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# ON THE ORBITAL STABILITY OF A BOUSSINESQ SYSTEM

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**Abstract** In this paper we establish the orbital stability of ground state solitary waves for a nonlinear one-dimensional Boussinesq system that models the evolution of two dimensional long water waves with small amplitude in the presence of surface tension. We also discuss the well-posedness for the Boussinesq system, using some Strichartz type estimates associated with the system and lowering the Sobolev index obtained in some previous results.

**Keywords** Solitary waves, ground state solutions, well-posedness, Strichartz estimates, water waves.

**MSC(2010)** 35E15, 35C08, 377K45, 35Q35.

## 1. Introduction

In the present work we study the nonlinear orbital stability of solitary wave solutions of the one-dimensional *abad*-Boussinesq system

$$\begin{cases} \left(I - a\mu\partial_x^2\right)\eta_t + \partial_x^2\Phi - b\mu\partial_x^4\Phi + \epsilon\partial_x\left(\eta\partial_x\Phi\right) = 0, \\ \left(I - a\mu\partial_x^2\right)\Phi_t + \eta - d\mu\partial_x^2\eta + \frac{\epsilon}{2}\left(\partial_x\Phi\right)^2 = 0, \end{cases} \tag{1.1}$$

that arise in the study of the evolution of small amplitude long water waves in the presence of surface tension (see [24, 29]). Here $\mu$ is the long-wave parameter (dispersion coefficient), $\epsilon$ is the amplitude parameter (nonlinear coefficient), and the functions $\eta(t,x)$ and $\Phi(t,x)$ denote the wave elevation and the potential velocity on the bottom $z = 0$, respectively; and the constants $a \geq 0$, $b > 0$, and $d > 0$ are such that

$$2a - (b + d) = \frac{1}{3} - \sigma,$$

where $\sigma^{-1}$ is known as the Bond number, associated with the surface tension.

As happens in many water waves models, a clever fact to analyze the orbital stability/instability is the existence of a Hamiltonian structure which characterizes solitary waves as critical points of the action functional. For this particular *abad*-

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Boussinesq system (1.1), the Hamiltonian is given by

$$\mathcal{H}\begin{pmatrix}\eta\\ \Phi\end{pmatrix}=\frac{1}{2}\int_{\mathbb{R}}\left(\eta^2+d\mu(\partial_x\eta)^2+(\partial_x\Phi)^2+b\mu(\partial_x^2\Phi)^2+\epsilon\eta(\partial_x\Phi)^2\right)dx,$$

and the Hamiltonian type structure is given by

$$\begin{pmatrix}\eta_t\\ \Phi_t\end{pmatrix}=\mathcal{J}\mathcal{H}'\begin{pmatrix}\eta\\ \Phi\end{pmatrix},\qquad \mathcal{J}=\begin{pmatrix}0 & \left(I-a\mu\partial_x^2\right)^{-1}\\ -\left(I-a\mu\partial_x^2\right)^{-1} & 0\end{pmatrix}.$$

In addition, by Noether's Theorem, there is a functional $\mathcal{Q}$ (named the Charge) which is conserved in time for classical solutions defined formally as

$$\begin{aligned}\mathcal{Q}\begin{pmatrix}\eta\\ \Phi\end{pmatrix}&=\frac{1}{2}\left\langle\mathcal{J}^{-1}\partial_x\begin{pmatrix}\eta\\ \Phi\end{pmatrix},\begin{pmatrix}\eta\\ \Phi\end{pmatrix}\right\rangle\\ &=\int_{\mathbb{R}}(I-a\mu\partial_x^2)\partial_x\eta\Phi\,dx=-\int_{\mathbb{R}}(\eta\partial_x\Phi+a\partial_x\eta\partial_x^2\Phi)\,dx.\end{aligned}$$

From this Hamiltonian structure, we have that solitary waves of wave speed $\omega$ for the Boussinesq system (1.1) correspond to stationary solutions of the modulated system

$$\begin{pmatrix}\eta_t\\ \Phi_t\end{pmatrix}=\mathcal{J}\mathcal{F}_\omega'\begin{pmatrix}\eta\\ \Phi\end{pmatrix},$$

where $\mathcal{F}_\omega=\mathcal{H}+\omega\mathcal{Q}$.

The orbital stability has been intensively studied for a long time, for example in recent years we have the works done by X. Zheng, J. Xin and X. Peng for generalized long-short wave equations (see [32]), Y. Cho and M. Lee for inhomogeneous nonlinear Schrödinger equations with singular potential (see [5]), M. Fontaine, M. Lemou and F. Méhats for the HMF Poisson model (see [9]), E. Csobo for a nonlinear Schrödinger equation with inverse square potential on the half-line (see [7]), F. Cristófani, F. Natali and A. Pastor for regularized dispersive equations (see [6]), J. Angulo and N. Goloshchapova for NLS equations with the $\delta'$-interaction (see [2]).

Regarding the stability issue, we need to recall that M. Grillakis, J. Shatah and W. Strauss in [11] established a general result to analyze the orbital stability of solitary waves for a class of abstract Hamiltonian systems. In this case, solitary waves of least energy $Y_\omega$ are minimums of the action functional $\mathcal{F}_\omega$ and the stability analysis depends on the positiveness of the symmetric operator $\mathcal{F}_\omega''(Y_\omega)$ in a neighborhood of the solitary wave $Y_\omega$, except possibly in two directions, and also the strict convexity of the real function

$$d_1(\omega)=\inf\{\mathcal{F}_\omega(Y):Y\in\mathcal{M}_\omega\},$$

where $\mathcal{M}_\omega$ is a suitable set. For 1D models like the KdV equation, Benney-Luke equation, the Benjamin-Ono equation, for example, in which there are explicit travelling waves, the verification of the positiveness of $\mathcal{F}_\omega''(Y_\omega)$ is much simpler due to the fact that the spectral analysis for the operator $\mathcal{F}''_\omega(Y_\omega)$ is reduced to studying the eigenvalues of a ordinary differential equation which at $\pm$ infinity becomes to a constant coefficients ordinary differential equation (see [4, 22, 26]). In the case for the *abad*-Boussinesq system, we have a harder task to overcome using Grillakis et. al. approach since the spectral analysis is not straightforward due to the lack of an explicit formula for travelling waves.

In order to avoid using Grillakis et. al. approach which requires the spectral analysis, we used a direct approach to prove orbital stability of ground state solitary wave solutions of the system (1.1) in the case of wave speed $\omega$ is near $1^-$, using strongly the variational characterization of $d_1$, as done for 2D models: see J. Shatah for nonlinear Klein Gordon equations [31], J. Quintero for the 2D-Benney-Luke equation [23], J. C. Saut for the KP equation [8], R. Fukuizumi for the nonlinear Schrödinger equation with harmonic potential [10] and Y. Liu for the generalized KP equation [19], among others.

One ingredient needed in our development on the stability of solitary waves is the well-posedness of the Cauchy problem associated to (1.1). In the absence of at least a local existence result in a suitable function class that includes the solitary wave solutions, the question of stability has no clear significance. On account of its structure, we see directly that the functional $\mathcal{H}$ is well defined when $\eta, \Phi_x \in H^1(\mathbb{R})$. This condition already characterizes the natural space (energy space) in which we consider the well-posedness and the existence and stability of solitary wave solutions. For instance, J. Quintero and A. Montes established in [30] the existence of solitary wave solutions which propagate with speed of wave $\omega > 0$, i. e. solutions of the form

$$\eta(t,x) = u\,(x-\omega t)\,, \quad \Phi(t,x) = v\,(x-\omega t)\,,$$

in the Hilbert space $X = H^1 \times \mathcal{V}^2$ with respect to norm

$$\|(u,v)\|_X^2 = \|u\|_{H^1}^2 + \|v\|_{\mathcal{V}^2}^2,$$

where the usual Sobolev space $H^1 = H^1(\mathbb{R})$, is the space defined as the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm

$$\|u\|_{H^1}^2 = \int_{\mathbb{R}} \left(u^2 + (u')^2\right) dx,$$

and the space $\mathcal{V}^2 = \mathcal{V}^2(\mathbb{R})$ is defined as the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm given by

$$\|v\|_{\mathcal{V}^2}^2 = \int_{\mathbb{R}} \left((v')^2 + (v'')^2\right) dx = \|v'\|_{H^1}^2$$

and the corresponding inner product

$$(v,w)_{\mathcal{V}^2} = (v', w')_{H^1}.$$

In addition, J. Quintero and A. Montes showed the well-posedness associated to the system (1.1) in the space $X^s = H^s \times \mathcal{V}^{s+1}$, where $H^s$ is the usual Sobolev space of order $s$ defined as the completion of the Schwartz class $\mathcal{S}(\mathbb{R})$ with respect to the norm

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi,$$

and $\mathcal{V}^{s+1}$ denotes the completion of the Schwartz class with respect to the norm

$$\|f\|^2_{\mathcal{V}^{s+1}} = \int_{\mathbb{R}} |\xi|^2(1+|\xi|^2)^s|\widehat{f}(\xi)|^2 d\xi,$$

where the Fourier transform of a function $w$ defined on $\mathbb{R}$ is given by

$$(\mathcal{F}w)(\xi) = \widehat{w}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}w(x)dx.$$

Using a bilinear estimate obtained by J. Bona and N. Tzvetkov in [3] as a key ingredient, J. Quintero and A. Montes showed the local well posedness for $s \geq 0$ in the case $a > 0$ and the global well-posedness in the energy space, $X = H^1 \times \mathcal{V}^2$, when the initial date is small enough; and also showed the local well-posedness for $s > \frac{3}{2}$ in the case $a = 0$, using the estimates for the Commutator of Kato (see works by Kato [12–14]).

In this paper we will show the local well-posedness for $s > \frac{3}{4}$ in the case for $a = 0$, which includes the energy space with $s = 1$. To prove this local result, we follow the ideas developed in the well-posedness for the KdV equation, Boussinesq equation, Ostrovsky equation, among others (see [15, 17, 18]). We will discuss the so-called $L^p - L^q$ smoothing effect properties of Strichartz type for solutions of the linear problem associated to (1.1). Then we will use them to obtain the local result for solutions of the nonlinear problem. The global well-posedness in the energy space $X = H^1 \times \mathcal{V}^2$, when the initial date is small enough, follows from the conservation in time of the Hamiltonian, a Sobolev type inequality and the use of energy estimates (Theorem 3.2 in [30]).

Finally, a special characteristic on the Boussinesq system (1.1) is that some well known water wave models as the one-dimensional Benney-Luke equation,

$$\Phi_{tt} - \Phi_{xx} + a\Phi_{xxxx} - b\Phi_{xxtt} + \Phi_t\Phi_{xx} + 2\Phi_x\Phi_{xt} = 0,$$

and the Korteweg-de Vries type equation,

$$u_t + \left(\sigma - \tfrac{1}{3}\right)u_{xxx} - 3uu_x = 0,$$

emerges from the *abad*-Boussinesq system (up to some order with respect to $\epsilon$), making the system (1.1) very interesting from the physical and numerical view points. Moreover, for small wave speed and large surface tension, we will see that a suitable (renormalized) family of solitary waves of the Boussinesq system (1.1) converges to a nontrivial solitary wave for a KdV type equation. We will use this fact in the stability analysis.

This paper is organized as follows. In section 2, we establish some estimates of type Strichartz for solutions of the linear problem associated to system (1.1) for the case $a = 0$. In section 3, we show the well-posedness for the Cauchy problem associated to the *abad*-Boussinesq system (1.1) for $s > \frac{3}{4}$ in the case $a = 0$. In section 4, we present some preliminaries for the stability result, related with the existence of solitary wave solutions for the system Boussinesq (1.1) and the link between solitary waves for the system (1.1) and the KdV equation. In section 5, we prove the strict convexity of $d_1$ for $c \in (0, 1)$, but near 1. In section 6, we establish the orbital stability result.

## 2. Linear Strichartz estimates for the case $a = 0$

In this section we will establish some estimates of type Strichartz for solutions of the linear Cauchy problem associated to (1.1) in the case for $a = 0$. These estimates will be the main ingredient in the proof of local well-posedness of the nonlinear Cauchy problem associated to (1.1). To prove these estimates we rely on the theory of oscillatory integral established by Kenig, Ponce and Vega in [15, 16].

In order to simplify the computation, we rescale the parameters $\mu$ and $\epsilon$ from the *abad*-Boussinesq system (1.1) by defining

$$\widetilde{\eta}(t,x) = \frac{1}{\epsilon}\eta\left(\frac{t}{\sqrt{\mu}}, \frac{x}{\sqrt{\mu}}\right), \quad \widetilde{\Phi}(t,x) = \frac{\mu}{\epsilon}\Phi\left(\frac{t}{\sqrt{\mu}}, \frac{x}{\sqrt{\mu}}\right).$$

Then we consider the following Cauchy problem,

$$\begin{cases}\eta_t + \Phi_{xx} - b\Phi_{xxxx} + (\eta\Phi_x)_x = 0\\ \Phi_t + \eta - d\eta_{xx} + \frac{1}{2}(\Phi_x)^2 = 0\\ \eta(x,0) = \eta_0(x),\ \Phi(x,0) = \Phi_0(x).\end{cases} \tag{2.1}$$

We see that the solution for the linear Cauchy problem associated to (2.1), with initial data $\Psi_0 = (\eta_0, \Phi_0) \in H^s \times \mathcal{V}^{s+1}$, is given by

$$\Psi(t) = (\eta(t), \Phi(t)) = S(t)\Psi_0, \tag{2.2}$$

where the semigroup $S(t)$ is defined as

$$S(t)(\eta,\Phi) = \Big(S_1(t)(\eta,\Phi), S_2(t)(\eta,\Phi)\Big),$$

where

$$S_1(t)(\eta,\Phi) = \int_{\mathbb{R}} e^{ix\xi}\Big[\cos(\phi(\xi)t)\widehat{\eta}(\xi) + |\xi|\varphi(\xi)\sin(\phi(\xi)t)\widehat{\Phi}(\xi)\Big]d\xi \equiv U(t)\eta + V(t)\Phi,$$

$$S_2(t)(\eta,\Phi) = \int_{\mathbb{R}} e^{ix\xi}\Big[\frac{-\sin(\phi(\xi)t)\widehat{\eta}(\xi)}{|\xi|\varphi(\xi)} + \cos(\phi(\xi)t)\widehat{\Phi}(\xi)\Big]d\xi \equiv W(t)\eta + U(t)\Phi,$$

and the functions $\phi, \varphi$ are given by

$$\phi(\xi) = |\xi|\sqrt{(1+b|\xi|^2)(1+d|\xi|^2)}, \quad \varphi(\xi) = \sqrt{\frac{1+b|\xi|^2}{1+d|\xi|^2}}.$$

From the following lemma we see that $S(t)$ is a bounded linear operator from $H^s \times \mathcal{V}^{s+1}$ into $H^s \times \mathcal{V}^{s+1}$.

