0(s(\tau,\lambda+\delta))\right)}{(1-\lambda)^{\gamma}} \right| d\lambda$$

$$+ \int_0^{1-\delta} \left| \frac{\tilde{k}_u\left(\tau, s(\tau,\lambda+\delta), u_0(s(\tau,\lambda+\delta))\right)}{(1-\lambda)^{\gamma}} - \frac{\tilde{k}_u\left(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))\right)}{(1-\lambda)^{\gamma}} \right| d\lambda$$

$$\leq \sup_{0\leq\tau,\lambda\leq 1} \left| \tilde{k}_u\left(\tau, s(\tau,\lambda+\delta), u_0(s(\tau,\lambda+\delta))\right) \right| \int_0^{1-\delta} \left| \frac{1}{(1-(\lambda+\delta))^{\gamma}} - \frac{1}{(1-\lambda)^{\gamma}} \right| d\lambda$$

$$+ \int_0^{1-\delta} \left| \frac{1}{(1-\lambda)^{\gamma}} \right| d\lambda \sup_{0\leq\tau,\lambda\leq 1} \left| \tilde{k}_u\left(\tau, s(\tau,\lambda+\delta), u_0(s(\tau,\lambda+\delta))\right) \right.$$

$$\left. - \tilde{k}_u(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))) \right|$$

$$\leq A_1^1 \int_0^{1-\delta} \left| \frac{1}{(1-(\lambda+\delta))^{\gamma}} - \frac{1}{(1-\lambda)^{\gamma}} \right| d\lambda + A_1^2 \int_0^{1-\delta} \left| \frac{1}{(1-\lambda)^{\gamma}} \right| d\lambda,$$

where $A_1^1 = \sup_{0\leq\tau,\lambda\leq 1} \left| \tilde{k}_u\left(\tau, s(\tau,\lambda+\delta), u_0(s(\tau,\lambda+\delta))\right) \right|$ and
$A_1^2 = \sup_{0\leq\tau,\lambda\leq 1} \left| \tilde{k}_u\left(\tau, s(\tau,\lambda+\delta), u_0(s(\tau,\lambda+\delta))\right) - \tilde{k}_u\left(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))\right) \right|$.

Let $a_5 = \sup_{0\leq\tau,\lambda\leq 1} \left| \tilde{k}_u\left(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))\right) \right| < \infty$.

Evaluating A_1^1 with assumption (iii) and Lemma 3.2, we get

$$\begin{aligned}
A_1^1 = & \sup_{0\leq\tau,\lambda\leq 1} \left| \tilde{k}_u\left(\tau, s(\tau,\lambda+\delta), u_0(s(\tau,\lambda+\delta))\right) \right| \\
\leq & \sup_{0\leq\tau,\lambda\leq 1} \left| \tilde{k}_u\left(\tau, s(\tau,\lambda+\delta), u_0(s(\tau,\lambda+\delta))\right) - \tilde{k}_u\left(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))\right) \right| \\
& + \sup_{0\leq\tau,\lambda\leq 1} \left| \tilde{k}_u\left(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))\right) \right| \\
\leq & a_3 \sup_{0\leq\tau,\lambda\leq 1} |u_0(s(\tau,\lambda+\delta)) - u_0(s(\tau,\lambda))| + a_5 \\
\leq & (a_3 b\delta^{1-\gamma} + a_5) < \infty. \qquad (3.12)
\end{aligned}$$

Also

$$\int_0^{1-\delta} \left| \frac{1}{(1-\lambda)^{\gamma}} \right| d\lambda = \frac{1}{1-\gamma}[1-\delta^{1-\gamma}] < b < \infty. \qquad (3.13)$$

On combining estimates (3.12) and (3.13) with Lemma 3.4, we can write

$$\begin{aligned}
& \|W(\tau, s(\tau,\lambda+\delta), u(s(\tau,\lambda+\delta))) - W(\tau, s(\tau,\lambda), u(s(\tau,\lambda)))\|_{L^1(I_\delta)} \\
\leq & \frac{A_1^1}{1-\gamma}\left[2\delta^{1-\gamma}\right] + b(b\delta + b_1\delta^{1-\gamma}) \\
= & \mathcal{O}(\delta^{1-\gamma}). \qquad (3.14)
\end{aligned}$$

With the help of estimate (3.14), estimate (3.11) can be written as

$$\begin{aligned}
& \|W(\tau, s(\tau,.), u(s(\tau,.))) - v_\tau\|_{L^1[0,1]} \\
\leq & C_1 \sup_{0\leq\delta\leq h} \|W(\tau, s(\tau,\lambda+\delta), u(s(\tau,\lambda+\delta))) - W(\tau, s(\tau,\lambda), u(s(\tau,\lambda)))\|_{L^1(I_\delta)} \\
& \qquad (3.15)
\end{aligned}$$

$$\leq C_1 \sup_{0\leq\delta\leq h} \left[\delta^{1-\gamma}\right] \leq C_1 h^{1-\gamma} = \mathcal{O}(h^{1-\gamma}).$$

Thus, this proves the anticipated result. □

Note:

$$A_2 = \int_0^1 \left|\frac{1}{(1-\lambda)^\gamma}\right| d\lambda = \frac{1}{1-\gamma} < \infty. \tag{3.16}$$

Lemma 3.5. *Suppose the unique solution $u_0 \in C^r(0,\,1]$, $r \geq 1$ and a linear operator $\mathcal{K}^{'}(u_0)$ be the Fréchet derivative of a nonlinear operator $\mathcal{K}$ at u_0 then the following hold:*

(i) $\left\|\mathcal{K}^{'}(\mathcal{P}_n u_0) - \mathcal{K}^{'}(u_0)\right\|_{L^\infty} \to 0 \quad as \quad n \to \infty.$

(ii) $\left\|\mathcal{K}^{'}(u_0)\mathcal{P}_n - \mathcal{K}^{'}(u_0)\right\|_{L^\infty} \to 0 \quad as \quad n \to \infty.$

Proof. (i) Now for any $y \in L^\infty[0,1]$,

$$\begin{aligned}
&\left|\mathcal{K}^{'}(\mathcal{P}_n u_0)y(\tau) - \mathcal{K}^{'}(u_0)y(\tau)\right| \\
=&\left|\int_0^1 (1-\lambda)^{-\gamma}\tilde{k}_u\left(\tau, s(\tau,\lambda), \mathcal{P}_n u_0(s(\tau,\lambda))\right) y(s(\tau,\lambda))\, d\lambda\right. \\
&\left. - \int_0^1 (1-\lambda)^{-\gamma}\tilde{k}_u(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))y(s(\tau,\lambda))\, d\lambda\right| \\
=&\left|\int_0^1 (1-\lambda)^{-\gamma}(\tilde{k}_u\left(\tau, s(\tau,\lambda), \mathcal{P}_n u_0(s(\tau,\lambda))\right)\right. \\
&\left. - \tilde{k}_u(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))))y(s(\tau,\lambda))\, d\lambda\right|.
\end{aligned}$$

On using estimates (2.11), (3.16), and (3.3), we can write

$$\begin{aligned}
&\left|\mathcal{K}^{'}(\mathcal{P}_n u_0)y(\tau) - \mathcal{K}^{'}(u_0)y(\tau)\right| \\
\leq& a_3 \int_0^1 \left|(1-\lambda)^{-\gamma}\right| \left|(\mathcal{P}_n u_0 - u_0)(s(\tau,\lambda))\right| \left|y(s(\tau,\lambda))\right|\, d\lambda \\
\leq& a_3 \left\|(\mathcal{P}_n - I)u_0(s(\tau,.))\right\|_{L^\infty} \left\|y(s(\tau,.))\right\|_{L^\infty} \int_0^1 \left|(1-\lambda)^{-\gamma}\right|\, d\lambda \\
\leq& a_3 A_2 b n^{-r} \left\|y(s(\tau,.))\right\|_{L^\infty}.
\end{aligned} \tag{3.17}$$

