Journal of Applied Analysis and Computation
Volume 13, Number 3, June 2023, 1429–1448
Website:http://www.jaac-online.com
DOI:10.11948/20220212

EVOLUTIONARY BEHAVIOR OF THE INTERACTION SOLUTIONS FOR A (3+1)-DIMENSIONAL GENERALIZED BREAKING SOLITON EQUATION*

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Abstract The interaction solutions have attracted the attention of many scholars because of they are valuable in analyzing the nonlinear dynamics of waves in shallow water and can be used for forecasting the appearance of rogue waves. In this paper, we investigate the interaction and rational solutions of a (3+1)-dimensional generalized breaking soliton equation by employing the Hirota bilinear and parameter limit methods along with symbolic computations. By studying the Hirota bilinear form of the equation, abundant interaction and rational solutions are derived by choosing appropriate parameters of the test function. The evolutionary behavior of the interaction solutions is also analyzed theoretically and graphically. Compare with the published literatures, we get some completely new results of the equation in this paper.

Keywords A (3+1)-dimensional generalized breaking soliton equation, Hirota bilinear form, evolutionary behavior, interaction solution, rational solution.

MSC(2010) 35Q51, 35Q53, 74J30.

1. Introduction

Recently, the interaction, the lump-type and the rational solutions of nonlinear evolution equations (NLEEs) have attracted the attention of many researchers [1–3, 8–20, 22–29, 32, 33, 35]. For examples, Liu et al. [12] got the bi-soliton, breather and rogue wave solutions of the (2+1)-dimensional nonlinear Schrödinger equation using Exp-function method. Through the Hirota bilinear method, Ma [17] formulated lump solutions for a combined fourth-order nonlinear equation in (2+1)-dimensions. Utilizing the linear superposition method, Hosseini et al. [8] constructed the rational wave solutions of the (4+1)-dimensional Boiti-Leon-Manna-Pempinelli (4D-BLMP) equation. The dynamical behavior of the solutions to the 4D-BLMP equation was also analyzed graphically in [8] by considering the special values of the involved parameters. By employing the parameter limit method [26] and symbolic

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*The authors were supported by National Natural Science Foundation of China (No. 11461022) and Natural Science Foundation of Yunnan Province, China (No. 2014FA037).

computation, Tan et al. [27] studied some nonlinear phenomena such as the local oscillations and degeneration behavior of double breather solutions for a (2+1)-dimensional KdV system, and got a new lump solution from the double breather solutions. He and Meng [9] obtained new interaction solutions for the sixth-order Ramani equation via the three wave method, Shen et al. [25] derived the lump and its interaction solutions of the generalized (3+1)-dimensional nonlinear wave equation. Multiple soliton solutions for the generalized (2+1)-dimensional Camassa-Holm-Kadomtsev-Petviashvili equation were presented in [23] employing the multiple-order line rogue wave solutions method. Variety interaction between k-lump and k-kink solutions for the generalized Burgers equation with variable coefficients were dived in [16] by the bilinear analysis. Abundant exact lump and interaction lump with two types of typical local excitations for a third-order evolution equation were found in [18]. Ilhan et al. [10], Manafian and Lakestani [19] obtained some lump and interaction solutions of a variable-coefficient Kadomtsev-Petviashvili equation and a (2+1)-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation, respectively, by using the Hirota bilinear method and so on.

A (3+1)-dimensional generalized breaking soliton equation is given as

$$\begin{cases} \omega_t + \alpha u_{xxx} + \beta u_{xxy} + \gamma u u_x + \lambda u u_y + \delta u_x v = 0, \\ \omega_x = u_x + u_y + u_z, \quad u_y = v_x, \end{cases} \tag{1.1}$$

which is proposed by Gai et al. in [4], where $u = u(x, y, z, t), v = v(x, y, z, t)$ and $\omega = \omega(x, y, z, t)$ are real functions of x, y, z and t, $\alpha, \beta, \gamma, \lambda$ and δ are the relevant parameters. Eq. (1.1) is a generalization of the following (2+1)-dimensional generalized breaking soliton equation [5, 21, 30, 31, 34, 36]:

$$\begin{cases} u_t + \alpha u_{xxx} + \beta u_{xxy} + \gamma u u_x + \lambda u u_y + \delta u_x v = 0, \\ u_y = v_x, \end{cases} \tag{1.2}$$

which has been investigated via different techniques such as the singularity analysis [31], the simplified Hirotas method [30], the bilinear Bäcklund transformation [36], the Bell's polynomials and the Hirota's bilinear method [5, 21, 34].

When $\gamma = 6\alpha$ and $\lambda = \delta = 3\beta$, Eq. (1.1) becomes the following (3+1)-dimensional generalized breaking soliton equation [4, 6]:

$$\begin{cases} \omega_t + \alpha u_{xxx} + \beta u_{xxy} + 6\alpha u u_x + 3\beta u u_y + 3\beta u_x v = 0, \\ \omega_x = u_x + u_y + u_z, \quad u_y = v_x. \end{cases} \tag{1.3}$$

Gai et al. [4] got the following Hirota bilinear form of Eq. (1.3):

$$\left(D_x D_t + D_y D_t + D_z D_t + \alpha D_x^4 + \beta D_x^3 D_y\right)(f \cdot f) = 0, \tag{1.4}$$

under the second-order logarithmic derivative transformation

$$u = 2 (\ln f)_{xx}, \quad v = 2 (\ln f)_{xy}, \tag{1.5}$$

where $D_x^4, D_x^3, D_x, D_y, D_z$ and D_t are the Hirota's bilinear differential operators [7], f is a real function of x, y, z and t. Based on the (1.4), abundant lump-type solutions, rogue wave type solutions, breather lump wave solutions and interaction solutions of Eq. (1.3) were constructed in [4]. The multiwave, multicomplexiton and positive multicomplexiton solutions of Eq. (1.3) were dived in [6].

It should be pointed out that the authors [4–6, 21, 30, 31, 34, 36] have given some interaction and rational solutions to Eqs. (1.2) and (1.3), however, they have not obtained the relevant results of Eq. (1.1). In this paper, we derive new interaction solutions of Eq. (1.1) and investigate the evolutionary behavior of the interaction solutions to get the rational solutions of the equation.

2. Bilinear form of Eq. (1.1)

Eq. (1.1) can be rewritten as

$$\partial_x^{-1}\left(u_{xt}+u_{yt}+u_{zt}\right)+\alpha u_{xxx}+\beta u_{xxy}+\gamma u u_x+\lambda u u_y+\delta u_x \partial_x^{-1} u_y=0. \tag{2.1}$$

Let $\delta=\lambda=\frac{\beta\gamma}{2\alpha}$ and $u=u_0+\phi_x$, then Eq. (2.1) becomes

$$\begin{aligned}&\phi_{xt}+\phi_{yt}+\phi_{zt}+\alpha\phi_{xxxx}+\beta\phi_{xxxy}+\gamma u_0\phi_{xx}+\frac{\gamma}{2}\left((\phi_x)^2\right)_x\\&+\frac{\beta\gamma}{2\alpha}u_0\phi_{xy}+\frac{\beta\gamma}{2\alpha}\left(\phi_x\phi_y\right)_x=0.\end{aligned} \tag{2.2}$$

Using transformation $\phi=\frac{12\alpha}{\gamma}\left(\ln f\right)_x$, i.e.

$$u=u_0+\frac{12\alpha}{\gamma}\left(\ln f\right)_{xx}, \tag{2.3}$$

where $f=f(x,y,z,t)$ is a real function of x,y,z and t, Eq. (2.2) becomes

$$\begin{aligned}&(\ln f)_{xxt}+(\ln f)_{xyt}+(\ln f)_{xzt}+\alpha\left(\ln f\right)_{xxxxx}+\beta\left(\ln f\right)_{xxxxy}+\gamma u_0\left(\ln f\right)_{xxx}\\&+6\alpha\left(\left((\ln f)_{xx}\right)^2\right)_x+\frac{\beta\gamma}{2\alpha}u_0\left(\ln f\right)_{xxy}+6\beta\left((\ln f)_{xx}\left(\ln f\right)_{xy}\right)_x=0.\end{aligned} \tag{2.4}$$

Integrating (2.4) once with respect to x, we have

$$\begin{aligned}&(\ln f)_{xt}+(\ln f)_{yt}+(\ln f)_{zt}+\alpha\left(\ln f\right)_{xxxx}+\beta\left(\ln f\right)_{xxxy}+\gamma u_0\left(\ln f\right)_{xx}\\&+6\alpha\left((\ln f)_{xx}\right)^2+\frac{\beta\gamma}{2\alpha}u_0\left(\ln f\right)_{xy}+6\beta\left(\ln f\right)_{xx}\left(\ln f\right)_{xy}=0.\end{aligned} \tag{2.5}$$

From (2.5), we obtain the Hirota bilinear form of Eq. (1.1) as follows:

$$\left(\gamma u_0 D_x^2+\frac{\beta\gamma}{2\alpha}u_0 D_x D_y+D_xD_t+D_yD_t+D_zD_t+\alpha D_x^4+\beta D_x^3 D_y\right)(f\cdot f)=0. \tag{2.6}$$

Remark 2.1. The (1.4) is a special case of the (2.6) as $u_0=0$ and $\gamma=6\alpha$.

3. Interaction solutions and their evolutionary behavior for Eq. (1.1)

In order to search the interaction solutions of Eq. (1.1), we take

$$f=a_0+a_1\mathrm{e}^{\xi_1}+a_2\cos\left(\xi_2\right)+a_3\cosh\left(\xi_3\right), \tag{3.1}$$

where $\xi_i = p_i(b_{i1}x+b_{i2}y+b_{i3}z+b_{i4}t+b_{i5})$, p_1 is a real number, p_2, p_3 are real numbers or pure imaginary numbers, b_{ij} $(i = 1, 2, 3, j = 1, 2, 3, 4, 5)$ and a_k $(k = 0, 1, 2, 3)$ are are real numbers to be determined later.