**Lemma 2.1.** *Suppose $s \in \mathbb{R}$. Then there exists $C > 0$ such that for all $t \in \mathbb{R}$,*

$$\|U(t)f\|_{H^s} \le C\|f\|_{H^s}, \quad \|V(t)g\|_{H^s} \le C\|g\|_{\mathcal{V}^{s+1}},$$

*and*

$$\|W(t)f\|_{\mathcal{V}^{s+1}} \le C\|f\|_{H^s}, \quad \|U(t)g\|_{\mathcal{V}^{s+1}} \le C\|g\|_{\mathcal{V}^{s+1}}.$$

**Proof.** Note that there are constants $C_1, C_2 > 0$ such that $C_1 \leq \varphi^2 \leq C_2$. Then, if $f \in H^s(\mathbb{R})$ and $g \in \mathcal{V}^{s+1}(\mathbb{R})$ we have that

$$\begin{aligned}\|U(t)f\|_{H^s} &= \Big(\int_{\mathbb{R}} (1+|\xi|^2)^s |\cos(\phi(\xi)t)\widehat{f}(\xi)|^2 d\xi\Big)^{1/2}\\ &\leq \Big(\int_{\mathbb{R}} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi\Big)^{1/2}\\ &= \|f\|_{H^s},\end{aligned}$$

and also that

$$\begin{aligned}\|V(t)g\|_{H^s} &= \Big(\int_{\mathbb{R}} (1+|\xi|^2)^s |\xi|^2 |\varphi(\xi)\sin(\phi(\xi)t)\widehat{g}(\xi)|^2 d\xi\Big)^{1/2}\\ &\leq C\Big(\int_{\mathbb{R}} (1+|\xi|^2)^s |\xi|^2 |\widehat{g}(\xi)|^2 d\xi\Big)^{1/2}\\ &= C\|g\|_{\mathcal{V}^{s+1}}.\end{aligned}$$

Similarly we obtain the other inequalities. □

Now, we will establish the Strichartz estimates and smoothing effects for the operators $U(t)$, $V(t)$ and $W(t)$ and then for $S(t)$.

**Theorem 2.1.** *For $\gamma \geq 0$ define*

$$J_\gamma^{\pm}(t)f(x) = \int_{\mathbb{R}} e^{i(\pm\phi(\xi)t + x\xi)} |\phi''(\xi)|^{\gamma/2} \widehat{f}(\xi)d\xi.$$

*Then for any $\gamma \in [0,1]$,*

$$\Big(\int_{\mathbb{R}} \|J_{\gamma/2}^{\pm}(t)f\|_{L^p}^q dt\Big)^{1/q} \leq C\|f\|_{L^2},$$

$$\Big\|\int_0^t J_\gamma^{\pm}(t-\tau)g(\cdot,\tau)d\tau\Big\|_{L_t^q L_x^p} \leq C\|g\|_{L_t^{q'} L_x^{p'}},$$

*and*

$$\Big\|\int_{\mathbb{R}} J_{\gamma/2}^{\pm}(-\tau)g(\cdot,\tau)d\tau\Big\|_{L_x^2} \leq C\|g\|_{L_t^{q'} L_x^{p'}},$$

*where $q = \frac{4}{\gamma}$, $p = \frac{2}{1-\gamma}$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.*

**Proof.** It is not difficult to prove that $\phi$ belongs to the general class $\mathcal{A}$ defined in [15]. In particular,

$$C_1|\xi|^2 \leq |\phi'(\xi)| \leq C_2|\xi|^2, \quad C_1|\xi| \leq |\phi''(\xi)| \leq C_2|\xi|, \quad \text{for} \quad \xi \neq 0.$$

Then using Theorem 2.1 in [15] we have the estimates (see also Theorem 2.3 in [17]). □

**Lemma 2.2.** *There exists $C > 0$ such that*

$$\begin{aligned}\|D_x^{1/4}U(t)f\|_{L_T^4 L_x^\infty} &\leq C(1+T^{1/4})\|f\|_{L^2},\\ \|D_x^{1/4}V(t)g\|_{L_T^4 L_x^\infty} &\leq C(1+T^{1/4})\|g\|_{\mathcal{V}^1},\\ \|D_x^{5/4}W(t)f\|_{L_T^4 L_x^\infty} &\leq C(1+T^{1/4})\|f\|_{L^2},\end{aligned}$$

*and*

$$\|D_x^{5/4}U(t)g\|_{L_T^4 L_x^\infty} \leq C(1+T^{1/4})\|g\|_{\mathcal{V}^1}.$$

**Proof.** Let $\chi \in C_0^\infty(\mathbb{R})$ a cut-off function, i.e. $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \equiv 1$ if $|x| \leq 1$ and $\chi \equiv 0$ if $|x| \geq 2$. Using the regularity of $\chi$ we have that

$$\begin{aligned}
\Big\| \int_{\mathbb{R}} e^{ix\xi} \cos(\phi(\xi)t)|\xi|^{1/4}\widehat{f}(\xi)\chi(\xi)d\xi \Big\|_{L^\infty} &\leq C \Big\| \int_{\mathbb{R}} e^{ix\xi} \cos(\phi(\xi)t)|\xi|^{1/4}\widehat{f}(\xi)\chi(\xi)d\xi \Big\|_{H^1} \\
&\leq C\Big( \int_{\mathbb{R}} (1+|\xi|^2)^s \Big| |\xi|^{1/4}\widehat{f}(\xi)\chi(\xi)\Big|^2 d\xi\Big)^{1/2} \\
&= C\|D_x^{1/4}(f * \chi^\vee)\|_{H^1} \\
&\leq C\|f\|_{L^2}.
\end{aligned}$$

Then we see that

$$\Big( \int_0^T \Big\| \int_{\mathbb{R}} e^{ix\xi} \cos(\phi(\xi)t)|\xi|^{1/4}\widehat{f}(\xi)\chi(\xi)d\xi \Big\|_{L^\infty}^4 dt\Big)^{1/4} \leq CT^{1/4}\|f\|_{L^2}.$$

On the other hand, using the Minkowski's inequality and Theorem 2.1 with $\gamma = 1$ we obtain that

$$\begin{aligned}
&\Big( \int_0^T \Big\| \int_{\mathbb{R}} e^{ix\xi} \cos(\phi(\xi)t)|\xi|^{1/4}\widehat{f}(\xi)(1-\chi(\xi))d\xi \Big\|_{L^\infty}^4 dt\Big)^{1/4} \\
\leq &\frac{1}{2}\Big( \int_0^T \Big\| \int_{\mathbb{R}} e^{i(\phi(\xi)t+x\xi)}|\phi''(\xi)|^{1/4}\frac{|\xi|^{1/4}}{|\phi''(\xi)|^{1/4}}\widehat{f}(\xi)(1-\chi(\xi))d\xi \Big\|_{L^\infty}^4 dt\Big)^{1/4} \\
&+ \frac{1}{2}\Big( \int_0^T \Big\| \int_{\mathbb{R}} e^{i(-\phi(\xi)t+x\xi)}|\phi''(\xi)|^{1/4}\frac{|\xi|^{1/4}}{|\phi''(\xi)|^{1/4}}\widehat{f}(\xi)(1-\chi(\xi))d\xi \Big\|_{L^\infty}^4 dt\Big)^{1/4} \\
\leq &C\Big\| \frac{|\xi|^{1/4}\widehat{f}(\xi)(1-\chi(\xi))}{|\phi''(\xi)|^{1/4}} \Big\|_{L^2} \leq C\|f\|_{L^2},
\end{aligned}$$

where we have used that $\frac{|\xi|^{1/4}(1-\chi(\xi))}{|\phi''(\xi)|^{1/4}} \in L^\infty(\mathbb{R})$. Now, since

$$\begin{aligned}
D_x^{1/4}U(t)f(x) = &\int_{\mathbb{R}} e^{ix\xi} \cos(\phi(\xi)t)|\xi|^{1/4}\widehat{f}(\xi)\chi(\xi)d\xi \\
&+ \int_{\mathbb{R}} e^{ix\xi} \cos(\phi(\xi)t)|\xi|^{1/4}\widehat{f}(\xi)(1-\chi(\xi))d\xi,
\end{aligned}$$

then we see that

$$\Big( \int_0^T \Big\| D_x^{1/4}U(t)f \Big\|_{L^\infty}^4 dt\Big)^{1/4} \leq C(1+T^{1/4})\|f\|_{L^2}.$$

In a similar fashion, we have that

$$\begin{aligned}
&\Big( \int_0^T \Big\| \int_{\mathbb{R}} e^{ix\xi}|\xi|\varphi(\xi)\sin(\phi(\xi)t)|\xi|^{1/4}\widehat{g}(\xi)\chi(\xi)d\xi \Big\|_{L^\infty}^4 dt\Big)^{1/4} \\
&\leq CT^{1/4}\|D_x^{1/4}(\partial_x g * \chi^\vee)\|_{H^1} \\
&\leq CT^{1/4}\|\partial_x g\|_{L^2} \\
&= CT^{1/4}\|g\|_{\mathcal{V}^1},
\end{aligned}$$

and also that

$$\Big( \int_0^T \Big\| \int_{\mathbb{R}} e^{ix\xi}|\xi|\varphi(\xi)\sin(\phi(\xi)t)|\xi|^{1/4}\widehat{g}(\xi)(1-\chi(\xi))d\xi \Big\|_{L^\infty}^4 dt\Big)^{1/4}$$

$$
\begin{aligned}
\leq&\frac{1}{2}\Big(\int_0^T\Big\|\int_{\mathbb{R}} e^{i(\phi(\xi)t+x\xi)}|\phi''(\xi)|^{1/4}\frac{\varphi(\xi)|\xi|^{1/4}}{|\phi''(\xi)|^{1/4}}|\xi|\widehat{g}(\xi)(1-\chi(\xi))d\xi\Big\|_{L^\infty}^4 dt\Big)^{1/4}\\
&+\frac{1}{2}\Big(\int_0^T\Big\|\int_{\mathbb{R}} e^{i(-\phi(\xi)t+x\xi)}|\phi''(\xi)|^{1/4}\frac{\varphi(\xi)|\xi|^{1/4}}{|\phi''(\xi)|^{1/4}}|\xi|\widehat{g}(\xi)(1-\chi(\xi))d\xi\Big\|_{L^\infty}^4 dt\Big)^{1/4}\\
\leq& C\Big\|\frac{|\xi|^{1/4}\widehat{\partial_x g}(\xi)\varphi(\xi)(1-\chi(\xi))}{|\phi''(\xi)|^{1/4}}\Big\|_{L^2}\\
\leq& C\|g\|_{\mathcal{V}^1},
\end{aligned}
$$

where we have used that $\frac{\varphi(\xi)|\xi|^{1/4}(1-\chi(\xi))}{|\phi''(\xi)|^{1/4}}\in L^\infty(\mathbb{R})$. Now, since

$$
\begin{aligned}
D_x^{1/4}V(t)g(x)=&\int_{\mathbb{R}} e^{ix\xi}|\xi|\varphi(\xi)\sin(\phi(\xi)t)|\xi|^{1/4}\widehat{g}(\xi)\chi(\xi)d\xi\\
&+\int_{\mathbb{R}} e^{ix\xi}|\xi|\varphi(\xi)\sin(\phi(\xi)t)|\xi|^{1/4}\widehat{g}(\xi)(1-\chi(\xi))d\xi,
\end{aligned}
$$

follows the inequality

$$
\Big(\int_0^T\Big\|D_x^{1/4}V(t)g\Big\|_{L^\infty}^4 dt\Big)^{1/4}\leq C(1+T^{1/4})\|g\|_{\mathcal{V}^1}.
$$

Similarly we obtain the other inequalities. □

**Lemma 2.3.** *Define*

$$
I^{\pm}(t)f(x)=\int_{\mathbb{R}} e^{i(\pm t\phi(\xi)+x\xi)}\widehat{f}(\xi)d\xi.
$$

*Then there exists* $C>0$ *such that*

$$
\sup_x\int_{\mathbb{R}}|I^{\pm}(t)f(x)|^2dt\leq C\int_{\mathbb{R}}\frac{|\widehat{f}(\xi)|^2}{|\phi'(\xi)|}d\xi.
$$

**Proof.** First we write

$$
\int_{\mathbb{R}} e^{i(t\phi(\xi)+x\xi)}\widehat{f}(\xi)d\xi=\int_{\xi<0} e^{i(t\phi(\xi)+x\xi)}\widehat{f}(\xi)d\xi+\int_{\xi>0} e^{i(t\phi(\xi)+x\xi)}\widehat{f}(\xi)d\xi.
$$