Hence, with the help of estimates (3.1) and (3.17), we have

$$\left\|\mathcal{K}^{'}(\mathcal{P}_n u_0) - \mathcal{K}^{'}(u_0)\right\|_{L^\infty} \leq a_3 A_2 b n^{-r} \to 0 \text{ as } n \to \infty. \tag{3.18}$$

(ii) Next, for any $y \in L^\infty[0,1]$, by using Hölder's inequality, Lemma 3.3, Theorem 3.2, estimates (2.11), (2.20), and (2.22):

$$\left|\left[\mathcal{K}^{'}(u_0)\mathcal{P}_n - \mathcal{K}^{'}(u_0)\right]y(\tau)\right|$$

$$
\begin{aligned}
&= \left| \int_0^1 (1-\lambda)^{-\gamma} \tilde{k}_u\left(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))\right) \mathcal{P}_n y(s(\tau,\lambda))\, d\lambda \right. \\
&\quad \left. - \int_0^1 (1-\lambda)^{-\gamma} \tilde{k}_u(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))) y(s(\tau,\lambda))\, d\lambda \right| \\
&= \left| \int_0^1 (1-\lambda)^{-\gamma} \tilde{k}_u\left(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))\right) (\mathcal{P}_n - I) y(s(\tau,\lambda))\, d\lambda \right| \\
&= \left| \int_0^1 W(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))(\mathcal{P}_n - I) y(s(\tau,\lambda))\, d\lambda \right| \\
&\le \| W(\tau, s(\tau,.), u_0(s(\tau,.))) - \nu_\tau \|_{L^1_{(\Lambda_1)}} \|(\mathcal{P}_n - I) y(s(\tau,.))\|_{L^\infty_{(\Lambda_1)}} \\
&\quad + \sum_{i=2}^{n-1} \|(\mathcal{P}_n - I) W(\tau, s(\tau,.), u_0(s(\tau,.)))\|_{L^2_{(\Lambda_i)}} \|(\mathcal{P}_n - I) y(s(\tau,.))\|_{L^2_{(\Lambda_i)}} \\
&\quad + \| W(\tau, s(\tau,.), u_0(s(\tau,.))) - \nu_\tau \|_{L^1_{(\Lambda_n)}} \|(\mathcal{P}_n - I) y(s(\tau,.))\|_{L^\infty_{(\Lambda_n)}} \qquad (3.19) \\
&\le \| W(\tau, s(\tau,.), u_0(s(\tau,.))) - \nu_\tau \|_{L^1_{(\Lambda_1)}} \|(\mathcal{P}_n - I) y(s(\tau,.))\|_{L^\infty_{(\Lambda_1)}} \\
&\quad + h_i^{\frac{1}{2}} \sum_{i=2}^{n-1} \|(\mathcal{P}_n - I) W(\tau, s(\tau,.), u_0(s(\tau,.)))\|_{L^\infty_{(\Lambda_i)}} h_i^{\frac{1}{2}} \|(\mathcal{P}_n - I) y(s(\tau,.))\|_{L^\infty_{(\Lambda_i)}} \\
&\quad + \| W(\tau, s(\tau,.), u_0(s(\tau,.))) - \nu_\tau \|_{L^1_{(\Lambda_n)}} \|(\mathcal{P}_n - I) y(s(\tau,.))\|_{L^\infty_{(\Lambda_n)}} \\
&\le C_1\ h_1^{1-\gamma}(1+\rho)\, \|y\|_{L^\infty} + C \sum_{i=2}^{n-1} h_i^r h_i^{\frac{1}{2}} \left(\left\| \frac{\partial^r W(\tau, s(\tau,.), u_0(s(\tau,.)))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} \right) \\
&\quad h_i^{\frac{1}{2}} \|(\mathcal{P}_n - I) y(s(\tau,.))\|_{L^\infty_{(\Lambda_i)}} + C_1\ h_n^{1-\gamma}(1+\rho)\, \|y\|_{L^\infty} \\
&\le C_1\ h_1^{1-\gamma}(1+\rho)\, \|y\|_{L^\infty} + C \sum_{i=2}^{n-1} h_i^{r+1} \left(\left\| \frac{\partial^r W(\tau, s(\tau,.), u_0(s(\tau,.)))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} \right) \\
&\quad \|(\mathcal{P}_n - I) y(s(\tau,.))\|_{L^\infty_{(\Lambda_i)}} + C_1\ h_n^{1-\gamma}(1+\rho)\, \|y\|_{L^\infty} \\
&\le C_1\ n^{-r}(1+\rho)\, \|y\|_{L^\infty} + C \sum_{i=2}^{n-1} n^{-(r+1)} \left(\left\| \frac{\partial^r W(\tau, s(\tau,.), u_0(s(\tau,.)))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} \right) \\
&\quad (1+\rho)\, \|y\|_{L^\infty} + C_1\ n^{-r}(1+\rho)\, \|y\|_{L^\infty} \\
&\le b\ n^{-r}\, \|y\|_{L^\infty}\,. \qquad (3.20)
\end{aligned}
$$

Hence,

$$
\left\| \mathcal{K}'(u_0)\mathcal{P}_n - \mathcal{K}'(u_0) \right\|_{L^\infty} \le b\ n^{-r} \ \to\ 0 \quad \text{as} \quad n \to\ \infty. \qquad (3.21)
$$

□

4. Convergence Analysis

The convergence results of the approximate solutions of Galerkin and iterated Galerkin methods are examined in this section.

The following theorem helps to establish the existence and uniqueness of the approximate solution of iterated Galerkin method.

Theorem 4.1. *Suppose a linear operator $\mathcal{K}^{'}(u_0)$ be the Fréchet derivative of a nonlinear operator $\mathcal{K}$ at u_0 and 1 is not an eigenvalue of $\mathcal{K}^{'}(u_0)$. Then the inverse of integral operator $(I - S_n^{'}(u_0))$ exists in the uniform norm i.e. there exists a positive constant $\mathcal{V} < \infty$ and $n_0 \in \mathbb{N}$ such that $\left\|(I - S_n^{'}(u_0))^{-1}\right\|_{L^\infty} \leq \mathcal{V}\ \forall\ n \geq n_0$.*

Proof. Here we use a result from Ahues et al. [1] i.e. let $\mathbb{X}$ be a Banach space and $T^{'}(u_0)$, $S_n^{'}(u_0) \in BL(\mathbb{X})$. If $S_n^{'}(u_0)$ is norm convergent to $T^{'}(u_0)$ and $(I - T^{'}(u_0))^{-1}$ exists and bounded on $\mathbb{X}$, then for sufficiently large value of n, $\exists\ \mathcal{V} > 0$ s.t. $(I - S_n^{'}(u_0))^{-1}$ exists and uniformly bounded on $\mathbb{X}$

i.e.

$$\left\|(I - S_n^{'}(u_0))^{-1}\right\|_{L^\infty} \leq \mathcal{V} < \infty.$$

So, here we only need to show that $\left\|S_n^{'}(u_0) - T^{'}(u_0)\right\|_{L^\infty} \to 0$ as $n \to \infty$.

$$\begin{aligned}
&\left\|S_n^{'}(u_0) - T^{'}(u_0)\right\|_{L^\infty} \\
&= \left\|\mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n - \mathcal{K}^{'}(u_0)\right\|_{L^\infty} \\
&= \left\|\mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n - \mathcal{K}^{'}(u_0)\mathcal{P}_n + \mathcal{K}^{'}(u_0)\mathcal{P}_n - \mathcal{K}^{'}(u_0)\right\|_{L^\infty} \\
&\leq \left\|\mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n - \mathcal{K}^{'}(u_0)\mathcal{P}_n\right\|_{L^\infty} + \left\|\mathcal{K}^{'}(u_0)\mathcal{P}_n - \mathcal{K}^{'}(u_0)\right\|_{L^\infty} \\
&\leq \left\|\mathcal{K}^{'}(\mathcal{P}_n u_0) - \mathcal{K}^{'}(u_0)\right\|_{L^\infty} \|\mathcal{P}_n\|_{L^\infty} + \left\|\mathcal{K}^{'}(u_0)\mathcal{P}_n - \mathcal{K}^{'}(u_0)\right\|_{L^\infty}. \qquad (4.1)
\end{aligned}$$

On combining Lemma 3.5 with Eq. (4.1), we get

$$\left\|S_n^{'}(u_0) - T^{'}(u_0)\right\|_{L^\infty} \to\ 0 \text{ as } n\ \to\ \infty. \qquad (4.2)$$

Hence the theorem is proved. □

In the forthcoming Theorem, the existence and uniqueness of the estimated solution of iterated Galerkin method is established.

