Journal of Applied Analysis and Computation
Volume 13, Number 3, June 2023, 1421–1428
Website:http://www.jaac-online.com
DOI:10.11948/20220206

ANOTHER IMPROVEMENT ON OSCILLATION CRITERIA FOR FIRST-ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract This article studies the oscillation of solutions to the delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0.$$

We improve the existing oscillation criteria, by lowering the existing bounds on $\limsup \int_\tau^t p$ that provide sufficient conditions for the oscillation of all solutions.

Keywords Oscillation of solutions, first-order delay differential equation, eventually positive solution.

MSC(2010) 34K11, 34C10.

1. Introduction

This article concerns the oscillation of solutions to the delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \ge t_0\,, \tag{1.1}$$

where $p, \tau \in C([t_0, \infty), [0, \infty))$, the delay argument τ is non-decreasing, $\tau(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \infty$. Establishing oscillation criteria for solutions to (1.1) has been the object of many studies, see for example the books [1,9,12] and the references cited therein.

By a solution, we mean a continuously differentiable function that satisfies (1.1). A solution is called oscillatory if it has arbitrarily large zeros, and otherwise it is called non-oscillatory.

Throughout this article we use the notation

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^t p(s)\,ds, \quad \text{and} \quad \beta = \limsup_{t\to\infty} \int_{\tau(t)}^t p(s)\,ds\,. \tag{1.2}$$

Composition of the function τ with itself is denoted by $\tau^{n+1}(t) = \tau^n(\tau(t))$, $\tau^1(t) = \tau(t)$, and $\tau^0(t) = t$. The solutions of the equation

$$\lambda = e^{\alpha\lambda}$$

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play an important role in the statement of oscillation criteria. So we consider the following cases: If $\alpha = 0$, then there is only one solution, $\lambda = 1$. If $0 < \alpha < 1/e$, then there are two solutions, $\lambda_1 < \lambda_2$; in this case λ_1 is a continuous and increasing function of α. If $\alpha = 1/e$, then there is only one solution $\lambda_1 = \lambda_2 = e$. If $\alpha > 1/e$, there is no solution.

A well known criterion for the oscillation of solutions to (1.1) states that if

$$\alpha > \frac{1}{e} \quad \text{or} \quad \beta > 1\,, \tag{1.3}$$

then every solution is oscillatory, see for example [1, 6, 12]. On the other hand if $\int_{\tau(t)}^{t} p(s)\,ds \le 1/e$ for all sufficiently large t, then there is a non-oscillatory solution, see [6, Corollary 2.1.1]. We are interested in the case $\alpha < \beta$. Stavroulakis [14] presented a summary of existing conditions and a table of bounds for β, in terms of α. Our goal is to improve the existing lower bounds for β; see Table 1 toward the end of this article.

Garab et al [7, Theorem 4] obtained the following result for the constant delay case, $\tau(t) = t - \tau_0$. Let p be a non-negative, bounded and uniformly continuous function, $0 < \alpha$, $1/e < \beta$, and the mapping $t \mapsto \int_{\tau(t)}^{t} p(s)\,ds$ be slowly varying at infinity. Then all solutions of (1.1) are oscillatory.

2. Results

We use additive and multiplicative estimates of solutions. Additive estimates are obtained by integrating (1.1) from $\tau(t)$ to t,

$$x(\tau(t)) = x(t) + \int_{\tau(t)}^{t} p(s) x(\tau(s))\,ds\,. \tag{2.1}$$

Then using this equality in the integrand we obtain an iterated integral. Repeating this process n times we have

$$\begin{aligned} x(\tau(t)) = x(t) &+ x(\tau(t)) \int_{\tau(t)}^{t} p(s_1) \int_{\tau(s_1)}^{\tau(t)} p(s_2)\,ds_2\,ds_1 \\ &+ x(\tau^2(t)) \int_{\tau(t)}^{t} p(s_1) \int_{\tau(s_1)}^{\tau(t)} p(s_2)\,ds_2\,ds_1 + \dots \\ &+ x(\tau^n(t)) \int_{\tau(t)}^{t} p(s_1) \int_{\tau(s_1)}^{\tau(t)} p(s_2) \cdots \int_{\tau(s_{n-1})}^{\tau^{n-1}(t)} p(s_n)\,ds_n \dots ds_1 \\ &+ R_{n+1}(t)\,, \end{aligned} \tag{2.2}$$

where

$$R_{n+1}(t) = \int_{\tau(t)}^{t} p(s_1) \int_{\tau(s_1)}^{\tau(t)} p(s_2) \cdots \int_{\tau(s_n)}^{\tau^n(t)} p(s_{n+1}) x(\tau(s_{n+1}))\,ds_{n+1} \dots ds_1\,.$$