Note that there exists $\psi$ such that $\psi(\phi(\xi))=\xi$, $\xi<0$. Then making the change of variable $\eta=\phi(\xi)$ we have that

$$
\int_{\xi<0} e^{i(t\phi(\xi)+x\xi)}\widehat{f}(\xi)d\xi=\int e^{i(t\eta+x\psi(\eta))}\widehat{f}_-(\psi(\eta))\frac{d\eta}{\phi'(\psi(\eta))},
$$

where $f_-(x)=f(x)$, for $x<0$ and equals 0 otherwise. Then, using Plancherel's identity and returning to the previous variables we have that

$$
\begin{aligned}
\int_{\mathbb{R}}\Big|\int e^{i(t\eta+x\psi(\eta))}\widehat{f}_-(\psi(\eta))\frac{d\eta}{\phi'(\psi(\eta))}\Big|^2dt=&\int\Big|\int_{\mathbb{R}} e^{i(t\eta+x\psi(\eta))}\widehat{f}_-(\psi(\eta))\frac{dt}{\phi'(\psi(\eta))}\Big|^2d\eta\\
=&\,C\int\Big|\frac{e^{ix\psi(\eta)}\widehat{f}_-(\psi(\eta))}{\phi'(\psi(\eta))}\Big|^2d\eta
\end{aligned}
$$

$$
\begin{aligned}
&= C\int_{\mathbb{R}} \frac{|\widehat{f}_{-}(\psi(\eta))|^2}{|\phi'(\psi(\eta))|^2} d\eta \\
&= C\int_{\mathbb{R}} \frac{|\widehat{f}_{-}(\xi)|^2}{|\phi'(\xi)|} d\xi.
\end{aligned}
$$

A similar argument can be used to obtain

$$
\int_{\mathbb{R}} \Big| \int_{\xi>0} e^{i(t\phi(\xi)+x\xi)} \widehat{f}(\xi) d\xi \Big|^2 dt = C\int_{\mathbb{R}} \frac{|\widehat{f}_{+}(\xi)|^2}{|\phi'(\xi)|} d\xi,
$$

where $f_+(x) = f(x)$, for $x > 0$ and equals 0 otherwise. So that we have the estimate for $I^+(t)f$. The proof for $I^-(t)f$ is analogous. □

**Lemma 2.4.** *There exists $C > 0$ such that*

$$
\begin{aligned}
&\|\partial_x U(t)f\|_{L^\infty_x L^2_T} \le C(1 + T^{1/2})\|f\|_{L^2}, \\
&\|\partial_x V(t)g\Big\|_{L^\infty_x L^2_T} \le C(1 + T^{1/2})\|g\|_{\mathcal{V}^1}, \\
&\|\partial_x^2 W f\|_{L^\infty_x L^2_T} \le C(1 + T^{1/2})\|f\|_{L^2},
\end{aligned}
$$

*and*

$$
\|\partial_x^2 U(t)g\|_{L^\infty_x L^2_T} \le C(1 + T^{1/2})\|g\|_{\mathcal{V}^1}.
$$

**Proof.** We take $\chi \in C_0^\infty(\mathbb{R})$ a cut-off function, and write

$$
\partial_x U(t)f = \int_{\mathbb{R}} ie^{ix\xi} \cos(\phi(\xi)t)\xi\widehat{f}(\xi)\chi(\xi)d\xi + \int_{\mathbb{R}} ie^{ix\xi} \cos(\phi(\xi)t)\xi\widehat{f}(\xi)(1 - \chi(\xi))d\xi.
$$

Sobolev's Lemma and the regularity of $\chi$ imply that

$$
\sup_x \Big( \int_0^T \Big| \int_{\mathbb{R}} ie^{ix\xi} \cos(\phi(\xi)t)\xi\widehat{f}(\xi)\chi(\xi)d\xi \Big|^2 \Big)^{1/2} \le C\|\partial_x f * \chi\|_{H^1} \le CT^{1/2}\|f\|_{L^2}.
$$

In addition, using the Lemma 2.3 we obtain that

$$
\begin{aligned}
&\sup_x \Big( \int_0^T \Big| \int_{\mathbb{R}} e^{ix\xi} \cos(\phi(\xi)t)i\xi\widehat{f}(\xi)(1 - \chi(\xi))d\xi \Big|^2 \Big)^{1/2} \\
\le & C\Big( \int_{\mathbb{R}} \frac{|\xi|^2|\widehat{f}(\xi)(1 - \chi(\xi))|^2}{|\phi'(\xi)|} d\xi \Big)^{1/2} \le C\|f\|_{L^2}.
\end{aligned}
$$

Then we have that

$$
\|\partial_x U(t)f\|_{L^\infty_x L^2_T} \le C(1 + T^{1/2})\|f\|_{L^2}.
$$

In a similar fashion, we have the other estimates. □

**Lemma 2.5.** *Let $s > \frac{3}{4}$, then there exists $C > 0$ such that*

$$
\begin{aligned}
&\|U(t)f\|_{L^2_x L^\infty_T} \le C(1 + T)^{1/2}\|f\|_{H^s}, \\
&\|V(t)g\|_{L^2_x L^\infty_T} \le C(1 + T)^{1/2}\|g\|_{\mathcal{V}^{s+1}}, \\
&\|\partial_x W(t)f\|_{L^2_x L^\infty_T} \le C(1 + T)^{1/2}\|f\|_{H^s},
\end{aligned}
$$

*and*

$$
\|\partial_x U(t)g\|_{L^2_x L^\infty_T} \le C(1 + T)^{1/2}\|g\|_{\mathcal{V}^{s+1}}.
$$

**Proof.** We can proceed as in Lemma 3.4 in [18] to obtain a function $H_k \in L^1(\mathbb{R})$, $k \in \mathbb{Z}$, satisfying

$$|I_k^{\pm}(t,x)| \leq H_k(x),$$

for any $x \in \mathbb{R}$ and $|t| \leq T$ and such that

$$\|H_k\|_{L^1} \leq C \begin{cases} (1+T)^{1/2} 2^{3k/2}, & k \geq 1 \\ (1+T), & -1 \leq k \leq 0 \\ (1+T)^{1/2} 2^{-k/2}, & k \leq -2, \end{cases}$$

where

$$\Omega_k = (-2^{k+1}, -2^{k-1}) \cup (2^{k-1}, 2^{k+1}), \quad k \in \mathbb{Z},$$

is an open covering of $\mathbb{R} - \{0\}$ and a subordinated partition of unity $\{\varphi_k\}_{k=-\infty}^{\infty}$ and

$$I_k^{\pm}(t,x) = C \int_{\mathbb{R}-\{0\}} e^{i(\pm t\phi(\xi) + x\xi)} \varphi_k(\xi) d\xi.$$

Then we argue as in [16, 18] to obtain the estimates. □

# 3. Well posedness for the Cauchy problem

This section is devoted proving well-posedness for the Cauchy problem (2.1) with initial date $(\eta_0, \Phi_0) \in H^s \times \mathcal{V}^{s+1}$. We will see as usual that the local well-posedness follows by the linear estimates of section 1, Leibniz rule and Banach fixed point theorem.

**Theorem 3.1.** *Let $s > \frac{3}{4}$. Then for all $(\eta_0, \Phi_0) \in H^s \times \mathcal{V}^{s+1}$ there exist $T > 0$ depending only on $\|(\eta_0, \Phi_0)\|_{H^s \times \mathcal{V}^{s+1}}$ and a unique solution*

$$(\eta, \Phi) \in C\left([0,T] : H^s \times \mathcal{V}^{s+1}\right)$$

*of the Cauchy problem* (2.1).

**Proof.** Let $s > \frac{3}{4}$ and $M > 0$. Define the space

$$X^T = \{(\eta, \Phi) \in C([0,T], H^s \times \mathcal{V}^{s+1}) \, : \, |||(\eta, \Phi)||| < \infty\},$$

where

$$\begin{aligned} |||(\eta, \Phi)||| = & \|\eta\|_{L_T^\infty H^s} + \|\Phi\|_{L_T^\infty \mathcal{V}^{s+1}} + \|\eta\|_{L_x^2 L_T^\infty} + \|\Phi_x\|_{L_x^2 L_T^\infty} \\ & + \|\eta_x\|_{L_T^4 L_x^\infty} + \|\Phi_{xx}\|_{L_T^4 L_x^\infty} + \|D_x^s \eta_x\|_{L_x^\infty L_T^2} + \|D_x^s \Phi_{xx}\|_{L_x^\infty L_T^2}. \end{aligned}$$

We consider the operator

$$\Gamma(\eta, \Phi) = (\Gamma_1(\eta, \Phi), \Gamma_2(\eta, \Phi)),$$

where

$$\Gamma_1(\eta, \Phi) = U(t)(\eta_0) + V(t)(\Phi_0) - \int_0^t \Big[ U(t-\tau)((\eta\Phi_x)_x)(\tau) + V(t-\tau)\left(\tfrac{1}{2}(\Phi_x)^2\right)(\tau) \Big] d\tau$$

and

$$\Gamma_2(\eta,\Phi)=W(t)(\eta_0)+U(t)(\Phi_0)-\int_0^t\Big[W(t-\tau)((\eta\Phi_x)_x)(\tau)+U(t-\tau)\left(\tfrac{1}{2}(\Phi_x)^2\right)(\tau)\Big]d\tau.$$

We will show that for $M$ and $T$ suitable positive numbers the map $\Gamma$ defines a contraction in

$$X_M^T=\{(\eta,\Phi)\in X^T \quad : \quad |||(\eta,\Phi)|||\le M\}.$$

First, we estimate the norm of $\Gamma(\eta,\Phi)$ in $H^s\times\mathcal{V}^{s+1}$. Using the Minkowski inequality, semigroup properties and Hölder inequality we see that

$$\begin{aligned}\|\Gamma_1(\eta,\Phi)(t)\|_{L^2}&\le C\left(\|\eta_0\|_{L^2}+\|\Phi_0\|_{\mathcal{V}^1}\right)+C\int_0^T\left(\|(\eta\Phi_x)_x\|_{L^2}+\|(\Phi_x)^2\|_{\mathcal{V}^1}\right)d\tau\\
&\le C(\|\eta_0\|_{L^2}+\|\Phi_0\|_{\mathcal{V}^1})\\
&\quad+CT^{3/4}\Big(\sup_{[0,T]}\|\eta\|_{L^2_x}\|\Phi_{xx}\|_{L^4_TL^\infty_x}\\
&\quad+\sup_{[0,T]}\|\Phi_x\|_{L^2_x}\|\eta_x\|_{L^4_TL^\infty_x}+\sup_{[0,T]}\|\Phi_x\|_{L^2_x}\|\Phi_{xx}\|_{L^4_TL^\infty_x}\Big)\end{aligned}$$

and

$$\begin{aligned}&\|\Gamma_2(\eta,\Phi)(t)\|_{\mathcal{V}^1}\\
\le&C(\|\eta_0\|_{L^2}+\|\Phi_0\|_{\mathcal{V}^1})\\
&+CT^{3/4}\big(\sup_{[0,T]}\|\eta\|_{L^2_x}\|\Phi_{xx}\|_{L^4_TL^\infty_x}+\sup_{[0,T]}\|\Phi_x\|_{L^2_x}\|\eta_x\|_{L^4_TL^\infty_x}+\sup_{[0,T]}\|\Phi_x\|_{L^2_x}\|\Phi_{xx}\|_{L^4_TL^\infty_x}\big).\end{aligned}$$

On the other hand, using Cauchy-Schwarz's inequality and Leibniz's rule we have that

$$\begin{aligned}&\|D_x^s\Gamma_1(\eta,\Phi)(t)\|_{L^2}\\
\le&C\left(\|D_x^s\eta_0\|_{L^2}+\|D_x^s\Phi_0\|_{\mathcal{V}^1}\right)+\left\|D_x^s\int_0^tU(t-\tau)(\eta\Phi_x)_x(\tau)d\tau\right\|_{L^2}\\
&+\left\|D_x^s\int_0^tV(t-\tau)(\Phi_x)^2(\tau)d\tau\right\|_{L^2}\\
\le&C(\|D_x^s\eta_0\|_{L^2}+\|D_x^s\Phi_0\|_{\mathcal{V}^1})+CT^{1/2}\left(\|D_x^s(\eta\Phi_x)_x\|_{L^2_xL^2_T}+\|D_x^s(\Phi_x)^2_x\|_{L^2_xL^2_T}\right)\\
\le&C(\|D_x^s\eta_0\|_{L^2}+\|D_x^s\Phi_0\|_{\mathcal{V}^1})\\
&+CT^{1/2}\left(\|D_x^s\eta\|_{L^4_TL^2_x}\|\Phi_{xx}\|_{L^4_TL^\infty_x}+\|D_x^s\Phi_{xx}\|_{L^\infty_xL^2_T}\|\eta\|_{L^2_xL^\infty_T}\right)\\
&+CT^{1/2}\left(\|D_x^s\Phi_x\|_{L^4_TL^2_x}\|\eta_x\|_{L^4_TL^\infty_x}+\|D_x^s\eta_x\|_{L^\infty_xL^2_T}\|\Phi_x\|_{L^2_xL^\infty_T}\right)\\
&+CT^{1/2}\left(\|D_x^s\Phi_x\|_{L^4_TL^2_x}\|\Phi_{xx}\|_{L^4_TL^\infty_x}+\|D_x^s\Phi_{xx}\|_{L^\infty_xL^2_T}\|\Phi_x\|_{L^2_xL^\infty_T}\right)\\
\le&C(\|D_x^s\eta_0\|_{L^2}+\|D_x^s\Phi_0\|_{\mathcal{V}^1})\\
&+CT^{3/4}\|D_x^s\eta\|_{L^\infty_TL^2_x}\|\Phi_{xx}\|_{L^4_TL^\infty_x}+CT^{1/2}\|D_x^s\Phi_{xx}\|_{L^\infty_xL^2_T}\|\eta\|_{L^2_xL^\infty_T}\\
&+CT^{3/4}\|D_x^s\Phi_x\|_{L^\infty_TL^2_x}\|\eta_x\|_{L^4_TL^\infty_x}+CT^{1/2}\|D_x^s\eta_x\|_{L^\infty_xL^2_T}\|\Phi_x\|_{L^2_xL^\infty_T}\\
&+CT^{3/4}\|D_x^s\Phi_x\|_{L^\infty_TL^2_x}\|\Phi_{xx}\|_{L^4_TL^\infty_x}+CT^{1/2}\|D_x^s\Phi_{xx}\|_{L^\infty_xL^2_T}\|\Phi_x\|_{L^2_xL^\infty_T}.\end{aligned}$$

