Journal of Applied Analysis and Computation
Volume 13, Number 3, June 2023, 1449–1474
Website:http://www.jaac-online.com
DOI:10.11948/20220317

A RELIABLE APPROACH FOR ANALYSING THE NONLINEAR KDV EQUATION OF FRACTIONAL ORDER

N. Ghanbari[1], K. Sayevand[1,†] and I. Masti[1]

Abstract Its applications in many domains, along with its challenging analytical solution, have led to several studies of the Korteweg-de Vries (KdV) equation over the past decade. Due to difficulties or impossibility with the analytical solution to this equation, the paper presents a numerical solution using the Crank-Nicolson difference method. A study of the stability and solvency of this method has been undertaken. In this paper, we prove that the scheme is first order convergent in space and $\min\{2-\nu, r\nu\}$ order convergent in time, where r refers to a gradation parameter and ν represents the fractional derivative. The results are then presented in numerical applications, looking at how it compares with other sophisticated schemes in the literature. The main benefit of the proposed scheme is the efficiency with regard to accuracy as compared to other available schemes.

Keywords Korteweg-de Vries fractional time equation, Crank-Nicolson difference method, Caputo fractional derivative.

MSC(2010) 65J15, 65G99, 35B35, 35R11.

1. Background and introduction

Researchers are able to greatly benefit from mathematical models. In order to investigate how infectious diseases are transmitted, many researchers have been used theoretical frameworks and numerical simulations. In [4], a stochastic seiaqhr model is examined for the transmission of Coronavirus 2019, which has been a major problem for a number of communities recently. The mathematics model of a brain tumor is introduced in [13]. Modeling glioma growth and diffusion using simple two-dimensional math is an extension of an earlier two-dimensional model.

The mathematical model of nonlinear waves in shallow water has also been studied by many scientists because of its importance and its ability to describe numerous physical phenomena in the fields of physics, math, and engineering. Mathematical models are developed by Korteweg and de Vries to study these nonlinear waves. A well-known equation of this type is KdV [10, 15].

This equation can simulate red spots on the surface of Jupiter. In addition, it describes non-linear waves in rotating fluids as well as other aspects of single waves

[†]The corresponding author.
Email: nasim.ghanbary@stu.malayeru.ac.ir (N. Ghanbari), ksayehvand@malayeru.ac.ir (K. Sayevand), iman.masti@stu.malayeru.ac.ir (I. Masti)
[1]Faculty of Mathematical Sciences and Statistics, Malayer University, Malayer, Iran

in plasma. Due to temperature differences in the ocean, there are sometimes gigantic waves that can destroy sea ships. These waves can also be described by the KdV equation. In addition, destructive ocean waves (tsunamis) are also described by the KdV equation. Pulse waves propagating through blood vessels provide another example for this equation. This equation is solved by the soliton after its discovery [21].

However, in all these modelings, the KdV equation with derivatives of the order is correct. In recent years, numerous partial derivative equations have been revised by changing their classical derivatives to fractional derivatives.

Integrals and derivatives in fractional calculation can have arbitrary orders because fractional calculus is a specialty of pure mathematics. Differential calculus and fractional integral tools have been applied to model many phenomena in economics, physics, chemistry, engineering, and other applied sciences. Mathematics has developed numerous definitions to fit the idea of incorrect integrals and derivatives through the years, using its own symbolism and approach. Riemann-Liouville, Grunwald-Letnikov, and Caputo are the most commonly formulations which were from fractional derivatives [5, 6, 16, 22–24].

In such a context, the fractional KdV equation, as well as many other fractional PBDEs, have been also investigated by different methods, either analytic or numerical. In [19], the adomian decomposition method is employed to resolve the fractional equation of KdV with the Caputo derivative. In [31], the same method is expanded to derive explicitly and numerically solutions from the KdV-Burgers fraction equations. In [11], a numerical method based on the Taylor series formula for explicit and approximate solution of the nonlinear equation of the KdV-Burgers fraction with time -place fraction derivatives was proposed and discussed. Since most of the work is super-analytical or semi-analytical, in the present work, we propose to solve the fractional KdV equation numerically, utilizing a finite difference method, and to examine some special properties of this solution.

Presenting a finite difference scheme for the solution of nonlinear Caputo time-fractional derivative equations is the purpose of this paper. Occasionally, fractional equations inherit the weak singularity kernel introduced in fractional derivative definitions. Here we consider a finite difference scheme on graded meshes and consider the singularity of solution in the initial layer. Moreover, the scheme is examined for stability and convergence. The well known $L1$ scheme on graded meshes has been used for time discretization to reduce the problem of weak singularities caused by fractional derivatives in the initial layer. we prove that the scheme is first order convergent in space and $\min\{2-\nu, r\nu\}$ order convergent in time. As well, the optimization of the computation of the presented scheme has been considered in order to increase its computational efficiency.

In light of the above statements the Korteweg–de Vries equation (KdV) is known to be useful. Briefly for the study of the time-evolution of long, unidirectional, weakly nonlinear waves over shallow fluid surfaces; long internal waves within densely laminated oceans; the motion of waves on crystal networks; or ionic acoustic waves in plasma dynamics. An approach to solving nonlinear KdV equations is developed in this work

$$\begin{cases} {}_0^c D_t^\nu \psi(x,t) - \lambda\psi(x,t) + \theta\psi(x,t)\psi_x(x,t) + \psi_{xxx}(x,t) = f(x,t), & x \in (0,L), t \in (0,T], \\ \psi(x,0) = \vartheta(x), \quad x \in [0,L], \\ \psi(0,t) = 0, \quad \psi(L,t) = 0, \quad \psi_x(L,t) = 0, \quad t \in (0,T], \end{cases} \tag{1.1}$$

where for $0 < \nu < 1$, $\vartheta(0) = \vartheta(L) = \vartheta'(L) = 0$, ${}_0^c D_t^\nu \psi(x,t)$ is Caputo's fractional derivative of order ν, and constants λ and θ are provided.

This paper has the following organization. Section 2 represents implement the method and outline. In Section 3, the existence of numerical solution of the proposed method is proved. In Sections 4,5 stability and convergence analysis are discussed, respectively. In Section 6 , we consider the optimizing the computation of the presented scheme. Section 7 illustrates the performance of the method for various examples. Finally, Section 8 presents the main conclusions.

2. Implement the method and outline

The nonlinear difference scheme is introduced in this section. Think of $\mathcal{M}$ and $\mathcal{N}$ as two positive integers. In this case, we assume that:

$$\begin{cases} x_j = jh & j = 0, 1, \cdots, \mathcal{M}, \\ t_n = \left(\dfrac{n}{\mathcal{N}}\right)^r T & n = 0, 1, ..., \mathcal{N}, \end{cases} \tag{2.1}$$

where $h = \dfrac{L}{\mathcal{M}}$ is the uniform space step size. Also, $r \geq 1$. Note that a quasi-uniform mesh pattern is applied at $r = 1$. Furthermore, the temporal meshes are graded while r increasing. That shows smaller initial step sizes compared to those with uniform step sizes.

Define $\tau_n = t_n - t_{n-1}$, $\Omega_h = \{x_j | j = 0, 1, ..., \mathcal{M}\}$ and $\Omega_\tau = \{t_n | n = 0, 1, ..., \mathcal{N}\}$, $\tilde{\Psi}_h = \{\psi \,|\psi = (\psi_0, \psi_1, ..., \psi_\mathcal{M})\}$, $\tilde{\Psi}_h^0 = \{\psi \,|\psi = (\psi_0, \psi_1, ..., \psi_\mathcal{M}), \psi_0 = \psi_\mathcal{M} = 0\}$.

Below, we discuss the Caputo fractional derivative approximation that is necessary for our analysis.

Definition 2.1 ([12]). A left-sided and right-sided Caputo derivative of order $\nu > 0$ can be defined as follows:

$$\begin{aligned} D_{a,t}^\nu f(t) &= \frac{1}{\Gamma(n-\nu)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\nu-1+n}} ds, \ t > a, \\ D_{t,b}^\nu f(t) &= \frac{(-1)^n}{\Gamma(n-\nu)} \int_u^b \frac{f^{(n)}(s)}{(s-t)^{\nu-1+n}} ds, \ t < b, \end{aligned} \tag{2.2}$$

n is a positive integer and holds in $n - 1 < \nu < n, \quad n \in \mathbb{N}$.

Definition 2.2 ([27]). According to Caputo's fractional derivative of order ν, bivariate functions have the following formula:

$$D_u^\nu f(u,t) = \frac{1}{\Gamma(n-\nu)} \int_0^u \frac{f^{(n)}(s,t)}{(u-s)^{\nu-1+n}} ds, \quad n - 1 < \nu < n, \ \ n \in \mathbb{N}. \tag{2.3}$$

Theorem 2.1 ([30]). *Assume $n-1<\nu\leq n$. If $f(t)$ has an $(n+1)th$ order derivative, then*

$$\lim_{\nu\to n^-} D^{\nu}_{a,t} f(t) = f^{(n)}(t), \tag{2.4}$$

while

$$\lim_{\nu\to (n-1)^+} D^{\nu}_{a,t} f(t) = f^{(n-1)}(t) - f^{(n-1)}(a). \tag{2.5}$$

Theorem 2.2 ([3]). *Suppose $n-1<\nu<n\in\mathbb{N}$ and $\tilde{\nu}>0$. Then*

$$\begin{aligned} D^{\nu}_{a,t}(x-a)^{\tilde{\nu}-1} &= \frac{\Gamma(\tilde{\nu})}{\Gamma(\tilde{\nu}-\nu)}(x-a)^{\tilde{\nu}-\nu-1}, \quad \tilde{\nu}>n,\\ D^{\nu}_{t,b}(b-x)^{\tilde{\nu}-1} &= \frac{\Gamma(\tilde{\nu})}{\Gamma(\tilde{\nu}-\nu)}(b-x)^{\tilde{\nu}-\nu-1}, \quad \tilde{\nu}>n. \end{aligned} \tag{2.6}$$

In addition

$$D^{\nu}_{a,t}(x-a)^k = 0 \text{ and } D^{\nu}_{t,b}(b-x)^k = 0,\ k=0,1,\cdots,n-1. \tag{2.7}$$

Based on the known $L1$ scheme, the Caputo fractional derivative can be approximated by the following lemma.

Lemma 2.1 ([2]). *Let $0<\nu<1, f(t)\in C^2[0,t_n]$, it holds that*

$$\begin{aligned} &\left|\frac{1}{\Gamma(1-\nu)}\int_0^{t_n}\frac{f'(s)ds}{(t_n-s)^{\nu}} - \frac{1}{\mu_n}\left[a_0^{\nu}f(t_n) - \sum_{k=1}^{n-1}(a^{\nu}_{n-k-1}-a^{\nu}_{n-k})f(t_k) - a^{\nu}_{n-1}f(t_0)\right]\right| \\ \leq &\frac{1}{\Gamma(2-\nu)}\left[\frac{1-\nu}{12}+\frac{2^2-\nu}{2-\nu}-(1+2^{-\nu})\right]\max_{0\leq t\leq t_n}|f''(t)|\,\tau^{2-\nu}, \end{aligned} \tag{2.8}$$

where

$$\begin{cases} \tau_n = t_n - t_{n-1},\\ \mu_n = \Gamma(2-\nu)\tau_n^{\nu},\\ a_k^{\nu} = (k+1)^{1-\nu}-k^{1-\nu},\ k=0,1,\cdots,n-1. \end{cases} \tag{2.9}$$

Due to the Lemma 2.1, consider the definition of a fractional derivative as follows:

$$\begin{aligned} D_t^{\nu} f(t_n) =& \frac{1}{\mu_n}\left[a_0^{\nu}\left(\frac{f(t_n)+f(t_{n-1})}{2}\right)\right.\\ &\left. - \sum_{k=1}^{n-1}(a^{\nu}_{n-k-1}-a^{\nu}_{n-k})\left(\frac{f(t_k)+f(t_{k-1})}{2}\right) - a^{\nu}_{n-1}f(t_0)\right]. \end{aligned} \tag{2.10}$$

Theorem 2.3. *Suppose $f\in C[0,T]\cap C^2(0,T]$, the constant γ is such that*

$$|f^{'}(t)|\leq\gamma t^{\nu-1}, \qquad |f^{''}(t)|\leq \gamma t^{\nu-2}, \qquad t\in(0,T]. \tag{2.11}$$

Thus there is a constant η that can be written as follows

$$|{}_0^cD_t^{\nu} f(t_n) - D_t^{\nu} f(t_n)| \leq \frac{\eta}{n^{min\{r\nu,2-\nu\}}}, \qquad n=1,2,\cdots,\mathcal{N}. \tag{2.12}$$