Substituting (3.1) into (2.6) and equating all coefficients of $\sin(\xi_2)e^{\xi_1}$, $\cos(\xi_2)e^{\xi_1}$, $\sinh(\xi_3)e^{\xi_1}$, $\cosh(\xi_3)e^{\xi_1}$, e^{ξ_1}, $\sin(\xi_2)\sinh(\xi_3)$, $\cos(\xi_2)\cosh(\xi_3)$, $\cos(\xi_2)$, $\cosh(\xi_3)$ and the constant term to zero, we obtain

$$\left\{\begin{aligned}&a_3 = 0, b_{12} = -\frac{\alpha b_{11}}{\beta}, b_{13} = -\frac{b_{11}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})}{2\beta b_{24}}, b_{14} = \frac{b_{11}b_{24}}{b_{21}},\\ &b_{22} = -\frac{\alpha b_{21}}{\beta}, b_{23} = -\frac{b_{21}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})}{2\beta b_{24}},\end{aligned}\right\} \tag{3.2}$$

with $\beta b_{21} b_{24} \neq 0$.

$$\left\{\begin{aligned}&a_3 = 0, b_{14} = \frac{p_2 b_{24}\sqrt{2}}{p_1}, b_{21} = -\frac{\beta b_{22}}{\alpha}, b_{23} = -\frac{b_{22}(2\alpha^2 b_{24} + \beta^2\gamma u_0 b_{22} - 2\alpha\beta b_{24})}{2\alpha^2 b_{24}},\\ &b_{11} = -\frac{p_2\beta b_{22}\sqrt{2}}{p_1\alpha}, b_{12} = \frac{p_2 b_{22}\sqrt{2}}{p_1}, b_{13} = \frac{p_2 b_{22}(2\alpha^2 b_{24} + \beta^2\gamma u_0 b_{22} - 2\alpha\beta b_{24})\sqrt{2}}{2p_1\alpha^2 b_{24}},\end{aligned}\right\} \tag{3.3}$$

with $p_1\alpha b_{24} \neq 0$.

$$\left\{a_1 = 0, b_{22} = -\frac{\alpha b_{21}}{\beta}, b_{23} = \frac{(\alpha-\beta)b_{21}}{\beta}, b_{32} = -\frac{\alpha b_{31}}{\beta}, b_{33} = \frac{(\alpha-\beta)b_{31}}{\beta}, u_0 = 0,\right\} \tag{3.4}$$

with $\beta \neq 0$.

$$\left\{\begin{aligned}&a_1 = 0, b_{22} = -\frac{\alpha b_{21}}{\beta}, b_{23} = -\frac{b_{21}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})}{2\beta b_{24}}, b_{32} = -\frac{\alpha b_{31}}{\beta},\\ &b_{33} = -\frac{b_{31}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})}{2\beta b_{24}}, b_{34} = \frac{b_{24}b_{31}}{b_{21}},\end{aligned}\right\} \tag{3.5}$$

with $\beta b_{21} b_{24} \neq 0$.

$$\left\{\begin{aligned}&a_0 = 0, b_{12} = -\frac{\alpha b_{11}}{\beta}, b_{13} = \frac{(\alpha-\beta)b_{11}}{\beta}, b_{22} = -\frac{\alpha b_{21}}{\beta}, b_{23} = \frac{(\alpha-\beta)b_{21}}{\beta},\\ &b_{32} = -\frac{\alpha b_{31}}{\beta}, b_{33} = \frac{(\alpha-\beta)b_{31}}{\beta}, u_0 = 0,\end{aligned}\right\} \tag{3.6}$$

with $\beta \neq 0$.

Case 1.

Let p_1 and $p_2 \in \mathbb{R}$, then using (2.3), (3.1) and (3.2), we obtain a interaction solution of Eq. (1.1) as follows:

$$u = u_0 + \frac{12\alpha}{\gamma}\left(\frac{p_1^2 a_1 b_{11}^2 e^{\eta_1} - p_2^2 a_2 b_{21}^2 \cos(\eta_2)}{a_0 + a_1 e^{\eta_1} + a_2\cos(\eta_2)} - \left(\frac{p_1 a_1 b_{11} e^{\eta_1} - p_2 a_2 b_{21}\sin(\eta_2)}{a_0 + a_1 e^{\eta_1} + a_2\cos(\eta_2)}\right)^2\right), \tag{3.7}$$

where

$$\eta_1 = p_1\left(b_{11}x - \frac{\alpha b_{11}}{\beta}y - \frac{b_{11}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})}{2\beta b_{24}}z + \frac{b_{11}b_{24}}{b_{21}}t + b_{15}\right),$$

$$\eta_2 = p_2\left(b_{21}x - \frac{\alpha b_{21}}{\beta}y - \frac{b_{21}(-2\alpha b_{24}+2\beta b_{24}+\beta\gamma u_0 b_{21})}{2\beta b_{24}}z + b_{24}t + b_{25}\right),$$

$\alpha, \beta(\neq 0), \gamma(\neq 0), u_0, a_0, a_1, a_2, b_{11}, b_{15}, b_{21}(\neq 0), b_{24}(\neq 0)$ and $b_{25} \in \mathbb{R}$.

To emerge rational solution from (3.7), the parameters a_i $(i = 0, 1, 2)$ need satisfy [27]:

$$\lim_{p_j \to 0}(a_0 + a_1 + a_2) = 0, \ \ j = 1, 2.$$

Therefore, if taking $p_2 = p_1, a_0 = -2, a_1 = 1, a_2 = \cos(p_2)$ and $p_1 \to 0$ in (3.7), then solution (3.7) emerges the following second-order rational solution:

$$u = u_0 - \frac{12\alpha}{\gamma}\left(\frac{G_1}{H_1}\right)^2, \tag{3.8}$$

where $G_1 = 2\beta b_{11}b_{21}b_{24}$, $H_1 = -2\beta b_{11}b_{21}b_{24}x + 2\alpha b_{11}b_{21}b_{24}y + b_{11}b_{21}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})z - 2\beta b_{11}b_{24}^2 t - 2\beta b_{15}b_{21}b_{24}$.

Selecting $\alpha = 1, \beta = 1, \gamma = 1, u_0 = 1$, $b_{11} = 1, b_{15} = 1, b_{21} = 2, b_{24} = -1, z = 1$ and $t = 1$, three-dimensional and contour plots of (3.8) are shown in Figure 1. Selecting $\alpha = 1, \beta = 1$, $\gamma = 1, u_0 = 1$, $a_0 = -2, a_1 = 1$, $b_{11} = 1, b_{15} = 1, b_{21} = 2, b_{24} = -1, b_{25} = 1, z = 1$ and $t = 1$, the limiting process of (3.7) is similar to that in Figure 2.

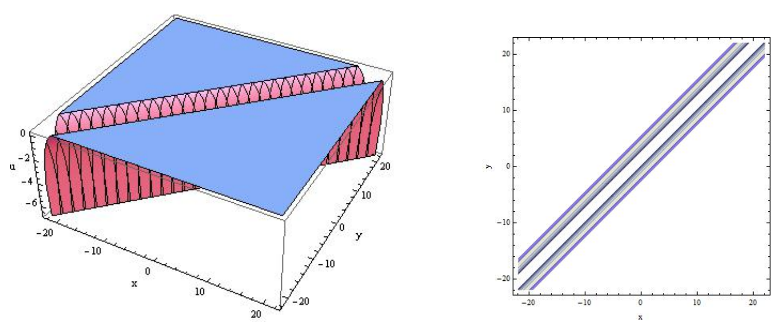


Figure 1. Three-dimensional and contour plots of (3.8).

Let $p_1 \in \mathbb{R}$ and $p_2 = \bar{p}_2 I$, here $\bar{p}_2 \in \mathbb{R}, I = \sqrt{-1}$, then using (2.3), (3.1) and (3.2), we obtain a interaction solution of Eq. (1.1) as follows:

$$u = u_0 + \frac{12\alpha}{\gamma}\left(\frac{p_1^2 a_1 b_{11}^2 e^{\eta_1} + \bar{p}_2^2 a_2 b_{21}^2 \cosh(\eta_3)}{a_0 + a_1 e^{\eta_1} + a_2\cosh(\eta_3)} - \left(\frac{p_1 a_1 b_{11} e^{\eta_1} + \bar{p}_2 a_2 b_{21}\sinh(\eta_3)}{a_0 + a_1 e^{\eta_1} + a_2\cosh(\eta_3)}\right)^2\right), \tag{3.9}$$

where $\eta_3 = \bar{p}_2\left(b_{21}x - \frac{\alpha b_{21}}{\beta}y - \frac{b_{21}(-2\alpha b_{24}+2\beta b_{24}+\beta\gamma u_0 b_{21})}{2\beta b_{24}}z + b_{24}t + b_{25}\right)$ and η_1 is given in (3.7). Moreover, if taking $\bar{p}_2 = p_1, a_0 = -2, a_1 = 1, a_2 = \cosh(\bar{p}_2)$ and $p_1 \to 0$ in (3.9), then solution (3.9) emerges the second-order rational solution (3.8).

Selecting $\alpha = 1, \beta = 1$, $\gamma = 1, u_0 = 1$, $a_0 = -2, a_1 = 1$, $b_{11} = 1, b_{15} = 1, b_{21} = 2, b_{24} = -1, b_{25} = 1, z = 1$ and $t = 1$, the limiting process of (3.9) is similar to that in Figure 3.

Case 2.

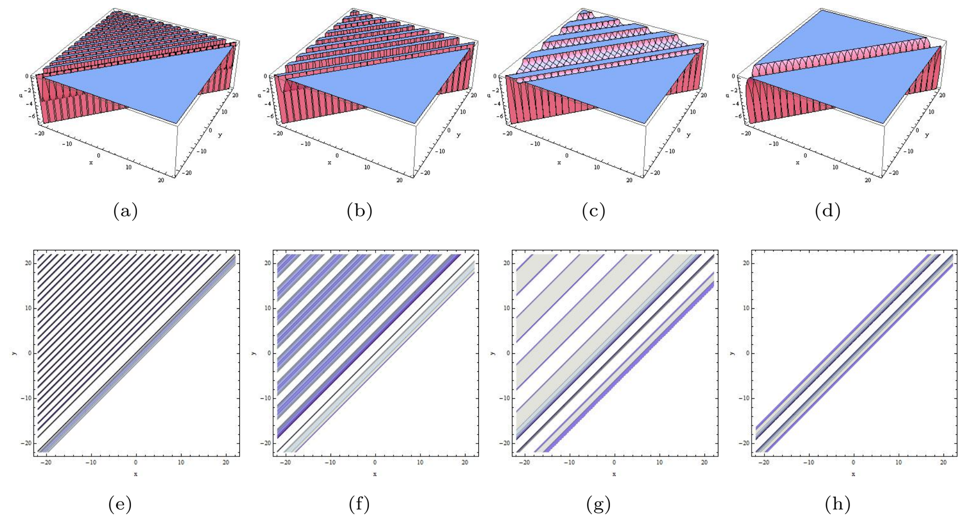


Figure 2. The limiting precess of (3.7) tends to (3.8) when $p_1 \to 0$. Parameters: $p_1 = 1, p_2 = 1, a_2 = \cos(1)$ in (a) and (e), $p_1 = 0.6, p_2 = 0.6, a_2 = \cos(0.6)$ in (b) and (f), $p_1 = 0.3, p_2 = 0.3, a_2 = \cos(0.3)$ in (c) and (g), $p_1 = 0.01, p_2 = 0.01, a_2 = \cos(0.01)$ in (d) and (h), respectively.