Multiplicative estimates are obtained for positive solutions. Dividing (1.1) by $x(t)$ and then integrating yields

$$-\int_{u}^{v} \frac{x'(s)}{x(s)}\,ds = \int_{u}^{v} p(s) \frac{x(\tau(s))}{x(s)}\,ds\,.$$

The integral on the left-hand side is $\ln(x)$, so that

$$x(u) = x(v)\exp\Big(\int_u^v p(s)\frac{x(\tau(s))}{x(s)}\,ds\Big)\,. \tag{2.3}$$

This estimate has been used by many authors for estimating $x(\tau(t))/x(t)$; see for example [1] and [6, Lemma 2.1].

Note that if x is an eventually positive solution of (1.1), then there exists t_1 such that both $x(t)$ and $x(\tau(t))$ are positive for all $t \ge t_1$. In this case, from (1.1) it follows that $x'(t) \le 0$, hence x is non-increasing. Applying (2.3) to $R_{n+1}(t)$ with $u = \tau(s_{n+1})$ and $v = \tau^{n+1}(t)$, and assuming that $x(\tau^{n+2}(t))$ is positive, we obtain

$$\begin{aligned} R_{n+1}(t) :=& x(\tau^{n+1}(t))\int_{\tau(t)}^{t} p(s_1)\int_{\tau(s_1)}^{\tau(t)} p(s_2)\,ds_2 \cdots \int_{\tau(s_n)}^{\tau^n(t)} p(s_{n+1}) \\ &\times \exp\Big(\int_{\tau(s_{n+1})}^{\tau^{n+1}(t)} p(s_{n+2})\frac{x(\tau(s_{n+2}))}{x(s_{n+2})}\,ds_{n+2}\Big)\,ds_{n+1}\dots ds_1\,. \end{aligned} \tag{2.4}$$

In particular

$$R_1(t) = x(\tau(t))\int_{\tau(t)}^{t} p(s_1)\exp\Big(\int_{\tau(s_1)}^{\tau(t)} p(s_2)\frac{x(\tau(s_2))}{x(s_2)}\,ds_2\Big)\,ds_1\,. \tag{2.5}$$

To abbreviate notation, we introduce the functions

$$\begin{aligned} &\rho_1(t) = \int_{\tau(t)}^{t} p(s_1)\,ds_1, \quad \rho_2(t) = \int_{\tau(t)}^{t} p(s_1)\int_{\tau(s_1)}^{\tau(t)}, p(s_2)\,ds_2\,ds_1,\ \dots, \\ &\rho_n(t) = \int_{\tau(t)}^{t} p(s_1)\int_{\tau(s_1)}^{\tau(t)} p(s_2)\cdots\int_{\tau(s_{n-1})}^{\tau^{n-1}(t)} p(s_n)\,ds_n\dots ds_1\,, \end{aligned} \tag{2.6}$$

so that (2.2) becomes

$$x(\tau(t)) = x(t) + x(\tau(t))\rho_1(t) + x(\tau^2(t))\rho_2(t) + \cdots + x(\tau^n(t))\rho_n(t) + R_{n+1}(t)\,. \tag{2.7}$$

This expression, without $R_{n+1}(t)$, yields an inequality used in [4,15] for establishing oscillation criteria. Next we estimate $x(\tau(t))/x(t)$ using the roots $\lambda_1 < \lambda_2$ of $\lambda = \exp(\alpha\lambda)$.