In a similar fashion, we also have that

$$\begin{aligned}
&\|D_x^s\Gamma_2(\eta,\Phi)(t)\|_{\mathcal{V}^1}\\
\leq &C(\|D_x^s\eta_0\|_{L^2}+\|D_x^s\Phi_0\|_{\mathcal{V}^1})\\
&+CT^{3/4}\|D_x^s\eta\|_{L_T^\infty L_x^2}\|\Phi_{xx}\|_{L_T^4L_x^\infty}+CT^{1/2}\|D_x^s\Phi_{xx}\|_{L_x^\infty L_T^2}\|\eta\|_{L_x^2L_T^\infty}\\
&+CT^{3/4}\|D_x^s\Phi_x\|_{L_T^\infty L_x^2}\|\eta_x\|_{L_T^4L_x^\infty}+CT^{1/2}\|D_x^s\eta_x\|_{L_x^\infty L_T^2}\|\Phi_x\|_{L_x^2L_T^\infty}\\
&+CT^{3/4}\|D_x^s\Phi_x\|_{L_T^\infty L_x^2}\|\Phi_{xx}\|_{L_T^4L_x^\infty}+CT^{1/2}\|D_x^s\Phi_{xx}\|_{L_x^\infty L_T^2}\|\Phi_x\|_{L_x^2L_T^\infty}.
\end{aligned}$$

From previous computations, we conclude that

$$\sup_{[0,T]}\|\Gamma(\eta,\Phi)(t)\|_{H^s\times\mathcal{V}^{s+1}}\leq C(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}})+CT^{1/2}(1+T^{1/4})|||(\eta,\Phi)|||^2.$$

Now, using Lemma 2.5 and the previous argument we have that

$$\begin{aligned}
&\|\Gamma_1(\eta,\Phi)\|_{L_x^2L_T^\infty}\\
\leq &C(1+T)^{1/2}(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}})+\left\|U(t)\int_0^t U(-\tau)((\eta\Phi_x)_x)(\tau)d\tau\right\|_{L_x^2L_T^\infty}\\
&+\left\|V(t)\int_0^t V(-\tau)(\Phi_x)^2(\tau)d\tau\right\|_{L_x^2L_T^\infty}\\
\leq &C(1+T)^{1/2}\left(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}}\right)\\
&+C(1+T)^{1/2}T^{1/2}\left(\|(\eta\Phi_x)_x\|_{H^sL_T^2}+\|(\Phi_x)_x^2\|_{H^sL_T^2}\right)\\
\leq &C(1+T)^{1/2}(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}})\\
&+C(1+T)^{1/2}T^{1/2}\Big(T^{1/4}\|\eta\|_{L_T^\infty H^s}\|\Phi_{xx}\|_{L_T^4L_x^\infty}+\|D_x^s\Phi_{xx}\|_{L_x^\infty L_T^2}\|\eta\|_{L_x^2L_T^\infty}\\
&+T^{1/4}\|\Phi_x\|_{L_T^\infty H^s}\|\eta_x\|_{L_T^4L_x^\infty}+\|D_x^s\eta_x\|_{L_x^\infty L_T^2}\|\Phi_x\|_{L_x^2L_T^\infty}\\
&+T^{1/4}\|\Phi_x\|_{L_T^\infty H^s}\|\Phi_{xx}\|_{L_T^4L_x^\infty}+\|D_x^s\Phi_{xx}\|_{L_x^\infty L_T^2}\|\Phi_x\|_{L_x^2L_T^\infty}\Big)\\
\leq &C(1+T)^{1/2}(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}})+C(1+T)^{1/2}T^{1/2}(1+T^{1/4})|||(\eta,\Phi)|||^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_x\Gamma_2(\eta,\Phi)\|_{L_x^2L_T^\infty}\\
\leq &C(1+T)^{1/2}(\|\eta_0\|_{H^s}+\|\eta_0\|_{\mathcal{V}^{s+1}})+\left\|\partial_xW(t)\int_0^t W(-\tau)((\eta\Phi_x)_x)(\tau)d\tau\right\|_{L_x^2L_T^\infty}\\
&+\left\|\partial_xU(t)\int_0^t U(-\tau)(\Phi_x)^2(\tau)d\tau\right\|_{L_x^2L_T^\infty}\\
\leq &C(1+T)^{1/2}\left(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}}\right)\\
&+C(1+T)^{1/2}T^{1/2}\left(\|(\eta\Phi_x)_x\|_{H^sL_T^2}+\|(\Phi_x)_x^2\|_{H^sL_T^2}\right)\\
\leq &C(1+T)^{1/2}(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}})+C(1+T)^{1/2}T^{1/2}(1+T^{1/4})|||(\eta,\Phi)|||^2.
\end{aligned}$$

Using Lemma 2.2 and Cauchy-Schwarz's inequality we obtain that

$$\|\partial_x\Gamma_1(\eta,\Phi)\|_{L_T^4L_x^\infty}\leq\|D_x^{1/4}U(t)D^{3/4}\eta_0\|_{L_T^4L_x^\infty}+\|D_x^{1/4}V(t)D^{3/4}\eta_0\|_{L_T^4L_x^\infty}$$

$$
\begin{aligned}
&+ \left\| D_x^{1/4} U(t) \int_0^t U(-\tau) D_x^{3/4}((\eta\Phi_x)_x)(\tau) d\tau \right\|_{L_T^4 L_x^\infty} \\
&+ \left\| D_x^{1/4} V(t) \int_0^t V(-\tau) D_x^{3/4}(\Phi_x)^2(\tau) d\tau \right\|_{L_T^4 L_x^\infty} \\
\leq & C(1+T^{1/4})(\|\eta_0\|_{H^s} + \|\Phi_0\|_{\mathcal{V}^{s+1}}) \\
&+ C(1+T^{1/4})T^{1/2}(\|(\eta\Phi_x)_x\|_{H^s L_T^2} + \|(\Phi_x)^2\|_{\mathcal{V}^{s+1} L_T^2}) \\
\leq & C(1+T^{1/4})(\|\eta_0\|_{H^s} + \|\Phi_0\|_{\mathcal{V}^{s+1}}) \\
&+ C(1+T^{1/4})T^{1/2}(1+T^{1/4})|||(\eta,\Phi)|||^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|\partial_x^2 \Gamma_2(\eta,\Phi)\|_{L_T^4 L_x^\infty} \leq & \|D_x^{5/4} W(t) D^{3/4}\eta_0\|_{L_T^4 L_x^\infty} + \|D_x^{5/4} U(t) D^{3/4}\eta_0\|_{L_T^4 L_x^\infty} \\
&+ \left\| D_x^{5/4} W(t) \int_0^t W(-\tau) D_x^{3/4}((\eta\Phi_x)_x)(\tau) d\tau \right\|_{L_T^4 L_x^\infty} \\
&+ \left\| D_x^{5/4} U(t) \int_0^t U(-\tau) D_x^{3/4}(\Phi_x)^2(\tau) d\tau \right\|_{L_T^4 L_x^\infty} \\
\leq & C(1+T^{1/4})(\|\eta_0\|_{H^s} + \|\Phi_0\|_{\mathcal{V}^{s+1}}) \\
&+ C(1+T^{1/4})T^{1/2}(1+T^{1/4})|||(\eta,\Phi)|||^2.
\end{aligned}
$$

Finally, from Lemma 2.4 and Leibniz's rule it follows that

$$
\begin{aligned}
&\|D_x^s \partial_x \Gamma_1(\eta,\Phi)\|_{L_x^\infty L_T^2} \\
\leq & C(1+T^{1/2}) \left(\|D_x^s \eta_0\|_{L^2} + \|D_x^s \Phi_0\|_{\mathcal{V}^1}\right) \\
&+ \left\| \partial_x U(t) \int_0^t U(-\tau) D_x^s (\eta\Phi_x)_x(\tau) d\tau \right\|_{L_x^\infty L_T^2} \\
&+ \left\| \partial_x V(t) \int_0^t V(-\tau) D_x^s (\Phi_x)^2(\tau) d\tau \right\|_{L_x^\infty L_T^2} \\
\leq & C(1+T^{1/2})(\|\eta_0\|_{H^s} + \|\Phi_0\|_{\mathcal{V}^{s+1}}) \\
&+ C(1+T^{1/2})T^{1/2}\left(\|D_x^s(\eta\Phi_x)_x\|_{L_x^2 L_T^2} + \|D_x^s(\Phi_x)_x^2\|_{L_x^2 L_T^2}\right) \\
\leq & C(1+T^{1/2})(\|\eta_0\|_{H^s} + \|\Phi_0\|_{\mathcal{V}^{s+1}}) \\
&+ C(1+T^{1/2})T^{1/2}\Big(\|D_x^s \eta\|_{L_T^4 L_x^2} \|\Phi_{xx}\|_{L_T^4 L_x^\infty} + \|D_x^s \Phi_{xx}\|_{L_x^\infty L_T^2} \|\eta\|_{L_x^2 L_T^\infty} \\
&+ \|D_x^s \Phi_x\|_{L_T^4 L_x^2} \|\eta_x\|_{L_T^4 L_x^\infty} + \|D_x^s \eta_x\|_{L_x^\infty L_T^2} \|\Phi_x\|_{L_x^2 L_T^\infty} \\
&+ \|D_x^s \Phi_x\|_{L_T^4 L_x^2} \|\Phi_{xx}\|_{L_T^4 L_x^\infty} + \|D_x^s \Phi_{xx}\|_{L_x^\infty L_T^2} \|\Phi_x\|_{L_x^2 L_T^\infty}\Big) \\
\leq & C(1+T^{1/2})(\|\eta_0\|_{H^s} + \|\Phi_0\|_{\mathcal{V}^{s+1}}) \\
&+ C(1+T^{1/2})T^{1/2}\Big(T^{1/4}\|D_x^s \eta\|_{L_T^\infty L_x^2} \|\Phi_{xx}\|_{L_T^4 L_x^\infty} + \|D_x^s \Phi_{xx}\|_{L_x^\infty L_T^2} \|\eta\|_{L_x^2 L_T^\infty} \\
&+ T^{1/4}\|D_x^s \Phi_x\|_{L_T^\infty L_x^2} \|\eta_x\|_{L_T^4 L_x^\infty} + \|D_x^s \eta_x\|_{L_x^\infty L_T^2} \|\Phi_x\|_{L_x^2 L_T^\infty} \\
&+ T^{1/4}\|D_x^s \Phi_x\|_{L_T^\infty L_x^2} \|\Phi_{xx}\|_{L_T^4 L_x^\infty} + \|D_x^s \Phi_{xx}\|_{L_x^\infty L_T^2} \|\Phi_x\|_{L_x^2 L_T^\infty}\Big) \\
\leq & C(1+T^{1/2})(\|\eta_0\|_{H^s} + \|\Phi_0\|_{\mathcal{V}^{s+1}}) + C(1+T^{1/2})T^{1/2}(1+T^{1/4})|||(\eta,\Phi)|||^2,
\end{aligned}
$$

and also that

$$\begin{aligned}
\|D_x^s\partial_x^2\Gamma_2(\eta,\Phi)\|_{L_x^\infty L_T^2} \leq& C(1+T^{1/2})(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}})\\
&+\left\|\partial_x^2 W(t)\int_0^t W(-\tau)D_x^s((\eta\Phi_x)_x)(\tau)d\tau\right\|_{L_x^\infty L_T^2}\\
&+\left\|\partial_x^2 U(t)\int_0^t U(-\tau)D_x^s(\Phi_x)^2(\tau)d\tau\right\|_{L_x^\infty L_T^2}\\
\leq& C(1+T^{1/2})(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}})\\
&+C(1+T^{1/2})T^{1/2}\Big(\|D_x^s(\eta\Phi_x)_x\|_{L_x^2L_T^2}+\|D_x^s(\Phi_x)_x^2\|_{L_x^2L_T^2}\Big)\\
\leq& C(1+T^{1/2})(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}})\\
&+C(1+T^{1/2})T^{1/2}(1+T^{1/4})|||(\eta,\Phi)|||^2.
\end{aligned}$$

From previous estimates we obtain that

$$\begin{aligned}
&|||\Gamma(\eta,\Phi)|||\\
&\leq C(1+T^{1/2})(1+T^{1/4})\Big(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}}\Big)+CT^{1/2}(1+T^{1/2})(1+T^{1/4})^2|||(\eta,\Phi)|||^2.
\end{aligned}$$

If we choose $M=2C(1+T^{1/2})(1+T^{1/4})\,(\|\eta_0\|_{H^s}+\|\Phi_0\|_{\mathcal{V}^{s+1}})$ and $T>0$ such that

$$CT^{1/2}(1+T^{1/2})(1+T^{1/4})^2M<1/2,$$

we have that $\Gamma$ is a contraction in $X_M^T$. Thus, the contraction mapping principle guarantees the existence of a unique $(\eta,\Phi)$ in $X_M^T$ solving the associated integral problem to system (2.1). To show the continuous dependence we follow a similar argument as the one described above. The uniqueness follows using a standard argument and so we will omit it. $\square$

Since in [30] was proved the local well-posedness of the Cauchy problem associated with the *abad*-Boussinesq system (1.1) for $s\geq 0$ in the case $a\neq 0$ (see Theorem 2.2 in [30]), we have the following corollary.