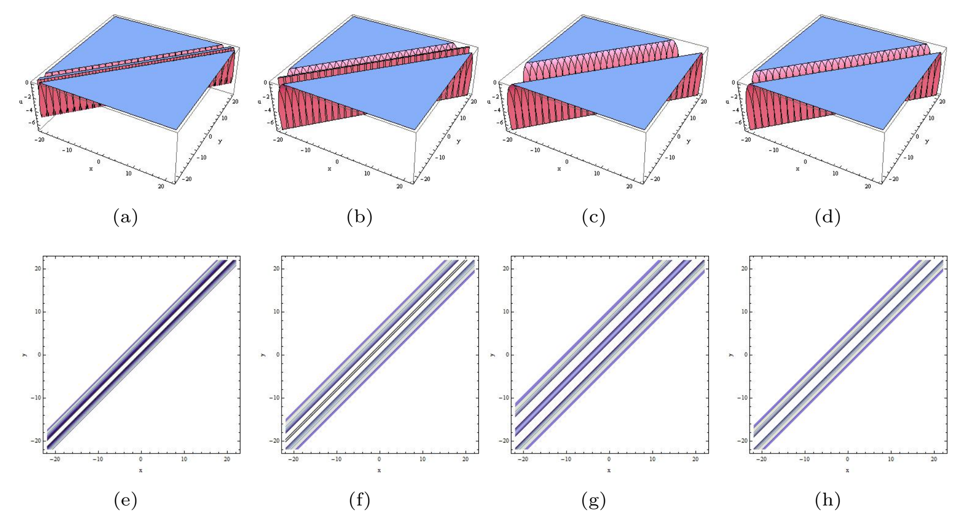


Figure 3. The limiting precess of (3.9) tends to (3.8) when $p_1 \to 0$. Parameters: $p_1 = 0.6, \bar{p}_2 = 0.6, a_2 = \cosh(0.6)$ in (a) and (e), $p_1 = 0.2, \bar{p}_2 = 0.2, a_2 = \cosh(0.2)$ in (b) and (f), $p_1 = 0.07, \bar{p}_2 = 0.07, a_2 = \cosh(0.07)$ in (c) and (g), $p_1 = 0.001, \bar{p}_2 = 0.001, a_2 = \cosh(0.001)$ in (d) and (h), respectively.

Let $p_1(\neq 0)$ and $p_2 \in \mathbb{R}$, then using (2.3), (3.1) and (3.3), we obtain a interaction solution of Eq. (1.1) as follows:

$$u = u_0 + \frac{12p_2^2\beta^2 b_{22}^2}{\alpha\gamma}\left(\frac{2a_1\mathrm{e}^{\eta_4} - a_2\cos(\eta_5)}{a_0 + a_1\mathrm{e}^{\eta_4} + a_2\cos(\eta_5)} - \left(\frac{a_1\sqrt{2}\mathrm{e}^{\eta_4} - a_2\sin(\eta_5)}{a_0 + a_1\mathrm{e}^{\eta_4} + a_2\cos(\eta_5)}\right)^2\right), \tag{3.10}$$

where

$$\eta_4 = p_1\left(-\tfrac{p_2\beta b_{22}\sqrt{2}}{p_1\alpha}x + \tfrac{p_2 b_{22}\sqrt{2}}{p_1}y + \tfrac{p_2 b_{22}(2\alpha^2 b_{24}+\beta^2\gamma u_0 b_{22}-2\alpha\beta b_{24})\sqrt{2}}{2p_1\alpha^2 b_{24}}z + \tfrac{p_2 b_{24}\sqrt{2}}{p_1}t + b_{15}\right),$$

$$\eta_5 = p_2\left(-\frac{\beta b_{22}}{\alpha}x + b_{22}y - \frac{b_{22}(2\alpha^2 b_{24}+\beta^2\gamma u_0 b_{22}-2\alpha\beta b_{24})}{2\alpha^2 b_{24}}z + b_{24}t + b_{25}\right),$$

$\alpha(\neq 0), \beta, \gamma(\neq 0), u_0, a_0, a_1, a_2, b_{15}, b_{22}, b_{24}(\neq 0)$ and $b_{25} \in \mathbb{R}$.

Moreover, if taking $p_2 = p_1, a_0 = -2, a_1 = 1, a_2 = \cos(p_2)$ and $p_1 \to 0$ in (3.10), then solution (3.10) emerges the following second-order rational solution:

$$u = u_0 - \frac{12\alpha}{\gamma}\left(\frac{G_2}{H_2}\right)^2, \tag{3.11}$$

where $G_2 = 2\alpha\beta b_{22}b_{24}$, $H_2 = -2\alpha\beta b_{22}b_{24}x + 2\alpha^2 b_{22}b_{24}y + b_{22}(2\alpha^2 b_{24} - 2\alpha\beta b_{24} + \beta^2\gamma u_0 b_{22})z + 2\alpha^2 b_{24}^2 t + \alpha^2 b_{15}b_{24}\sqrt{2}$.

Selecting $\alpha = 1, \beta = 1, \gamma = 1, u_0 = 1,\ b_{15} = 1, b_{22} = 1, b_{24} = 1, z = 1$ and $t = 9$, three-dimensional and contour plots of (3.11) are shown in Figure 4. Selecting $\alpha = 1, \beta = 1,\ \gamma = 1, u_0 = 1,\ a_0 = -2, a_1 = 1,\ b_{15} = 1, b_{22} = 1, b_{24} = 1, b_{25} = 1, z = 1$ and $t = 9$, the limiting process of (3.10) is similar to that in Figure 5.

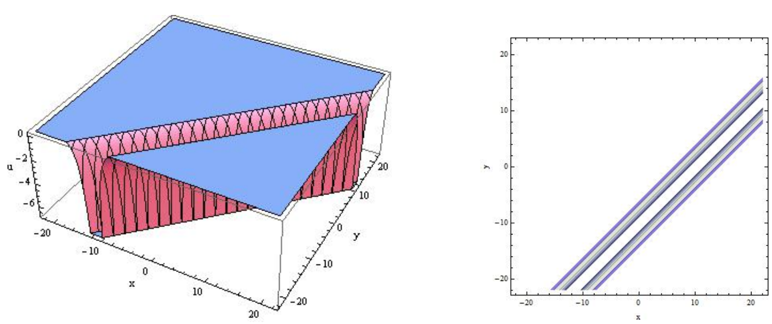


Figure 4. Three-dimensional and contour plots of (3.11).

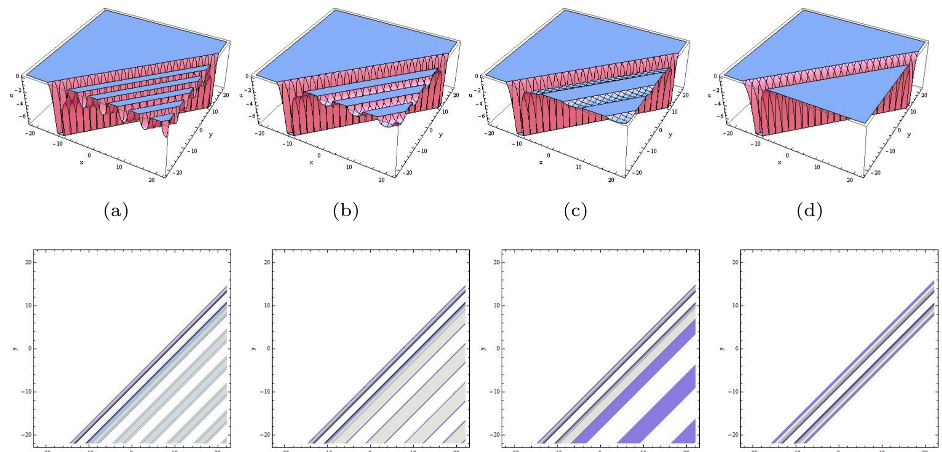
Case 3.

Let p_2 and $p_3 \in \mathbb{R}$, then using (2.3), (3.1) and (3.4), we obtain a interaction solution of Eq. (1.1) as follows:

$$u = -\tfrac{12\alpha}{\gamma}\left(\tfrac{p_2^2 a_2 b_{21}^2\cos(\eta_6) - p_3^2 a_3 b_{31}^2\cosh(\eta_7)}{a_0 + a_2\cos(\eta_6) + a_3\cosh(\eta_7)} + \left(\tfrac{p_2 a_2 b_{21}\sin(\eta_6) - p_3 a_3 b_{31}\sinh(\eta_7)}{a_0 + a_2\cos(\eta_6) + a_3\cosh(\eta_7)}\right)^2\right), \tag{3.12}$$

where $\eta_6 = p_2\left(b_{21}x - \tfrac{\alpha b_{21}}{\beta}y + \tfrac{(\alpha-\beta)b_{21}}{\beta}z + b_{24}t + b_{25}\right)$, $\eta_7 = p_3\left(b_{31}x - \tfrac{\alpha b_{31}}{\beta}y + \tfrac{(\alpha-\beta)b_{31}}{\beta}z + b_{34}t + b_{35}\right)$, $\alpha, \beta(\neq 0), \gamma(\neq 0)$, a_0, a_2, a_3, $b_{21}, b_{24}, b_{25}, b_{31}, b_{34}$ and $b_{35} \in \mathbb{R}$. Moreover, if taking $p_3 = p_2, a_0 = -2, a_2 = \cos(p_2), a_3 = \cosh(p_3)$ and $p_2 \to 0$ in (3.12), then solution (3.12) emerges the following fourth-order rational solution:

$$u = \frac{12\alpha}{\gamma}\left(\frac{G_{31}}{H_3} - \left(\frac{G_{32}}{H_3}\right)^2\right), \tag{3.13}$$



(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 5. The limiting precess of (3.10) tends to (3.11) when $p_1 \to 0$. Parameters: $p_1 = 1, p_2 = 1, a_2 = \cos(1)$ in (a) and (e), $p_1 = 0.7, p_2 = 0.7, a_2 = \cos(0.7)$ in (b) and (f), $p_1 = 0.5, p_2 = 0.5, a_2 = \cos(0.5)$ in (c) and (g), $p_1 = 0.05, p_2 = 0.05, a_2 = \cos(0.05)$ in (d) and (h), respectively.

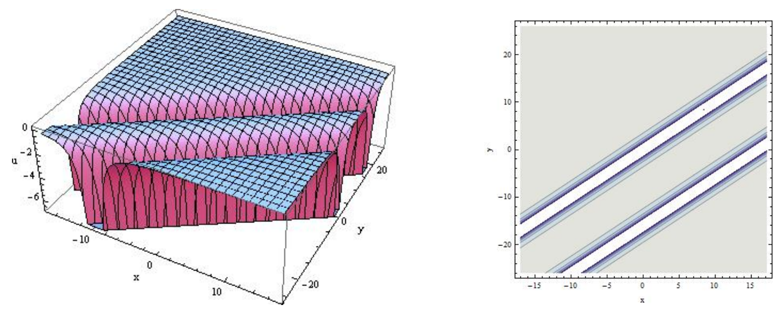


Figure 6. Three-dimensional and contour plots of (3.13).

where $G_{31} = 2\beta^2(b_{21}^2 - b_{31}^2)$, $G_{32} = 2\beta((b_{21}^2 - b_{31}^2)(\beta x - \alpha y + (\alpha-\beta)z) + \beta((b_{21}b_{24} - b_{31}b_{34})t + b_{21}b_{25} - b_{31}b_{35}))$, $H_3 = ((b_{21} + b_{31})(\beta x - \alpha y + (\alpha - \beta)z) + \beta((b_{24} + b_{34})t + b_{25} + b_{35})) \times ((b_{21} - b_{31})(\beta x - \alpha y + (\alpha - \beta)z) + \beta((b_{24} - b_{34})t + b_{25} - b_{35}))$.

Selecting $\alpha = 1, \beta = 1, \gamma = 1$, $b_{21} = 1, b_{24} = 1, b_{25} = 1, b_{31} = -2, b_{34} = 1, b_{35} = 1, z = 1$ and $t = 7$, three-dimensional and contour plots of (3.13) are shown in Figure 6. Selecting $\alpha = 1, \beta = 1$, $\gamma = 1$, $a_0 = -2$, $b_{21} = 1, b_{24} = 1, b_{25} = 1, b_{31} = -2, b_{34} = 1, b_{35} = 1, z = 1$, and $t = 7$, the limiting process of (3.12) is similar to that in Figure 7.