Theorem 4.2. *Suppose u_0 be the unique solution of the Eq. (2.8) in $C^r(0, 1]$, $r \geq 1$. For sufficiently large n, there exists a positive constant q lying in the open interval $(0,1)$, and unique solution $\tilde{u}_n$ of equation $u = S_n(u)$ lying in an open ball $B(u_0, \delta)$ with centre u_0 and radius $\delta > 0$ such that*

$$\frac{\kappa_n}{(1+q)} \leq \|\tilde{u}_n - u_0\|_\infty \leq \frac{\kappa_n}{(1-q)},$$

where $\kappa_n = \left\|(I - S_n^{'}(u_0))^{-1}(S_n(u_0) - T(u_0))\right\|_{L^\infty}$.

Proof. Now using estimates (2.11), (3.16) and Lemma 3.1 for any $w \in B(u_0, \delta)$, we have

$$\left\|S_n^{'}(w) - S_n^{'}(u_0)\right\|_{L^\infty} = \left\|\mathcal{K}^{'}(\mathcal{P}_n w)\mathcal{P}_n - \mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n\right\|_{L^\infty},$$

$$\begin{aligned}
&\left|\mathcal{K}^{'}(\mathcal{P}_n w)\mathcal{P}_n y - \mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n y\right| \\
&= \Bigg| \int_0^1 (1-\lambda)^{-\gamma}\Big[\tilde{k}_u\left(\tau, s(\tau,\lambda), \mathcal{P}_n w(s(\tau,\lambda))\right)
\end{aligned}$$

$$
\begin{aligned}
&- \tilde{k}_u(\tau, s(\tau,\lambda), \mathcal{P}_n u_0(s(\tau,\lambda)))\Big] \mathcal{P}_n y(s(\tau,\lambda))\, d\lambda \Bigg| \\
&\le a_3 \int_0^1 \left|(1-\lambda)^{-\gamma}\right| |\mathcal{P}_n(w-u_0)(s(\tau,\lambda))| \, |\mathcal{P}_n y(s(\tau,\lambda))| \, d\lambda \\
&\le a_3 A_2 \|\mathcal{P}_n(w-u_0)\|_{L^\infty} \|\mathcal{P}_n y\|_{L^\infty} \\
&\le a_3 A_2 \rho^2 \|(w-u_0)\|_{L^\infty} \|y\|_{L^\infty} \\
&\le a_3 A_2 \rho^2 \delta \|y\|_{L^\infty} . \qquad (4.3)
\end{aligned}
$$

So, with the help of the foregoing Theorem 4.1 and estimate (4.3), we can write

$$\sup_{w \in B(u_0,\delta)} \left\|(I - S_n^{'}(u_0))^{-1}(S_n^{'}(w) - S_n^{'}(u_0))\right\|_{L^\infty} \le \mathcal{V} a_3 A_2 \rho^2 \delta \le q \text{ (say)}.$$

In this case, we choose δ such that $q \in (0,1)$ which satisfy Eq. (3.5).

$$\kappa_n = \left\|(I - S_n^{'}(u_0))^{-1}(S_n(u_0) - T(u_0))\right\|_{L^\infty} \le \mathcal{V} \|\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0)\|_{L^\infty} . \quad (4.4)$$

Using assumption (ii), estimates (3.16) and (3.3), we obtain

$$
\begin{aligned}
&\left|\left[\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0)\right] y(\tau)\right| \\
=&\Bigg| \int_0^1 (1-\lambda)^{-\gamma} \Big[\tilde{k}\,(\tau, s(\tau,\lambda), \mathcal{P}_n u_0(s(\tau,\lambda))) \\
&- \tilde{k}(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))\Big] y(s(\tau,\lambda))\, d\lambda \Bigg| \\
\le& a_1 \int_0^1 \left|(1-\lambda)^{-\gamma}\right| |(\mathcal{P}_n u_0 - u_0)(s(\tau,\lambda))| \, |y(s(\tau,\lambda))| \, d\lambda \\
\le& a_1 A_2 \|\mathcal{P}_n u_0 - u_0\|_{L^\infty} \|y(s(\tau,.))\|_{L^\infty} \\
\le& a_1 A_2 b n^{-r} \|y(s(\tau,.))\|_{L^\infty} . \qquad (4.5)
\end{aligned}
$$

Thus,

$$\|\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0)\|_{L^\infty} \le a_1 A_2 b n^{-r} \;\to\; 0 \text{ as } n \;\to\; \infty. \qquad (4.6)$$

Thus, by choosing n large enough such that $\kappa_n \le \delta(1-q)$ and using Theorem 3.1, we get

$$\frac{\kappa_n}{(1+q)} \le \|\tilde{u}_n - u_0\|_{L^\infty} \le \frac{\kappa_n}{(1-q)},$$

where $\kappa_n = \left\|(I - S_n^{'}(u_0))^{-1}(S_n(u_0) - T(u_0))\right\|_{L^\infty}$, which implies $\tilde{u}_n$ uniquely exist in an open ball $B(u_0,\delta)$.
This completes the proof. □

In the next theorem, we discuss the convergence result for the iterated Galerkin approximate solution $\tilde{u}_n$.

Theorem 4.3. *Suppose the unique solution* $u_0 \in C^r(0,\, 1]$ *and the source function* $g \in C^r[0,\, 1]$, $r \ge 1$. $\tilde{u}_n$ *is the approximated iterated Galerkin solution defined by Eq.* (2.28), *then it holds*

$$\|\tilde{u}_n - u_0\|_\infty = \mathcal{O}(n^{-2r}).$$

Proof. It follows from the Theorem 4.2 that

$$\frac{\kappa_n}{(1+q)} \le \|\tilde{u}_n - u_0\|_{L^\infty} \le \frac{\kappa_n}{(1-q)},$$

where $\kappa_n = \left\|(I - S_n'(u_0))^{-1}(S_n(u_0) - T(u_0))\right\|_{L^\infty}$.

Hence using Theorem 4.1, we get

$$\begin{aligned}
\|\tilde{u}_n - u_0\|_{L^\infty} \le \frac{\kappa_n}{1-q} &\le \frac{1}{1-q}\left\|(I - S_n'(u_0))^{-1}(S_n(u_0) - T(u_0))\right\|_{L^\infty} \\
&\le \frac{1}{1-q}\left\|(I - S_n'(u_0))^{-1}\right\|_{L^\infty}\|(S_n(u_0) - T(u_0))\|_{L^\infty} \\
&\le \frac{\mathcal{V}}{1-q}\|(S_n(u_0) - T(u_0))\|_{L^\infty} \\
&= \frac{\mathcal{V}}{1-q}\left\|\left[\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0)\right]\right\|_{L^\infty}. \qquad (4.7)
\end{aligned}$$

On applying Mean value theorem and definition of projection operator in estimate (4.7), we have

$$\begin{aligned}
&\left|\left[\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0)\right]y(\tau)\right| \\
&= \left|\int_0^1 (1-\lambda)^{-\gamma}\left[\tilde{k}\,(\tau, s(\tau,\lambda), \mathcal{P}_n u_0(s(\tau,\lambda))) - \tilde{k}(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))\right]y(s(\tau,\lambda))\,d\lambda\right| \\
&= \Bigg|\int_0^1 (1-\lambda)^{-\gamma}\left[\tilde{k}_u\,(\tau, s(\tau,\lambda), \{u_0 + \zeta(\mathcal{P}_n u_0 - u_0)\}(s(\tau,\lambda)))\,(u_0 - \mathcal{P}_n u_0)(s(\tau,\lambda))\right] \\
&\quad \times y(s(\tau,\lambda))\,d\lambda\Bigg| \\
&\le \Bigg|\int_0^1 (1-\lambda)^{-\gamma}\Big[\Big(\tilde{k}_u(\tau, s(\tau,\lambda), \{u_0 + \zeta(\mathcal{P}_n u_0 - u_0)\}(s(\tau,\lambda))) \\
&\quad - \tilde{k}_u(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))\Big)\,(u_0 - \mathcal{P}_n u_0)(s(\tau,\lambda))\Big]y(s(\tau,\lambda))\,d\lambda\Bigg| \\
&\quad + \left|\int_0^1 (1-\lambda)^{-\gamma}\tilde{k}_u\,(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))\,(u_0 - \mathcal{P}_n u_0)(s(\tau,\lambda))y(s(\tau,\lambda))\,d\lambda\right|,
\end{aligned} \qquad (4.8)$$

where $0 \le \zeta \le 1$.