**Corollary 3.1.** *Suppose that $s$ is such that if $a\neq 0$, $s\geq 0$ and if $a=0$, $s>\frac{3}{4}$. Then for all $(\eta_0,\Phi_0)\in H^s\times\mathcal{V}^{s+1}$ there exist $T>0$ depending only on $\|(\eta_0,\Phi_0)\|_{H^s\times\mathcal{V}^{s+1}}$ and a unique solution $(\eta,\Phi)\in C\left([0,T]:H^s\times\mathcal{V}^{s+1}\right)$ of the Cauchy problem associated with (1.1) and initial condition $(\eta_0,\Phi_0)$.*

Moreover, we have the global well-posedness in the energy space $X=H^1\times\mathcal{V}^2$.

**Theorem 3.2.** *There is $\delta>0$ such that for $(\eta_0,\Phi_0)\in H^1\times\mathcal{V}^2$ with*

$$\|(\eta_0,\Phi_0)\|_{H^1\times\mathcal{V}^2}\leq\delta,$$

*the Cauchy problem associated with (1.1) and initial condition $(\eta_0,\Phi_0)$ has a unique global solution $(\eta,\Phi)\in C\left([0,\infty),H^1\times\mathcal{V}^2\right)$.*

**Proof.** See Theorem 3.2 in [30]. $\square$

# 4. Preliminaries for stability

We start mentioning that solitary waves correspond to pair of functions $(u, v)$ satisfying the system

$$\begin{cases} bv'''' - v'' + \omega\,(u' - au''') - (uv')' = 0, \\ u - du'' - \omega\,(v' - cv''') + \dfrac{1}{2}\,(v')^2 = 0. \end{cases} \tag{4.1}$$

It is straightforward to see that such pair $(u, v)$ is characterized as critical points of the functional $J_\omega$ given by

$$J_\omega(u, v) = I_\omega(u, v) + G(u, v),$$

where the functionals $I_\omega$ and $G$ are defined on the space $X = H^1 \times \mathcal{V}^2$ by

$$\begin{aligned} I_\omega(u, v) &= I_1(u, v) + I_{2,\omega}(u, v), \\ I_1(u, v) &= \int_{\mathbb{R}} \left[u^2 + d(u')^2 + (v')^2 + b(v'')^2\right] dx, \\ I_{2,\omega}(u, v) &= -2\omega \int_{\mathbb{R}} (uv' + au'v'')\, dx, \\ G(u, v) &= \int_{\mathbb{R}} u(v')^2 dx. \end{aligned}$$

As done by J. Quintero and A. Montes in [29] in the 2-dimensional case, existence of solitary waves for the Boussinesq system (1.1) for $0 < |\omega| < \min\left\{1, \frac{b}{a}, \frac{d}{a}\right\}$ can be established by using the Concentration-Compactness Principle. The strategy is to consider the following minimization problem

$$\mathcal{I}_\omega := \inf\left\{I_\omega(u, v) \;:\; (u, v) \in X \quad \text{with} \quad G(u, v) = 1\right\}, \tag{4.2}$$

and then the existence of solitary wave solutions is consequence of the following results, which we will use throughout this work (see Lemma 3.1, Theorem 3.1 and Theorem 3.3 in [29]). Hereafter, we assume wave speed $\omega$ satisfying $0 < |\omega| < \min\left(1, \frac{b}{a}, \frac{d}{a}\right)$.

**Lemma 4.1.** *The functional $I_\omega$ is nonnegative and there are positive constants $M_1(a, b, d, \omega)$ and $M_2(a, b, d, \omega)$ such that*

$$M_1\|(u, v)\|_X^2 \le I_\omega(u, v) \le M_2\|(u, v)\|_X^2. \tag{4.3}$$

*Furthermore, $\mathcal{I}_\omega$ is finite and positive.*

**Theorem 4.1.** *If $(u_0, v_0)$ is a minimizer for problem (4.2), then $(u, v) = -k(u_0, v_0)$ is a nontrivial solution of (4.1) for $k = \frac{2}{3}\mathcal{I}_\omega$.*

**Theorem 4.2.** *If $(u_m, v_m)$ is a minimizing sequence for (4.2), then there is a subsequence (which we denote the same), a sequence of points $(y_m) \in \mathbb{R}$, and a minimizer $(u_0, v_0) \in X$ of (4.2), such that the translated functions*

$$(\tilde{u}_m, \tilde{v}_m) = (u_m(\cdot + y_m), v_m(\cdot + y_m))$$

*converge to $(u_0, v_0)$ strongly in $X$.*

We point out that J. Quintero and A. Montes in [30] using the Mountain Pass Lemma established the existence of solitary waves of wave speed $\omega$ with $0 < |\omega| < \min\left(1, \frac{b}{a}, \frac{d}{a}\right)$ for the Boussinesq system (1.1).

On the other side, a very interesting fact is that can be established a link between solitary waves for the Boussinesq system (1.1) and the KdV equation. As done in the 2-dimensional case by J. Quintero and A. Montes for example in [24, 29] can be proved that a renormalized family of solitons of the Boussinesq system converges to a nontrivial soliton for a KdV equation, assuming $\omega$ is close to $1^-$ ot $\epsilon \to 0^+$, and balancing the effects of nonlinearity and dispersion. More precisely, set $\sigma > \frac{1}{3}$, $\epsilon > 0$, $\omega^2 = 1 - \epsilon$ and for a given couple $(u, v) \in X$ define the functions $z$ and $w$ by

$$u(x) = \epsilon^{\frac{1}{6}} z(y), \quad v(x) = \epsilon^{-\frac{1}{3}} w(y), \quad y = \epsilon^{\frac{1}{2}} x. \tag{4.4}$$

Then a simple calculation shows that

$$I_1(u, v) = \epsilon^{\frac{5}{6}} I^{1,\epsilon}(z, w), \quad I_{2,\omega}(u, v) = \epsilon^{\frac{5}{6}} I^{2,\epsilon}(z, w),$$

and also that

$$I_{\omega(\epsilon)}(u, v) = \epsilon^{\frac{5}{6}} I^{\epsilon}(z, w), \quad G(u, v) = G^{\epsilon}(z, w),$$

where $I^1$, $I^{2,\epsilon}$, $I^\epsilon$ and $G^\epsilon$ are given by

$$\begin{aligned}
I^{\epsilon}(z, w) &= I^{1,\epsilon}(z, w) + I^{2,\epsilon}(z, w),\\
I^{1,\epsilon}(z, w) &= \int_{\mathbb{R}} \left(\epsilon^{-1} z^2 + d(z')^2 + \epsilon^{-1}(w')^2 + b(w'')^2\right) dy,\\
I^{2,\epsilon}(z, w) &= -2\omega \int_{\mathbb{R}} (\epsilon^{-1} z w' + a z' w'')\, dy,\\
G^{\epsilon}(z, w) &= \int_{\mathbb{R}} z(w')^2\, dy.
\end{aligned}$$

Note that if $0 < |\omega| < \min\left(1, \frac{b}{a}, \frac{d}{a}\right)$, then $I^\epsilon(z, w) > 0$ and there is a family $(u_\omega, v_\omega)_\omega$ such that

$$I_\omega(u_\omega, v_\omega) = \mathcal{I}_\omega, \quad G(u_\omega, v_\omega) = 1.$$

Thus, if we denote

$$\mathcal{I}^\epsilon := \inf\left\{I^\epsilon(z, w) \;:\; (z, w) \in X \quad \text{with} \quad G^\epsilon(z, w) = 1\right\},$$

there is a correspondent family $(z^\epsilon, w^\epsilon)_\epsilon$ such that

$$\mathcal{I}^\epsilon = I^\epsilon(z^\epsilon, w^\epsilon), \quad G^\epsilon(z^\epsilon, w^\epsilon) = 1, \quad \mathcal{I}_\omega = \epsilon^{\frac{5}{6}} \mathcal{I}^\epsilon.$$

We also have that $(z^\epsilon, w^\epsilon)$ is a solution, in the sense of distributions, of the system

$$\begin{cases} b\epsilon w'''' - w'' + \omega\left(z' - a\epsilon z'''\right) + \dfrac{2}{3}\epsilon^{\frac{1}{6}} \mathcal{I}_\omega \left(z w'\right)' = 0,\\[2mm] z - d\epsilon z'' - \omega\left(w' - a\epsilon w'''\right) - \dfrac{1}{3}\epsilon^{\frac{1}{6}} \mathcal{I}_\omega \left(w'\right)^2 = 0. \end{cases} \tag{4.5}$$

We are interested in relating the family $(z^\epsilon, w^\epsilon)_\epsilon$ with the solitons for the KdV equation, as $\epsilon \to 0$. To do this, we define in $\mathcal{V}^2$ the functionals

$$J^\epsilon(w) = I^\epsilon(\omega w_x, w), \quad K^\epsilon(w) = G(\omega w_x, w). \tag{4.6}$$

We also define the number $\mathcal{J}^\epsilon$

$$\mathcal{J}^\epsilon = \inf\{J^\epsilon(w) : w \in \mathcal{Z},\ K^\epsilon(w) = 1\},$$

where following the same approach in [24, 29], we set the Banach space $\mathcal{Z}$ as the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm given by

$$||w||_{\mathcal{Z}}^2 = \int_{\mathbb{R}} (w_x^2 + w_{xx}^2)\, dx,$$

and obtain the following result.

**Lemma 4.2.** *Let $\sigma > \frac{1}{3}$ and $0 < |\omega| < \min\left(1, \frac{b}{a}, \frac{d}{a}\right)$. Then we have that*

$$\lim_{\epsilon\to 0^+} \mathcal{I}^\epsilon = \lim_{\epsilon\to 0^+} I^\epsilon(z^\epsilon, w^\epsilon) = \mathcal{J}^0 > 0, \qquad \lim_{\epsilon\to 0^+} K^\epsilon(w^\epsilon) = \lim_{\epsilon\to 0^+} G(z^\epsilon, w^\epsilon) = 1,$$

*where*

$$\begin{aligned} &\mathcal{J}^0 = \inf\{J^0(w) : w \in \mathcal{Z},\ G^0(w) = 1\},\\ &J^0(w) = \int_{\mathbb{R}} \left(w_x^2 + \left(\sigma - \tfrac{1}{3}\right) w_{xx}^2\right) dx,\\ &G^0(w) = \int_{\mathbb{R}} w_x^3 dx. \end{aligned}$$

Now, we study the main result in this section. We see that a translate subsequence of the renormalized sequence $(z^\epsilon, w^\epsilon)$ converges weakly to a couple $(z_0, w_0)$ that satisfies the system (4.1), and so $z_0 = \partial_x w_0$ is a weak solution of a KdV type equation. Before we go further, we have the following characterization of solitary waves for the KdV equation.

**Theorem 4.3.** *Let $\sigma > \frac{1}{3}$ and let $(w_m)_m$ be a minimizing sequence for $\mathcal{J}^0$, then there exist a subsequence (denoted the same) and a nonzero distribution $w_0 \in \mathcal{Z}$ such that*

$$J^0(w_0) = \mathcal{J}^0,$$

*and there exists a sequence of points $(y_n)_m \subset \mathbb{R}$ such that $w_m(\cdot + y_m) \to w_0$ in $\mathcal{Z}$. Moreover, $w_0$ is a distributional solution of the equation*

$$w_{xx} - \left(\sigma - \frac{1}{3}\right) w_{xxxx} + 2\mathcal{J}^0 w_x w_{xx} = 0, \tag{4.7}$$

*and so $w = \left(\frac{2}{3}\mathcal{J}^0\right) \partial_x w_0$ is a nontrivial solitary wave solution in the sense of distributions for the KdV type equation*

$$\partial_x w - \left(\sigma - \frac{1}{3}\right) \partial_x^3 w + 3w\partial_x w = 0. \tag{4.8}$$

Then we have the coming result.