Let $p_3 \in \mathbb{R}$ and $p_2 = \bar{p}_2 I$, here $\bar{p}_2 \in \mathbb{R}, I = \sqrt{-1}$, then using (2.3), (3.1) and (3.4), we obtain a interaction solution of Eq. (1.1) as follows:

$$u = \frac{12\alpha}{\gamma}\left(\frac{\bar{p}_2^2 a_2 b_{21}^2 \cosh(\eta_8) + p_3^2 a_3 b_{31}^2 \cosh(\eta_7)}{a_0 + a_2\cosh(\eta_8) + a_3\cosh(\eta_7)} - \left(\frac{\bar{p}_2 a_2 b_{21}\sinh(\eta_8) + p_3 a_3 b_{31}\sinh(\eta_7)}{a_0 + a_2\cosh(\eta_8) + a_3\cosh(\eta_7)}\right)^2\right), \tag{3.14}$$

where $\eta_8 = \bar{p}_2\left(b_{21}x - \frac{\alpha b_{21}}{\beta}y + \frac{(\alpha-\beta)b_{21}}{\beta}z + b_{24}t + b_{25}\right)$ and η_7 is given in (3.12). Moreover, if taking $p_3 = \bar{p}_2, a_0 = -2, a_2 = \cosh(\bar{p}_2), a_3 = \cosh(p_3)$ and $\bar{p}_2 \to 0$ in

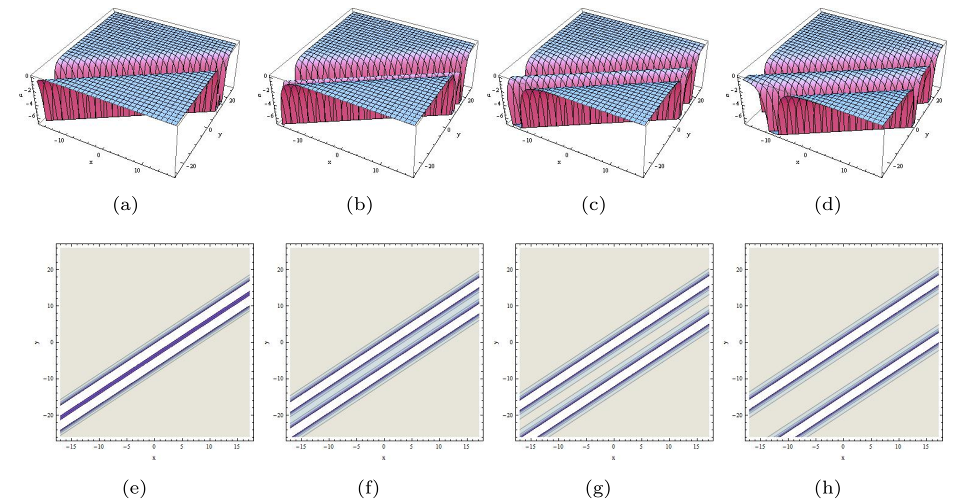


Figure 7. The limiting precess of (3.12) tends to (3.13) when $p_2 \to 0$. Parameters: $p_2 = 0.35, p_3 = 0.35, a_2 = \cos(0.35), a_3 = \cosh(0.35)$ in (a) and (e), $p_2 = 0.22, p_3 = 0.22, a_2 = \cos(0.22), a_3 = \cosh(0.22)$ in (b) and (f), $p_2 = 0.12, p_3 = 0.12, a_2 = \cos(0.12), a_3 = \cosh(0.12)$ in (c) and (g), $p_2 = 0.01, p_3 = 0.01, a_2 = \cos(0.01), a_3 = \cosh(0.01)$ in (d) and (h), respectively.

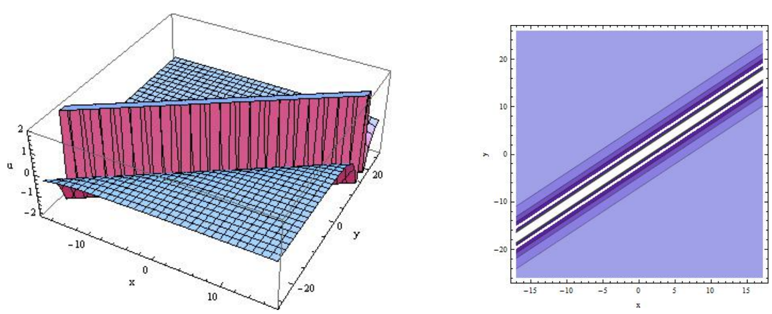


Figure 8. Three-dimensional and contour plots of (3.15).

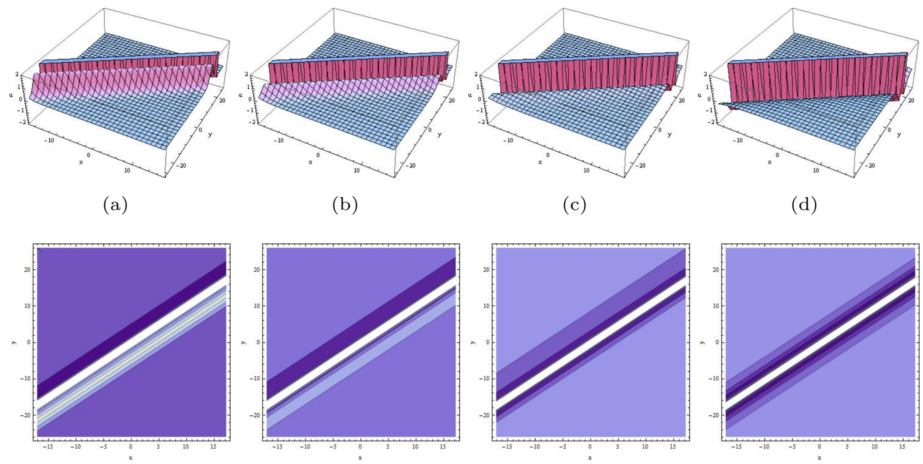
(3.14), then solution (3.14) emerges the following fourth-order rational solution:

$$u = \frac{12\alpha}{\gamma}\left(\frac{\bar{G}_{31}}{\bar{H}_3} - \left(\frac{\bar{G}_{32}}{\bar{H}_3}\right)^2\right), \tag{3.15}$$

where $\bar{G}_{31} = 2\beta^2(b_{21}^2 + b_{31}^2)$, $\bar{G}_{32} = (2\beta((b_{21}^2 + b_{31}^2)(\beta x - \alpha y + (\alpha - \beta)z) + \beta((b_{21}b_{24} + b_{31}b_{34})t + b_{21}b_{25} + b_{31}b_{35})))^2$, $\bar{H}_3 = \beta^2(b_{21}^2 + b_{31}^2)x^2 + 2\beta((b_{21}^2 + b_{31}^2)(-\alpha y + (\alpha - \beta)z) + \beta((b_{21}b_{24} + b_{31}b_{34})t + b_{21}b_{25} + b_{31}b_{35}))x + \alpha^2(b_{21}^2 + b_{31}^2)y^2 - 2\alpha((\alpha - \beta)(b_{21}^2 + b_{31}^2)z + \beta((b_{21}b_{24} + b_{31}b_{34})t + b_{21}b_{25} + b_{31}b_{35}))y + (\alpha - \beta)^2(b_{21}^2 + b_{31}^2)z^2 + (2\beta(\alpha - \beta)((b_{21}b_{24} + b_{31}b_{34})t + b_{21}b_{25} + b_{31}b_{35}))z + \beta^2((b_{24}^2 + b_{34}^2)t^2 + 2(b_{24}b_{25} + b_{34}b_{35})t + b_{25}^2 + b_{35}^2)$.

Selecting $\alpha = 1, \beta = 1, \gamma = 1, b_{21} = 1, b_{24} = 1, b_{25} = 1, b_{31} = -2, b_{34} = 1, b_{35} = 1, z = 1$ and $t = 1$, three-dimensional and contour plots of (3.15) are shown in Figure 8. Selecting $\alpha = 1, \beta = 1, \gamma = 1, a_0 = -2, b_{21} = 1, b_{24} = 1, b_{25} = 1, b_{31} = -2, b_{34} = 1, b_{35} = 1, z = 1$ and $t = 1$, the limiting process of (3.14) is similar to that in Figure 9.

Let $p_2 \in \mathbb{R}$ and $p_3 = \bar{p}_3 I$, here $\bar{p}_3 \in \mathbb{R}, I = \sqrt{-1}$, then using (2.3), (3.1) and



(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 9. The limiting precess of (3.14) tends to (3.15) when $\bar{p}_2 \to 0$. Parameters: $\bar{p}_2 = 0.77, p_3 = 0.77, a_2 = \cosh(0.77), a_3 = \cosh(0.77)$ in (a) and (e), $\bar{p}_2 = 0.6, p_3 = 0.6, a_2 = \cosh(0.6), a_3 = \cosh(0.6)$ in (b) and (f), $\bar{p}_2 = 0.4, p_3 = 0.4, a_2 = \cosh(0.4), a_3 = \cosh(0.4)$ in (c) and (g), $\bar{p}_2 = 0.02, p_3 = 0.02, a_2 = \cosh(0.02), a_3 = \cosh(0.02)$ in (d) and (h), respectively.

(3.4), we obtain a interaction solution of Eq. (1.1) as follows:

$$u = -\frac{12\alpha}{\gamma}\left(\frac{p_2^2 a_2 b_{21}^2 \cos(\eta_6)+\bar{p}_3^2 a_3 b_{31}^2 \cos(\eta_9)}{a_0+a_2\cos(\eta_6)+a_3\cos(\eta_9)} + \left(\frac{p_2 a_2 b_{21}\sin(\eta_6)+\bar{p}_3 a_3 b_{31}\sin(\eta_9)}{a_0+a_2\cos(\eta_6)+a_3\cos(\eta_9)}\right)^2\right), \tag{3.16}$$

where $\eta_9 = \bar{p}_3\left(b_{31}x - \frac{\alpha b_{31}}{\beta}y + \frac{(\alpha-\beta)b_{31}}{\beta}z + b_{34}t + b_{35}\right)$ and η_6 is given in (3.12). Moreover, if taking $\bar{p}_3 = p_2, a_0 = -2, a_2 = \cos(p_2), a_3 = \cos(\bar{p}_3)$ and $p_2 \to 0$ in (3.16), then solution (3.16) emerges the fourth-order rational solution (3.15).

Selecting $\alpha = 1, \beta = 1$, $\gamma = 1$, $a_0 = -2$, $b_{21} = 1, b_{24} = 1, b_{25} = 1, b_{31} = -2, b_{34} = 1, b_{35} = 1, z = 1$ and $t = 1$, the limiting process of (3.16) is similar to that in Figure 10.