Proof. See [28]. □

We place points (x_j, t_n) in Eq. (1.1). So we will have:

$$\begin{aligned}&{}_0^cD_t^{\nu}\psi(x_j,t_n)-\lambda\psi(x_j,t_n)+\theta\psi(x_j,t_n)\psi_x(x_j,t_n)+\psi_{xxx}(x_j,t_n)\\ =&f(x_j,t_n),\ 1\le j\le \mathcal{M}-1,\ 1\le n\le \mathcal{N}.\end{aligned} \tag{2.13}$$

Using Eqs. (2.9-2.10) to approximate the time-fractional derivative, we now derive the Crank-Nicolson difference scheme for the problem (1.1).

$$\begin{aligned}&\frac{1}{\mu_n}\left[a_0^{\nu}\left(\frac{\psi(x_j,t_n)+\psi(x_j,t_{n-1})}{2}\right)-\sum_{k=1}^{n-1}(a_{n-k-1}^{\nu}-a_{n-k}^{\nu})\left(\frac{\psi(x_j,t_k)+\psi(x_j,t_{k-1})}{2}\right)\right.\\ &\left.-\,a_{n-1}^{\nu}\psi(x_j,t_0)\right]-\lambda\left(\frac{\psi(x_j,t_n)+\psi(x_j,t_{n-1})}{2}\right)+\theta\left(\frac{\psi(x_j,t_n)+\psi(x_j,t_{n-1})}{2}\right)\\ &\times\psi_x\left(\frac{\psi(x_j,t_n)+\psi(x_j,t_{n-1})}{2}\right)+\psi_{xxx}\left(\frac{\psi(x_j,t_n)+\psi(x_j,t_{n-1})}{2}\right)\\ =&f(x_j,t_n),\quad 1\le j\le \mathcal{M}-1,\quad 1\le n\le \mathcal{N}.\end{aligned} \tag{2.14}$$

For $0 \le j \le \mathcal{M}$ and $0 \le n \le \mathcal{N}$, define grid functions:

$$\begin{cases}\psi_j^n=\psi(x_j,t_n),\\ f_j^n=f(x_j,t_n),\\ \vartheta_j=\vartheta(x_j).\end{cases} \tag{2.15}$$

To continue the process, if $\psi, \xi \in \tilde{\Psi}_h$ is a grid function, consider the following definitions:

$$\begin{cases}\Delta_x\psi_j=\dfrac{\psi_{j+1}-\psi_{j-1}}{2h}, & 1\le j\le \mathcal{M}-1,\\ \chi(\psi,\xi)_j=\dfrac{1}{3}\left(\psi_j\Delta_x(\psi)_j+\Delta_x(\psi\xi)_j\right), & 1\le j\le \mathcal{M}-1,\\ \delta_x\psi_{j+\frac12}=\dfrac{\psi_{j+1}-\psi_j}{h}, & 1\le j\le \mathcal{M},\\ \delta_x^2\psi_j=\dfrac{\psi_{j-1}-2\psi_j+\psi_{j+1}}{h^2}, & 1\le j\le \mathcal{M}-1,\\ \delta_x^3\psi_{j+\frac12}=\delta_x^2(\delta_x\psi_{j+\frac12})=\dfrac{1}{h^3}\left(\psi_{j+2}-3\psi_{j+1}+3\psi_j-\psi_{j-1}\right), & 1\le j\le \mathcal{M}-2,\\ \psi_j^{n-\frac12}=\frac{\psi_j^n+\psi_j^{n-1}}{2}, & 1\le j\le \mathcal{M}-1,\ 1\le n\le \mathcal{N}.\end{cases} \tag{2.16}$$

To discretize ψ_{xxx}, we write the Taylor expansion at points ψ_{j+2}^n, ψ_{j+1}^n, ψ_j^n and ψ_{j-1}^n at points (x_j, t_n). If we consider the first four sentences in each of the above extensions:

$$\psi_{xxx}(x_j,t_n)=\frac{1}{h^3}\left(\psi(x_{j+2},t_n)-3\psi(x_{j+1},t_n)+3\psi(x_j,t_n)-\psi(x_{j-1},t_n)\right). \tag{2.17}$$

Since the $\psi(L,t) = \psi_x(L,t) = 0$ will have $\psi(x_{\mathcal{M}+1}, t_n) = \psi(x_{\mathcal{M}-1}, t_n)$, therefore

$$\psi_{xxx}(x_{\mathcal{M}-1},t_n)=\frac{1}{h^3}\left(\psi(x_{\mathcal{M}-1},t_n)-\psi(x_{\mathcal{M}-2},t_n)-3(\psi(x_{\mathcal{M}},t_n)-\psi(x_{j-1},t_n))\right). \tag{2.18}$$

Hence for $j = 1, 2, ...\mathcal{M} - 1$, we have

$$\psi_{xxx}|_j^n = \frac{1}{h^3}\left(\psi_{j+2}^n - 3\psi_{j+1}^n + 3\psi_j^n - \psi_{j-1}^n\right),\ 1 \le j \le \mathcal{M}-2,\ \ 1 \le n \le \mathcal{N}, \tag{2.19}$$

$$\psi_{xxx}|_{\mathcal{M}-1}^n = \frac{1}{h^3}\left(\psi_{\mathcal{M}-1}^n - \psi_{\mathcal{M}-2}^n - 3\left(\psi_{\mathcal{M}}^n - \psi_{j-1}^n\right)\right),\ i=\mathcal{M}-1,\ \ 1 \le n \le \mathcal{N}, \tag{2.20}$$

$$\frac{1}{\mu_n}\left[a_0^\nu \Psi_j^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}\left(a_{n-k-1}^\nu - a_{n-k}^\nu\right)\Psi_j^{k-\frac{1}{2}} - a_{n-1}^\nu \Psi_j^0\right] - \lambda \Psi_j^{n-\frac{1}{2}} + \theta\chi(\Psi^{n-\frac{1}{2}}, \Psi^{n-\frac{1}{2}})_j$$
$$+ \delta_x^3 \Psi_{j+\frac{1}{2}}^{n-\frac{1}{2}} - f_j^{n-\frac{1}{2}} = (R_x)_j^n + (R_t)_j^n,\ \ 1 \le j \le \mathcal{M}-2,\ \ 1 \le n \le \mathcal{N}, \tag{2.21}$$

and for $j = \mathcal{M} - 1$, we can get

$$\begin{aligned}&\frac{1}{\mu_n}\left[a_0^\nu \Psi_{\mathcal{M}-1}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}\left(a_{n-k-1}^\nu - a_{n-k}^\nu\right)\Psi_{\mathcal{M}-1}^{k-\frac{1}{2}} - a_{n-1}^\nu \Psi_{\mathcal{M}-1}^0\right] - \lambda \Psi_{\mathcal{M}-1}^{n-\frac{1}{2}}\\ &+ \theta\chi(\Psi^{n-\frac{1}{2}}, \Psi^{n-\frac{1}{2}})_{\mathcal{M}-1} + \frac{1}{h^2}\left(\delta_x \Psi_{\mathcal{M}-\frac{3}{2}}^{n-\frac{1}{2}} - 3\delta_x \Psi_{\mathcal{M}-\frac{1}{2}}^{n-\frac{1}{2}}\right) - f_{\mathcal{M}-1}^{n-\frac{1}{2}}\\ &= (R_x)_{\mathcal{M}-1}^n + (R_t)_{\mathcal{M}-1}^n,\ \ 1 \le n \le \mathcal{N}.\end{aligned} \tag{2.22}$$

A truncation error in time and space is represented respectively by $(R_x)_j^n$ and $(R_t)_j^n$. This is achieved by reusing the Theorem 2.3 and Eqs. (2.19-2.20) there will be a positive constant η such that:

$$|(R_t)_j^n| \le \frac{\eta}{n^{\min\{r\nu, 2-\nu\}}},\ |(R_x)_j^n| \le \eta h,\ j = 0, 1, 2, ..., \mathcal{M},\ n = 0, 1, 2, ..., \mathcal{N}. \tag{2.23}$$

As an initial boundary condition, the following conditions apply:

$$\begin{cases}\Psi_j^0 = \vartheta(x_j) = \vartheta_j, & 1 \le j \le \mathcal{M}-1,\\ \Psi_0^n = 0, \quad \Psi_{\mathcal{M}}^0 = 0, & 1 \le n \le \mathcal{N}.\end{cases} \tag{2.24}$$

Now with eliminating truncated errors $(R_x)_j^n$ and $(R_t)_j^n$, and by substituting the exact solution Ψ_j^n with the numerical equivalent ψ_j^n in Equations (2.21) and (2.22), we propose the following Crank-Nicolson difference scheme as follows:

$$\begin{aligned}&\frac{1}{\mu_n}\left[a_0^\nu \psi_j^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}\left(a_{n-k-1}^\nu - a_{n-k}^\nu\right)\psi_j^{k-\frac{1}{2}} - a_{n-1}^\nu \psi_j^0\right] - \lambda \psi_j^{n-\frac{1}{2}} + \theta\chi(\psi^{n-\frac{1}{2}}, \psi^{n-\frac{1}{2}})_j\\ &+ \delta_x^3 \psi_{j+\frac{1}{2}}^{n-\frac{1}{2}} - f_j^{n-\frac{1}{2}} = 0,\ \ 1 \le j \le \mathcal{M}-2,\ \ 1 \le n \le \mathcal{N},\end{aligned} \tag{2.25}$$

and for $j = \mathcal{M} - 1$

$$\begin{aligned}&\frac{1}{\mu_n}\left[a_0^\nu \psi_{\mathcal{M}-1}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}\left(a_{n-k-1}^\nu - a_{n-k}^\nu\right)\psi_{\mathcal{M}-1}^{k-\frac{1}{2}} - a_{n-1}^\nu \psi_{\mathcal{M}-1}^0\right] - \lambda \psi_{\mathcal{M}-1}^{n-\frac{1}{2}}\\ &+ \theta\chi(\psi^{n-\frac{1}{2}}, \psi^{n-\frac{1}{2}})_{\mathcal{M}-1} + \frac{1}{h^2}\left(\delta_x \psi_{\mathcal{M}-\frac{3}{2}}^{n-\frac{1}{2}} - 3\delta_x \psi_{\mathcal{M}-\frac{1}{2}}^{n-\frac{1}{2}}\right) - f_{\mathcal{M}-1}^{n-\frac{1}{2}} = 0,\ \ 1 \le n \le \mathcal{N},\end{aligned} \tag{2.26}$$

$$\begin{cases}\psi_j^0 = \vartheta(x_j) = \vartheta_j, & 1 \le j \le \mathcal{M}-1,\\ \psi_0^n = 0, \quad \psi_{\mathcal{M}}^0 = 0, & 1 \le n \le \mathcal{N}.\end{cases} \tag{2.27}$$

The computational molecule of its numerical scheme is shown in Figures 1 and 2 for clarity. Vertical and horizontal axes represent time and space discretization, respectively, and $h = \tau$ was also assumed as a convenience.

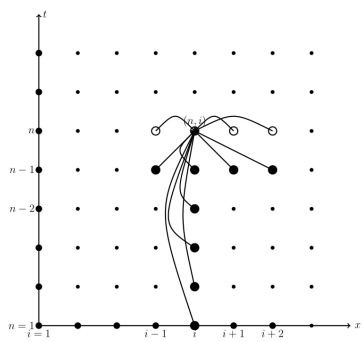

Figure 1. when $1 \le j \le \mathcal{M}-2, \;\; 1 \le n \le \mathcal{N}$.

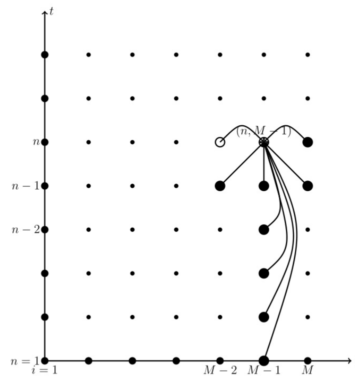

Figure 2. when $j = \mathcal{M}-1, \;\; 1 \le n \le \mathcal{N}$.

3. Analysis of the existence of numerical solution

Let $\xi = (\xi_0, \xi_1, ..., \xi_{\mathcal{M}}) \in \tilde{\Psi}^0$, considering the [26], this is not a difficult task

$$h\sum_{j=1}^{\mathcal{M}-2}(\delta_x^2\delta_x\xi_{j+\frac{1}{2}})\xi_j + \frac{1}{h}(\delta_x\xi_{\mathcal{M}-\frac{3}{2}} - 3\delta_x\xi_{\mathcal{M}-\frac{1}{2}})\xi_{\mathcal{M}-1} = \frac{1}{2}h^2\sum_{j=1}^{\mathcal{M}-2}(\delta_x^2\xi_j)^2 + \frac{1}{2}(\delta_x\xi_{\frac{1}{2}})^2 + \frac{3}{2}(\delta_x\xi_{\mathcal{M}-\frac{1}{2}})^2. \tag{3.1}$$

Lemma 3.1 ([29]). *Assume that $\psi, \xi \in \tilde{\Psi}^0$. Then $(\chi(\psi,\xi),\psi)$ is equal to 0.*

Firstly, the Browder theorem is presented so that we can prove existence and uniqueness. Within this paper, we will use the following notation for convenience $\lambda^* := \max\{\lambda, 0\}$.