**Lemma 4.3.** *Let $\sigma > \frac{1}{3}$ and $0 < |\omega| < \min\left(1, \frac{b}{a}, \frac{d}{a}\right)$. For any sequence $\epsilon_j \to 0^+$ there is a translate subsequence (denoted the same) of $(z^{\epsilon_j}, w^{\epsilon_j})_j$ and there exist nontrivial distributions $w_0 \in \mathcal{Z}$ and $z_0 \in H^1$ such that as $j \to \infty$,*

$$w^{\epsilon_j} \to w_0 \quad \text{in } \mathcal{Z}, \qquad z^{\epsilon_j} - \partial_x w^{\epsilon_j} \to 0, \qquad z^{\epsilon_j} \to z_0 \quad \text{in } H^1.$$

*Moreover,* $(z_0, w_0)$ *is a nontrivial weak solution of the system*

$$\begin{aligned} &z = \partial_x w \\ &\partial_{xx} w - \left(\sigma - \frac{1}{3}\right) \partial_{xxxx} w + 3 w_x \partial_{xx} w = 0. \end{aligned} \tag{4.9}$$

*In other words,* $z_0 = \partial_x w_0 \in H^1$*, with* $\partial_x w_0$ *being a solution of the solitary wave equation for a KdV equation in the sense of distributions.*

**Proof.** Let $(\epsilon_j)_j$ be a sequence of positive number such that $\epsilon_j \to 0^+$. We note that $\left\{ \left(G^0\left(w^{\epsilon_j}\right)\right)^{-\frac{1}{3}} w^{\epsilon_j} \right\}_j$ is a minimizing sequence for $\mathcal{J}^0$ and also that

$$G^0\left(w^{\epsilon_j}\right) \to 1.$$

Using this fact and Theorem 4.3, we have that there exist a translate sequence of $(z^{\epsilon_j}, w^{\epsilon_j})_j$ (denote the same) and there exist a nonzero distribution $w_0 \in \mathcal{Z}$ such that $w^{\epsilon_j} \to w_0$ in $\mathcal{Z}$ and $w_0$ is a solution of the equation (4.7). Then there exist an on trivial distribution $z_0 \in H^1$ such that $z^{\epsilon_j} \to z_0$ $H^1$. Thus, we obtain also that $z_0 = \partial w_0$. Now, we note that the solitary wave system (4.5) can be rewritten, after taking the $x$-derivative and multiplying for $\omega_j$ the second equation, as

$$\begin{aligned} &\epsilon_j^{-1}(\omega_j (z^{\epsilon_j})' - (w^{\epsilon_j})'') + (b(w^{\epsilon_j})'''' - a\omega_j (z^{\epsilon_j})''') \\ &+ \frac{2}{3}\mathcal{I}^{\epsilon_j}\left((z^{\epsilon_j})(w^{\epsilon_j})'' + (z^{\epsilon_j})'(w^{\epsilon_j})'\right) = 0, \\ &\epsilon_j^{-1}(\omega_j (z^{\epsilon_j})' - \omega_j^2 (w^{\epsilon_j})'') + \omega_j (a\omega_j (w^{\epsilon_j})'''' - d(z^{\epsilon_j})''') - \frac{2}{3}\omega_j \mathcal{I}^{\epsilon_j}\left((w^{\epsilon_j})'(w^{\epsilon_j})''\right) = 0. \end{aligned}$$

Using $\omega_j^2 = 1 - \epsilon_j$ and by subtracting the second from first equation, we get that

$$\begin{aligned} &- (w^{\epsilon_j})'' - \omega_j (a\omega_j (w^{\epsilon_j})'''' - d(z^{\epsilon_j})''') + \frac{2}{3}\omega_j \mathcal{I}^{\epsilon_j}\left((w^{\epsilon_j})'(w^{\epsilon_j})''\right) \\ &= - (b(w^{\epsilon_j})'''' - a\omega_j (z^{\epsilon_j})''') - \frac{2}{3}\mathcal{I}^{\epsilon_j}\left((z^{\epsilon_j})(w^{\epsilon_j})'' + (z^{\epsilon_j})'(w^{\epsilon_j})'\right). \end{aligned} \tag{4.10}$$

Then, using that $\mathcal{I}^{\epsilon_j} \to \mathcal{I}^0$, $w^{\epsilon_j} \to w_0$ in $\mathcal{Z}$, $z^{\epsilon_j} \to z_0$ in $H^1$, and $z_0 = \partial w_0$, for any test function $\psi \in C_0^\infty(\mathbb{R})$, we have that

$$\lim_{j\to\infty} \left\langle \frac{2\omega_j}{3}\mathcal{I}^{\epsilon_j}((w_j^{\epsilon_j})'(w_j^{\epsilon_j})'') + \frac{2}{3}\mathcal{I}^{\epsilon_j}((z_j^{\epsilon_j})(w_j^{\epsilon_j})')', \psi \right\rangle = 2\mathcal{I}^0 \langle w_0' w_0'', \psi\rangle.$$

Thus, we see that

$$\begin{aligned} &\lim_{j\to\infty} \left\langle -(w_j^{\epsilon_j})'' + (b - a\omega_j^2)(w^{\epsilon_j})'''' + \omega_j (d-a)(z^{\epsilon_j})''', \psi \right\rangle \\ &= \left\langle -w_0'' + \left(\sigma - \frac{1}{3}\right) w_0'''', \psi \right\rangle, \end{aligned}$$

since $2a - (b+d) = \frac{1}{3} - \sigma$. Therefore, from (4.10) we concluded that $w_0$ is a nontrivial solution of the equation

$$w_{xx} - \left(\sigma - \frac{1}{3}\right) w_{xxxx} + 2\mathcal{J}^0 w_x w_{xx} = 0.$$

In particular, the pair $(z^0, w^0) = -\left(\frac{2}{3}\mathcal{J}^0\right)(z_0, w_0)$ is a nontrivial solution of the system (4.9). In other words, $z^0 = \partial_x w^0$ is a solution for of the KdV solitary wave equation (4.8) in distributional sense. $\square$

We will use the Lemma 4.2 and Lemma 4.3 in our proof of stability.

# 5. Ground state solutions and convexity of $d_1$

Recall that the solitary waves for the Boussinesq system (1.1) are characterized as critical points of the functional defined on $X = H^1 \times \mathcal{V}^2$ by

$$J_\omega(u,v) = I_\omega(u,v) + G(u,v).$$

In particular, if

$$K_\omega(u,v) = \langle J'_\omega(u,v), (u,v)\rangle$$

we have that

$$K_\omega(u,v) = 2I_\omega(u,v) + 3G(u,v) = 2J_\omega(u,v) + G(u,v).$$

Now, define the set

$$\mathcal{M}_\omega = \{(u,v) \in X \ : \ K_\omega(u,v) = 0,\ (u,v) \neq 0\}.$$

Note that $\mathcal{M}_\omega$ is just the "artificial constrain" for minimizing the functional $J_\omega$ on $X$. We will see that the analysis of the orbital stability of ground states solutions depends upon some properties of the function $d$ defined by

$$d(\omega) = \inf\{J_\omega(u,v) : \ (u,v) \in \mathcal{M}_\omega\}.$$

A ground state solution is a solitary wave which minimizes the action functional $J_\omega$ among all the nonzero solutions of (4.1). Moreover, the set of ground state solutions

$$\mathcal{G}_\omega = \{(u,v) \in \mathcal{M}_\omega : \ d(\omega) = J_\omega(u,v)\}$$

can be characterized as

$$\mathcal{G}_\omega = \left\{(u,v) \in X \setminus \{0\} \ : \ d(\omega) = \frac{1}{3}I_\omega(u,v) = -\frac{1}{2}G(u,v)\right\} \subset \mathcal{M}_\omega.$$

In the next lemmas we present important properties of $d(\omega)$.

**Lemma 5.1.** *Let* $0 < |\omega| < \min\left(1, \frac{b}{a}, \frac{d}{a}\right)$ *and* $\sigma > \frac{1}{3}$. *Then*

1. $d(\omega)$ *exist and is positive.*
2. $d(\omega) = \inf\left\{\frac{1}{3}I_\omega(u,v) \ : \ K_\omega(u,v) \leq 0,\ (u,v) \neq 0\right\}$.

**Proof.** 1. Let $(u,v) \in \mathcal{M}_\omega$, then we have that

$$J_\omega(u,v) = \frac{1}{3}I_\omega(u,v) \geq 0.$$

This implies that $d(\omega)$ exists. Now, Using the Young inequality and that the embedding $H^1(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ is continuous for $q \geq 2$, we see that there is a constant $C > 0$ such that

$$|G(u,v)| \leq C\Big(\|u\|^3_{H^1} + \|v'\|^3_{H^1}\Big).$$

Thus, we see that

$$J_\omega(u,v) = \frac{1}{3}I_\omega(u,v) = -\frac{1}{2}G(u,v) \leq C\|(u,v)\|^3_X \leq C\left(I_\omega(u,v)\right)^{\frac{3}{2}}.$$

Then follows that $\frac{1}{3}I_\omega(u,v) \geq C$, and this implies that $d(\omega) \geq C > 0$.

2. For $(u,v) \in X$ such that $K_\omega(u,v) \leq 0$ we have that $G(u,v) < 0$. Define $\alpha \in [0,1)$ by

$$\alpha = -\frac{2I_\omega(u,v)}{3G(u,v)}.$$

Then a direct computation shows that $K_\omega(\alpha(u,v)) = 0$. In other words, $\alpha(u,v) \in \mathcal{M}_\omega$. So that,

$$d(\omega) \leq J_\omega(\alpha(u,v)) = \frac{\alpha^2}{3}I_\omega(u,v) \leq \frac{1}{3}I_\omega(u,v).$$

Hence, we obtain that

$$d(\omega) \leq \inf\Big\{\frac{1}{3}I_\omega(u,v) \ : \ K_\omega(u,v) \leq 0\Big\}.$$

If $(u,v) \in \mathcal{M}_\omega$, we see that $J_\omega(u,v) = \frac{1}{3}I_\omega(u,v)$ and also that

$$\begin{aligned}\inf\Big\{\frac{1}{3}I_\omega(u,v) \ : \ K_\omega(u,v) \leq 0,\ (u,v) \neq 0\Big\} &\leq \inf\big\{J_\omega(u,v) \ : \ (u,v) \in \mathcal{M}_\omega\big\} \\ &= d(\omega),\end{aligned}$$

meaning that the statement 2 of lemma follows. □

**Lemma 5.2.** *Let $0 < |\omega| < \min\left(1, \frac{b}{a}, \frac{d}{a}\right)$ and $\sigma > \frac{1}{3}$. Then*

1. *If $(u_m, v_m)$ is a minimizing sequence of $d(\omega)$, then there is a subsequence, which we denote the same, a sequence of points $(y_m) \in \mathbb{R}$, and $(u^\omega, v^\omega) \in X \setminus \{0\}$ such that the translated functions*

$$(u_m(\cdot + y_m), v_m(\cdot + y_m))$$

*converge to $(u^\omega, v^\omega)$ strongly in $X$, $(u^\omega, v^\omega) \in \mathcal{M}_\omega$, $d(\omega) = J_\omega(u^\omega, v^\omega)$ and $(u^\omega, v^\omega)$ is a solution of (4.1). Moreover,*

$$d(\omega) = \frac{4}{27}\mathcal{I}_\omega^3, \tag{5.1}$$

*where $\mathcal{I}_\omega = \inf\{I_\omega(u,v) \ : \ G(u,v) = 1,\ (u,v) \in X\}$.*

2. *Let $(u_m, v_m)$ be a sequence in $X$ such that*

$$\frac{1}{3}I_\omega(u_m, v_m) \to d(\omega) \quad \textit{and} \quad J_\omega(u_m, v_m) \to d_1 \leq d(\omega).$$

*Then there exist a subsequence of $(u_m, v_m)$ which denote the same, a sequence $(y_m) \in \mathbb{R}^2$ and $(u^\omega, v^\omega) \in \mathcal{M}_\omega$ such that the translated functions*

$$(u_m(\cdot + y_m), v_m(\cdot + y_k))$$

*converge to $(u^\omega, v^\omega)$ strongly in $X$ and $d_1 = d(\omega) = \frac{1}{3}I_\omega(u^\omega, v^\omega)$.*

**Proof.** The first part of this result is consequence of the Theorems (4.1)-(4.2) and the following argument. Let $(u,v) \in X \setminus \{0\}$ be such that $K_\omega(u,v) = 0$, then

$$I_\omega(u,v) = -\frac{3}{2}G(u,v) = \frac{3}{2}|G(u,v)| = 3J_\omega(u,v).$$

Consider the couple

$$(w,z)=\frac{1}{G^{\frac{1}{3}}(u,v)}(u,v).$$

Then $G(w,z)=1$. Thus,

$$\mathcal{I}_\omega\le I_\omega(w,z)=\frac{1}{G^{\frac{2}{3}}(u,v)}I_\omega(u,v)=\left(\frac{3}{2}\right)^{\frac{2}{3}}I_\omega^{\frac{1}{3}}(u,v)=\left(\frac{3}{2}\right)^{\frac{2}{3}}\Big(3J_\omega(u,v)\Big)^{\frac{1}{3}}.$$

So that, we concluded

$$\frac{4}{27}\mathcal{I}_\omega^3\le d(\omega).$$

Now, suppose that $(u,v)\neq 0$ such that $G(u,v)=1$. Take $t$ such that

$$K_\omega(tu,tv)=0.$$

In this case, $2I_\omega(u,v)+3t=0$. Therefore

$$t^2=\frac{4}{9}I_\omega^2(u,v).$$

Then we obtain,

$$d(\omega)\le J_\omega(tu,tv)=t^2\left(I_\omega(u,v)+t\right)=\frac{4}{27}I_\omega^3(u,v).$$

Thus, we have shown that

$$d(\omega)\le\frac{4}{27}\left(\mathcal{I}_\omega\right)^3.$$

This proves (5.1). Now, we show the second part. Since $K_\omega=2I_\omega+3G$, then we see that

$$J_\omega(u_m,v_m)=\frac{1}{3}\left(I_\omega(u_m,v_m)+K_\omega(u_m,v_m)\right)\to d_1\le d(\omega).$$

Then for $m$ large enough we have that $K_\omega(u_m,v_m)\le 0$. This fact implies that the sequence $(u_m,v_m)$ is a minimizing sequence for $d(\omega)$. Then using the part 1 we have that there exist a subsequence of $(u_m,v_m)$, which denote the same, a sequence $(y_m)\in\mathbb{R}$ and $(u^\omega,v^\omega)\in\mathcal{M}_\omega$ such that

$$(u_m(\cdot+y_m),v_m(\cdot+y_m))\to(u^\omega,v^\omega)\quad\text{in}\quad X.$$

In particular $K_\omega(u^\omega,v^\omega)=0$ and $d_1=d(\omega)=\frac{1}{3}I_\omega(u^\omega,v^\omega)$. □