Lemma 2.1 ([11, Lemma 1]). *Let $0 < \alpha$ and x be an eventually positive solution of* (1.1). *Then $0 < \alpha \le 1/e$ and*

$$\lambda_1 \le \liminf_{t\to\infty}\frac{x(\tau(t))}{x(t)} \le \lambda_2\,. \tag{2.8}$$

Let

$$0 < \hat\alpha \le \alpha = \frac{1}{e}\,. \tag{2.9}$$

Then the smaller solution $\hat\lambda$ of $\lambda = \exp(\hat\alpha\lambda)$ satisfies the following: $\hat\lambda$ depends continuously on $\hat\alpha$, and $\hat\lambda$ approaches λ_1 (the smaller solution of $\lambda = e^{\alpha\lambda}$) as $\hat\alpha$ approaches α.

From Lemma 2.1, there exists $t_2 \geq t_1$ such that

$$\hat{\lambda} \leq \frac{x(\tau(t))}{x(t)}, \quad \hat{\lambda} \leq \frac{x(\tau^2(t))}{x(\tau(t))}, \dots \quad \forall t \geq t_2\,. \tag{2.10}$$

Furthermore assuming $x(\tau^i(t))$ is positive for t large enough,

$$\frac{x(\tau^i(t))}{x(t)} = \frac{x(\tau^i(t))}{x(\tau^{i-1}(t))}\frac{x(\tau^{i-1}(t))}{x(\tau^{i-2}(t))}\cdots\frac{x(\tau(t))}{x(t)} \geq \hat{\lambda}^i\,. \tag{2.11}$$

To estimate R_{n+1} from below we use the following assumption: There exists $\omega > 0$ such that for all $u \leq v$,

$$\int_{\tau(u)}^{\tau(v)} p(s)\,ds \geq \omega \int_u^v p(s)\,ds\,. \tag{2.12}$$

This assumption was also used in [2, 11].

Remark 2.1. (1) If $\omega > 1$, then $\lim_{t\to\infty}\int_{\tau(t)}^t p = 0$. In which case $\int_{\tau(t)}^t p \leq 1/e$ for t large enough, then there exists a non-oscillatory solution, see [6, Corollary 3.11].

To prove that the limit is zero, for each t large enough, we define n as the largest integer such that $t_0 \leq \tau^n(t) =: t_1$. Then by (2.12),

$$\int_{\tau(t_1)}^{t_1} p = \int_{\tau^{n+1}(t)}^{\tau^n(t)} p \geq \omega \int_{\tau^n(t)}^{\tau^{n-1}(t)} p \geq \omega^2 \int_{\tau^{n-1}(t)}^{\tau^{n-2}(t)} p \geq \cdots \geq \omega^n \int_{\tau(t)}^t p\,.$$

Note that $n \to \infty$ as $t \to \infty$, and that $\omega^n \to \infty$. Meanwhile the left-hand side of the above inequality remains bounded. Therefore $\int_{\tau(t)}^t p$ must approach zero as $t \to \infty$.

(2) If $\omega = 1$, then the mapping $t \mapsto \int_{\tau(t)}^t p$ is non-increasing and non-negative; therefore it converges as $t \to \infty$. When $\int_{\tau(t)}^t p$ converges to a value greater than $1/e$, every solution is oscillatory, see (1.3). When $\int_{\tau(t)}^t p$ converges to a value less than $1/e$, there is a non-oscillatory solution, see [6, Corollary 3.11]. When $\int_{\tau(t)}^t p$ converges to $1/e$, assuming that $0 \leq \int_{\tau(t)}^t p(s)\,ds - 1/e$ and that it decays to 0, it can be shown that every solution is oscillatory. However the assumptions for [5, Lemma 2], and the assumptions for [10, [Lemma 2.1] seem to be insufficient for their conclusions.

That $f(t) = \int_{\tau(t)}^t p$ is non-increasing can be shown as follows: For $\omega = 1$, and $u \leq v$, we have

$$f(v) - f(u) = \int_{\tau(u)}^u p - \int_{\tau(v)}^v p = \int_u^v p - \int_{\tau(u)}^{\tau(v)} p \leq 0\,.$$

This proof was suggested by the anonymous referee.

(3) We are interested in the oscillation of solutions, so we restrict our attention to the case $0 < \omega < 1$.