Theorem 3.1 ([7]). *Assume that $\mathcal{H}$ is a finite-dimensional space with the inner product $(.,.)_{\mathcal{H}}$ and the induced norm $\|.\|_{\mathcal{H}}$, and $\Pi : \mathcal{H} \to \mathcal{H}$ is a continuous operator. If there exists a constant $\Xi > 0$ such that*

$$\mathcal{R}(\Pi(Z), Z)_{\mathcal{H}} \ge 0, \forall Z \in \mathcal{H} : \|Z\|_{\mathcal{H}} = \Xi,$$

there exists an element $Z^* \in \mathcal{H}$ with $\|Z^*\|_H \le \Xi$ such that $\Pi(Z^*) = 0$.

Theorem 3.2. *Assume $\lambda^*\mu_n\tau^{\nu}_{n-\frac{1}{2}} < \frac{1}{2}$. Then there's a solution of (2.25 -2.27).*

Proof. Based on (2.27), ψ^0 is given.
Now, let's assume that $\{\psi^k \,\big|\, \psi^k \in \tilde{\Psi}^0_h, 0 \le k \le n-1\}$ has been determined. Let $\psi^{n-\frac{1}{2}} \in \tilde{\Psi}^0_h$ and define

$$\Pi(\psi^{n-\frac{1}{2}})_j = \frac{1}{\mu_n}\left[a_0^{\nu}\psi_j^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}\left(a^{\nu}_{n-k-1} - a^{\nu}_{n-k}\right)\psi_j^{k-\frac{1}{2}} - a^{\nu}_{n-1}\psi_j^0\right] \tag{3.2}$$

$$- \lambda\psi_j^{n-\frac{1}{2}} + \theta\chi(\psi^{n-\frac{1}{2}}, \psi^{n-\frac{1}{2}})_j + \delta_x^2(\delta_x\psi_{j+\frac{1}{2}}^{n-\frac{1}{2}}), \quad 1 \le j \le \mathcal{M}-2,$$

$$\Pi(\psi^{n-\frac{1}{2}})_{\mathcal{M}-1} = \frac{1}{\mu_n}\left[a_0^\nu\psi_{\mathcal{M}-1}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}\left(a_{n-k-1}^\nu - a_{n-k}^\nu\right)\psi_{\mathcal{M}-1}^{k-\frac{1}{2}} - a_{n-1}^\nu\psi_{\mathcal{M}-1}^0\right] \tag{3.3}$$

$$- \lambda\psi_{\mathcal{M}-1}^{n-\frac{1}{2}} + \theta\chi(\psi^{n-\frac{1}{2}}, \psi^{n-\frac{1}{2}})_{\mathcal{M}-1} + \frac{1}{h^2}(\delta_x\psi_{\mathcal{M}-\frac{3}{2}}^{n-\frac{1}{2}} - 3\delta_x\psi_{\mathcal{M}-\frac{3}{2}}^{n-\frac{1}{2}}),$$

with $\Pi(\psi^{n-\frac{1}{2}})_0 = \Pi(\psi^{n-\frac{1}{2}})_\mathcal{M} = 0$.

$$\begin{aligned}
&(\Pi(\psi^{n-\frac{1}{2}}), \psi^{n-\frac{1}{2}})\\
=&\frac{1}{\mu_n}\left[{a_0}^\nu(\psi^{n-\frac{1}{2}}, \psi^{n-\frac{1}{2}}) - \sum_{k=1}^{n-1}(a_{n-k-1}^\nu - a_{n-k}^\nu)(\psi^{k-\frac{1}{2}}, \psi^{n-\frac{1}{2}}) - {a_{n-1}}^\nu(\psi^0, \psi^{n-\frac{1}{2}})\right]\\
&- \lambda(\psi^{n-\frac{1}{2}}, \psi^{n-\frac{1}{2}}) + \theta(\chi(\psi^{n-\frac{1}{2}}, \psi^{n-\frac{1}{2}}), \psi^{n-\frac{1}{2}})_{\mathcal{M}-1} + h[\sum_{i=1}^{\mathcal{M}-2}\psi_j^{n-\frac{1}{2}}\delta_x^2(\delta_x\psi_{j+\frac{1}{2}}^{n-\frac{1}{2}})\\
&+ \frac{1}{h^2}\psi_{\mathcal{M}-1}^{n-\frac{1}{2}}(\delta_x\psi_{\mathcal{M}-\frac{3}{2}}^{n-\frac{1}{2}} - 3\delta_x\psi_{\mathcal{M}-\frac{1}{2}}^{n-\frac{1}{2}})].
\end{aligned} \tag{3.4}$$

Now, implementation (3.1) and Lemma 3.1 we get

$$\begin{aligned}
&(\Pi(\psi^{n-\frac{1}{2}}), \psi^{n-\frac{1}{2}})\\
\ge&\frac{1}{2}D_t^\nu(\left\|\psi^{n-\frac{1}{2}}\right\|^2) - \lambda\left\|\psi^{n-\frac{1}{2}}\right\|^2\\
=&\frac{1}{2\mu_n}\left[a_0^\nu\left\|\psi^{n-\frac{1}{2}}\right\|^2 - \sum_{k=1}^{n-1}(a_{n-k-1}^\nu - a_{n-k}^\nu)\left\|\psi^{n-\frac{1}{2}}\right\|^2 - a_{n-1}^\nu\left\|\psi^0\right\|^2\right] - \lambda\left\|\psi^{n-\frac{1}{2}}\right\|^2\\
\ge&\frac{1}{2\mu_n}\left\{a_0^\nu\left\|\psi^{n-\frac{1}{2}}\right\|^2 - \left[\sum_{k=1}^{n-1}(a_{n-k-1}^\nu - a_{k-1}^\nu + a_{n-1})\right]\max_{0\le k\le n-1}\left\|\psi^{k-\frac{1}{2}}\right\|^2\right\}\\
&- \lambda\left\|\psi^{n-\frac{1}{2}}\right\|^2\\
=&\frac{1}{2\mu_n}\left\{[a_0^\nu - 2\lambda\mu_n]\left\|\psi^{k-\frac{1}{2}}\right\|^2 - a_0^\nu\max_{0\le k\le n-1}\left\|\psi^{k-\frac{1}{2}}\right\|^2\right\},
\end{aligned} \tag{3.5}$$

when

$$\left\|\psi^{n-\frac{1}{2}}\right\|^2 = \frac{a_0^\nu}{a_0^\nu - 2\lambda\mu_n}\max_{0\le k\le n-1}\left\|\psi^{k-\frac{1}{2}}\right\|^2 = \frac{1}{1 - 2\lambda\mu_n\tau_{n-\frac{1}{2}}^\nu}\max_{0\le k\le n-1}\left\|\psi^{k-\frac{1}{2}}\right\|^2, \tag{3.6}$$

and we have

$$(\Pi(\psi^{n-\frac{1}{2}}), \psi^{n-\frac{1}{2}}) \ge 0. \tag{3.7}$$

With assumption $\lambda^*\mu_n\tau_n^\nu < \frac{1}{2}$ and Theorem 3.1, there exists a $\psi^{n-\frac{1}{2}} \in \tilde{\Psi}^0$ such that $\Pi(\psi^{n-\frac{1}{2}}) = 0$ and

$$\left\|\psi^{n-\frac{1}{2}}\right\| \le \frac{\max_{0\le k\le n-1}\left\|\psi^{k-\frac{1}{2}}\right\|^2}{\sqrt{1 - 2\lambda\mu_n\tau_{n-\frac{1}{2}}^\nu}}. \tag{3.8}$$

□

4. Analysis on the stability

Assume that $f^k = f(t_k)$ and the $\nabla_\tau f^k = f^k - f^{k-1}$ difference operator for $k \geq 1$. Using the nonuniform $L1$ calculations, the Caputo derivative is given by

$$D_N^\nu f^n = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \omega_{1-\nu}(t_n - w) \nabla_\tau \frac{f^k}{\tau_k} dw = \sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau f^k, \tag{4.1}$$

based on the kernel $\omega_\nu(t) = \frac{t^\nu}{\Gamma(\nu+1)}, t > 0$. In this case, $a_{n-k}^{(n)}$ represents the time-level-dependent convolution coefficient

$$a_{n-k}^{(n)} = \int_{t_{k-1}}^{t_k} \frac{\omega_{1-\nu}(t_n - w)}{\tau_k} dw = \frac{\omega_{2-\nu}(t_n - t_{k-1}) - \omega_{2-\nu}(t_n - t_k)}{\tau_k}, \quad 1 \leq k \leq n. \tag{4.2}$$

It follows that the integrated mean-value theorem says that $L1$ coefficient (4.2) satisfies (refer to [32])

$$a_{n-k+1}^{(n)} < \omega_{1-\nu}(t_n - t_{k-1}) < a_{n-k}^{(n)}, \quad 1 \leq k \leq n. \tag{4.3}$$

Here is a numerical analysis of the nonuniform $L1$ scheme's numerical stability (1.1). In order to do that, the $L1$ coefficient (4.2) is used for recursively defining a convolutional coefficient for $n \geq 1$, in the following way

$$Q_{n-k}^{(n)} = \frac{1}{a_0^{(k)}} \begin{cases} 1, & k = n, \\ \sum\limits_{j=k+1}^{n} (a_{j-k-1}^{(j)} - a_{j-k}^{(j)}) Q_{n-j}^{(n)}, & 1 \leq k \leq n-1. \end{cases} \tag{4.4}$$

The next lemma will be a numerical simulation of two properties of the convolutional kernel $\omega_\nu(t - w)$:

$$\sum_{j=k}^{n} \int_{t_{j-1}}^{t_j} \omega_\nu(t_n - w) \omega_{1-\nu}(w - t_{k-1}) dw = 1, \quad n \geq k \geq 1, \tag{4.5}$$

$$\sum_{j=k}^{n} \int_{t_{j-1}}^{t_j} \omega_\nu(t_n - w) \omega_{1+m\nu-\nu}(w) dw = \omega_{1+m\nu}(t_n), \quad m \geq 0, \ n \geq 1. \tag{4.6}$$

For the property (4.5) form to be preserved, the equality $\sum\limits_{j=k}^{n} Q_{n-j}^{(n)} a_{j-k}^{(j)} \equiv 1$ must be enforced discrete (refer to Lemma 4.1), thereby delivering the Definition (4.4) of $Q_{n-j}^{(n)}$. For this and all subsequent summations, if the upper summation index is lower than the lower summation index, the sums will always be zero.

Lemma 4.1 ([17]). *$Q_{n-k}^{(n)}$ defined in (4.4) is a discrete coefficient that is well defined by*

$$0 < Q_{n-k}^{(n)} < \Gamma(2-\nu) \tau_k^\nu, \quad 1 \leq k \leq n. \tag{4.7}$$

Furthermore, the following hold:
(I) we have

$$\sum_{j=k}^{n} Q_{n-j}^{(n)} a_{j-k}^{(j)} = 1, \quad n \leq k \leq 1.$$

(II) With respect to any nonnegative integer $0 \le m \le [\frac{1}{\nu}]$,

$$\sum_{j=k}^{n} Q_{n-j}^{(n)} \omega_{1+m\nu-\nu}(t_j) \le \omega_{1+m\nu}(t_n), \quad n \ge 1.$$

(III) If $\tau_{k-1} \le \tau_k, 2 \le k \le \mathcal{N}$ *is satisfied by the nonuniform grid, for any integer* $m \ge 1$,

$$\sum_{j=1}^{n-1} Q_{n-j}^{(n)} \omega_{1+m\nu-\nu}(t_j) \le \omega_{1+m\nu}(t_n), \quad n \ge 1.$$

Lemma 4.2 ([22]). *Assuming*

$$a_k^{\nu} = (n+1)^{\nu} - n^{\nu}, \quad n = 0, 1, 2, \dots, \tag{4.8}$$

the following properties apply to $a_k^{\nu} = (k+1)^{\nu} - k^{\nu}, \quad k = 0, 1, 2, \cdots,$

1. $a_0^{\nu} = 1, a_0^{\nu} > 0, \quad k = 0, 1, 2, \cdots,$
2. $a_k^{\nu} > a_{k+1}^{\nu}, \quad k = 0, 1, 2, \cdots.$

Lemma 4.3. *[17] Assume that there is a constant* $\mathcal{F} \ge 0$, *and* $\mathcal{P}^n$, $\mathcal{J}^n$ *and* $\mathcal{K}^n$ *are non-negative sequences, so that*

$$D_t^{\nu}(\mathcal{P})^2 \le \mathcal{F}(\mathcal{P}^n)^2 + \mathcal{P}^n(\mathcal{J}^n + \mathcal{K}^n), \quad n \ge 1. \tag{4.9}$$