**Lemma 5.3.** *Let $0<|\omega|<\min\left(1,\frac{b}{a},\frac{d}{a}\right)$ and $\sigma>\frac{3}{8}$. Then*

1. *If $0<\omega_1<\omega_2<1$ and $(u,v)\in\mathcal{G}_\omega$, then we have that $d(\omega)$ and $I_{2,\omega}(u,v)$ are uniformly bounded functions on $[\omega_1,\omega_2]$.*
2. *If $\omega_1<\omega_2$ and $(u^{\omega_i},v^{\omega_i})\in\mathcal{G}_{\omega_i}$, we have the following inequalities*

$$d(\omega_1)\le d(\omega_2)-\left(\frac{\omega_2-\omega_1}{\omega_2}\right)I_{2,\omega_2}(u^{\omega_2},v^{\omega_2})+o(\omega_2-\omega_1),$$
$$d(\omega_2)\le d(\omega_1)+\left(\frac{\omega_2-\omega_1}{\omega_1}\right)I_{2,\omega_1}(u^{\omega_1},v^{\omega_1})+o(\omega_2-\omega_1).$$

*3. If $0<\omega_1<\omega_2<1$, $(u^{\omega_1},v^{\omega_1})\in\mathcal{G}_{\omega_1}$ and $I_{2,\omega_1}(u^{\omega_1},v^{\omega_1})\le 0$, then*

$$d(\omega_2)\le d(\omega_1)+\frac{(\omega_2-\omega_1)}{3\omega_1}I_{2,\omega_1}(u^{\omega_1},v^{\omega_1}).$$

*In particular, $d$ is a strictly decreasing function on $(\omega_1,1)$.*

**Proof.** 1. Let $\omega_1,\omega_2$ be such that $0<\omega_1<\omega_2<1$ and let $(u,v)\in X$ be such that $G(u,v)\neq 0$. Define $t_\omega$ by

$$t_\omega=-\frac{2}{3}\frac{I_\omega(u,v)}{G(u,v)}.$$

Then we have that $K_\omega(t_\omega(u,v))=0$, $J_\omega(t_\omega(u,v))=\frac{t_\omega^2}{3}I_\omega(u,v)$ and there exist $C>0$ such that for all $\omega\in[\omega_1,\omega_2]$,

$$d(\omega)\le J_\omega(t_\omega(u,v))=\frac{4}{27}\frac{I_\omega^3(u,v)}{G^2(u,v)}\le C\frac{\|(u,v)\|_X^6}{G^2(u,v)}.$$

Now, let $(w,z)\in\mathcal{G}_\omega$, then we see that $2I_\omega(w,z)+3G(w,z)=0$. Moreover,

$$C_1(\omega_1,\omega_2)\|(w,z)\|_X^2\le 2I_\omega(w,z)=3|G(w,z)|\le C\|(w,z)\|_X^3.$$

Then we conclude that

$$C_1(\sigma,\omega_1,\omega_2)\le\|(w,z)\|_X\le C_2(\omega_1,\omega_2)\left(\frac{1}{3}I_\omega(w,z)\right)^{\frac{1}{2}}.$$

Thus, we have shown that

$$d(\omega)\ge\left(\frac{C_1(\omega_1,\omega_2)}{C_2(\omega_1,\omega_2)}\right)^2.$$

Hence, if $(u,v)\in\mathcal{G}_\omega$ we obtain that $I_\omega(u,v)$ and $G(u,v)$ are uniformly bounded on $[\omega_1,\omega_2]$ since

$$d(\omega)=\frac{1}{3}I_\omega(u,v)=-\frac{1}{2}G(u,v),$$

which implies that $I_{2,\omega}(u,v)$ is also uniformly bounded because $K_\omega(u,v)=0$ and

$$I_1(u,v)\cong\|(u,v)\|_X^2.$$

2. Let $(w,z)$ be defined by $(w,z)=t(u^{\omega_2},v^{\omega_2})$. We want $t$ such that $K_{\omega_1}(w,z)=0$. Note that

$$\begin{aligned}K_{\omega_1}(w,z)&=2t^2I_{\omega_1}(u^{\omega_2},v^{\omega_2})+3t^3G(u^{\omega_2},v^{\omega_2})\\&=t^2\left(2I_{\omega_2}(u^{\omega_2},v^{\omega_2})-\frac{2(\omega_2-\omega_1)}{\omega_2}I_{2,\omega_2}(u^{\omega_2},v^{\omega_2})\right)+3t^3G(u^{\omega_2},v^{\omega_2})\\&=t^2\left(3tG(u^{\omega_2},v^{\omega_2})-3G(u^{\omega_2},v^{\omega_2})-\frac{2(\omega_2-\omega_1)}{\omega_2}I_{2,\omega_2}(u^{\omega_2},v^{\omega_2}\right).\end{aligned}$$

Thus, $t$ has to be such that

$$tG(u^{\omega_2},v^{\omega_2})=G(u^{\omega_2},v^{\omega_2})+\frac{2(\omega_2-\omega_1)}{3\omega_2}I_{2,\omega_2}(u^{\omega_2},v^{\omega_2})$$

or equivalently

$$t = 1 + \frac{2(\omega_2-\omega_1)}{3\omega_2}\left(\frac{I_{2,\omega_2}(u^{\omega_2},v^{\omega_2})}{G(u^{\omega_2},v^{\omega_2})}\right) = 1 - \frac{(\omega_2-\omega_1)}{3\omega_2}\left(\frac{I_{2,\omega_2}(u^{\omega_2},v^{\omega_2})}{d(\omega_2)}\right).$$

Then for this $t$, we conclude that $K_{\omega_1}(w,z)=0$. Now,

$$\begin{aligned} d(\omega_1) \le J_{\omega_1}(w,z) &= t^2\left(I_{\omega_1}(u^{\omega_2},v^{\omega_2}) + tG(u^{\omega_2},v^{\omega_2})\right) \\ &= t^2\left(I_{\omega_2}(u^{\omega_2},v^{\omega_2}) + \frac{\omega_1-\omega_2}{\omega_2}I_{2,\omega_2}(u^{\omega_2},v^{\omega_2}) + tG(u^{\omega_2},v^{\omega_2})\right) \\ &= t^2\left(d(\omega_2) - \frac{\omega_2-\omega_1}{3\omega_2}I_{2,\omega_2}(u^{\omega_2},v^{\omega_2})\right). \end{aligned}$$

But we have that

$$\begin{aligned} t^2 &= \left(1 - \frac{(\omega_2-\omega_1)}{3\omega_2}\left(\frac{I_{2,\omega_2}(u^{\omega_2},v^{\omega_2})}{d(\omega_2)}\right)\right)^2 \\ &= 1 - \frac{2(\omega_2-\omega_1)}{3\omega_2}\left(\frac{I_{2,\omega_2}(u^{\omega_2},v^{\omega_2})}{d(\omega_2)}\right) + O\left((\omega_2-\omega_1)^2\right). \end{aligned}$$

Then we see that

$$\begin{aligned} &t^2\left(d(\omega_2) - \frac{(\omega_2-\omega_1)}{3\omega_2}I_{2,\omega_2}(u^{\omega_2},v^{\omega_2})\right) \\ =&d(\omega_2) - \frac{(\omega_2-\omega_1)}{\omega_2}I_{2,\omega_2}(u^{\omega_2},v^{\omega_2}) + O\left((\omega_2-\omega_1)^2\right), \end{aligned}$$

which implies the desired result,

$$d(\omega_1) \le d(\omega_2) - \left(\frac{\omega_2-\omega_1}{\omega_2}\right)I_{2,\omega_2}(u^{\omega_2},v^{\omega_2}) + o(\omega_2-\omega_1).$$

Now, let $(w,z)$ be defined by $(w,z)=t(u^{\omega_1},v^{\omega_1})$. As before, we want $t$ such that $K_{\omega_2}(w,z)=0$. In this case,

$$t = 1 - \frac{2(\omega_2-\omega_1)}{3\omega_1}\left(\frac{I_{2,\omega_1}(u^{\omega_1},v^{\omega_1})}{G(u^{\omega_1},v^{\omega_1})}\right) = 1 + \frac{(\omega_2-\omega_1)}{3\omega_1}\left(\frac{I_{2,\omega_1}(u^{\omega_1},v^{\omega_1})}{d(\omega_1)}\right).$$

Since $K_{\omega_1}(w,z)=0$, we see that

$$d(\omega_2) \le J_{\omega_2}(w,z) = t^2\left(d(\omega_1) + \frac{\omega_2-\omega_1}{3\omega_1}I_{2,\omega_1}(u^{\omega_1},v^{\omega_1})\right).$$

Then, as above, we have that

$$t^2 = 1 + \frac{2(\omega_2-\omega_1)}{3\omega_1}\left(\frac{I_{2,\omega_1}(u^{\omega_1},v^{\omega_1})}{d(\omega_1)}\right) + O\left((\omega_2-\omega_1)^2\right).$$

Using this we conclude that

$$t^2\left(d(\omega_1) + \frac{(\omega_2-\omega_1)}{3\omega_1}I_{2,\omega_1}(u^{\omega_1},v^{\omega_1})\right)$$

$$=d(\omega_1) + \frac{(\omega_2 - \omega_1)}{\omega_1} I_{2,\omega_1}(u^{\omega_1}, v^{\omega_1}) + O\left((\omega_2 - \omega_1)^2\right),$$

which implies the other inequality.

3. Assume that $K_{\omega_1}(u^{\omega_1}, v^{\omega_1}) = 0$. Hence we see that $G(u^{\omega_1}, v^{\omega_1}) \leq 0$. Now, if $I_{2,\omega_1}(u^{\omega_1}, v^{\omega_1}) \leq 0$ then for $\omega_1 < \omega_2$ we have that

$$K_{\omega_2}(u^{\omega_1}, v^{\omega_1}) = K_{\omega_1}(u^{\omega_1}, v^{\omega_1}) + \frac{2(\omega_2 - \omega_1)}{\omega_1} I_{2,\omega_1}(u^{\omega_1}, v^{\omega_1}) \leq 0.$$

Thus, we obtain that

$$\begin{aligned} d(\omega_2) \leq \frac{1}{3} I_{\omega_2}(u^{\omega_1}, v^{\omega_1}) &= \frac{1}{3}\left(I_{\omega_1}(u^{\omega_1}, v^{\omega_1}) + \frac{\omega_2 - \omega_1}{\omega_1} I_{2,\omega_1}(u^{\omega_1}, v^{\omega_1})\right) \\ &\leq d(\omega_1) + \frac{\omega_2 - \omega_1}{3\omega_1} I_{2,\omega_1}(u^{\omega_1}, v^{\omega_1}). \end{aligned}$$

This also implies that $d(\omega_2) < d(\omega_1)$, provided that $0 < \omega_1 < \omega_2 < 1$. □

Now, we will prove that the function $d$ is strictly convex on $(\omega_0, 1)$ with $\omega_0 > 0$ near 1. To do this, we compute $d'$ and analyze the behavior of $d$ and $d'$ near $1^-$. We have the following results.

**Lemma 5.4.** *If $(u^\omega, v^\omega) \in \mathcal{G}_\omega$, then we have that*

$$d'(\omega) = \frac{I_{2,\omega}(u^\omega, v^\omega)}{\omega} = \mathcal{Q}(u^\omega, v^\omega). \tag{5.2}$$

**Proof.** Note that $d'$ can be computed by taking appropriate limits in part 2 of Lemma 5.3. □

**Theorem 5.1.** *Let $0 < |\omega| < \min\left(1, \frac{b}{a}, \frac{d}{a}\right)$, $\sigma > \frac{1}{3}$, and $(u^\omega, v^\omega) \in \mathcal{G}_\omega$. Then we have that*

$$\lim_{\omega \to 1^-} d(\omega) = 0 \quad \textit{and} \quad I_{2,\omega}(u^\omega, v^\omega) < 0 \quad \textit{for } \omega \textit{ near } 1^-.$$

**Proof.** From Lemma 4.2 and (5.1) we obtain the first part. Now, using the same notation as Section 5 we have that

$$\epsilon I^{2,\epsilon}(z^\epsilon, w^\epsilon) = -2\omega \int_{\mathbb{R}} \left(z^\epsilon (w^\epsilon)' + a\epsilon (z^\epsilon)' (w^\epsilon)''\right) dy.$$

Then using Lemma 4.3 we see that

$$\lim_{\epsilon \to 0^+} \epsilon I^{2,\epsilon}(z^\epsilon, w^\epsilon) < 0,$$

meaning for $\epsilon$ near $0^+$ that $I^{2,\epsilon}(z^\epsilon, w^\epsilon) < 0$, which implies for $\omega$ near $1^-$,

$$I_{2,\omega}(u^\omega, v^\omega) < 0.$$

□

**Theorem 5.2.** *Let $0 < |\omega| < \min\left(1, \frac{b}{a}, \frac{d}{a}\right)$ and $\sigma > \frac{1}{3}$. Then there exist $0 < \omega_0 < 1$ enough near 1 such that $d$ is a decreasing function on $(\omega_0, 1)$. Furthermore, $\lim_{\omega \to 1^-} d'(\omega) = 0$.*

**Proof.** From Theorem 5.1 we have that $d$ is a decreasing function for $\omega$ near $1^-$ and we also have that $\lim_{\omega\to 1^-} \|(u^\omega, v^\omega)\|_X = 0$ for any $(u^\omega, v^\omega) \in X$ such that $d(\omega) = \frac{1}{3} I_\omega(u^\omega, v^\omega)$, since

$$\|(u^\omega, v^\omega)\|_X^2 \le C(\sigma) I_\omega(u^\omega, v^\omega) = C(\sigma) d(\omega).$$

Thus, from (5.2) and definition of $I_{2,\omega}$ we conclude that

$$|d'(\omega)| \le C \left(\|u^\omega\|_{L^2(\mathbb{R}^2)} \|(v^\omega)'\|_{L^2(\mathbb{R}^2)} + \|(u^\omega)'\|_{L^2(\mathbb{R}^2)} \|(v^\omega)''\|_{L^2(\mathbb{R}^2)}\right) \le C \|(u^\omega, v^\omega)\|_X^2.$$

Therefore

$$\lim_{\omega\to 1^-} d'(\omega) = 0.$$

□

From previous results we have the following Corollary.