Lemma 2.2. *Let $\hat{\alpha}$ and $\hat{\lambda}$ be as defined above, and x be a solution of* (1.1). *If* (2.12) *holds and $x(\tau^2(t))$ is positive for all t large enough, then*

$$R_1(t) \geq x(\tau(t))\frac{1}{\hat{\lambda}\omega}\Big[\exp\Big(\hat{\lambda}\omega\int_{\tau(t)}^t p(s)\,ds\Big) - 1\Big]. \tag{2.13}$$

Proof. Inequalities (2.10) and (2.12) provide the following lower bound for the inner-most integral in $R_1(t)$,

$$\int_{\tau(s_1)}^{\tau(t)} p(s_2)\frac{x(\tau(s_2))}{x(s_2)}\, ds_2 \geq \hat{\lambda}\omega \int_{s_1}^{t} p(s_2)\, ds_2\,. \tag{2.14}$$

Then

$$R_1(t) \geq x(\tau(t)) \int_{\tau(t)}^{t} p(s_1) e^{\hat{\lambda}\omega \int_{s_1}^{t} p(s_2)\, ds_2}\, ds_1.$$

Integrating by substitution with $u(s_1) = \hat{\lambda}\omega \int_{s_1}^{t} p(s_2)\, ds_2$ and $u'(s_1) = -\hat{\lambda}\omega p(s_1)$ we obtain (2.13). □

Lemma 2.3 ([4, Lemma 2.4])**.** *Let* $0 < \alpha \leq 1/e$*, and* x *be an eventually positive solution of* (1.1)*. Then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(\tau(t))} \geq 1 - \alpha - \frac{1}{\lambda_1}\,. \tag{2.15}$$

The proof of [4, Lemma 2.4] uses the inequality obtained from (2.2) by omitting the remainder $R_{n+1}(t)$. This does not affect the result because as $t \to \infty$, we have that $n \to \infty$ and $R_{n+1}(t) \to 0$. We omit the proof of the lemma here. Lemma 2.3 was also proved in [11, Lemma 2] and [14, Remark 2.3] under assumption (2.12).

From (2.2) with $n = 0$, and (2.13), we have

$$x(\tau(t)) \geq x(t) + x(\tau(t))\frac{1}{\hat{\lambda}\omega}\Big[\exp\Big(\hat{\lambda}\omega \int_{\tau(t)}^{t} p(s)\, ds\Big) - 1\Big].$$

Assuming that $x(\tau(t))$ is positive for t large enough, we can divide the above inequality by $x(\tau(t))$ and obtain

$$1 \geq \frac{x(t)}{x(\tau(t))} + \frac{1}{\hat{\lambda}\omega}\Big[e^{\hat{\lambda}\omega \int_{\tau(t)}^{t} p} - 1\Big]. \tag{2.16}$$

Then taking limits as $t \to \infty$, using that $\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\, ds = \beta$, and (2.15), we have

$$\begin{aligned} 1 &\geq \liminf_{t\to\infty} \frac{x(t)}{x(\tau(t))} + \limsup_{t\to\infty} \frac{1}{\hat{\lambda}\omega}\Big[e^{\hat{\lambda}\omega \int_{\tau(t)}^{t} p} - 1\Big] \\ &\geq 1 - \alpha - \frac{1}{\lambda_1} + \frac{1}{\hat{\lambda}\omega}\big[e^{\hat{\lambda}\omega^{n+1}\beta} - 1\big]. \end{aligned}$$

Note that as $\hat{\alpha} \to \alpha$, we have that $\hat{\lambda} \to \lambda_1$, and

$$\beta \leq \frac{1}{\lambda_1\omega} \ln\big(1 + \omega(\alpha\lambda_1 + 1)\big). \tag{2.17}$$

Theorem 2.1. *Assume* $0 < \alpha \leq 1/e$*,* (2.12) *holds with* $0 < \omega < 1$*, and*

$$\beta > \frac{1}{\lambda_1\omega} \ln\big(1 + \omega(\alpha\lambda_1 + 1)\big)\,. \tag{2.18}$$

Then every solution of (1.1) *is oscillatory.*

Proof. To obtain a contradiction, we assume that x is a non-oscillatory solution of (1.1). First assume that x is eventually positive so that (2.17) holds for t sufficiently large which contradicts (2.18). Therefore x cannot be eventually positive. Now if y is a eventually negative solution, by the linearity of (1.1), we see that $x(t) = -y(t)$ is a positive solution. Then we can use the above argument to obtain a contradiction. This completes the proof. $\square$

Remark 2.2. Certainly ω cannot equal 1, because by Remark 2.3(2), $\omega = 1$ makes $\alpha = \beta$. In turn, this equality makes the following oscillation criteria not satisfiable, for the typical case $\alpha = 1/e$ and $\lambda_1 = e$: inequality (2.18) above, inequality $L > \sqrt{7-2e}/e$ in [13, Remark 3], inequality (12) in [11, Remark 2], and inequality (3.8) in [4].