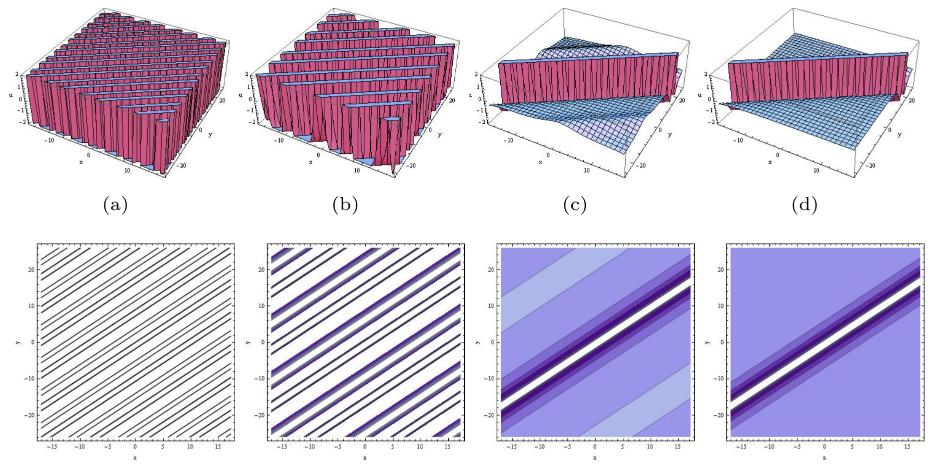
Case 4.

Let p_2 and $p_3 \in \mathbb{R}$, then using (2.3), (3.1) and (3.5), we obtain a interaction solution of Eq. (1.1) as follows:

$$\begin{aligned} u =& u_0 - \frac{12\alpha}{\gamma}\left(\frac{p_2^2 a_2 b_{21}^2\cos(\eta_{10}) - p_3^2 a_3 b_{31}^2\cosh(\eta_{11})}{a_0 + a_2\cos(\eta_{10}) + a_3\cosh(\eta_{11})}\right. \\ &\left. + \left(\frac{p_2 a_2 b_{21}\sin(\eta_{10}) - p_3 a_3 b_{31}\sinh(\eta_{11})}{a_0 + a_2\cos(\eta_{10}) + a_3\cosh(\eta_{11})}\right)^2\right), \end{aligned} \tag{3.17}$$

where

$$\eta_{10} = p_2\left(b_{21}x - \frac{\alpha b_{21}}{\beta}y - \frac{b_{21}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})}{2\beta b_{24}}z + b_{24}t + b_{25}\right),$$
$$\eta_{11} = p_3\left(b_{31}x - \frac{\alpha b_{31}}{\beta}y - \frac{b_{31}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})}{2\beta b_{24}}z + \frac{b_{24}b_{31}}{b_{21}}t + b_{35}\right),$$



(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 10. The limiting precess of (3.16) tends to (3.15) when $p_2 \to 0$. Parameters: $p_2 = 0.7, \bar{p}_3 = 0.7, a_2 = \cos(0.7), a_3 = \cos(0.7)$ in (a) and (e), $p_2 = 0.4, \bar{p}_3 = 0.4, a_2 = \cos(0.4), a_3 = \cos(0.4)$ in (b) and (f), $p_2 = 0.12, \bar{p}_3 = 0.12, a_2 = \cos(0.12), a_3 = \cos(0.12)$ in (c) and (g), $p_2 = 0.02, \bar{p}_3 = 0.02, a_2 = \cos(0.02), a_3 = \cos(0.02)$ in (d) and (h), respectively.

$\alpha, \beta(\neq 0), \gamma(\neq 0), u_0, a_0, a_2, a_3, b_{21}, b_{24}, b_{25}, b_{31}$ and $b_{35} \in \mathbb{R}$. Moreover, if taking $p_3 = p_2, a_0 = -2, a_2 = \cos(p_2), a_3 = \cosh(p_3)$ and $p_2 \to 0$ in (3.17), then solution (3.17) emerges the following fourth-order rational solution:

$$u = u_0 + \frac{12\alpha}{\gamma}\left(\frac{G_{41}}{H_4} - \left(\frac{G_{42}}{H_4}\right)^2\right), \tag{3.18}$$

where $G_{41} = 8\beta^2 b_{21}^2 b_{24}^2(b_{21}^2 - b_{31}^2)$, $G_{42} = -4\beta b_{21}b_{24}((b_{21}^2 - b_{31}^2)(-2\beta b_{21}b_{24}x + 2\alpha b_{21}b_{24}y + (-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})b_{21}z - 2\beta b_{24}^2 t) - 2\beta b_{21}b_{24}(b_{21}b_{25} - b_{31}b_{35}))$, $H_4 = ((b_{21} + b_{31})(-2\beta b_{21}b_{24}x + 2\alpha b_{21}b_{24}y + (-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})b_{21}z - 2\beta b_{24}^2 t) - 2\beta b_{21}b_{24}(b_{25} + b_{35})) \times ((b_{21} - b_{31})(-2\beta b_{21}b_{24}x + 2\alpha b_{21}b_{24}y + (-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})b_{21}z - 2\beta b_{24}^2 t) - 2\beta b_{21}b_{24}(b_{25} - b_{35}))$.

Selecting $\alpha = 1, \beta = 1, \gamma = 1, u_0 = 1, b_{21} = 1, b_{24} = 1, b_{25} = 1, b_{31} = -2, b_{35} = 1, z = 1$ and $t = 15$, three-dimensional and contour plots of (3.18) are shown in Figure 11. Selecting $\alpha = 1, \beta = 1, \gamma = 1, u_0 = 1, a_0 = -2, b_{21} = 1, b_{24} = 1, b_{25} = 1, b_{31} = -2, b_{35} = 1, z = 1$ and $t = 15$, the limiting process of (3.17) is similar to that in Figure 12.

Let $p_3 \in \mathbb{R}$ and $p_2 = \bar{p}_2 I$, here $\bar{p}_2 \in \mathbb{R}, I = \sqrt{-1}$, then using (2.3), (3.1) and (3.5), we obtain a interaction solution of Eq. (1.1) as follows:

$$\begin{aligned} u =& u_0 + \frac{12\alpha}{\gamma}\Bigg(\frac{\bar{p}_2^2 a_2 b_{21}^2 \cosh(\eta_{12}) + p_3^2 a_3 b_{31}^2 \cosh(\eta_{11})}{a_0 + a_2\cosh(\eta_{12}) + a_3\cosh(\eta_{11})} \\ & - \left(\frac{\bar{p}_2 a_2 b_{21}\sinh(\eta_{12}) + p_3 a_3 b_{31}\sinh(\eta_{11})}{a_0 + a_2\cosh(\eta_{12}) + a_3\cosh(\eta_{11})}\right)^2\Bigg), \end{aligned} \tag{3.19}$$

where $\eta_{12} = \bar{p}_2\left(b_{21}x - \frac{\alpha b_{21}}{\beta}y - \frac{b_{21}(-2\alpha b_{24}+2\beta b_{24}+\beta\gamma u_0 b_{21})}{2\beta b_{24}}z + b_{24}t + b_{25}\right)$, η_{11} is given in (3.17). Moreover, if taking $p_3 = \bar{p}_2, a_0 = -2, a_2 = \cosh(\bar{p}_2), a_3 = \cosh(p_3)$ and

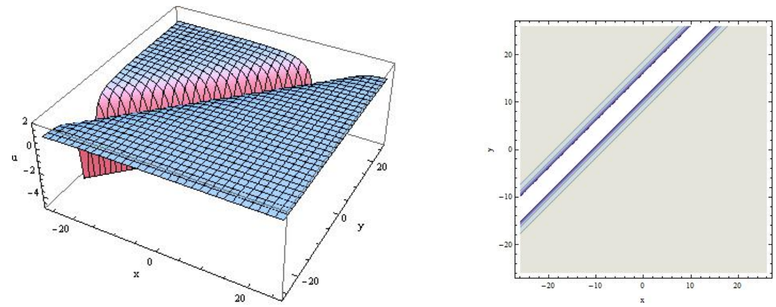


Figure 11. Three-dimensional and contour plots of (3.18).

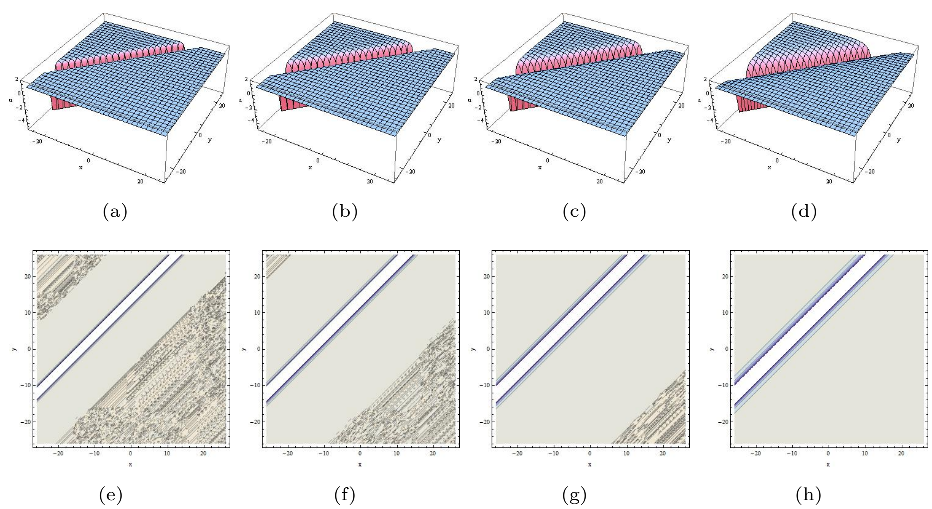


Figure 12. The limiting precess of (3.17) tends to (3.18) when $p_2 \to 0$. Parameters: $p_2 = 1, p_3 = 1, a_2 = \cos(1), a_3 = \cosh(1)$ in (a) and (e), $p_2 = 0.6, p_3 = 0.6, a_2 = \cos(0.6), a_3 = \cosh(0.6)$ in (b) and (f), $p_2 = 0.4, p_3 = 0.4, a_2 = \cos(0.4), a_3 = \cosh(0.4)$ in (c) and (g), $p_2 = 0.04, p_3 = 0.04, a_2 = \cos(0.04), a_3 = \cosh(0.04)$ in (d) and (h), respectively.

$\bar{p}_2 \to 0$ in (3.19), then solution (3.19) emerges the following fourth-order rational solution:

$$u = u_0 + \frac{12\alpha}{\gamma}\left(\frac{\bar{G}_{41}}{\bar{H}_4} - \left(\frac{\bar{G}_{42}}{\bar{H}_4}\right)^2\right), \tag{3.20}$$

where $\bar{G}_{41} = 8\beta^2 b_{21}^2 b_{24}^2(b_{21}^2 + b_{31}^2)$, $\bar{G}_{42} = -4\beta b_{21}b_{24}((b_{21}^2 + b_{31}^2)(-2\beta b_{21}b_{24}x + 2\alpha b_{21}b_{24}y + (-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})b_{21}z - 2\beta b_{24}^2 t) - 2\beta b_{21}b_{24}(b_{21}b_{25} + b_{31}b_{35}))$, $\bar{H}_4 = 4\beta^2 b_{21}^2 b_{24}^2(b_{21}^2 + b_{31}^2)x^2 + 4\beta b_{21}b_{24}((b_{21}^2 + b_{31}^2)(-2\alpha b_{21}b_{24}y - b_{21}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})z + 2\beta b_{24}^2 t) + 2\beta b_{21}b_{24}(b_{21}b_{25} + b_{31}b_{35}))x + 4\alpha^2 b_{21}^2 b_{24}^2(b_{21}^2 + b_{31}^2)y^2 + 4\alpha b_{21}b_{24}((b_{21}^2 + b_{31}^2)(b_{21}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})z - 2\beta b_{24}^2 t) - 2\beta b_{21}b_{24}(b_{21}b_{25} + b_{31}b_{35}))y + b_{21}^2(b_{21}^2 + b_{31}^2)(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})^2 z^2 - 4\beta b_{21}b_{24}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})(b_{24}(b_{21}^2 + b_{31}^2)t + b_{21}(b_{21}b_{25} + b_{31}b_{35}))z + 4\beta^2 b_{24}^2(b_{21}^2 + b_{31}^2)t^2 + 8\beta^2 b_{21}b_{24}^3(b_{21}b_{25} + b_{31}b_{35})t + 4\beta^2 b_{21}^2 b_{24}^2(b_{25}^2 + b_{35}^2)$.