**Corollary 5.1.** *Let $\sigma > \frac{1}{3}$. Then $d$ is strictly convex for $\omega$ near $1^-$.*

## 6. Orbital stability of the solitary waves

We first consider the modulated system associated with the system (4.1) on $X$. In other words, we assume that the solution $(\eta(t), \Phi(t))$ of the *abad*-Boussinesq system (1.1) has the form

$$\eta(t,x) = u(t, x - \omega t), \quad \Phi(t,x) = v(t, x - \omega t).$$

Then we see that $(u(t), v(t))$ satisfies the modulated system

$$\begin{cases} \left(I - a\partial_x^2\right) u_t - \omega \left(I - a\partial_x^2\right) u_x + \partial_x^2 v - b\partial_x^4 v + \partial_x \left(u \partial_x v\right) = 0, \\ \left(I - a\partial_x^2\right) v_t - \omega \left(I - a\partial_x^2\right) v_x + u - d\partial_x^2 u + \dfrac{1}{2} \left(\partial_x v\right)^2 = 0. \end{cases} \tag{6.1}$$

We note that the modulated Hamiltonian for this system has the form

$$\mathcal{H}_\omega(u,v) = \frac{1}{2} J_\omega(u,v) = \mathcal{H}(u,v) + \frac{1}{2} I_{2,\omega}(u,v).$$

We also observe that $\mathcal{H}_\omega$ is conserved in time on solutions since

$$\begin{aligned} \left(I - a\partial_x^2\right) u_t &= \partial_x \mathcal{H}_{v,\omega}(w,z) = \omega\Big(I - a\partial_x^2\Big) u_x + b\partial_x^2 v - \partial_x^2 v - \partial_x \left(u\partial_x v\right), \\ -\left(I - a\partial_x^2\right) v_t &= \partial_x \mathcal{H}_{u,\omega}(w,z) = -\omega\Big(I - a\partial_x^2\Big) v_x + u - d\partial_x^2 u + \frac{1}{2} (\partial_x v)^2, \end{aligned}$$

where

$$\partial_x \mathcal{H}_\omega(u,v) = \left(\partial_x \mathcal{H}_{u,\omega}(u,v), \partial_x \mathcal{H}_{v,\omega}(u,v)\right).$$

Now we introduce the regions $\mathcal{R}_\omega^i, i = 1, 2$, in the energy space $X$ by

$$\begin{aligned} \mathcal{R}_\omega^1 &= \left\{ (w,z) \in X \ : \ \mathcal{H}_\omega(w,z) < \frac{1}{2} d(\omega), \ \frac{1}{3} I_\omega(w,z) < d(\omega) \right\}, \\ \mathcal{R}_\omega^2 &= \left\{ (w,z) \in X \ : \ \mathcal{H}_\omega(w,z) < \frac{1}{2} d(\omega), \ \frac{1}{3} I_\omega(w,z) > d(\omega) \right\}, \end{aligned}$$

and have the following result.

**Lemma 6.1.** *$\mathcal{R}_\omega^1, \mathcal{R}_\omega^2$ are invariant regions under the flow for the modulated system (6.1).*

**Proof.** Let $(u_0, v_0) \in \mathcal{R}_\omega^1$. Suppose that $(w(t), z(t))$ satisfies the modulated system (6.1) with initial condition

$$w(0) = u_0, \quad z(0) = v_0.$$

By characterization of $d(\omega)$ and definition of $\mathcal{R}_\omega^1$, we must have that

$$K_\omega(u_0, v_0) > 0.$$

In fact, suppose that $K_\omega(u_0, v_0) \leq 0$. Then we see that $d(\omega) \leq \frac{1}{3} I_\omega(u_0, v_0)$. Moreover, if $(w(t), z(t)) \in \mathcal{R}_\omega^1$ for some $t > 0$, we have that $K_\omega(w(t), z(t)) > 0$. Now, suppose that there exists a minimum $t_0$ such that $K_\omega(w(t), z(t)) > 0$ for $t \in [0, t_0)$ and $K_\omega(w(t_0), z(t_0)) = 0$. We note by the characterization of $d(\omega)$ that

$$\begin{aligned}
d(\omega) &\leq \frac{1}{3} I_\omega(w(t_0), z(t_0)) \\
&\leq \liminf_{t \to t_0^-} \left( \frac{1}{3} I_\omega(w(t), z(t)) + \frac{1}{3} K_\omega(w(t), z(t)) \right) \\
&\leq \liminf_{t \to t_0^-} J_\omega(w(t), z(t)) \leq 2 \liminf_{t \to t_0^-} \mathcal{H}_\omega(w(t), z(t)) \\
&\leq 2\mathcal{H}_\omega(u_0, v_0) < d(\omega).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d(\omega) &> 2\mathcal{H}_\omega(w(t), z(t)) = 2\mathcal{H}_\omega(u_0, v_0) = J_\omega(u_0, v_0) \\
&= \frac{1}{3} I_\omega(u_0, v_0) + \frac{1}{3} K_\omega(u_0, v_0) > \frac{1}{3} I_\omega(u_0, v_0),
\end{aligned}$$

which shows that $\mathcal{R}_\omega^1$ is invariant under the flow for the modulated system (6.1). In a similar fashion we have that $\mathcal{R}_\omega^2$ is also invariant under the flow for the modulated system (6.1). □

The following lemma will be used to obtain stability with respect to the ground state solutions. We will use the notation $U^\omega = (u^\omega, v^\omega)$ for "ground state solution", that is, $d(\omega) = J_\omega(U^\omega)$.

**Lemma 6.2.** *Let $\sigma > \frac{1}{3}$ and $0 < \omega_0 < 1$ be near 1. If $U(t) = (\eta(t), \Phi(t))$ is a global solution of abad-Boussinesq system (1.1) with initial condition $U(0) = U_0 \in X$, then for every $M$, there is $\delta(M)$ such that if*

$$\|U_0 - U^{\omega_0}\|_X < \delta(M).$$

*Then we have*

$$d\left(\omega_0 + \frac{1}{M}\right) \leq \frac{1}{3} I_{\omega_0}(U(t)) \leq d\left(\omega_0 - \frac{1}{M}\right), \quad \textit{for all } t \in \mathbb{R}.$$

**Proof.** Let $M > 0$ be fixed and define $\omega_1 = \omega_0 - \frac{1}{M}$ and $\omega_2 = \omega_0 + \frac{1}{M}$. Now, let $(z^i(t), w^i(t))$ be defined by the formulas

$$\eta(t, x) = u^i(t, x - \omega_i t), \quad \Phi(t, x) = v^i(t, x - \omega_i t), \quad i = 1, 2.$$

Then the couple $(u^i(t), v^i(t))$ satisfies the modulated system (6.1) with initial condition

$$(u^i(0), v^i(0)) = U(0).$$

For this solution we have that the modulated Hamiltonian is conserved in time, in other words

$$\mathcal{H}_{\omega_i}(U(t)) = \mathcal{H}_{\omega_i}(U(0)).$$

Now, using the hypothesis we conclude for small $\delta$ that

$$I_{\omega_i}(U^{\omega_0}) = I_{\omega_i}(U(0)) + O(\delta).$$

Since $d$ is a strictly decreasing function such that $d(\omega_0) = \frac{1}{3} I_{\omega_0}(U^{\omega_0})$, we can choose $\delta$ small enough in such a way that

$$d(\omega_2) < \frac{1}{3} I_{\omega_0}(U(0)) < d(\omega_1).$$

We also have that

$$\begin{aligned} J_{\omega_i}(U(0)) &= J_{\omega_i}(U^{\omega_0}) + O(\delta) \\ &= J_{\omega_0}(U^{\omega_0}) + \frac{\omega_i - \omega_0}{\omega_0} I_{2,\omega_0}(U^{\omega_0}) + O(\delta) \\ &= d(\omega_0) + (\omega_i - \omega_0) d'(\omega_0) + O(\delta). \end{aligned}$$

Using that $d$ is twice differentiable, we have for some $\tilde{\omega}$ between $\omega_i$ and $\omega_0$ that

$$d(\omega_i) = d(\omega_0) + (\omega_i - \omega_0) d'(\omega_0) + \frac{1}{2}(\omega_i - \omega_0)^2 d''(\omega_0),$$

where we are using Taylor expansion on $\omega_0$. So, replacing this in previous inequality, we conclude that

$$J_{\omega_i}(U(0)) = d(\omega_i) - \frac{1}{2}(\omega_i - \omega_0)^2 d''(\omega_0) + O(\delta).$$

So, choosing $\delta$ small enough such that

$$-\frac{1}{2}(\omega_i - \omega_0)^2 d''(\omega_0) + O(\delta) < 0,$$

we conclude that

$$2\mathcal{H}_{\omega_i}(U(0)) = J_{\omega_i}(U(0)) < d(\omega_i). \tag{6.2}$$

Then, using Lemma 6.1, we have for all $t \in \mathbb{R}$ that

$$\mathcal{H}_{\omega_i}(U(t)) < \frac{1}{2} d(\omega_i), \quad d\Big(\omega_0 + \frac{1}{M}\Big) \le \frac{1}{3} I_{\omega_0}(U(t)) \le d\Big(\omega_0 - \frac{1}{M}\Big).$$

$\square$

Finally we establish the main result in this work.

**Theorem 6.1** (Orbital stability)**.** *Let $\sigma > \frac{1}{3}$; $\frac{b}{a}, \frac{d}{a} > 1$, and $0 < \omega_0 < 1$ be near 1. Then the solitary wave solutions $U^{\omega_0}$ (ground state solitary wave solutions) of the abad-Boussinesq system (1.1) are stable in the following sense: Given $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that if $U_0 \in X$ satisfies*

$$\|U_0 - U^{\omega_0}\|_X < \delta(\varepsilon),$$

*then there exist a unique solution $U(t)$ of the Cauchy problem associated to Boussinesq system (1.1) with initial condition $U_0$ such that*

$$\inf_{V\in\mathcal{G}_{\omega_0}} \|U(t)-V\|_X < \varepsilon, \quad \textit{for all} \;\; t\in\mathbb{R}.$$

**Proof.** We will argue by contradiction. Suppose that there exist a positive number $\varepsilon_0$, and sequences $(t_k)\subset\mathbb{R}$ and $(U_0^k)\subset X$, such that

$$\lim_{k\to\infty}\|U_0^k-U^{\omega_0}\|_X=0, \qquad \inf_{V\in\mathcal{G}_{\omega_0}}\|U^k(t_k)-V\|_X>\varepsilon_0,$$

where $U^k$ denotes the unique solution of system (1.1) with initial condition $U^k(0)=U_0^k$. Now, from the Lemma 6.2 and the assumption, given $m>0$ we have the existence of $\delta(m)$ and a subsequence $k_m$ such that

$$\|U_0^{k_m}-U^{\omega_0}\|_X<\delta(m)$$

and

$$d\Big(\omega_0+\frac{1}{k_m}\Big)\le\frac{1}{3}I_{\omega_0}\left(U^{k_m}(t_{k_m})\right)\le d\Big(\omega_0-\frac{1}{k_m}\Big),$$

meaning that there exist a subsequence of $(U^k(t_k))$, which we denote the same, such that

$$d\Big(\omega_0+\frac{1}{k}\Big)\le\frac{1}{3}I_{\omega_0}\left(U^{k}(t_{k})\right)\le d\Big(\omega_0-\frac{1}{k}\Big).$$

In particular, we have that $\frac{1}{3}I_{\omega_0}\left(U^k(t_k)\right)\longrightarrow d(\omega_0)$ as $k\to\infty$. Now, we consider $\omega_2=\omega_0+\frac{1}{k}$ and $V^{k,2}(t)$ defined as $U^k(t,x)=V^{k,2}(t,x-\omega_2 t)$. Then as in proof of previous lemma (see (6.2)), we obtain that

$$2\mathcal{H}_{\omega_2}\left(U^k(t_k)\right)=J_{\omega_2}\left(U^k(t_k)\right)<d(\omega_2)<d(\omega_0)<d\Big(\omega_0-\frac{1}{k}\Big).$$

On the other hand,

$$\begin{aligned}J_{\omega_2}\left(U^k(t_k)\right)&=J_{\omega_0}\left(U^k(t_k)\right)+\left(\frac{\omega_2-\omega_0}{\omega_0}\right)I_{2,\omega_0}\left(U^k(t_k)\right)\\&=J_{\omega_0}\left(U^k(t_k)\right)+\left(\frac{1}{k\omega_0}\right)I_{2,\omega_0}\left(U^k(t_k)\right).\end{aligned}$$

But note that

$$\lim_{k\to\infty}\left(\frac{1}{k\omega_0}\right)\left|I_{2,\omega_0}\left(U^k(t_k)\right)\right|\le\lim_{k\to\infty}\frac{C}{k}\|U^k(t_k)\|_X^2=0.$$

Using this fact, we conclude that

$$J_{\omega_0}\left(U^k(t_k)\right)\longrightarrow d_1\le d(\omega_0).$$

Then by Lemma 5.2, there exist $U_{\omega_0}\in\mathcal{G}_{\omega_0}$ such that

$$U^k(t_k)\longrightarrow U_{\omega_0}\quad\text{in}\quad X,\qquad \frac{1}{3}I_{\omega_0}\left(U^k(t_k)\right)\longrightarrow d(\omega_0)=d_1,\qquad k\to\infty,$$

and also that $J_{\omega_0}\left(U^k(t_k)\right)\longrightarrow d(\omega_0)$. But this contradicts the assumption of instability

$$\inf_{V\in\mathcal{G}_{\omega_0}}\|U^k(t_k)-V\|_X>\varepsilon_0.$$

□

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