By using estimates (2.11), (3.3), (3.16) and definition of projection operator, we evaluate the first term of the above estimate (4.8),

$$\begin{aligned}
&\Bigg|\int_0^1 (1-\lambda)^{-\gamma}\Big[\tilde{k}_u\Big(\tau, s(\tau,\lambda), \{u_0 + \zeta(\mathcal{P}_n u_0 - u_0)\}(s(\tau,\lambda))) \\
&\quad - \tilde{k}_u(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))\Big)(u_0 - \mathcal{P}_n u_0)(s(\tau,\lambda))\Big]y(s(\tau,\lambda))\,d\lambda\Bigg| \\
&\le a_3\int_0^1 \left|(1-\lambda)^{-\gamma}\right|\left|(\mathcal{P}_n u_0 - u_0)(s(\tau,\lambda))\right|\left|(u_0 - \mathcal{P}_n u_0)(s(\tau,\lambda))\right|\left|y(s(\tau,\lambda))\right|\,d\lambda
\end{aligned}$$

$$\leq a_3 A_2 \left\|(\mathcal{P}_n u_0 - u_0)\right\|_{L^\infty}^2 \|y\|_{L^\infty}$$
$$\leq a_3 A_2 b n^{-2r} \|y\|_{L^\infty} . \tag{4.9}$$

By applying Lemma 3.3, second term of the estimate (4.8) is as follows:

$$\begin{aligned}
&\left| \int_0^1 (1-\lambda)^{-\gamma} \tilde{k}_u \left(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))\right) (u_0 - \mathcal{P}_n u_0)(s(\tau,\lambda))\, d\lambda \right| \\
=& \left| \int_0^1 W(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda)))(\mathcal{P}_n - I) u_0(s(\tau,\lambda))\, d\lambda \right| \\
\leq& \left\|W(\tau, s(\tau,.), u_0(s(\tau,.))) - \nu_\tau\right\|_{L^1_{(\Lambda_1)}} \left\|(\mathcal{P}_n - I)u_0(s(\tau,.))\right\|_{L^\infty_{(\Lambda_1)}} \\
&+ \sum_{i=2}^{n-1} \left\|(\mathcal{P}_n - I)W(\tau, s(\tau,.), u_0(s(\tau,.)))\right\|_{L^2_{(\Lambda_i)}} \left\|(\mathcal{P}_n - I)u_0(s(\tau,.))\right\|_{L^2_{(\Lambda_i)}} \\
&+ \left\|W(\tau, s(\tau,.), u_0(s(\tau,.))) - \nu_\tau\right\|_{L^1_{(\Lambda_n)}} \left\|(\mathcal{P}_n - I)u_0(s(\tau,.))\right\|_{L^\infty_{(\Lambda_n)}} .
\end{aligned} \tag{4.10}$$

Using estimate (3.3) and Theorem 3.2 in the sub-interval $\Lambda_1 = [0, \tau_1]$, we have

$$\left\|W(\tau, s(\tau,.), u_0(s(\tau,.))) - \nu_\tau\right\|_{L^1_{(\Lambda_1)}} \leq C_1 h_1^{1-\gamma} \leq d_1 n^{-r}, \tag{4.11}$$

and

$$\left\|(\mathcal{P}_n - I)u_0(s(\tau,.))\right\|_{L^\infty_{(\Lambda_1)}} \leq d_2 n^{-r}. \tag{4.12}$$

In the sub-intervals $\Lambda_i = [\tau_{i-1}, \tau_i],\ l = 2, 3, \ldots, n-1$, using the estimates (2.22) and Lemma 3.1, we obtain

$$\begin{aligned}
&\left\|(\mathcal{P}_n - I)W(\tau, s(\tau,.), u_0(s(\tau,.)))\right\|_{L^2_{(\Lambda_i)}} \\
\leq& h_i^{\frac{1}{2}} \left\|(\mathcal{P}_n - I)W(\tau, s(\tau,.), u_0(s(\tau,.)))\right\|_{L^\infty_{(\Lambda_i)}} \\
\leq& C h_i^r h_i^{\frac{1}{2}} \left\| \frac{\partial^r W(\tau, s(\tau,.), u_0(s(\tau,.)))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} \\
\leq& C h_i^{r+\frac{1}{2}} \left\| \frac{\partial^r W(\tau, s(\tau,.), u_0(s(\tau,.)))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} \\
\leq& C n^{-(r+\frac{1}{2})} \left\| \frac{\partial^r W(\tau, s(\tau,.), u_0(s(\tau,.)))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} ,
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
\left\|(\mathcal{P}_n - I)u_0(s(\tau,.))\right\|_{L^2_{(\Lambda_i)}} &\leq h_i^{\frac{1}{2}} \left\|(\mathcal{P}_n - I)u_0(s(\tau,.))\right\|_{L^\infty_{(\Lambda_i)}} \\
&\leq C h_i^r h_i^{\frac{1}{2}} \left\| \frac{\partial^r u_0(s(\tau,.))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} \\
&\leq C h_i^{r+\frac{1}{2}} \left\| \frac{\partial^r u_0(s(\tau,.))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} \\
&\leq C n^{-(r+\frac{1}{2})} \left\| \frac{\partial^r u_0(s(\tau,.))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} .
\end{aligned} \tag{4.14}$$

In the sub-intervals $\Lambda_n = [\tau_{n-1}, 1]$, using the estimates (2.20) and Lemma 3.1 and Theorem 3.2, we obtain

$$\|W(\tau, s(\tau,.), u_0(s(\tau,.))) - \nu_\tau\|_{L^1_{(\Lambda_n)}} \leq C_1 h_n^{1-\gamma} \leq d_3 n^{-r}, \tag{4.15}$$

and

$$\begin{aligned}\|(\mathcal{P}_n - I)u_0(s(\tau,.))\|_{L^\infty_{(\Lambda_n)}} &\leq C h_n^r \left\| \frac{\partial^r u_0(s(\tau,.))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_n)}} \\ &\leq C n^{-(rq)} \left\| \frac{\partial^r u_0(s(\tau,.))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_n)}} \\ &\leq C n^{-(\frac{r^2}{1-\gamma})} \left\| \frac{\partial^r u_0(s(\tau,.))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_n)}}. \end{aligned} \tag{4.16}$$

For $i = 2, 3, \dots, n$, set

$$\tilde{Q}_1^i = \left\| \frac{\partial^r W(\tau, s(\tau,.), u_0(s(\tau,.)))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} < \infty,$$

and

$$\tilde{Q}_2^i = \left\| \frac{\partial^r u_0(s(\tau,.))}{\partial \lambda^r} \right\|_{L^\infty_{(\Lambda_i)}} < \infty.$$

Then using estimates from (4.9)–(4.16) in (4.7), we obtain

$$\begin{aligned}&\|\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0)\|_{L^\infty} \\ \leq& a_3 A_2 b n^{-2r} + d_1 d_2 n^{-2r} + C^2 \sum_{i=2}^{n-1} n^{-(2r+1)} \tilde{Q}_1^i \tilde{Q}_2^i + C d_3 n^{-(r+\frac{r^2}{1-\gamma})} \tilde{Q}_2^n \\ =& d_4 n^{-2r} + n^{-2r} C^2 \sum_{i=2}^{n-1} n^{-1} \tilde{Q}_1^i \tilde{Q}_2^i + n^{-2r} C d_3 n^{-(\frac{r^2}{1-\gamma}-r)} \tilde{Q}_2^n \\ \leq& b n^{-2r} \max(1, n^{-1}, n^{-(\frac{r^2}{1-\gamma}-r)}) \\ \leq& \tilde{b} n^{-2r},\end{aligned}$$

where $d_4 = a_3 A_2 b + d_1 d_2,$ and $\tilde{b} = b \max(1, n^{-1}, n^{-(\frac{r^2}{1-\gamma}-r)})$.