Specifying any finite time, $t_n = T > 0$, *with* $\tau_{\max} \le (\frac{1}{2\mathcal{F}\Gamma(2-\nu)})^{\frac{1}{\nu}}$,

$$\mathcal{P}^n \le 2E_{\nu}(2\mathcal{F}t_n^{\nu}) \left(\mathcal{P}^0 + \max_{1 \le j \le n} \sum_{l=1}^{j} Q_{j-l}^{(j)} \mathcal{J}^l + \omega_{\nu}(t_n) \max_{1 \le j \le n} \mathcal{K}^j \right), \quad 1 \le n \le \mathcal{N}, \tag{4.10}$$

where $E_{\nu}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(k\nu+1)}$ *is a Mittag-Leffler function,* $\omega_{\nu}(t) = \frac{t^{\nu}}{\Gamma(\nu+1)}, t > 0$, *and the discrete convolution kernel* $Q_{n-k}^{(n)}$ *is defined by*

$$Q_{n-k}^{(n)} = \begin{cases} \Gamma(2-\nu)\tau_n^{\nu}, \\ (1-\nu)\tau_k^{\nu} \sum\limits_{j=k+1}^{n} (a_{k+1}^{(j,\nu)} - a_k^{(j,\nu)}) Q_{n-j}^{(n)}, \quad k = n-1, n-2, \cdots, 1. \end{cases} \tag{4.11}$$

Theorem 4.1. *Assume that* $\left\{ \psi_j^{n-\frac{1}{2}} \middle| 0 \le j \le \mathcal{M}, 0 \le n \le \mathcal{N} \right\}$ *is the solution of the scheme (2.25)-(2.27), we have*

$$\left\| \psi^{n-\frac{1}{2}} \right\| \le E_{\nu}(4\lambda^* t_{n-\frac{1}{2}}^{\nu}) \left\| \vartheta \right\|, \quad 1 \le n \le \mathcal{N}. \tag{4.12}$$

Proof. Multiplying (2.25) and (2.26) by $h{\psi_j}^{n-\frac{1}{2}}$ and $h{\psi_{\mathcal{M}-1}}^{n-\frac{1}{2}}$ respectively, and with summing up for j, we get

$$\frac{1}{\mu_n} \left[a_0^{\nu} \left\| \psi^{n-\frac{1}{2}} \right\|^2 - \sum_{k=1}^{n-1} (a_{n-k-1}^{\nu} - a_{n-k}^{\nu})(\psi^{k-\frac{1}{2}}, \psi^{n-\frac{1}{2}}) - a_{n-1}^{\nu}(\psi^0, \psi^{n-\frac{1}{2}}) \right]$$

$$-\lambda\left\|\psi^{n-\frac{1}{2}}\right\|^2+\theta(\chi(\psi^{n-\frac{1}{2}},\psi^{n-\frac{1}{2}}),\psi^{n-\frac{1}{2}})+h\sum_{j=1}^{\mathcal{M}-2}\delta_x^2(\delta_x\psi_{j+\frac{1}{2}}^{n-\frac{1}{2}})\psi_j^{n-\frac{1}{2}}$$
$$+\frac{1}{h}(\delta_x\psi_{\mathcal{M}-\frac{3}{2}}^{n-\frac{1}{2}}-3\delta_x\psi_{\mathcal{M}-\frac{1}{2}}^{n-\frac{1}{2}})\psi_{\mathcal{M}-1}^{n-\frac{1}{2}}=0. \tag{4.13}$$

By (3.1) and Lemma 3.1, we get

$$\begin{aligned}&h\sum_{j=1}^{\mathcal{M}-2}\delta_x^2(\delta_x\psi_{j+\frac{1}{2}}^{n-\frac{1}{2}})\psi_j^{n-\frac{1}{2}}+\frac{1}{h}(\delta_x\psi_{\mathcal{M}-\frac{3}{2}}^{n-\frac{1}{2}}-3\delta_x\psi_{\mathcal{M}-\frac{1}{2}}^{n-\frac{1}{2}})\psi_{\mathcal{M}-1}^{n-\frac{1}{2}}\\&=\frac{1}{2}h^2\sum_{j=1}^{\mathcal{M}-2}(\delta^2{}_x\psi_j^{n-\frac{1}{2}})+\frac{1}{2}(\delta_x\psi_{\frac{1}{2}}^{n-\frac{1}{2}})+\frac{3}{2}(\delta_x\psi_{\mathcal{M}-\frac{1}{2}}^{n-\frac{1}{2}}),\end{aligned} \tag{4.14}$$

$$(\chi(\psi^{n-\frac{1}{2}},\psi^{n-\frac{1}{2}}),\psi^{n-\frac{1}{2}})=0. \tag{4.15}$$

If we substitute (4.14) and (4.15) into (4.13), we get

$$\begin{aligned}&\frac{1}{\mu_n}\left[a_0^\nu\left\|\psi^{n-\frac{1}{2}}\right\|^2-\sum_{k=1}^{n-1}(a_{n-k-1}^\nu-a_{n-k}^\nu)(\psi^{k-\frac{1}{2}},\psi^{n-\frac{1}{2}})-a_{n-1}^\nu(\psi^0,\psi^{n-\frac{1}{2}})\right]\\&\leq\lambda^*\left\|\psi^{n-\frac{1}{2}}\right\|^2,\end{aligned} \tag{4.16}$$

Using Lemma 4.2, we can obtain

$$\begin{aligned}&\frac{1}{\mu_n}\left[a_0^\nu\left\|\psi^{n-\frac{1}{2}}\right\|^2-\sum_{k=1}^{n-1}(a_{n-k-1}^\nu-a_{n-k}^\nu)(\psi^{k-\frac{1}{2}},\psi^{n-\frac{1}{2}})-a_{n-1}^\nu(\psi^0,\psi^{n-\frac{1}{2}})\right]\\&\geq\frac{1}{2\mu_n}[a_0^\nu\left\|\psi^{n-\frac{1}{2}}\right\|^2-\sum_{k=1}^{n-1}(a_{n-k-1}^\nu-a_{n-k}^\nu)\left\|\psi^{n-\frac{1}{2}}\right\|^2-a_{n-1}^\nu\left\|\psi^{n-\frac{1}{2}}\right\|^2\\&\quad+a_0^\nu\left\|\psi^{n-\frac{1}{2}}\right\|^2-\sum_{k=1}^{n-1}(a_{n-k-1}^\nu-a_{n-k}^\nu)\left\|\psi^{k-\frac{1}{2}}\right\|^2-a_{n-1}^\nu\left\|\psi^0\right\|^2]\\&=\frac{1}{2}D_t^\nu\left\|\psi^{n-\frac{1}{2}}\right\|^2.\end{aligned} \tag{4.17}$$

Combining with (4.16), we know

$$D_t^\nu\left\|\psi^{n-\frac{1}{2}}\right\|^2\leq2\lambda^*\left\|\psi^{n-\frac{1}{2}}\right\|^2,\quad1\leq n\leq\mathcal{N}. \tag{4.18}$$

By utilizing Lemma 4.3, we find the following

$$\left\|\psi^{n-\frac{1}{2}}\right\|^2\leq2E_\nu(4\lambda^*t_{n-\frac{1}{2}}^\nu)\left\|\vartheta\right\|,\quad0\leq n\leq\mathcal{N}. \tag{4.19}$$

□

5. Conducting a convergence analysis

In order to have a new property for the $L1$ coefficient $a_{n-k}^{(n)}$, we describe it below.

Lemma 5.1. *In the nonuniform grid case $\tau_{k-1} \le \tau_k, 2 \le k \le \mathcal{N}$, the $L1$ coefficient (4.2) satisfies, while*

$$0 < \omega_{1-\nu}(t_n - t_k) - \omega_{1-\nu}(t_n - t_{k-1}) \le a^{(n)}_{n-k-1} - a^{(n)}_{n-k}, \quad 1 \le k \le n-1. \tag{5.1}$$

Proof. Observing $\omega'_\beta(t) = \omega_{\beta-1}(t)$, $\int_0^t \omega_\beta(\alpha)d\alpha = \omega_{\beta+1}(t)$ we apply the Definition (4.2) of the $L1$ coefficient $a^{(n)}_{n-k}$ to find

$$\begin{aligned} a^{(n)}_{n-k} - \omega_{1-\nu}(t_n - t_{k-1}) &= \int_{t_{k-1}}^{t_k} \frac{\omega_{1-\nu}(t_n - w) - \omega_{1-\nu}(t_n - t_{k-1})}{\tau_k} dw \\ &= -\frac{1}{\tau_k}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{w} \omega_{-\nu}(t_n - \alpha)d\alpha dw \\ &= \int_{t_{k-1}}^{t_k} \frac{\alpha - t_k}{\tau_k}\omega_{-\nu}(t_n - \alpha)d\alpha > 0, \quad 1 \le k \le n. \end{aligned} \tag{5.2}$$

In the previous equality, the order of integration has been switched, and $\omega_{-\nu}(t_n - \alpha) < 0$ for $0 < \nu < 1$. Taking auxiliary functions into consideration now would be appropriate:

$$\begin{aligned} \Upsilon_1(\Theta) &= \int_{t_{k-1}}^{t_{k-1}+\Theta\tau_k} \frac{\alpha - t_{k-1} - \Theta\tau_k}{\tau_k}\omega_{-\nu}(t_n - \alpha)d\alpha, \quad 1 \le k \le n, \\ \Upsilon_2(\Theta) &= \int_{t_{k-1}}^{t_{k-1}+\Theta\tau_{k+1}} \frac{\alpha - t_{k-1} - \Theta\tau_{k+1}}{\tau_{k+1}}\omega_{-\nu}(t_n - \alpha)d\alpha, \quad 1 \le k \le n-1, \end{aligned} \tag{5.3}$$

such that $\Upsilon_1(0) = \Upsilon'_1(0) = \Upsilon_2(0) = \Upsilon'_2(0) = 0$. Thanks to the Cauchy differential mean-value theorem, there exists $\kappa_k, \varsigma_k \in (0,1)$ such that

$$\begin{aligned} \frac{a^{(n)}_{n-k} - \omega_{1-\nu}(t_n - t_{k-1})}{a^{(n)}_{n-k-1} - \omega_{1-\nu}(t_n - t_k)} &= \frac{\Upsilon_1(1) - \Upsilon_1(0)}{\Upsilon_2(1) - \Upsilon_2(0)} = \frac{\Upsilon'_1(\kappa_k)}{\Upsilon'_2(\kappa_k)} \\ &= \frac{\Upsilon'_1(\kappa_k) - \Upsilon'_1(0)}{\Upsilon'_2(\kappa_k) - \Upsilon'_2(0)} = \frac{\Upsilon''_1(\varsigma_k)}{\Upsilon''_2(\varsigma_k)} \\ &= \frac{\tau_k\omega_{-\nu}(t_n - t_{k-1} - \varsigma_k\tau_k)}{\tau_{k+1}\omega_{-\nu}(t_n - t_k - \varsigma_k\tau_{k+1})} \le \frac{(t_n - t_k - \varsigma_k\tau_{k+1})^{\nu+1}}{(t_n - t_k + \tau_k - \varsigma_k\tau_k)^{\nu+1}} \\ &\le \frac{(t_n - t_k)^{\nu+1}}{(t_n - t_k + (1-\varsigma_k)\tau_k)^{\nu+1}} \le 1, \quad 1 \le k \le n-1, \end{aligned} \tag{5.4}$$

where the basic assumption $\tau_{k-1} \le \tau_k, 2 \le k \le \mathcal{N}$ is used in the first inequality. □

Lemma 5.2 ([26]). *Assume that $f \in \mathcal{C}^2((0,T])$ and there exists a constant $\mathcal{C}_f > 0$ such that*

$$|f''(t)| \le C_f(1 + t^{\varepsilon-2}), \quad 0 < t \le T, \tag{5.5}$$

where $\varepsilon \in (0,1) \cup (1,2)$ is a regularity parameter. If the nonuniform grid fulfills

$\tau_{k-1} \leq \tau_k, 2 \leq k \leq \mathcal{N}$, then

$$\begin{aligned}
&\sum_{j=1}^{n} Q_{n-j}^{(n)} \left| {}_0^C D_t^{\nu} f(t_j) - D_t^{\nu} f(t_j) \right| \\
\leq &\sum_{k=1}^{n} \frac{2Q_{n-k}^{(n)}}{\Gamma(2-\nu)} \int_{t_{k-1}}^{t_k} (\alpha - t_{k-1}) \left| f''(\alpha) \right| d\alpha \\
\leq &C_f \left(\frac{\tau_1}{\varepsilon} + \frac{1}{1-\nu} \max_{2 \leq k \leq n} (t_k - t_1)^{\nu} t_{k-1}^{\varepsilon-2} \tau_k^{2-\nu} \right), \quad n \geq 1.
\end{aligned} \tag{5.6}$$

In this case, the discrete convolution kernel $Q_{n-j}^{(n)}$ is defined by (4.4).