Selecting $\alpha = 1, \beta = 1, \gamma = 1, u_0 = 1, b_{21} = 1, b_{24} = 1, b_{25} = 1, b_{31} = -2, b_{35} = 1, z = 1$ and $t = -5$, three-dimensional and contour plots of (3.20) are shown in Figure 13. Selecting $\alpha = 1, \beta = 1, \gamma = 1, u_0 = 1, a_0 = -2, b_{21} = 1, b_{24} = 1, b_{25} =$

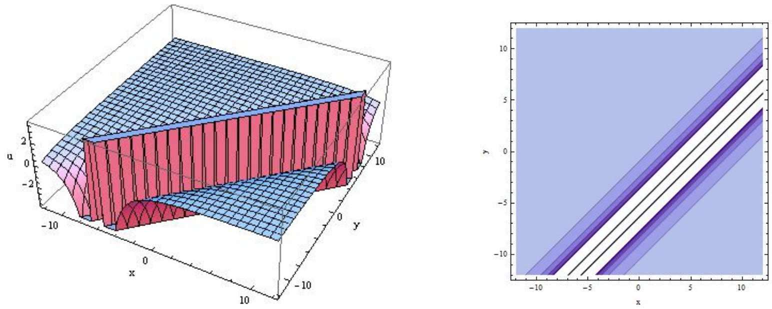


Figure 13. Three-dimensional and contour plots of (3.20).

$1, b_{31} = -2, b_{35} = 1, z = 1$ and $t = -5$, the limiting process of (3.19) is similar to that in Figure 14.

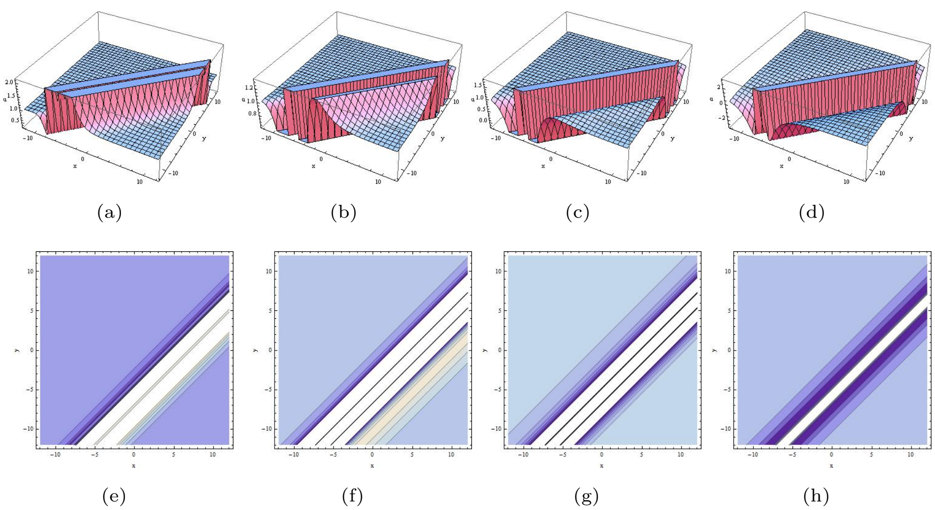


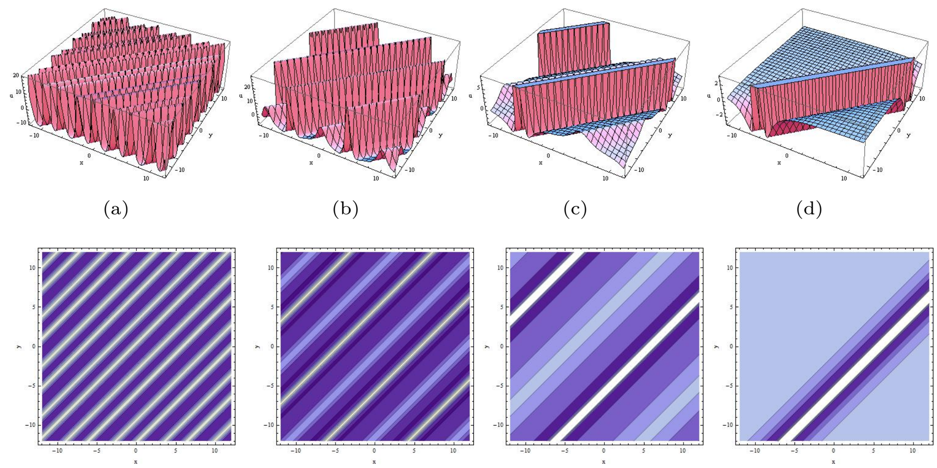
Figure 14. The limiting precess of (3.19) tends to (3.20) when $\bar{p}_2 \to 0$. Parameters: $\bar{p}_2 = 1, p_3 = 1, a_2 = \cosh(1), a_3 = \cosh(1)$ in (a) and (e), $\bar{p}_2 = 0.6, p_3 = 0.6, a_2 = \cosh(0.6), a_3 = \cosh(0.6)$ in (b) and (f), $\bar{p}_2 = 0.45, p_3 = 0.45, a_2 = \cosh(0.45), a_3 = \cosh(0.45)$ in (c) and (g), $\bar{p}_2 = 0.05, p_3 = 0.05, a_2 = \cosh(0.05), a_3 = \cosh(0.05)$ in (d) and (h), respectively.

Let $p_2 \in \mathbb{R}$ and $p_3 = \bar{p}_3 I$, here $\bar{p}_3 \in \mathbb{R}, I = \sqrt{-1}$, then using (2.3), (3.1) and (3.5), we obtain a interaction solution of Eq. (1.1) as follows:

$$\begin{aligned} u =& u_0 - \frac{12\alpha}{\gamma}\left(\frac{p_2^2 a_2 b_{21}^2 \cos(\eta_{10}) + \bar{p}_3^2 a_3 b_{31}^2 \cos(\eta_{13})}{a_0 + a_2\cos(\eta_{10}) + a_3\cos(\eta_{13})}\right. \\ &\left. + \left(\frac{p_2 a_2 b_{21}\sin(\eta_{10}) + \bar{p}_3 a_3 b_{31}\sin(\eta_{13})}{a_0 + a_2\cos(\eta_{10}) + a_3\cos(\eta_{13})}\right)^2\right), \end{aligned} \tag{3.21}$$

where $\eta_{13} = \bar{p}_3\left(b_{31}x - \frac{\alpha b_{31}}{\beta}y - \frac{b_{31}(-2\alpha b_{24} + 2\beta b_{24} + \beta\gamma u_0 b_{21})}{2\beta b_{24}}z + \frac{b_{24}b_{31}}{b_{21}}t + b_{35}\right)$, η_{10} is given in (3.17). Moreover, if taking $\bar{p}_3 = p_2, a_0 = -2, a_2 = \cos(p_2), a_3 = \cos(\bar{p}_3)$ and $p_2 \to 0$ in (3.21), then solution (3.21) emerges the fourth-order rational solution (3.20).

Selecting $\alpha = 1, \beta = 1, \gamma = 1, u_0 = 1, a_0 = -2, b_{21} = 1, b_{24} = 1, b_{25} = 1, b_{31} = -2, b_{35} = 1, z = 1$ and $t = -5$, the limiting process of (3.21) is similar to that in Figure 15.



(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 15. The limiting precess of (3.21) tends to (3.20) when $p_2 \to 0$. Parameters: $p_2 = 1, \bar{p}_3 = 1, a_2 = \cos(1), a_3 = \cos(1)$ in (a) and (e), $p_2 = 0.6, \bar{p}_3 = 0.6, a_2 = \cos(0.6), a_3 = \cos(0.6)$ in (b) and (f), $p_2 = 0.3, \bar{p}_3 = 0.3, a_2 = \cos(0.3), a_3 = \cos(0.3)$ in (c) and (g), $p_2 = 0.05, \bar{p}_3 = 0.05, a_2 = \cos(0.05), a_3 = \cos(0.05)$ in (d) and (h), respectively.

Case 5.

Let p_1, p_2 and $p_3 \in \mathbb{R}$, then using (2.3), (3.1) and (3.6), we obtain a interaction solution of Eq. (1.1) as follows:

$$\begin{aligned} u =& \frac{12\alpha}{\gamma}\Bigg(\frac{p_1^2 a_1 b_{11}^2 \mathrm{e}^{\eta_{14}} - p_2^2 a_2 b_{21}^2 \cos(\eta_{15}) + p_3^2 a_3 b_{31}^2 \cosh(\eta_{16})}{a_1 \mathrm{e}^{\eta_{14}} + a_2 \cos(\eta_{15}) + a_3 \cosh(\eta_{16})} \\ & - \left(\frac{p_1 a_1 b_{11} \mathrm{e}^{\eta_{14}} - p_2 a_2 b_{21} \sin(\eta_{15}) + p_3 a_3 b_{31} \sinh(\eta_{16})}{a_1 \mathrm{e}^{\eta_{14}} + a_2 \cos(\eta_{15}) + a_3 \cosh(\eta_{16})}\right)^2\Bigg), \end{aligned} \tag{3.22}$$

where

$$\eta_{14} = p_1\left(b_{11}x - \frac{\alpha b_{11}}{\beta}y + \frac{(\alpha-\beta)b_{11}}{\beta}z + b_{14}t + b_{15}\right),$$
$$\eta_{15} = p_2\left(b_{21}x - \frac{\alpha b_{21}}{\beta}y + \frac{(\alpha-\beta)b_{21}}{\beta}z + b_{24}t + b_{25}\right),$$
$$\eta_{16} = p_3\left(b_{31}x - \frac{\alpha b_{31}}{\beta}y + \frac{(\alpha-\beta)b_{31}}{\beta}z + b_{34}t + b_{35}\right),$$

$\alpha, \beta(\neq 0), \gamma(\neq 0), a_1, a_2, a_3, b_{11}, b_{14}, b_{15}, b_{21}, b_{24}, b_{25}, b_{31}, b_{34}$ and $b_{35} \in \mathbb{R}$. Moreover, if taking $p_2 = p_1, p_3 = p_1, a_1 = -2, a_2 = \cos(p_2), a_3 = \cosh(p_3)$ and $p_1 \to 0$ in (3.22), then solution (3.22) emerges the following second-order rational solution:

$$u = -\frac{12\alpha}{\gamma}\left(\frac{G_5}{H_5}\right)^2, \tag{3.23}$$

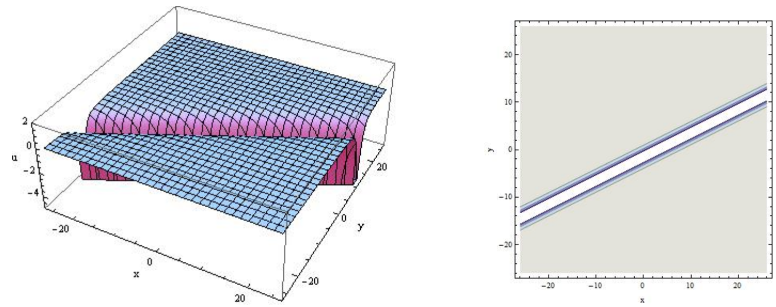


Figure 16. Three-dimensional and contour plots of (3.23).

where $G_5 = \beta b_{11}$, $H_5 = \beta b_{11}x - \alpha b_{11}y + (\alpha - \beta)b_{11}z + \beta b_{14}t + \beta b_{15}$.