Thus,

$$\|\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0)\|_{L^\infty} = O(n^{-2r}). \tag{4.17}$$

By (4.7) and (4.17), we can conclude

$$\begin{aligned}\|\tilde{u}_n - u_0\|_{L^\infty} &\leq \frac{\mathcal{V}}{1-q} \left\| \left[\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0)\right] \right\|_{L^\infty} \\ &= \mathcal{O}(n^{-2r}).\end{aligned} \tag{4.18}$$

Thus, this proves the anticipated result. □

The following theorem concerns the convergence result for the approximate Galerkin solution.

Theorem 4.4. *Suppose the unique solution $u_0 \in C^r(0, 1]$ and the source function $g \in C^r[0, 1]$, $r \geq 1$. u_n is the Galerkin approximate solution of (2.24), then it holds*

$$\|u_n - u_0\|_{L^\infty} = \mathcal{O}(n^{-r}).$$

Proof. By using $\mathcal{P}_n \tilde{u}_n = u_n$, we obtain

$$\begin{aligned} u_n - u_0 &= u_n - \mathcal{P}_n u_0 + \mathcal{P}_n u_0 - u_0 \\ &= \mathcal{P}_n \tilde{u}_n - \mathcal{P}_n u_0 + \mathcal{P}_n u_0 - u_0 \\ &= \mathcal{P}_n(\tilde{u}_n - u_0) + (\mathcal{P}_n - I)u_0. \\ \|u_n - u_0\|_{L^\infty} &\leq \|\mathcal{P}_n(\tilde{u}_n - u_0)\|_{L^\infty} + \|(\mathcal{P}_n - I)u_0\|_{L^\infty} \\ &\leq \rho \|\tilde{u}_n - u_0\|_{L^\infty} + \|(\mathcal{P}_n - I)u_0\|_{L^\infty}. \end{aligned} \tag{4.19}$$

By applying Theorem 4.3 and estimate (3.3)

$$\begin{aligned} \|u_n - u_0\|_{L^\infty} &\leq b\rho\, n^{-2r} + bn^{-r} \\ &= \mathcal{O}(n^{-r}). \end{aligned} \tag{4.20}$$

Thus, this proves the anticipated result. □

Remark 4.1. The Galerkin method provides optimal order of convergence in Theorem 4.4 whereas iterated Galerkin method provides superconvergence that can be seen by Theorem 4.3.

In the next section, we discuss the multi-Galerkin and iterated multi-Galerkin methods.

5. Multi-Galerkin Method

In this part, we obtain approximate solution for the non-linear weakly singular Volterra-Urysohn integral equation using the multi-Galerkin and iterated multi-Galerkin methods and also achieve superconvergence results. Following that, we show that the iterated multi-Galerkin approach achieves a better superconvergence rate.

The multi projection operator on $\mathbb{X}$ is defined by

$$\begin{aligned} \mathcal{K}_n^M(u) &= \mathcal{P}_n\mathcal{K}(u) + \mathcal{K}(\mathcal{P}_n u) - \mathcal{P}_n\mathcal{K}(\mathcal{P}_n u) \\ &= \mathcal{P}_n\mathcal{K}(u) + (I - \mathcal{P}_n)\mathcal{K}(\mathcal{P}_n u). \end{aligned} \tag{5.1}$$

For Eq. (2.8), the multi-Galerkin approach seeks an approximate solution $u_n^M \in \mathbb{X}$ such that

$$u_n^M - \mathcal{K}_n^M(u_n^M) = g. \tag{5.2}$$

Let the operator

$$T_n^M(u) = g + \mathcal{K}_n^M(u), \quad u \in \mathbb{X},$$

then Eq. (5.2) becomes

$$T_n^M(u_n^M) = u_n^M.$$

For a more precise approximation solution, we define

$$\tilde{u}_n^M = \mathcal{K}(u_n^M) + g. \tag{5.3}$$

This is called iterated multi-Galerkin approximate solution.

The linear operator $T_n^{M^{'}}(u)$ at u_0 is defined by:

$$\begin{aligned} T_n^{M^{'}}(u_0) &= \mathcal{K}_n^{M^{'}}(u_0) \\ &= \mathcal{P}_n\mathcal{K}^{'}(u_0) + \mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n - \mathcal{P}_n\mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n \\ &= \mathcal{P}_n\mathcal{K}^{'}(u_0) + (I-\mathcal{P}_n)\mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n. \end{aligned}$$

Here, we provide a theorem that helps us to prove the existence and uniqueness of a multi-Galerkin solution.

Theorem 5.1. *Suppose a linear operator $\mathcal{K}^{'}(u_0)$ be the Fréchet derivative of a nonlinear operator $\mathcal{K}$ at u_0 and 1 is not an eigenvalue of $\mathcal{K}^{'}(u_0)$. Then the inverse of the integral operator $(I - T_n^{M^{'}}(u_0))$ exists in the uniform norm i.e. there exists a positive constant $\mathcal{V}_1 < \infty$ and $n_0 \in \mathbb{N}$ such that $\left\|(I - T_n^{M^{'}}(u_0))^{-1}\right\|_{L^\infty} \leq \mathcal{V}_1 \ \forall \ n \geq n_0$.*

Proof. Using Theorem 4.1, we get

$$\begin{aligned} \left\|T_n^{M^{'}}(u_0) - T^{'}(u_0)\right\|_{L^\infty} &= \left\|\mathcal{P}_n\mathcal{K}^{'}(u_0) + (I-\mathcal{P}_n)\mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n - \mathcal{K}^{'}(u_0)\right\|_{L^\infty} \\ &= \left\|(\mathcal{P}_n - I)\mathcal{K}^{'}(u_0) - (\mathcal{P}_n - I)\mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n\right\|_{L^\infty} \\ &= \left\|(\mathcal{P}_n - I)\big[\mathcal{K}^{'}(u_0) - \mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n\big]\right\|_{L^\infty} \\ &\leq (1+\rho)\left\|\mathcal{K}^{'}(u_0) - \mathcal{K}^{'}(\mathcal{P}_n u_0)\mathcal{P}_n\right\|_{L^\infty} \qquad (5.4) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This implies $T_n^{M^{'}}(u_0)$ is norm convergent to $T^{'}(u_0)$ in uniform norm.

Using a result from Ahues et al. [1], $\exists$ a positive constant $\mathcal{V}_1 < \infty$ and $n_0 \in \mathbb{N}$ s.t. $\left\|(I - T_n^{M^{'}}(u_0))^{-1}\right\|_{L^\infty} \leq \mathcal{V}_1 \ \forall \ n > n_0$.

This establishes the anticipated result. □

Next, we discuss the existence and uniqueness of the multi-Galerkin solution in the following theorem.