The Lemma 5.2 provides an easy way to calculate the global temporal error of the $L1$ formula on non-uniform meshes.

Let us recall that Lemma 5.2 holds for most nonuniform meshes under a weak assumption $\tau_{k-1} \leq \tau_k, 2 \leq k \leq \mathcal{N}$. Furthermore, when considering a uniform mesh where $\tau = T\mathcal{N}^{-1}$ and $t_k = k\tau$, Lemma 5.2 states that

$$\begin{aligned}
&\sum_{j=1}^{n} Q_{n-j}^{(n)} \left| {}_0^C D_t^{\nu} f(t_j) - D_t^{\nu} f(t_j) \right| \\
\leq &\frac{C_f \tau^{\varepsilon}}{\varepsilon} + C_f \tau^{\min\{\varepsilon, 2-\nu\}} \max_{2 \leq k \leq n} t_{k-1}^{\varepsilon-2+\nu} \tau^{2-\nu-\min\{\varepsilon,2-\nu\}} \\
\leq &C_f \left(\frac{\tau^{\varepsilon}}{\varepsilon} + \frac{1}{1-\nu} T^{\varepsilon - \min\{\varepsilon,2-\nu\}} \tau^{\min\{\varepsilon,2-\nu\}} \right), \quad n \geq 1.
\end{aligned} \tag{5.7}$$

If the initial regularity of the solution for $\varepsilon \leq 2-\nu$ improves, the convergence order increase, but the accuracy barrier lies at $O(\tau^{2-\nu})$. The corollary is thus as follows.

Corollary 5.1. *Let's assume that f fulfills the regularity assumption(5.5). In the case of uniform time meshing, the approximate error of $L1$ formula (4.2) can be bounded by (5.7). In other words the uniform time grid is optimal if $\varepsilon \in [2-\nu, 2)$.*

According to [8, 9, 14, 20] the smoothly graded time mesh $t_k \leq T(\frac{k}{\mathcal{N}})^{\zeta}$ is now considered when the grading parameter $\zeta > 1$. By verifying the time-step $\tau_k \leq \zeta k^{\zeta-1} T\mathcal{N}^{-\zeta}$ and

$$t_k - t_1 = T\mathcal{N}^{-\zeta}(k^{\zeta} - 1) \leq \zeta k^{\zeta-1}(k-1)T\mathcal{N}^{-\zeta}, \quad 2 \leq k \leq \mathcal{N}. \tag{5.8}$$

Additionally, it states:

$$\begin{aligned}
&(t_k - t_1)^{\nu} t_{k-1}^{\varepsilon-2} \tau_k^{2-\nu} \leq \zeta^2 T^{\varepsilon} k^{2(\zeta-1)} (k-1)^{\zeta\varepsilon - 2\zeta + \nu} \mathcal{N}^{-\zeta\varepsilon} \leq \zeta^2 4^{\zeta-1} T^{\varepsilon} (k-1)^{\zeta\varepsilon - 2 + \nu} \mathcal{N}^{-\zeta\nu} \\
= &\zeta^2 4^{\zeta-1} T^{\varepsilon} (k-1)^{\min\{\zeta\varepsilon, 2-\nu\} - (2-\nu)} \left(\frac{k-1}{\mathcal{N}}\right)^{\zeta\varepsilon - \min\{\zeta\varepsilon, 2-\nu\}} \mathcal{N}^{-\min\{\zeta\varepsilon, 2-\nu\}} \\
\leq &\zeta^2 4^{\zeta-1} T^{\varepsilon} (k-1)^{\min\{\zeta\varepsilon, 2-\nu\}}, \quad 2 \leq k \leq \mathcal{N}.
\end{aligned} \tag{5.9}$$

Accordingly, inequality (5.6) in Lemma 5.2 results in

$$\sum_{j=1}^{\mathcal{N}} Q_{n-j}^{(n)} \left| {}_0^C D_t^{\nu} f(t_j) - D_t^{\nu} f(t_j) \right|$$

$$\leq C_f(\varepsilon^{-1}T^{\varepsilon}\mathcal{N}^{-\zeta\varepsilon}+\frac{\zeta^2}{1-\nu}4^{\zeta-1}T^{\varepsilon}\mathcal{N}^{-\min\{\zeta\varepsilon,2-\nu\}}),\quad n\geq 1. \tag{5.10}$$

This means that the error of the nonuniform $L1$ formula (4.1) can be bounded by (5.10).

Theorem 5.1. *Assume that the original problem (1.1) has a solution. Then the numerical solution* $\Psi_j^{n-\frac{1}{2}}$ *of the difference scheme (2.25-2.27) is convergent to the true solution* $\psi(x,t)$.
Denote

$$\mathcal{E}_j^n=\Psi_j^{n-\frac{1}{2}}-\psi_j^{n-\frac{1}{2}},\quad 0\leq j\leq\mathcal{M},0\leq n\leq\mathcal{N}. \tag{5.11}$$

Therefore it is possible for there to be a constant C *that depends neither on* $\mathcal{M}$ *nor* $\mathcal{N}$*, in such a way that*

$$\|\mathcal{E}^n\|\leq C(\mathcal{N}^{-\min\{r\nu,2-\nu\}}+h),\quad 0\leq n\leq\mathcal{N}. \tag{5.12}$$

Proof. We can obtain the error equations by subtracting (2.25)-(2.27) from (2.21), (2.22) and (2.24)

$$\frac{1}{\mu_n}[a_0^{\nu}\mathcal{E}_j^n-\sum_{k=1}^{n-1}(a_{n-k-1}^{\nu}-a_{n-k}^{\nu})\mathcal{E}_j^k-a_{n-1}^{\nu}\mathcal{E}_j^0]-\lambda\mathcal{E}_j^0+\theta[\chi(\Psi^{n-\frac{1}{2}},\Psi^{n-\frac{1}{2}})_j$$
$$-\chi(\psi^{n-\frac{1}{2}},\psi^{n-\frac{1}{2}})_j]+\delta_x^2(\delta_x\mathcal{E}_j^n)=(R_t)_j^{n-\frac{1}{2}}+(R_x)_j^{n-\frac{1}{2}}, \tag{5.13}$$

and

$$\frac{1}{\mu_n}[a_0^{\nu}\mathcal{E}_{\mathcal{M}-1}^n-\sum_{k=1}^{n-1}(a_{n-k-1}^{\nu}-a_{n-k}^{\nu})\mathcal{E}_{\mathcal{M}-1}^k-a_{n-1}^{\nu}\mathcal{E}_{\mathcal{M}-1}^0]-\lambda\mathcal{E}_{\mathcal{M}-1}^0$$
$$+\theta[\chi(\Psi^{n-\frac{1}{2}},\Psi^{n-\frac{1}{2}})_{\mathcal{M}-1}-\chi(\psi^{n-\frac{1}{2}},\psi^{n-\frac{1}{2}})_{\mathcal{M}-1}]+\frac{1}{h^2}\delta_x^2(\delta_x\mathcal{E}_{\mathcal{M}-\frac{3}{2}}^n-3\delta_x\mathcal{E}_{\mathcal{M}-\frac{1}{2}}^n)$$
$$=(R_t)_{\mathcal{M}-1}^{n-\frac{1}{2}}+(R_x)_{\mathcal{M}-1}^{n-\frac{1}{2}},\quad 1\leq n\leq\mathcal{N}, \tag{5.14}$$

where

$$\mathcal{E}_j^0=0,\quad 1\leq j\leq\mathcal{M}-1, \tag{5.15}$$
$$\mathcal{E}_0^n=0,\mathcal{E}_m^n=0,\quad 0\leq n\leq\mathcal{N}. \tag{5.16}$$

After multiplying (5.13) and (5.14) by $h\mathcal{E}_i^n$ and $h\mathcal{E}_i^{\mathcal{M}-1}$, and adding all terms together, we have

$$\frac{1}{\mu_n}\left[a_0^{\nu}\|\mathcal{E}^n\|^2-\sum_{k=1}^{n-1}(a_{n-k-1}^{\nu}-a_{n-k}^{\nu})(\mathcal{E}^k,\mathcal{E}^n)-a_{n-1}^{\nu}(\mathcal{E}^0,\mathcal{E}^n)\right]-\lambda\|\mathcal{E}^n\|^2$$
$$+\theta[(\chi(\Psi^{n-\frac{1}{2}},\Psi^{n-\frac{1}{2}})-\chi(\psi^{n-\frac{1}{2}},\psi^{n-\frac{1}{2}}),\mathcal{E}^n]+h\sum_{j=1}^{\mathcal{M}-2}(\delta_x^2\delta_x e_{j+\frac{1}{2}}^n)\mathcal{E}_j^n$$
$$+\frac{1}{h}(\delta_x\psi_{\mathcal{M}-\frac{3}{2}}^n-3\delta_x\psi_{\mathcal{M}-\frac{1}{2}}^{n-\frac{1}{2}})\mathcal{E}_{\mathcal{M}-1}^{n-\frac{1}{2}}=((R_t)_{\mathcal{M}-1}^n+(R_x)_{\mathcal{M}-1}^n,\mathcal{E}^n),\quad 0\leq n\leq\mathcal{N}. \tag{5.17}$$

By (3.1), we have

$$\begin{aligned} & h\sum_{j=1}^{\mathcal{M}-2}\delta_x^2(\delta_x\mathcal{E}_{j+\frac{1}{2}}^n)\mathcal{E}_j^n+\frac{1}{h}(\delta_x\mathcal{E}_{\mathcal{M}-\frac{3}{2}}^n-3\delta_x\mathcal{E}_{\mathcal{M}-\frac{1}{2}}^n)\mathcal{E}_{\mathcal{M}-1}^n \\ =&\frac{1}{2}h^2\sum_{j=1}^{\mathcal{M}-2}(\delta_x^2\mathcal{E}_j^n)^2+\frac{1}{2}(\delta_x\mathcal{E}_{\frac{1}{2}}^n)^2+\frac{3}{2}(\delta_x\mathcal{E}_{\mathcal{M}-\frac{1}{2}}^n)^2. \end{aligned} \tag{5.18}$$

Similar to (4.17) , it holds for the left hand of (5.17), for the first term, that

$$\frac{1}{\mu_n}\left[a_0^\nu\|\mathcal{E}^n\|^2-\sum_{k=1}^{n-1}(a_{n-k-1}^\nu-a_{n-k}^\nu)(\mathcal{E}^k,\mathcal{E}^n)-a_{n-1}^\nu(\mathcal{E}^0,\mathcal{E}^n)\right]\geq\frac{1}{2}D_t^\nu\|\mathcal{E}^n\|^2. \tag{5.19}$$

According to Lemma 3.1, the third term on the left hand of (5.17) may be found as follows:

$$\begin{aligned} & \left(\chi(\Psi^{n-\frac{1}{2}},\Psi^{n-\frac{1}{2}})-\chi(\psi^{n-\frac{1}{2}},\psi^{n-\frac{1}{2}}),\mathcal{E}^n\right) \\ =&\left(\chi(\Psi^{n-\frac{1}{2}},\Psi^{n-\frac{1}{2}})-\chi(\Psi^{n-\frac{1}{2}}-\mathcal{E}^n,\Psi^{n-\frac{1}{2}}-\mathcal{E}^n),\mathcal{E}^n\right) \\ =&\left(\chi(\mathcal{E}^n,\Psi^{n-\frac{1}{2}})+\chi(\Psi^{n-\frac{1}{2}},\mathcal{E}^n)-\chi(\mathcal{E}^n,\mathcal{E}^n),\mathcal{E}^n\right) \\ =&\left(\chi(\mathcal{E}^n,\Psi^{n-\frac{1}{2}}),\mathcal{E}^n\right) \\ =&\frac{1}{3}h\sum_{j=1}^{\mathcal{M}-1}[\mathcal{E}_j^n\Delta_x\Psi_j^{n-\frac{1}{2}}+\Delta_x(\mathcal{E}\Psi)_j^{n-\frac{1}{2}}]\mathcal{E}_j^n \\ =&\frac{1}{3}h\sum_{j=1}^{\mathcal{M}-1}(\Delta_x\Psi_j^{n-\frac{1}{2}})(\mathcal{E}_j^n)^2+\frac{1}{6}\sum_{j=1}^{\mathcal{M}-1}(\mathcal{E}_{j+1}^n\Psi_{j+1}^{n-\frac{1}{2}}-\mathcal{E}_{j-1}^n\Psi_{j-1}^{n-\frac{1}{2}})\mathcal{E}_j^n \\ =&\frac{1}{3}h\sum_{j=1}^{\mathcal{M}-1}(\Delta_x\Psi_j^{n-\frac{1}{2}})(\mathcal{E}_j^n)^2+\frac{1}{6}h\sum_{j=1}^{\mathcal{M}-1}(\mathcal{E}_{j+1}^n\mathcal{E}_j^n\delta_x\Psi_{j+\frac{1}{2}}^{n-\frac{1}{2}}). \end{aligned} \tag{5.20}$$