Table 1 shows lower bounds for $\beta = \limsup_{t\to\infty} \int_{\tau(t)}^t p$, as ω approaches 1 from below. Note that the bound in Theorem 2.1 is slightly lower than the previous bounds.

Table 1. Oscillation criteria, in [4, 11, 13] and Theorem 2.1 with $\omega = 0.9999$

Ref.	$\alpha = 1/e$, $\lambda_1 = e$	$\alpha = 2\ln(e/2)/e$, $\lambda_1 = e/2$	$\alpha = 0$, $\lambda_1 = 1$
[11]	$\beta > 0.471518$	$\beta > 0.923057$	$\beta > 1$
[13]	$\beta > 0.459987$	$\beta > 0.741974$	$\beta > \sqrt{3} - 1 \approx 0.732750$
[4]	$\beta > 0.459188$	$\beta > 0.716267$	$\beta > \ln(2) \approx 0.693147$
Thm. 2.1	$\beta > 0.404157$	$\beta > 0.615009$	$\beta > \ln(2) \approx 0.693147$

Example 2.1. We consider the equation

$$x'(t) + x\big(t - (a + b\sin(\gamma t)\big) = 0, \quad t \ge 0\,, \tag{2.19}$$

where $p = 1$, $a = \frac{9}{40} + \frac{1}{2e}$, $b = \frac{9}{40} - \frac{1}{2e}$, $\tau(t) = t - (a + b\sin(\gamma t)$, $\omega = 0.999$, and $\gamma = (1-\omega)/b$. Note that $\tau(t) \le t$ because $a + b\sin(\gamma t) \ge 0$. We also have

$$\begin{aligned}
\alpha &= \liminf_{t\to\infty} \int_{\tau(t)}^t 1\,ds = \liminf_{t\to\infty} \big(a + b\sin(\gamma t)\big) = \frac{1}{e}, \\
\beta &= \limsup_{t\to\infty} \big(a + b\sin(\gamma t)\big) = a + b = 0.45\,.
\end{aligned}$$

Using the mean value theorem, with $\xi \in (\gamma u, \gamma v)$, we show that (2.12) is satisfied.

$$\begin{aligned}
\int_{\tau(u)}^{\tau(v)} p = \tau(v) - \tau(u) &= v - u + b\big(\sin(\gamma v) - \sin(\gamma u)\big) \\
&= \big[1 + b\gamma \frac{\sin(\gamma v) - \sin(\gamma u)}{\gamma v - \gamma u}\big](v - u) \\
&= \big[1 + b\gamma\cos(\xi)\big] \int_u^v p \\
&\ge [1 - b\gamma] \int_u^v p = \omega \int_u^v p\,.
\end{aligned}$$

Then β satisfies (2.18), and does not satisfy the other inequalities on the second column of Table 1. Therefore the oscillation criteria in the references do not apply to this example.

Remark 2.3. With the intent of improving bound (2.18), we extended Lemma 2.2 for $n \geq 1$, and obtained

$$\begin{aligned} &R_{n+1}(t) \\ \geq &x(\tau(t))\Big\{\frac{1}{\hat{\lambda}\omega^{(n+1)(n+2)/2}}\Big[\exp\Big(\hat{\lambda}\omega^{n+1}\int_{\tau(t)}^{t} p(s)\,ds\Big)-1\Big]-\frac{1}{\omega^{n(n+1)/2}}\rho_1(t) \\ &-\frac{\hat{\lambda}}{\omega^{(n-1)n/2}}\rho_2(t)-\cdots-\frac{\hat{\lambda}^{n-1}}{\omega^{(1)(2)/2}}\rho_n(t)\Big\}\,. \end{aligned} \tag{2.20}$$

However this expression does not lead to an explicit bound for β.

Also note that the optimal bound $\beta > 1/e$ has not been reached yet; it remains an open question.

Acknowledgments

The authors wish to thank the anonymous referees for making corrections to the original manuscript.

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