Theorem 5.2. *Suppose u_0 be the unique solution of the Eq. (2.8) in $C^r(0, 1]$. Then for sufficiently large n, there exists a positive constant q lying in the open interval $(0,1)$, and unique solution u_n^M of equation $u = T_n^M(u)$ lying in an $B(u_0, \delta)$ open ball with centre u_0 and radius $\delta > 0$ such that*

$$\frac{\alpha_n}{(1+q)} \leq \left\|u_n^M - u_0\right\|_{L^\infty} \leq \frac{\alpha_n}{(1-q)},$$

where

$$\alpha_n = \left\|(I - T_n^{M^{'}}(u_0))^{-1}(T_n^M(u_0) - T(u_0))\right\|_{L^\infty}.$$

Further, we achieve

$$\left\|u_n^M - u_0\right\|_{L^\infty} = \mathcal{O}(n^{-2r}).$$

Proof. It follows from Theorem 5.1, $\exists$ a positive constant $\mathcal{V}_1$ such that

$$\left\|(I - T_n^{M'}(u_0))^{-1}\right\|_{L^\infty} \leq \mathcal{V}_1 < \infty.$$

Now, for any $w \in B(u_0, \delta)$, we can write

$$\begin{aligned}
&\left\|T_n^{M'}(u_0) - T_n^{M'}(w)\right\|_{L^\infty} \\
&= \left\|\mathcal{P}_n\mathcal{K}'(u_0) + (I - \mathcal{P}_n)\mathcal{K}'(\mathcal{P}_n u_0)\mathcal{P}_n - \mathcal{P}_n\mathcal{K}'(w) - (I - \mathcal{P}_n)\mathcal{K}'(\mathcal{P}_n w)\mathcal{P}_n\right\|_{L^\infty} \\
&\leq \left\|\mathcal{P}_n\big[\mathcal{K}'(u_0) - \mathcal{K}'(w)\big]\right\|_{L^\infty} + \left\|(I - \mathcal{P}_n)\mathcal{K}'(\mathcal{P}_n u_0)\mathcal{P}_n - (I - \mathcal{P}_n)\mathcal{K}'(\mathcal{P}_n w)\mathcal{P}_n\right\|_{L^\infty}.
\end{aligned}$$

On simplifying first term with the estimates (2.11) and (3.16), we get

$$\begin{aligned}
\left\|\mathcal{P}_n\big[\mathcal{K}'(u_0) - \mathcal{K}'(w)\big]y\right\|_{L^\infty} &\leq \|\mathcal{P}_n\|_{L^\infty}\left\|\mathcal{K}'(u_0) - \mathcal{K}'(w)\right\|_{L^\infty}\|y\|_{L^\infty} \\
&\leq \rho a_3 A_2 \|u_0 - w\|_{L^\infty}\|y\|_{L^\infty}.
\end{aligned}$$

From the estimates (2.11), (3.16) and Lemma 3.1, the second term can be written as

$$\begin{aligned}
&\left\|(I - \mathcal{P}_n)\big[\mathcal{K}'(\mathcal{P}_n u_0)\mathcal{P}_n - \mathcal{K}'(\mathcal{P}_n w)\mathcal{P}_n\big]y\right\|_{L^\infty} \\
&\leq (1+\rho)a_3\left\|\big[\mathcal{K}'(\mathcal{P}_n u_0)\mathcal{P}_n - \mathcal{K}'(\mathcal{P}_n w)\mathcal{P}_n\big]y\right\|_{L^\infty} \\
&\leq (1+\rho)a_3 A_2\|\mathcal{P}_n(u_0 - w)\|_{L^\infty}\|\mathcal{P}_n y\|_{L^\infty} \\
&\leq (1+\rho)a_3 A_2\rho^2\|(u_0 - w)\|_{L^\infty}\|y\|_{L^\infty}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\|T_n^{M'}(u_0) - T_n^{M'}(w)\right\|_{L^\infty} &\leq \big[\rho + (1+\rho)\rho^2\big]a_3 A_2\|(u_0 - w)\|_{L^\infty} \\
&\leq \big[\rho + (1+\rho)\rho^2\big]a_3 A_2\delta.
\end{aligned}$$

So, we can write

$$\sup_{w\in B(u_0,\delta)}\left\|(I - T_n^{M'}(u_0))^{-1}\{T_n^{M'}(u_0) - T_n^{M'}(w)\}\right\|_{L^\infty} \leq \mathcal{V}_1\big[\rho + (1+\rho)\rho^2\big]a_3 A_2\delta \leq q(\text{say}).$$

In this case, we choose δ such that $q \in (0,1)$ which satisfying equation (3.5).
We use the Theorem 5.1 and estimate (4.6) to compute

$$\begin{aligned}
\alpha_n &= \left\|(I - T_n^{M'}(u_0))^{-1}(T_n^M(u_0) - T(u_0))\right\|_{L^\infty} \\
&\leq \mathcal{V}_1\|\mathcal{P}_n\mathcal{K}(u_0) + (I - \mathcal{P}_n)\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0)\|_{L^\infty} \\
&= \mathcal{V}_1\|(\mathcal{P}_n - I)\mathcal{K}(u_0) + (I - \mathcal{P}_n)\mathcal{K}(\mathcal{P}_n u_0)\|_{L^\infty} \\
&= \mathcal{V}_1\left\|(\mathcal{P}_n - I)\big[\mathcal{K}(u_0) - \mathcal{K}(\mathcal{P}_n u_0)\big]\right\|_{L^\infty} \\
&\leq \mathcal{V}_1(1+\rho)\|\mathcal{K}(u_0) - \mathcal{K}(\mathcal{P}_n u_0)\|_{L^\infty} \\
&\leq \mathcal{V}_1(1+\rho)a_1 A_2 b n^{-r} \\
&\to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

The equation (3.7) is satisfied by choosing n large enough such that $\alpha_n \leq \delta(1-q)$. As a result, by applying Theorem 3.1, we get

$$\frac{\alpha_n}{(1+q)} \leq \left\|u_n^M - u_0\right\|_{L^\infty} \leq \frac{\alpha_n}{(1-q)},$$

which implies u_n^M exists uniquely in an open ball $B(u_0, \delta)$.

Now,

$$\begin{aligned}\left\|u_n^M - u_0\right\|_{L^\infty} \leq \frac{\alpha_n}{1-q} &\leq \frac{1}{1-q}\left\|(I - T_n^{M'}(u_0))^{-1}(T_n^M(u_0) - T(u_0))\right\|_{L^\infty} \\ &\leq \frac{1}{1-q}\mathcal{V}_1 \left\|(T_n^M(u_0) - T(u_0))\right\|_{L^\infty} \\ &\leq \frac{1}{1-q}\mathcal{V}_1 \left\|(\mathcal{K}_n^M - \mathcal{K})u_0\right\|_{L^\infty}. \end{aligned} \tag{5.5}$$

Combining Eqs. (3.1) and (4.17),

$$\begin{aligned}\left\|(\mathcal{K}_n^M - \mathcal{K})u_0\right\|_{L^\infty} &= \|(\mathcal{P}_n\mathcal{K}u_0 + (I - \mathcal{P}_n)\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}u_0\|_{L^\infty} \\ &\leq (1+\rho)\|\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}u_0\|_{L^\infty} \\ &\leq bn^{-2r}. \end{aligned} \tag{5.6}$$

The estimates (5.5) and (5.6) lead to the conclusion

$$\left\|u_n^M - u_0\right\|_\infty = \mathcal{O}(n^{-2r}).$$

This completes the proof. □

Lemma 5.1. *Suppose $u_0 \in C^r(0, 1]$ be the unique solution of the Eq. (2.8) and a linear operator $\mathcal{K}'(u_0)$ be the Fréchet derivative of a nonlinear operator $\mathcal{K}(u)$ at u_0 then*

$$\left\|\mathcal{K}'(u_0)(I - \mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0))\right\|_{L^\infty} = \mathcal{O}(n^{-3r}).$$

Proof. For $\tau \in [0, 1]$, we have

$$\begin{aligned}&\left\|\mathcal{K}'(u_0)(I - \mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0))\right\|_{L^\infty} \\ &= \sup_{\tau,\,\lambda\in[0,1]}\left|\mathcal{K}'(u_0)(I - \mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0))\right|.\end{aligned}$$