Denote $\varpi=\max_{0\leq x\leq L,0\leq t\leq T}|\psi_x(x,t)|$. Afterward

$$\begin{aligned} & -\left(\chi(\Psi^{n-\frac{1}{2}},\Psi^{n-\frac{1}{2}})-\chi(\psi^{n-\frac{1}{2}},\psi^{n-\frac{1}{2}}),\mathcal{E}^n\right) \\ \leq&\frac{1}{3}\varpi\left[h\sum_{j=1}^{\mathcal{M}-1}(\mathcal{E}_j^n)^2+\frac{h}{2}\left|\mathcal{E}_{j+1}^n\mathcal{E}_j^n\right|\right]\leq\frac{1}{2}\varpi\|\mathcal{E}^n\|^2. \end{aligned} \tag{5.21}$$

Substituting (5.18)-(5.21) into (5.17), we obtain

$$D_t^\nu\|\mathcal{E}^n\|^2\leq(2\lambda^*+|\theta|\,\varpi)\|\mathcal{E}^n\|^2+2\,\|\mathcal{E}^n\|\left(\left\|(R_t)^{n-\frac{1}{2}}\right\|+\left\|(R_x)^{n-\frac{1}{2}}\right\|\right). \tag{5.22}$$

For the maximum time step size $\tau_{\mathcal{N}}\leq(\frac{1}{4\lambda\mu_n})^{\frac{1}{\nu}}$, apply Lemmas 4.3 and 5.2

$$\|\mathcal{E}^n\|\leq 2E_\nu(2(2\lambda^*+|\theta|\,\varpi)t_{n-\frac{1}{2}}^\nu)$$

$$\times\left(\left\|\mathcal{E}^0\right\|+2\max_{1\le k\le n}\sum_{l=1}^{k}Q_{k-l}^{(k)}\left\|(R_t)^l\right\|+2\omega_\nu(t_n)\max_{1\le k\le n}\left\|(R_x)^k\right\|\right)$$
$$\le 4E_\nu(2(2\lambda^*+|\theta|\varpi)t_{n-\frac{1}{2}}^{\nu})$$
$$\times\left(\frac{C}{\nu}\tau_1^{\nu}+\frac{C}{1-\nu}\max_{2\le k\le n}(t_k-t_1)^{\nu}t_{k-1}^{\nu-2}\tau_{k-1}^{\nu-2}+\frac{t_n^{\nu}}{\Gamma(1+\nu)}h\right)$$
$$\le 4CE_\nu(2(2\lambda^*+|\theta|\varpi)t_{n-\frac{1}{2}}^{\nu})$$
$$(T^{\nu}\mathcal{N}^{-r\nu}+r^2 4^{r-1}T^{\nu}\mathcal{N}^{-\min\{r\nu,2-\nu\}}+t_{n-\frac{1}{2}}^{\nu}h),\quad 1\le n\le\mathcal{N}. \tag{5.23}$$

Neither $\mathcal{M}$ nor $\mathcal{N}$ affect the constant C. □

6. Optimizing the computation

To improve the computational efficiency, an algorithm has been applied which was presented in [1]. In the method, a useful approximate is estimated to decrease the computational cost as well as Caputo fractional derivative's evaluation memory with $0<\nu<1$ order. For scheme (2.25-2.27), we can directly share the fast algorithm and error estimate here. Detailed information is provided in [1], [25].

Lemma 6.1 ([1])**.** *Let assume that tolerance to error as ε, the cut-off time limitation as δ, and the final time as T, as well as existing a natural number $\mathcal{N}$ and two positive numbers $\mathcal{N}_{exp}$ and ι_i, $i=1,...,\mathcal{N}_{\rm exp}$ such as*

$$\left|t^{-\nu}-\sum_{i=1}^{\mathcal{N}_{exp}}\ell_i e^{-\iota_i t}\right|\le\varepsilon,\quad\forall\quad\delta\le t\le T, \tag{6.1}$$

where $\mathcal{N}_{exp}=\mathcal{O}((\log\varepsilon^{-1})(\log\log\varepsilon^{-1}+\log(T\delta^{-1}))+(\log\delta^{-1})(\log\log\varepsilon^{-1}+\log\delta^{-1}))$.

In (6.1), $\sum_{i=1}^{\mathcal{N}_{exp}}\ell_j e^{-\iota_l t}$ is an approximation to $\int_0^{2^{\mathcal{N}}}e^{-\iota t}\iota^{\beta-1}d\iota$ with respect to $0<\delta\le t\le T$, a Gauss-Jacobi integral has been used for the integral on sub intervals $[0,2^{-\mathcal{M}}]$ and $[2^{-\mathcal{M}},2^{\mathcal{N}}]$, and where $\mathcal{M}=\mathcal{O}(\log T)$, and $\mathcal{N}=((\log\log\varepsilon^{-1})+\log\delta^{-1})$ are supplied integers. As a result, in (6.1) ι_i and ℓ_i denote the nodes and weights of the Gauss-Legendre or Gauss-Jacobi quadrature at various intervals as detailed in Theorem 2.1 in [14].

When $f(t)$, $0\le t\le T$, which contains $0<\nu<1$, then denote

$$\begin{aligned}&{}^{H}D_t^{\nu}f(t_n)=\frac{1}{\mu_n}\left[\sum_{i=1}^{\mathcal{N}_{exp}}\ell_i H_i^n+a_n^{(n,\nu)}(f(t_n)-f(t_{n-1}))\right],\ n\ge1,\\&H_i^n=e^{-\iota_j\tau_n}H_i^{n-1}+X_i^n(f(t_n)-f(t_{n-1})),\ n\ge2,\ H_i^1=0,\\&X_i^n=\frac{1}{\tau_{n-1}}\int_{t_{n-2}}^{t_{n-1}}e^{-\iota_i(t_n-\iota)}d\iota,\ 1\le i\le\mathcal{N}_{exp}.\end{aligned} \tag{6.2}$$

Lemma 6.2 ([18])**.** *Let*

$$|f^{'}(t)|\le\gamma t^{\nu-1},\qquad|f^{''}(t)|\le\gamma t^{\nu-2},\qquad t\in(0,T]. \tag{6.3}$$

we have

$$D_t^\nu f(t_n) = {}^H D_t^\nu f(t_n) + \mathcal{O}(n^{-\min\{r(1+\nu),2-\nu\}} + \varepsilon), \quad n = 1, 2, ..., \mathcal{N}. \tag{6.4}$$

Optimizing the calculations for Problem (1.1) is the next step. By evaluating the Eq (1.1), we have the following

$$D_t^\nu \psi(x_j, t_n) - \lambda\psi(x_j, t_n) + \theta\psi(x_j, t_n)\psi_x(x_j, t_n) + \psi_{xxx}(x_i, t_n) = 0. \tag{6.5}$$

On graded meshes, rather than relying on the $L1$ method in the previous scheme ,estimating $D_t^\nu \psi(x_j, t_n)$ with (6.2) yields

$$\begin{aligned}&\frac{1}{\mu_n}\left[\sum_{i=1}^{\mathcal{N}_{exp}} \ell_i H_{i,j}^n + a_n^{(n,\nu)}(\Psi_j^n - \Psi_j^{n-1})\right] - \lambda\Psi_j^n + \theta\chi(\Psi^n, \Psi^n)_j + \delta_x^2(\delta_x \Psi_{j+\frac{1}{2}}^n)\\ =&(R_t^H)_j^n + (R_x^H)_j^n, \quad 1 \le j \le \mathcal{M} - 2,\ 1 \le n \le \mathcal{N},\end{aligned} \tag{6.6}$$

$$\begin{aligned}&\frac{1}{\mu_n}\left[\sum_{i=1}^{\mathcal{N}_{exp}} \ell_i H_{i,\mathcal{M}-1}^n + a_n^{(n,\nu)}(\Psi_{\mathcal{M}-1}^n - \Psi_{\mathcal{M}-1}^{n-1})\right] - \lambda\Psi_{\mathcal{M}-1}^n + \theta\chi(\Psi^n, \Psi^n)_{\mathcal{M}-1}\\ &+ \frac{1}{h^2}(\delta_x \Psi_{\mathcal{M}-\frac{3}{2}}^n - 3\delta_x \Psi_{\mathcal{M}-\frac{3}{2}}^n) = (R_t^H)_{\mathcal{M}-1}^n + (R_x^H)_{\mathcal{M}-1}^n, \quad 1 \le j \le \mathcal{M} - 2.\end{aligned} \tag{6.7}$$

$$H_{i,j}^n = e^{-\iota_i \tau_n} H_{i,j}^{n-1} + X_i^n(\Psi_j^{n-1} - \Psi_j^{n-2}), \quad 1 \le i \le \mathcal{N}_{exp}, \tag{6.8}$$

$$H_{i,j}^1 = 0,\ i = 1, 2, ..., \mathcal{N}_{exp}, \quad 1 \le j \le \mathcal{M} - 1, \tag{6.9}$$

where

$$|(R_t^H)_j^n| \le \frac{\eta}{n^{\min\{r\nu,2-\nu\}} + \varepsilon},\ |(R_x^H)_j^n| \le \eta h,\ j = 1, 2, ..., \mathcal{M} - 1,\ n = 1, 2, ..., \mathcal{N}. \tag{6.10}$$

The above formula can be deduced from (2.19) and (2.20), as well as from Lemmas 5.2, 6.2, and Theorem.

With the truncation errors eliminated and initial boundaries considered

$$\begin{aligned}&\Psi_j^0 = \vartheta(x_j),\ 1 \le j \le \mathcal{M} - 1,\\ &\Psi_0^n = 0, \Psi_{\mathcal{M}}^n = 0,\ 0 \le n \le \mathcal{N}.\end{aligned} \tag{6.11}$$

as a scheme, we present the following

$$\begin{aligned}&\frac{1}{\mu_n}\left[\sum_{i=1}^{\mathcal{N}_{exp}} \ell_i h_{i,j}^n + a_n^{(n,\nu)}(\psi_j^n - \psi_j^{n-1})\right] - \lambda\psi_j^n + \theta\chi(\psi^n, \psi^n)_j + \delta_x^2(\delta_x \psi_{j+\frac{1}{2}}^n) = 0,\\ &1 \le j \le \mathcal{M} - 2, 1 \le n \le \mathcal{N},\end{aligned} \tag{6.12}$$

$$\begin{aligned}&\frac{1}{\mu_n}\left[\sum_{i=1}^{\mathcal{N}_{exp}} \ell_i h_{i,\mathcal{M}-1}^n + a_n^{(n,\nu)}(\psi_{\mathcal{M}-1}^n - \psi_{\mathcal{M}-1}^{n-1})\right]\\ &- \lambda\Psi_{\mathcal{M}-1}^n + \theta\chi(\psi^n, \psi^n)_{\mathcal{M}-1} + \frac{1}{h^2}(\delta_x \psi_{\mathcal{M}-\frac{3}{2}}^n - 3\delta_x \psi_{\mathcal{M}-\frac{3}{2}}^n) = 0 1 \le j \le \mathcal{M} - 2,\end{aligned} \tag{6.13}$$

$$h_{i,j}^{n} = e^{-\iota_i \tau_n} h_{i,j}^{n-1} + x_i^n(\psi_j^{n-1} - \psi_j^{n-2}), \tag{6.14}$$

$$h_{i,j}^{1} = 0, i = 1, 2, ..., \mathcal{N}_{exp}, \quad 1 \le j \le \mathcal{M} - 1, \tag{6.15}$$

$$\psi_j^0 = \vartheta(x_j), \quad 1 \le j \le \mathcal{M} - 1, \tag{6.16}$$

$$\psi_0^n = 0, \psi_{\mathcal{M}}^n = 0, \quad 0 \le n \le \mathcal{N}. \tag{6.17}$$

Theorem 6.1. *Let*

$$\left\{ \Psi_j^{n-\frac{1}{2}} \middle| 0 \le j \le \mathcal{M}, 0 \le n \le \mathcal{N} \right\},$$

is the solution to (1.1), and

$$\left\{ \psi_j^{n-\frac{1}{2}} \middle| 0 \le j \le \mathcal{M}, 0 \le n \le \mathcal{N} \right\},$$

is the solution to (2.25)-(2.27). Denote

$$\bar{\mathcal{E}}_j^n = \Psi_j^{n-\frac{1}{2}} - \psi_j^{n-\frac{1}{2}}, \quad 0 \le j \le \mathcal{M}, 0 \le n \le \mathcal{N}. \tag{6.18}$$

Therefore it is possible for there to be a constant C that depends neither on $\mathcal{M}$ nor $\mathcal{N}$, in such a way that

$$\left\| \bar{\mathcal{E}^n} \right\| \le C(\mathcal{N}^{-\min\{r\nu, 2-\nu\}} + h + \varepsilon), \quad 0 \le n \le \mathcal{N}. \tag{6.19}$$

The proof of the theorem is not presented as it would be achieved by following the same approach used for Theorem 5.1 by applying Lemma 6.2.