Selecting $\alpha = 2, \beta = 1$, $\gamma = 1$, $b_{11} = 1, b_{14} = 1, b_{15} = 1, z = 1$ and $t = -5$, three-dimensional and contour plots of (3.23) are shown in Figure 16. Selecting $\alpha = 2, \beta = 1$, $\gamma = 1$, $a_1 = -2$, $b_{11} = 1, b_{14} = 1, b_{15} = 1$, $b_{21} = 2, b_{24} = 1, b_{25} = 1$, $b_{31} = 3, b_{34} = 1, b_{35} = 1, z = 1$ and $t = -5$, the limiting process of (3.22) is similar to that in Figure 17.

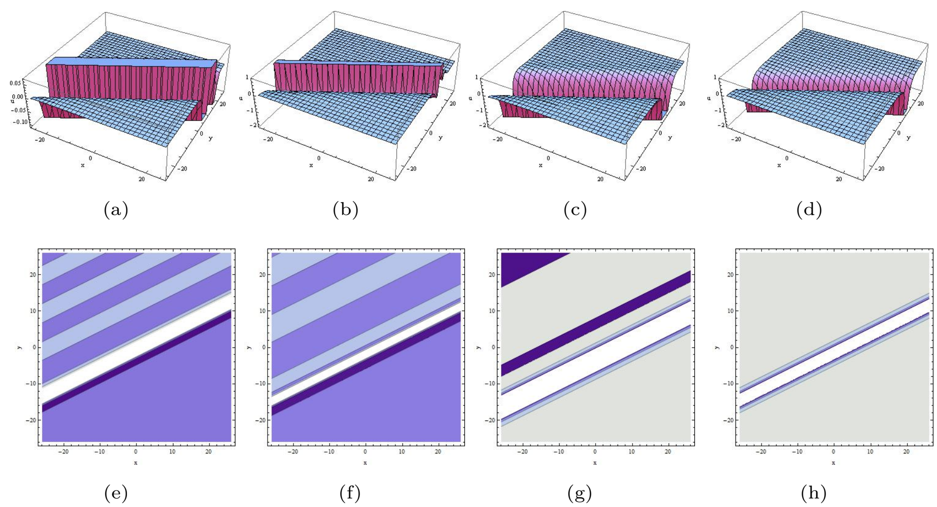


Figure 17. The limiting precess of (3.22) tends to (3.23) when $p_1 \to 0$. Parameters: $p_1 = 0.15, p_2 = 0.15, p_3 = 0.15, a_2 = \cos(0.15), a_3 = \cosh(0.15)$ in (a) and (e), $p_1 = 0.1, p_2 = 0.1, p_3 = 0.1, a_2 = \cos(0.1), a_3 = \cosh(0.1)$ in (b) and (f), $p_1 = 0.05, p_2 = 0.05, p_3 = 0.05, a_2 = \cos(0.05), a_3 = \cosh(0.05)$ in (c) and (g), $p_1 = 0.001, p_2 = 0.001, p_3 = 0.001, a_2 = \cos(0.001), a_3 = \cosh(0.001)$ in (d) and (h), respectively.

Let $p_1, p_3 \in \mathbb{R}$ and $p_2 = \bar{p}_2 I$, here $\bar{p}_2 \in \mathbb{R}, I = \sqrt{-1}$, then using (2.3), (3.1) and (3.6), we obtain a interaction solution of Eq. (1.1) as follows:

$$
\begin{aligned}
u =& \frac{12\alpha}{\gamma}\Bigg(\frac{p_1^2 a_1 b_{11}^2 e^{\eta_{14}} + \bar{p}_2^2 a_2 b_{21}^2 \cosh(\eta_{17}) + p_3^2 a_3 b_{31}^2 \cosh(\eta_{16})}{a_1 e^{\eta_{14}} + a_2\cosh(\eta_{17}) + a_3\cosh(\eta_{16})} \\
& - \left(\frac{p_1 a_1 b_{11} e^{\eta_{14}} + \bar{p}_2 a_2 b_{21}\sinh(\eta_{17}) + p_3 a_3 b_{31}\sinh(\eta_{16})}{a_1 e^{\eta_{14}} + a_2\cosh(\eta_{17}) + a_3\cosh(\eta_{16})}\right)^2\Bigg),
\end{aligned}
\tag{3.24}
$$

where $\eta_{17} = \bar{p}_2\left(b_{21}x - \frac{\alpha b_{21}}{\beta}y + \frac{(\alpha-\beta)b_{21}}{\beta}z + b_{24}t + b_{25}\right)$, η_{14} and η_{16} are given in

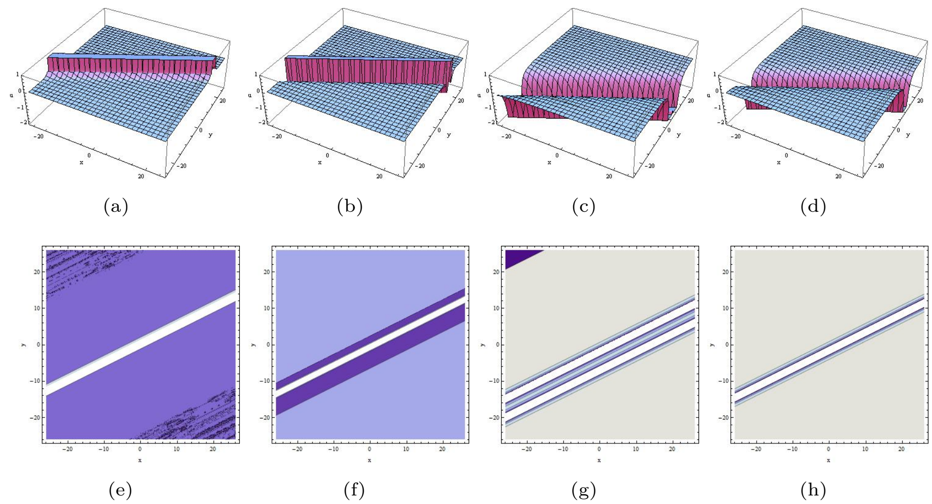


Figure 18. The limiting precess of (3.24) tends to (3.23) when $p_1 \to 0$. Parameters: $p_1 = 0.7, \bar{p}_2 = 0.7, p_3 = 0.7, a_2 = \cosh(0.7), a_3 = \cosh(0.7)$ in (a) and (e), $p_1 = 0.2, \bar{p}_2 = 0.2, p_3 = 0.2, a_2 = \cosh(0.2), a_3 = \cosh(0.2)$ in (b) and (f), $p_1 = 0.02, \bar{p}_2 = 0.02, p_3 = 0.02, a_2 = \cosh(0.02), a_3 = \cosh(0.02)$ in (c) and (g), $p_1 = 0.001, \bar{p}_2 = 0.001, p_3 = 0.001, a_2 = \cosh(0.001), a_3 = \cosh(0.001)$ in (d) and (h), respectively.

(3.22). Moreover, if taking $\bar{p}_2 = p_1, p_3 = p_1, a_1 = -2, a_2 = \cosh(\bar{p}_2), a_3 = \cosh(p_3)$ and $p_1 \to 0$ in (3.24), then solution (3.24) emerges the second-order rational solution (3.23).

Selecting $\alpha = 2, \beta = 1,\ \gamma = 1,\ a_1 = -2,\ b_{11} = 1, b_{14} = 1, b_{15} = 1,\ b_{21} = 2, b_{24} = 1, b_{25} = 1,\ b_{31} = 3, b_{34} = 1, b_{35} = 1, z = 1$ and $t = -5$, the limiting process of (3.24) is similar to that in Figure 18.

Let $p_1, p_2 \in \mathbb{R}$ and $p_3 = \bar{p}_3 I$, here $\bar{p}_3 \in \mathbb{R}, I = \sqrt{-1}$, then using (2.3), (3.1) and (3.6), we obtain a interaction solution of Eq. (1.1) as follows:

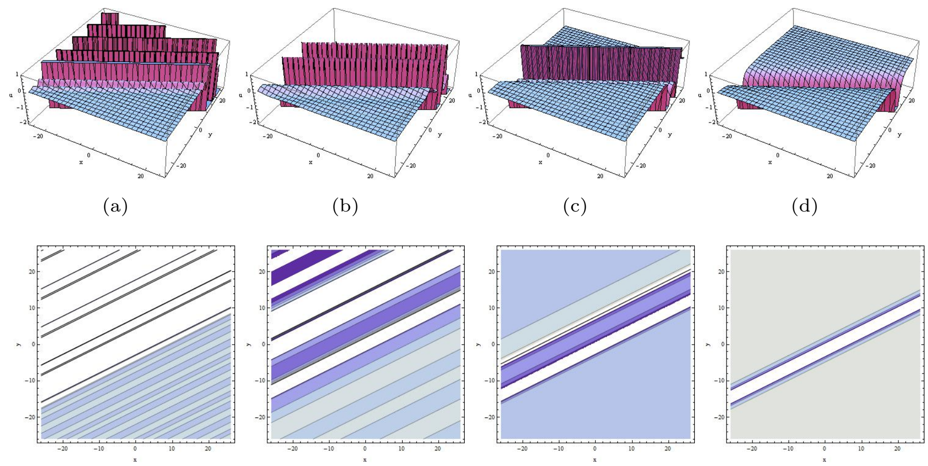
$$u = \frac{12\alpha}{\gamma}\left(\frac{p_1^2 a_1 b_{11}^2 \mathrm{e}^{\eta_{14}} - p_2^2 a_2 b_{21}^2 \cos(\eta_{15}) - \bar{p}_3^2 a_3 b_{31}^2 \cos(\eta_{18})}{a_1 \mathrm{e}^{\eta_{14}} + a_2 \cos(\eta_{15}) + a_3 \cos(\eta_{18})}\right. \\ \left. - \left(\frac{p_1 a_1 b_{11} \mathrm{e}^{\eta_{14}} - p_2 a_2 b_{21} \sin(\eta_{15}) - \bar{p}_3 a_3 b_{31} \sin(\eta_{18})}{a_1 \mathrm{e}^{\eta_{14}} + a_2 \cos(\eta_{15}) + a_3 \cos(\eta_{18})}\right)^2\right), \tag{3.25}$$

where $\eta_{18} = \bar{p}_3\left(b_{31}x - \frac{\alpha b_{31}}{\beta}y + \frac{(\alpha-\beta)b_{31}}{\beta}z + b_{34}t + b_{35}\right)$, η_{14} and η_{15} are given in (3.22). Moreover, if taking $p_2 = p_1, \bar{p}_3 = p_1, a_1 = -2, a_2 = \cos(p_2), a_3 = \cos(\bar{p}_3)$ and $p_1 \to 0$ in (3.25), then solution (3.25) emerges the second-order rational solution (3.23).