Now for any $v_\tau \in \mathbb{X}_n$, $\langle v_\tau, (\mathcal{P}_n - I)u_0(s(\tau, .))\rangle = 0$, by using Theorem 3.2, and Eq. (4.17), we get

$$\begin{aligned}&\left|\mathcal{K}'(u_0)(I - \mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0))\right| \\ &= \left|\int_0^1 (1-\lambda)^{-\gamma}\tilde{k}_u\left(\tau, s(\tau,\lambda), u_0(s(\tau,\lambda))\right)(I - \mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0))\, d\lambda\right| \\ &= |\langle W(\tau, s(\tau, .), u(s(\tau, .))), (I - \mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0))\rangle| \\ &= |\langle W(\tau, s(\tau, .), u(s(\tau, .))) - \nu_\tau, (I - \mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0))\rangle| \\ &\leq \|W(\tau, s(\tau, .), u(s(\tau, .))) - \nu_\tau\|_{L^1}\|(I - \mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0))\|_{L^\infty} \\ &\leq \|W(\tau, s(\tau, .), u(s(\tau, .))) - \nu_\tau\|_{L^1}(1+\rho)\|(\mathcal{K}(\mathcal{P}_n u_0) - \mathcal{K}(u_0))\|_{L^\infty}\end{aligned}$$

$$\leq bn^{-r}(1+\rho)n^{-2r} = b(1+\rho)n^{-3r}. \tag{5.7}$$

Therefore,

$$\left\|\mathcal{K}^{'}(u_0)(I-\mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0)-\mathcal{K}(u_0))\right\|_{L^\infty} = \mathcal{O}(n^{-3r}).$$

This establishes the anticipated result. □

Now, we analyze the superconvergence result for the iterated multi-Galerkin approximate solution in the following theorem.

Theorem 5.3. *Suppose the unique solution $u_0 \in C^r(0,1]$ and the source function $g \in C^r[0,1]$, $r \geq 1$. $\tilde{u}_n^M$ is the approximated iterated multi-Galerkin solution described by* (5.3)*, then it holds*

$$\left\|\tilde{u}_n^M - u_0\right\|_{L^\infty} = \mathcal{O}(n^{-3r}).$$

Proof. With a Lemma 6 from Mandal [20], we have

$$\begin{aligned}
&\left\|(\tilde{u}_n^M - u_0)\right\|_{L^\infty} \\
\leq&(d_0C_2 + M_1M_2)\left\|u_n^M - u_0\right\|^2_{L^\infty} + \left\|\mathcal{K}^{'}(u_0)(I-\mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0)-\mathcal{K}(u_0))\right\|_{L^\infty} \\
&+ M_1\left\|\mathcal{P}_n\mathcal{K}^{'}(u_0)(I-\mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0)-\mathcal{K}(u_0))\right\|_{L^\infty} \\
\leq&(d_0C_2+M_1M_2)\left\|u_n^M-u_0\right\|^2_{L^\infty}+(1+M_1\rho)\left\|\mathcal{K}^{'}(u_0)(I-\mathcal{P}_n)(\mathcal{K}(\mathcal{P}_n u_0)-\mathcal{K}(u_0))\right\|_{L^\infty}.
\end{aligned} \tag{5.8}$$

Combining results of Theorem 5.2 and Lemma 5.1 with Eq. (5.8), we obtain

$$\left\|\tilde{u}_n^M - u_0\right\|_{L^\infty} = \mathcal{O}(n^{-3r}). \tag{5.9}$$

Hence the proof follows. □

Remark 5.1. Both the multi-Galerkin method and the iterated multi-Galerkin method achieve superconvergence, although the iterated multi-Galerkin method converges faster than the multi-Galerkin method, as shown in Theorem 5.2 and Theorem 5.3.

6. Numerical results

In this part, we provide two numerical examples to demonstrate the efficacy of the proposed theoretical results. For the subspaces $\mathbb{X}_n$, piecewise polynomials are considered as the basis functions with respect to the graded mesh described by Eqs. (2.16) and (2.17). For both examples, we choose $r = 1$, implying that piecewise constants are chosen as the basis function. Afterward, the error bounds and order of convergence are discussed for the approximated solutions of Galerkin, multi-Galerkin, and their iterated versions in Table 1, Table 2, Table 3, and Table 4. The error bounds for Galerkin, iterated Galerkin, multi Galerkin, and iterated multi Galerkin methods are denoted by $||u - u_n||_{L^\infty} = \mathcal{O}(n^{-\beta})$, $||u - \tilde{u}_n||_{L^\infty} = \mathcal{O}(n^{-\alpha})$, $||u - u_n^M||_{L^\infty} = \mathcal{O}(n^{-\delta})$, and $||u - \tilde{u}_n^M||_{L^\infty} = \mathcal{O}(n^{-\eta})$ respectively, where β, α, δ and η are the order of convergence for respective methods.

Example 6.1. Consider the following second kind weakly singular Urysohn VIE as:

$$u(\tau) = \tau^4 - \frac{65536}{109395}\tau^{\frac{17}{2}} + \int_0^\tau (\tau - s)^{-\frac{1}{2}} u^2(s)\, ds, \quad s \in [0,1], \tag{6.1}$$

where the exact solution is $u(\tau) = \tau^4$.

With the transformation $s = \tau\lambda$ the transformed integral equation as follows:

$$u(\tau) = \tau^4 - \frac{65536}{109395}\tau^{\frac{17}{2}} + \int_0^1 \frac{\sqrt{\tau}}{\sqrt{(1-\lambda)}} u^2(\tau)\, d\lambda, \quad \lambda \in [0,1].$$

Now, for $r = 1$, $\gamma = \frac{1}{2}$, and $q_1 = \frac{r}{1-\gamma}$ the expected order of convergence are $\beta = 1$, $\alpha = 2$, $\delta = 2$ and $\eta = 3$.

In the following, Table 1 presents errors bounds and convergence rates for Galerkin and iterated Galerkin methods, and Table 2 presents the errors bounds and convergence rates for multi-Galerkin and iterated multi-Galerkin methods.

Table 1. Galerkin and iterated Galerkin methods

n	$\|u - u_n\|_{L^\infty}$	β	$\|u - \tilde{u}_n\|_{L^\infty}$	α
2	7.5204523625 $\times 10^{-1}$	0.99	5.6939337958 $\times 10^{-1}$	1.95
4	4.3315186164 $\times 10^{-1}$	1.01	1.75152298646 $\times 10^{-1}$	2.10
8	2.2561647472$\times 10^{-1}$	1.02	5.2240878745$\times 10^{-2}$	2.03
16	1.2452631362 $\times 10^{-1}$	0.98	1.2414505961 $\times 10^{-2}$	2.07
32	7.444760575 $\times 10^{-2}$	0.93	2.8952980356 $\times 10^{-3}$	2.09
64	5.4521010902$\times 10^{-2}$	1.04	2.4274747067$\times 10^{-3}$	2.15
128	1.2911593372$\times 10^{-2}$	1.04	2.1570795669$\times 10^{-4}$	2.02

Table 2. Multi Galerkin and iterated multi Galerkin methods

n	$\|u - u_n^M\|_{L^\infty}$	δ	$\|u - \tilde{u}_n^M\|_{L^\infty}$	η
2	5.64830289512$\times 10^{-1}$	1.98	4.312128602 $\times 10^{-1}$	2.92
4	1.87685840862 $\times 10^{-1}$	2.02	8.62670201 $\times 10^{-2}$	2.96
8	5.5736068337$\times 10^{-2}$	1.98	1.5230867832 $\times 10^{-2}$	2.88
16	1.34979966662$\times 10^{-2}$	2.03	1.7841305556 $\times 10^{-3}$	2.99
32	2.5933896965 $\times 10^{-3}$	2.13	2.3318538260 $\times 10^{-4}$	2.99
64	1.4201784830$\times 10^{-3}$	1.88	2.4455217002 $\times 10^{-5}$	3.05
128	2.8074204795$\times 10^{-4}$	1.96	3.4557854730$\times 10^{-6}$	3.02

From Table 1, one may observe that for all the values of n the order of convergence for Galerkin and iterated Galerkin methods are approximately 1 and 2 respectively. In addition, the order of convergence for multi-Galerkin and iterated multi-Galerkin methods are approximately 2 and 3, respectively, as shown in Table 2. Thus, obtained convergence rates for all the methods are similar to the proposed theoretical results.