7. Numerical examples

This section presents three numerical examples to demonstrate the efficiency and accuracy of the proposed method described previously. Each example shows a comparison of the results obtained with the proposed method and those obtained by other methods. In this analysis, numerical calculations are performed by running MATLAB R2016b programs on an Intel Core i7 laptop.

Example 7.1 ([26]). Let us consider that time fractional nonlinear KdV equation as follows:

$$\begin{cases} D_t^{\nu}\psi(x,t) - \dfrac{1}{20}\psi(x,t) + 2\psi(x,t)\psi_x(x,t) + \psi_{xxx}(x,t) = f(x,t), \ x \in (0,1), \ t \in (0,1], \\ \psi(x,0) = 0, \ x \in [0,1], \\ \psi(0,t) = 0, \ \psi(1,t) = 0, \ \psi_x(1,t) = 0, \ t \in (0,1]. \end{cases} \tag{7.1}$$

The analytical solution can be obtained by $\psi(x,t) = t^{\nu}(x-1)^4 \sin(\pi x)$. Assuming that the exact solution and the approximate solution are $\Psi(x,t)$ and $\psi(x,t)$, respectively and the error is defined as follows:

$$E(\mathcal{M}, N) = \underset{0 \le j \le \mathcal{N}}{Max} \sqrt{\|\Psi^j - \psi^j\|^2}. \tag{7.2}$$

Assuming $\mathcal{N} = [\mathcal{M}^{\frac{1}{\min\{r\nu, 2-\nu\}}}]$, $\nu = \{0.3, 0.6, 0.9\}$ and $r = \{1, 2, 3\}$. Tables (1–8) show the convergence orders for different values of $\mathcal{M}$ and r. Additionally, in Table

Table 1. Maximum error and convergence order of Example 7.1 for different values of ν and $\mathcal{N}$ with $r = 1$.

$\mathcal{M}$	$\nu=0.3$			$\nu=0.6$			$\nu=0.9$		
	$\mathcal{N}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$\mathcal{N}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$\mathcal{N}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$
16	10321	6.017×10^{-3}	-	101	3.970×10^{-3}	-	21	2.657×10^{-4}	-
32	104031	5.851×10^{-4}	0.857	322	8.012×10^{-4}	0.951	47	1.695×10^{-4}	0.997
64	1048576	1.115×10^{-4}	1.041	1024	5.861×10^{-5}	1.172	101	2.329×10^{-5}	1.296

Table 2. Maximum error and convergence order of Example 7.1 for different values of ν and $\mathcal{N}$ with $r = 2$.

$\mathcal{M}$	$\nu=0.3$			$\nu=0.6$			$\nu=0.9$		
	$\mathcal{N}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$\mathcal{N}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$\mathcal{N}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$
16	101	1.170×10^{-3}	-	10	8.382×10^{-4}	-	12	7.420×10^{-4}	-
32	322	6.531×10^{-4}	0.927	17	5.812×10^{-4}	0.871	23	8.734×10^{-5}	0.971
64	1024	2.502×10^{-4}	0.993	32	4.912×10^{-5}	0.948	43	3.438×10^{-5}	1.136
128	3250	4.726×10^{-5}	1.146	57	2.983×10^{-5}	1.225	82	7.032×10^{-6}	1.402

Table 3. Maximum error and convergence order of Example 7.1 for different values of ν and $\mathcal{N}$ with $r = 3$.

$\mathcal{M}$	$\nu=0.3$			$\nu=0.6$			$\nu=0.9$		
	$\mathcal{N}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$\mathcal{N}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$\mathcal{N}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$
16	21	2.517×10^{-3}	-	7	6.142×10^{-4}	-	12	5.740×10^{-4}	-
32	47	5.301×10^{-4}	0.941	11	4.358×10^{-4}	1.262	23	7.113×10^{-5}	1.309
64	101	1.625×10^{-4}	0.972	19	8.197×10^{-5}	1.479	43	8.251×10^{-6}	1.563
128	219	3.091×10^{-5}	1.056	32	1.973×10^{-5}	1.503	82	6.382×10^{-6}	1.726

Table 4. Comparison maximum errors of Example 7.1 for various values of $(\mathcal{M}, r)$ when $\nu = \{0.3, 0.9\}$.

$(\mathcal{M}, r)$	$\nu=0.3$		$\nu=0.9$	
	Ref. [26]	our method	Ref. [26]	our method
(32 , 1)	2.08×10^{-2}	5.811×10^{-4}	1.81×10^{-2}	1.695×10^{-4}
(64 , 1)	1.04×10^{-2}	1.115×10^{-4}	9.23×10^{-3}	2.329×10^{-5}
(64 , 2)	1.03×10^{-2}	2.502×10^{-4}	5.26×10^{-3}	3.438×10^{-5}
(128 , 2)	5.17×10^{-3}	4.726×10^{-5}	2.62×10^{-3}	7.032×10^{-6}
(64 , 3)	9.25×10^{-3}	1.625×10^{-4}	3.38×10^{-3}	8.251×10^{-6}
(128 , 3)	4.69×10^{-3}	3.091×10^{-5}	1.695×10^{-3}	6.382×10^{-6}

(4), we compare our the proposed method with that in Ref. [26]. Comparing the two methods, it is demonstrated that the method presented in the previous chapter is more efficient.

Table 4 shows a comparison between our method and the method presented in Ref. [26]. For this purpose, according to the error defined in Eq.(7.2) and in similar cases ν and $\mathcal{M}$, a comparison has been made between the errors reported in Ref. [26] and the errors reported in Tables 1-3. By examining this table, it can be concluded that our proposed method has higher accuracy and efficiency. It can also be seen that as the order of the fractional derivative increases, the accuracy of our method becomes much higher and in some cases it reaches twice the accuracy of Ref. [26]

method.

Example 7.2 ([26]). Take into account based on the following time fractional nonlinear equation for KdV below:

$$\begin{cases} D_t^{\nu}\psi(x,t) - \dfrac{4}{5}\psi(x,t) + 2\psi(x,t)\psi_x(x,t) + \psi_{xxx}(x,t) = f(x,t),\ \ x \in (0,1], t \in (0, 2^{-8}], \\ \psi(x,0) = x(x-1)^2,\ \ x \in [0,1], \\ \psi(0,t) = 0,\ \ \psi(1,t) = 0,\ \ \psi_x(1,t) = 0,\ \ t \in (0, 2^{-8}]. \end{cases} \tag{7.3}$$

Since we cannot obtain the exact solution in this example, here are the steps we take to calculate the space error:

$$E(h,\mathcal{N}) = \underset{0\le j\le \mathcal{M}}{Max} |\psi_j(h,\mathcal{N}) - \psi_{2j}(\frac{h}{2},\mathcal{N})|. \tag{7.4}$$

Additionally, the convergence order can be calculated as follows:

$$R_{\mathcal{M}} = \log_2\left(\frac{E(h,\mathcal{N})}{E(\frac{h}{2},\mathcal{N})}\right). \tag{7.5}$$

Tables (5-9), present the convergence orders and estimate the errors for different values of $r = \{1,2,3\}$, $\mathcal{M} = \{32, 64, 128, 256, 512\}$ and $\nu = \{0.3, 0.6, 0.9\}$. Figures 3-5 show the numerical solutions for $\nu = 0.3, 0.6, 0.9$ and $r = 1, 3$. By smaller fractional derivative order ν, they show that the singularity of the solution increases and it gets to zero faster.

Table 5. Estimate the errors and convergence order of Example 7.2 for various values of ν with $r = 1$.

$\mathcal{M}$	$\nu = 0.3$		$\nu = 0.6$		$\nu = 0.9$	
	$E(h,\mathcal{N})$	$R_{\mathcal{M}}$	$E(h,\mathcal{N})$	$R_{\mathcal{M}}$	$E(h,\mathcal{N})$	$R_{\mathcal{M}}$
32	$-$	-	$-$	-	$-$	-
64	9.123×10^{-5}	-	6.841×10^{-5}	-	5.561×10^{-5}	-
128	4.670×10^{-5}	0.982	3.317×10^{-5}	0.987	2.291×10^{-5}	0.992
256	3.516×10^{-5}	0.996	2.626×10^{-5}	0.991	1.238×10^{-5}	1.064
512	1.822×10^{-5}	0.999	1.063×10^{-5}	0.996	8.131×10^{-6}	1.105

Table 6. Estimate the errors and convergence order of Example 7.2 for different values of ν with $r = 2$.

$\mathcal{M}$	$\nu = 0.3$		$\nu = 0.6$		$\nu = 0.9$	
	$E(h,\mathcal{N})$	$R_{\mathcal{M}}$	$E(h,\mathcal{N})$	$R_{\mathcal{M}}$	$E(h,\mathcal{N})$	$R_{\mathcal{M}}$
32	$-$	-	$-$	-	$-$	-
64	5.431×10^{-5}	-	4.713×10^{-5}	-	2.780×10^{-5}	-
128	2.360×10^{-5}	0.991	1.142×10^{-5}	0.986	1.976×10^{-5}	0.999
256	1.876×10^{-5}	1.052	9.652×10^{-6}	0.996	5.146×10^{-6}	1.114
512	8.503×10^{-6}	1.086	7.112×10^{-6}	1.102	8.291×10^{-7}	1.086

Table 7. Estimate the errors and convergence order of Example 7.2 for different values of ν with $r = 3$.

$\mathcal{M}$	$\nu = 0.3$		$\nu = 0.6$		$\nu = 0.9$	
	$E(h,\mathcal{N})$	$R_{\mathcal{M}}$	$E(h,\mathcal{N})$	$R_{\mathcal{M}}$	$E(h,\mathcal{N})$	$R_{\mathcal{M}}$
32	–	-	–	-	–	-
64	2.063×10^{-5}	-	1.827×10^{-5}	-	2.426×10^{-6}	-
128	1.692×10^{-5}	0.998	4.317×10^{-6}	0.972	1.147×10^{-6}	0.992
256	9.786×10^{-6}	1.063	8.913×10^{-6}	0.986	9.400×10^{-7}	1.183
512	6.081×10^{-6}	1.107	5.112×10^{-7}	0.993	7.307×10^{-7}	1.265

Table 8. The CPU time of Example 7.1 for different values of ν, $\mathcal{M}$ and $\mathcal{N}$ with $r = 3$.

$\mathcal{M}$	$\nu = 0.3$		$\nu = 0.6$		$\nu = 0.9$	
	$\mathcal{N}$	CPU time(s)	$\mathcal{N}$	CPU time(s)	$\mathcal{N}$	CPU time(s)
16	21	73	7	25	12	41
32	47	118	11	38	23	54
64	101	157	19	62	43	109
128	219	256	32	143	82	199

Table 9. The CPU time of Example 7.2 for different values of ν and $\mathcal{M}$ with $r = 3$.

$\mathcal{M}$	$\nu = 0.3$	$\nu = 0.6$	$\nu = 0.9$
	CPU time(s)	CPU time(s)	CPU time(s)
32	76	84	93
64	135	147	170
128	177	201	226
256	276	316	343
512	342	375	394

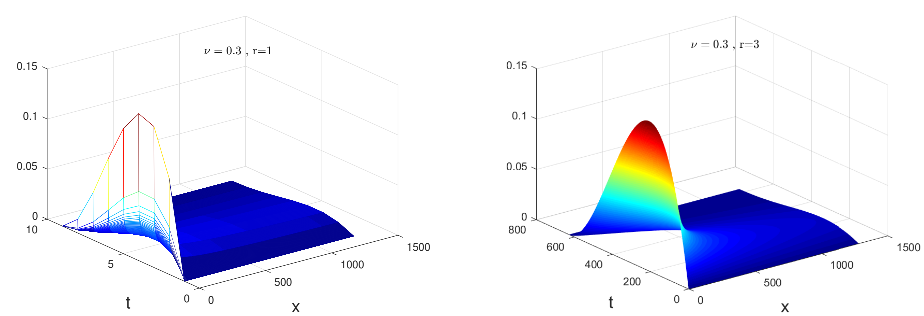

Figure 3. The numerical solutions of Example 7.2 for $\nu = 0.3, r = 1$ (left shape) and $\nu = 0.3, r = 3$ (right shape).