Selecting $\alpha = 2, \beta = 1,\ \gamma = 1,\ a_1 = -2,\ b_{11} = 1, b_{14} = 1, b_{15} = 1,\ b_{21} = 2, b_{24} = 1, b_{25} = 1,\ b_{31} = 3, b_{34} = 1, b_{35} = 1, z = 1$ and $t = -5$, the limiting process of (3.25) is similar to that in Figure 19.

4. Conclusion

In this paper, we first obtain Hirota bilinear form (2.6) of Eq. (1.1) by using the Hirota bilinear method. Based on the (2.6), we second formulate some interaction



(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 19. The limiting precess of (3.25) tends to (3.23) when $p_1 \to 0$. Parameters: $p_1 = 0.3, p_2 = 0.3, \bar{p}_3 = 0.3, a_2 = \cos(0.3), a_3 = \cos(0.3)$ in (a) and (e), $p_1 = 0.1, p_2 = 0.1, \bar{p}_3 = 0.1, a_2 = \cos(0.1), a_3 = \cos(0.1)$ in (b) and (f), $p_1 = 0.02, p_2 = 0.02, \bar{p}_3 = 0.02, a_2 = \cos(0.02), a_3 = \cos(0.02)$ in (c) and (g), $p_1 = 0.001, p_2 = 0.001, \bar{p}_3 = 0.001, a_2 = \cos(0.001), a_3 = \cos(0.001)$ in (d) and (h), respectively.

solutions of Eq. (1.1) by choosing appropriate test function and the parameters. Utilizing the parameter limit method along with symbolic computations, we third dive the rational solutions of Eq. (1.1) from the formulated interaction solutions in this paper. We last illustrate the correctness of the theoretical results through image simulations. Compare with the published literatures [4–6,21,30,31,34,36], we obtain some completely new results of Eq. (1.1) which include new Hirota bilinear form, new interaction and rational solutions etc. In fact, we only take the test function as (3.1) in this paper, other test functions may be selected. We believe that more and more new interaction and rational solutions of Eq. (1.1) will be presented in the future.

Acknowledgements. The authors would like to thank the editors and reviewers for their very valuable comments and suggestions.

Competing Interests. The authors declare that there are no competing interests regarding the publication of this paper.

References

[1] X. Cheng, W. Ma and Y. Yang, *Lump-type solutions of a generalized Kadomtsev-Petviashvili equation in (3+1)-dimensions,* Chin. Phys. B, 2019, 28(10), 100203.

[2] J. Gu, Y. Zhang and H. Dong, *Dynamic behaviors of interaction solutions of (3+1)-dimensional Shallow Water wave equation,* Comput. Math. Appl., 2018, 76(6), 1408–1419.

[3] X. Guan, W. Liu, Q. Zhou and A. Biswas, *Some lump solutions for a generalized (3+1)-dimensional Kadomtsev-Petviashvili equation,* Appl. Math. Comput., 2020, 366, 124757.

[4] L. Gai, W. Ma and M. Li, *Lump-type solutions, rogue wave type solutions and periodic lump-stripe interaction phenomena to a (3+1)-dimensional generalized breaking soliton equation,* Phys. Lett. A, 2020, 384(8), 126178.

[5] X. Hu, S. Lin and L. Wang, *Integrability, multiple-cosh , lumps and lump-soliton solutions to a (2+1)-dimensional generalized breaking soliton equation,* Commun. Nonlinear Sci. Numer. Simulat., 2020, 91, 105447.

[6] K. Hosseini, A. R. Seadawy, M. Mirzazadeh, M. Eslami, S. Radmehr and D. Baleanu, *Multiwave, multicomplexiton, and positive multicomplexiton solutions to a (3+1)-dimensional generalized breaking soliton equation,* Alex. Eng. J., 2020, 59(5), 3473–3479.

[7] R. Hirota, *The Direct Method in Soliton Theory,* Springer, Berlin, 1980.

[8] K. Hosseini, W. Ma, R. Ansari, M. Mirzazadeh, R. Pouyanmehr and F. Samadani, *Evolutionary behavior of rational wave solutions to the (4+1)-dimensional Boiti-Leon-Manna-Pempinelli equation,* Phys. Scr., 2020, 95(6), 065208.

[9] B. He and Q. Meng, *Bilinear form and new interaction solutions for the sixth-order Ramani equation,* Appl. Math. Lett., 2019, 98, 411–418.

[10] O. A. Ilhan, J. Manafian and M. Shahriari, *Lump wave solutions and the interaction phenomenon for a variable-coefficient Kadomtsev-Petviashvili equation,* Comput. Math. Appl., 2019, 78, 2429–2448.

[11] X. Liu, J. Yu and Z. Lou, *New interaction solutions from residual symmetry reduction and consistent Riccati expansion of the (2+1)-dimensional Boussinesq equation,* Nonlinear Dyn., 2018, 92, 1469–1479.

[12] C. Liu, M. Chen, P. Zhou and L. Chen, *Bi-solitons, breather solution family and rogue waves for the (2+1)-dimensional nonlinear Schrödinger equation,* J. Appl. Anal. Comput., 2016, 6(2), 367–375.

[13] J. Liu, W. Zhu and L. Zhou, *Interaction solutions and abundant exact solutions for the new (3+1)-dimensional generalized Kadomtsev-Petviashvili equation in fluid mechanics,* J. Appl. Anal. Comput., 2020, 10 (3), 960–971.

[14] W. Liu, A. M. Wazwaz and X. Zheng, *Families of semi-rational solutions to the Kadomtsev-Petviashvili I equation,* Commun. Nonlinear Sci. Numer. Simulat., 2019, 67, 480–491.

[15] J. Liu, J. Du, Z. Zeng and B. Nie, *New three-wave solutions for the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation,* Nonlinear Dyn., 2017, 88, 655–661.

[16] Z. Li, J. Manafian, N. Ibrahimov, A. Hajar, K. S. Nisar and W. Jamshed, *Variety interaction between k-lump and k-kink solutions for the generalized Burgers equation with variable coefficients by bilinear analysis,* Results Phys., 2021, 28, 104490.

[17] W. Ma, *A search for lump solutions to a combined fourth-order nonlinear PDE in (2+1)-dimensions,* J. Appl. Anal. Comput., 2019, 9(4), 1319–1332.

[18] J. Manafian, S. A. Mohammed, A. Alizadeh, H. M. Baskonus and W. Gao, *Investigating lump and its interaction for the third-order evolution equation arising propagation of long waves over shallow water,* Eur. J. Mech. B-Fluid., 2020, 84, 289–301.

[19] J. Manafian and M. Lakestani, *N-lump and interaction solutions of localized waves to the (2+1)-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation,* J. Geom. Phys., 2020, 150, 103598.

[20] J. Manafian, B. Mohammadi-Ivatloo and M. Abapour, *Lump-type solutions and interaction phenomenon to the (2+1)-dimensional Breaking Soliton equation,* Appl. Math. Comput., 2019, 356, 13–41.

[21] Y. Ma and B. Li, *Interactions between soliton and rogue wave for a (2+1)-dimensional generalized breaking soliton system: Hidden rogue wave and hidden soliton,* Comput. Math. Appl., 2019, 78 (3), 827–839.

[22] Q. Meng, *Rational solutions and interaction solutions for a fourth-order nonlinear generalized Boussinesq water wave equation,* Appl. Math. Lett., 2020, 110, 106580.

[23] Y. Qian, J. Manafian, S. Y. Mohyaldeen, L. S. Esmail, S. A. Gorovoy and G. Singh, *Multiple-order line rogue wave, lump and its interaction, periodic, and cross-kink solutions for the generalized CHKP equation,* Propuls. Power Res., 2021, 10(3), 277–293.

[24] B. Ren, W. Ma and J. Yu, *Rational solutions and their interaction solutions of the (2+1)-dimensional modified dispersive water wave equation,* Comput. Math. Appl., 2019, 77 (8), 2086–2095.

[25] G. Shen, J. Manafian, D. T. N. Huy, K. S. Nisar, M. Abotaleb and N. D. Trung, *Abundant soliton wave solutions and the linear superposition principle for generalized (3+1)-D nonlinear wave equation in liquid with gas bubbles by bilinear analysis,* Results Phys., 2022, 32, 105066.

[26] W. Tan, Z. Dai, J. Xie and D. Qiu, *Parameter limit method and its application in the (4+1)-dimensional Fokas equation,* Comput. Math. Appl., 2018, 75 (12), 4214–4220.

[27] W. Tan, W. Zhang and J. Zhang, *Evolutionary behavior of breathers and interaction solutions with M-solitons for (2+1)-dimensional KdV system,* Appl. Math. Lett., 2020, 101, 106063.

[28] S. Tian, D. Guo, X. Wang and T. Zhang, *Traveling wave, lump wave, rogue wave, multi-kink solitary wave and interaction solutions in a (3+1)-dimensional Kadomtsev-Petviashvili equation with Bäcklund transformation,* J. Appl. Anal. Comput., 2021, 11(1), 45–58.

[29] J. Wu, Y. Liu, L. Piao, J. Zhuang and D. Wang, *Nonlinear localized waves resonance and interaction solutions of the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation,* Nonlinear Dyn., 2020, 100, 1527–1541.

[30] A. M. Wazwaz, *A new integrable (2+1)-dimensional generalized breaking soliton equation: n-soliton solutions and traveling wave solutions,* Commun. Theor. Phys., 2016, 66(4), 385–388.

[31] G. Xu, *Integrability of a (2+1)-dimensional generalized breaking soliton equation,* Appl. Math. Lett., 2015, 50, 16–22.

[32] Y. Xu, X. Zheng and J. Xin, *Abundant new non-traveling wave solutions for the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation,* J. Appl. Anal. Comput., 2021, 11(4), 2052–2069.

[33] Y. Yin, W. Ma, J. Liu and X. Lü, *Diversity of exact solutions to a (3+1)-dimensional nonlinear evolution equation and its reduction,* Comput. Math. Appl., 2018, 76(6), 1275–1283.

[34] X. Yan, S. Tian, M. Dong, L. Zhou and T. Zhang,*Characteristics of solitary wave, homoclinic breather wave and rogue wave solutions in a (2+1)-dimensional generalized breaking soliton equation,* Comput. Math. Appl., 2018, 76(1), 179–186.

[35] Y. Zhou, S. Manukure and W. Ma, *Lump and lump-soliton solutions to the Hirota-Satsuma-Ito equation,* Commun. Nonlinear Sci. Numer. Simulat., 2019, 68, 56–62.

[36] Z. Zhao and B. Han, *Quasiperiodic wave solutions of a (2+1)-dimensional generalized breaking soliton equation via bilinear Bäcklund transformation,* Eur. Phys. J. Plus, 2016, 131, 128.