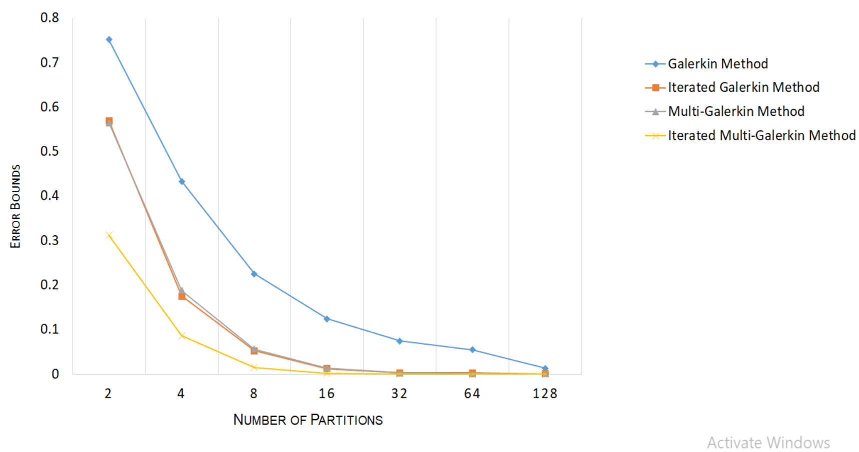

Figure 1. Comparison of error bounds among all the proposed methods

Figure 1 depicts a comparison of error bounds for all of the proposed methods, demonstrating that when the number of partitions increases, the associated error bounds decrease. Because of its linear convergence rate for piecewise constant, the Galerkin method provides a slightly higher error bound (i.e., 0.01291159, when $n = 128$), whereas the iterated multi-Galerkin method provides very lesser error bound (i.e. 0.0000034558, when $n = 128$). As a result, we may conclude that the iterated multi-Galerkin method has the best convergence rate among all other methods.

Example 6.2. Consider the following second kind weakly singular Urysohn VIE as:

$$u(\tau) = \sqrt{2}\frac{21}{32}\pi\tau^2 + \sqrt{\tau} - \int_0^\tau (\tau - s)^{-\frac{3}{4}} s^{\frac{1}{4}} u^3(s)\, ds, \quad s \in [0,1], \tag{6.2}$$

where the exact solution of the given integral equation (6.2) is $u(\tau) = \sqrt{\tau}$.

Now, by using transformation $s = \tau\lambda$, we obtain the following transformed integral equation as:

$$u(x) = \sqrt{2}\frac{21}{32}\pi\tau^2 + \sqrt{\tau} - \int_0^1 \frac{\sqrt{\tau}}{(1-\lambda)^{\frac{3}{4}}} \lambda^{\frac{1}{4}} u^3\, d\lambda, \quad \lambda \in [0,1].$$

For $r = 1$, $\gamma = \frac{1}{2}$, and $q_1 = \frac{r}{1-\gamma}$ the expected order of convergence are $\beta = 1$, $\alpha = 2$, $\delta = 2$ and $\eta = 3$.

Table 3 provides error bounds and convergence rates for Galerkin and iterated Galerkin methods, whereas Table 4 provides error bounds and convergence rates for multi-Galerkin and iterated multi-Galerkin methods.

Table 3. Galerkin and iterated Galerkin methods

n	$\|u - u_n\|_{L^\infty}$	β	$\|u - \tilde{u}_n\|_{L^\infty}$	α
2	$7.5824601737 \times 10^{-1}$	0.96	$5.6548431370 \times 10^{-1}$	1.98
4	$4.4767919716 \times 10^{-1}$	0.97	$1.79294994754 \times 10^{-1}$	2.07
8	$1.9850372001 \times 10^{-1}$	1.11	$4.4529009089 \times 10^{-2}$	2.14
16	$1.1444003538 \times 10^{-1}$	1.02	$1.24778799718 \times 10^{-2}$	2.07
32	$4.9736026522 \times 10^{-2}$	1.07	$3.1265205569 \times 10^{-3}$	2.06
64	$3.1011753053 \times 10^{-2}$	0.99	$2.5458996071 \times 10^{-4}$	1.98
128	$7.9523821572 \times 10^{-3}$	1.16	$3.2705312910 \times 10^{-4}$	1.92

Table 4. Multi Galerkin and iterated multi Galerkin methods

n	$\|u - u_n^M\|_{L^\infty}$	δ	$\|u - \tilde{u}_n^M\|_{L^\infty}$	η
2	$5.71263064562 \times 10^{-1}$	1.94	$3.9654850110 \times 10^{-1}$	3.21
4	$2.0252939941 \times 10^{-1}$	1.93	$8.1773416735 \times 10^{-2}$	3.02
8	$3.9113556511 \times 10^{-2}$	2.23	$1.2318774076 \times 10^{-2}$	3.03
16	$1.5210251041 \times 10^{-2}$	1.98	$2.1362333595 \times 10^{-3}$	2.99
32	$4.2395115410 \times 10^{-3}$	1.95	$1.8999708727 \times 10^{-4}$	3.07
64	$7.8727848300 \times 10^{-3}$	2.05	$3.1565217002 \times 10^{-5}$	2.98
128	$2.5154204795 \times 10^{-4}$	1.99	$3.2163854730 \times 10^{-6}$	3.03

Similar to Example 6.1, order of convergence for Galerkin and iterated Galerkin methods are approximately 1 and 2, respectively, for all values of n, as shown in Table 3. According to Table 4, the order of convergence for multi-Galerkin and iterated multi-Galerkin methods are approximately 2 and 3, respectively.

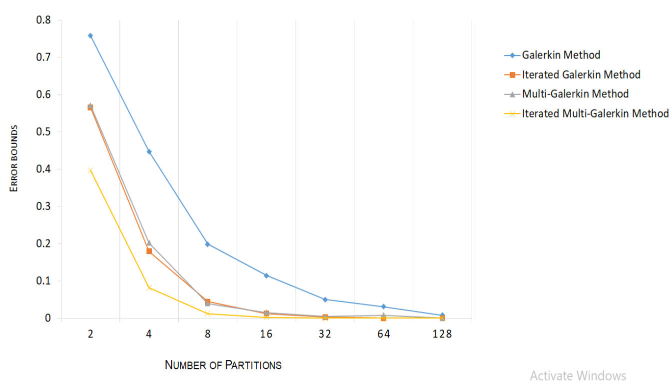

Figure 2. Comparison of error bounds among all the proposed methods

Figure 2 shows the comparison between errors bounds for all the proposed methods, in which one can easily say that when the number of partitions increases then corresponding errors bounds decrease. In this example also, among all these methods, the Galerkin method provides a slightly higher error bound (i.e., 0.00795238, when $n = 128$) whereas iterated multi-Galerkin method provides lesser error bound (i.e., 0.000003216, when $n = 128$). Therefore, we can say that iterated multi-Galerkin method provides the best solution in terms of convergence as well as error bounds.

6.1. Conclusion

In this work, we proposed Galerkin and multi-Galerkin methods for solving the second kind Volterra Urysohn integral equation with a weakly singular kernel, as well as iterated versions of these methods. We established error estimates for all these methods in L^∞ norm. Moreover, we also obtained the order of convergence for all the proposed methods and showed optimal order of convergence is achieved for Galerkin method while the rest of the methods obtain superconvergence rates. Approximate solutions of iterated Galerkin and multi-Galerkin methods converge with the same order. But in comparison to the iterated Galerkin method and multi-Galerkin method, the iterated multi-Galerkin method gives a better convergence result. Eventually, with the help of numerical examples theoretical results are verified and also demonstrated that as the number of partitions of an interval increases, the error bounds fall.

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