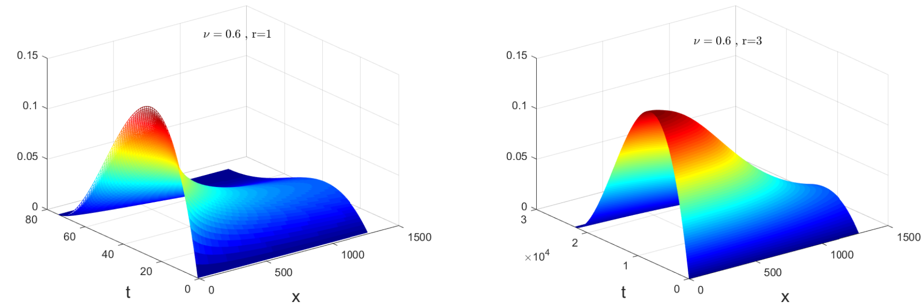

Figure 4. The numerical solutions of Example 7.2 for $\nu = 0.6, r = 1$ (left shape) and $\nu = 0.6, r = 3$ (right shape).

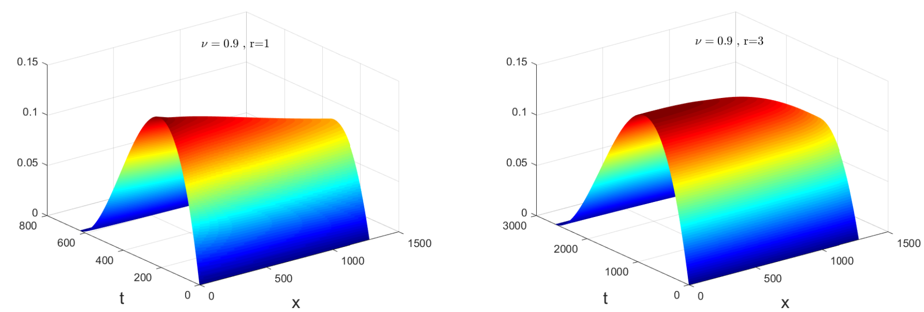

Figure 5. The numerical solutions of Example 7.2 for $\nu = 0.9, r = 1$ (left shape) and $\nu = 0.9, r = 3$ (right shape).

Example 7.3 ([10]). Consider

$$\begin{cases} D_t^{\nu}\psi(x,t) + 6\psi(x,t)\psi_x(x,t) + \psi_{xxx}(x,t) = f(x,t), \quad x \in (0,1), t \in (0,1], \\ \psi(0,t) = 0, \qquad t \in (0,1], \\ \psi(1,t) = 0, \qquad t \in (0,1], \\ \psi_x(1,t) = 0, \quad t \in (0,1]. \end{cases} \tag{7.6}$$

In this example, for

$$f(x,t) = \frac{-\Gamma(7)}{\Gamma(7-\nu)} t^{6-\nu} x(1-x)^2 + 6(1-t^6)^2 \left(3x^5 - 10x^4 + 12x^3 - 6x^2 + x\right) + 6(1-t^6),$$

analyze the solution provided by $\psi(x,t) = (1-t^6)x(1-x)^2$. Assuming that the exact solution and the approximate solution are $\Psi(x,t)$ and $\psi(x,t)$, respectively. The error is defined as follows:

$$E(\mathcal{M},\mathcal{N}) = \underset{0\leq j\leq \mathcal{N}}{Max} \sqrt{\|\Psi^j - \psi^j\|^2}. \tag{7.7}$$

Tables (10-12), by assuming $\mathcal{N} = \left[\mathcal{M}^{\frac{1}{\min\{r\nu, 2-\nu\}}}\right]$, present the convergence orders and maximum errors for different values of $\mathcal{M} = \{16, 32, 64, 128\}$, $\nu = \{0.2, 0.6, 0.9\}$ and $r = \{1, 2, 3\}$.

Table 10. Maximum error and convergence order of Example 7.3 for different values of ν and $\mathcal{N}$ with $r = 1$.

$\mathcal{M}$	$\nu = 0.2$		$\nu = 0.6$		$\nu = 0.9$	
	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$
16	7.35×10^{-4}	-	6.48×10^{-4}	-	5.42×10^{-5}	-
32	3.27×10^{-4}	1.28	4.12×10^{-4}	1.33	1.51×10^{-5}	1.34
64	1.96×10^{-4}	1.31	6.83×10^{-5}	1.37	9.23×10^{-6}	1.41
128	5.50×10^{-5}	1.37	2.57×10^{-5}	1.46	1.24×10^{-6}	1.53

Table 11. Maximum error and convergence order of Example 7.3 for different values of ν and $\mathcal{N}$ with $r = 2$.

$\mathcal{M}$	$\nu = 0.2$		$\nu = 0.6$		$\nu = 0.9$	
	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$
16	6.17×10^{-4}	-	8.38×10^{-4}	-	8.42×10^{-5}	-
32	1.53×10^{-4}	1.37	5.81×10^{-5}	1.39	7.73×10^{-5}	1.40
64	2.50×10^{-5}	1.39	4.91×10^{-5}	1.43	3.43×10^{-6}	1.45
128	4.72×10^{-5}	1.39	2.98×10^{-6}	1.44	7.03×10^{-7}	1.48

Table 12. Maximum error and convergence order of Example 7.3 for different values of ν and $\mathcal{N}$ with $r = 3$.

$\mathcal{M}$	$\nu = 0.3$		$\nu = 0.6$		$\nu = 0.9$	
	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$	$E(\mathcal{M},\mathcal{N})$	$R_{\mathcal{M}}$
16	2.51×10^{-4}	-	6.14×10^{-5}	-	5.74×10^{-5}	-
32	5.30×10^{-5}	1.38	4.35×10^{-5}	1.46	7.11×10^{-6}	1.56
64	1.62×10^{-5}	1.43	8.19×10^{-6}	1.47	8.25×10^{-7}	1.65
128	3.09×10^{-6}	1.48	6.97×10^{-7}	1.50	1.38×10^{-7}	1.72

8. Conclusion

By using the classical fractional $L1$ on classified meshes according to time and central differences for location decay, we have presented a scheme that has a first-time convergence over space and can have a $\min\{2 - \nu, r\}$ order convergence over time. Initially, Grunwald inequality which is useful for examining numerical stability and evaluating numerical methods of solving linear and nonlinear differential equations was thoroughly investigated. Continually implementing our proposed method and reviewing its stability and convergence, we claim that our method is effective. To explain the simplicity of our suggested method and to better understand the computational molecule reader, we have drawn our method. Finally, by comparing examples to previous works, we have presented encouraging results and concluded the paper. In our future research, we envisage studying and solving the nonlinear KdV equation using meshless methods. We will also apply the idea to solve this equation using weighted finite difference methods.

Declarations conflict of interest. The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] M. Abbaszadeh and M. Dehghan, *The Crank-Nicolson/interpolating stabilized element-free Galerkin method to investigate the fractional Galilei invariant advection-diffusion equation*, Mathematical Methods in the Applied Sciences, 2021, 44(4), 2752–2768.

[2] A. Babaei and S. Banihashemi, *Reconstructing unknown nonlinear boundary conditions in a time-fractional inverse reaction-diffusion-convection problem*, Numerical Methods for Partial Differential Equations, 2019, 35(3), 976–992.

[3] A. Babaei and S. Banihashemi, *Stable numerical approach to solve a time-fractional inverse heat conduction problem*, Iranian Journal of Science and Technology, Transactions A: Science, 2018, 42(4), 2225–2236.

[4] A. Babaei, H. Jafari, S. Banihashemi and M. Ahmadi, *Mathematical analysis of a stochastic model for spread of Coronavirus*, Chaos, Solitons and Fractals, 2021, 145, 110788.

[5] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional calculus: models and numerical methods,* World Scientific, Singapore, 2012.

[6] D. Baleanu and B. Shiri, *Nonlinear higher order fractional terminal value problems*, AIMS Mathematics, 2022, 7(5), 7489–7506.

[7] E. F. Browder, *Existence and uniqueness theorems for solutions of nonlinear boundary value problems*, Proceedings of the Symposia in Applied Mathematics, Providence, AMS, 1965, 1724–29.

[8] H. Brunner, *The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes*, Mathematics Computation, 1985, 45, 417–437.

[9] H. Brunner, L. Ling and M. Yamamoto, *Numerical simulations of 2D fractional subdiffusion problems*, Journal of Computational Physics, 2010, 229, 6613–6622.

[10] W. Cao, Y. Xu and Z. Zheng, *Finite difference/collocation method for a generalized time-fractional KdV equation*, Applied Sciences, 2018, 8(1), 42, https://doi.org/10.3390/app8010042.

[11] A. El-Ajou, O. A. Arqub and S. Momani, *Approximate analytical solution of the nonlinear fractional KdV-Burgers equation: A new iterative algorithm,* Journal of Computational Physics, 2015, 293, 81–95.

[12] R. Erfanifar, K. Sayevand, N. Ghanbari and H. Esmaeili, *A modified Chebyshev ϑ-weighted Crank-Nicolson method for analyzing fractional sub-diffusion equations*, Numerical Methods for Partial Differential Equations, 2021, 37(1), 614–625.

[13] R. M. Ganji, H. Jafari, S. P. Moshokoa and N. S. Nkomo, *A mathematical model and numerical solution for brain tumor derived using fractional operator*, Results in Physics, 2021, 28, 104671.

[14] S. Jiang, J. Zhang, Q. Zhang and Z. Zhang, *Fast evalution of the caputo fractional derivative and its applications to fractional diffusion equations*, Communications in Computational Physics, 2017, 21, 650–678.

[15] D. J. Kordeweg and G. Vries, *On the change of form of long waves advancing in a rectangular channel, and a new type of long stationary wave,* Philosophical Magazine, 1895, 39, 422–443.

[16] C. Li and M. Cai, *Theory and numerical approximations of fractional integrals and derivatives*, Society for Industrial and Applied Mathematics, 2019, https://doi.org/10.1137/1.9781611975888.

[17] H. Liao, D. Li and J. Zhang, *Sharp error estimate of the nonuniform l1 formula for linear reaction-subdiffusion equations*, SIAM Journal on Numerical Analysis, 2018, 56(2), 1112–1133.

[18] N. H. Luc, H. Jafari and P. Kumam, *On an initial value problem for time fractional pseudo-parabolic equation with Caputo derivarive*, Mathematical Methods in the Applied Sciences, 2021, https://doi.org/10.1002/mma.7204.

[19] S. Momani, *An explicit and numerical solutions of the fractional KdV equation*, Mathematics and Computer in Simulation, 2005, 70(2), 110–118.

[20] K. Mustapha, *An implicit finite difference time-stepping method for a subdiffusion equation with spatial discretization by finite elements*, IMA Journal of Numerical Analysis, 2011, 31, 719–739.

[21] M. Sajjadian, *Numerical solutions of Korteweg de Vries and Korteweg de Vries-Burger's equations using computer programming*, International Journal of Nonlinear Science, 2013, 15(1), 69–79.

[22] K. Sayevand, N. Ghanbari and I. Masti, *A robust computational framework for analyzing the Bloch-Torrey equation of fractional order*, Computational and Applied Mathematics, 2021, 40(4), 1–21.

[23] K. Sayevand, J. T. Machado and I. Masti, *Analysis of dual Bernstein operators in the solution of the fractional convection-diffusion equation arising in underground water pollution*, Journal of Computational and Applied Mathematics, 2022, 399, 113729.

[24] K. Sayevand, J. T. Machado and I. Masti, *On dual Bernstein polynomials and stochastic fractional integro-differential equations*, Mathematical Methods in the Applied Sciences, 2020, 43(17), 9928–9947.

[25] Y. Shen, Z. Sun and R. Du, *Fast finite difference schemes for the time-fractional diffusion equations with a weak singularity at the initial time*, East Asian Journal on Applied Mathematics, 2018, 8(4), 834–858.

[26] J. Shen, Z. Sun and W. Cao, *A finite difference scheme on graded meshes for time-fractional nonlinear Korteweg-de Vries equation*, Applied Mathematics and Computation, 2019, 361, 752–765.

[27] A. Shidfar and A. Babaei, *The Sinc-Galerkin method for solving an inverse parabolic problem with unknown source term*, Numerical Methods for Partial Differential Equations, 2013, 29, 64–78.

[28] M. Stynes, E. ORiordan and J. L. Gracia, *Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation*, SIAM Journal on Numerical Analysis, 2015, 55(2), 1057–1079.

[29] H. Sun and Z. Sun, *On two linearized difference schemes for burgers equation, International Journal of Computer Mathematics*, 2015, 92(6), 1160–1179.

[30] A. Taghavi, A. Babaei and A. Mohammadpour, *A stable numerical scheme for a time-fractional inverse parabolic equation*, Inverse Problems in Science and Engineering, 2017, 25(10), 1474–1491.

[31] Q. Wang, *Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method*, Applied Mathematics and Computation, 2006, 182, 1048–1055.

[32] Y. Zhang, Z. Sun and H. Liao, *Finite difference methods for the time fractional diffusion equation on nonuniform meshes*, Journal of Computational Physics, 2014, 265, 195–210.