Journal of Applied Analysis and Computation
Volume 13, Number 3, June 2023, 1505–1534
Website:http://www.jaac-online.com
DOI:10.11948/20220334

# QUASI-PERIODIC SOLUTIONS FOR 1D NONLINEAR WAVE EQUATION*

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**Abstract** In this paper, one-dimensional (1D) nonlinear wave equation

$$u_{tt} - u_{xx} + mu + u^7 = 0$$

on the finite $x$-interval $[0, \pi]$ with Dirichlet boundary conditions is considered. It is proved that there are many 3-dimensional elliptic invariant tori, and thus quasi-periodic solutions for the above equation. This is an extension of the previous work [11] by the same author, where many 2-dimensional elliptic invariant tori for the above equation are obtained. The proof is based on infinite-dimensional KAM theory and partial Birkhoff normal form.

**Keywords** Invariant tori, quasi-periodic solutions, KAM theory, nonlinear wave equation, Birkhoff normal form.

**MSC(2010)** 37K55, 35B10, 35B20.

## 1. Introduction and main results

In this paper, we are going to investigate the nonlinear wave equation

$$u_{tt} - u_{xx} + mu + u^7 = 0, \quad (t, x) \in \mathbb{R} \times [0, \pi] \tag{1.1}$$

under Dirichlt boundary conditions

$$u(t, 0) = 0 = u(t, \pi),$$

where the parameter $m$ is real and positive, sometimes referred to as the "mass". We prove that the above equation admits many small amplitude quasi-periodic solutions corresponding to 3-dimensional invariant tori of an associated infinite dimensional dynamical systems. This result extends the existence of 2-dimensional invariant tori for the same equation in [11]. Also see the existence of $b \geq 2$ and 2-dimensional invariant tori for the quintic nonlinear wave equation in [12] and [13] respectively.

We study (1.1) as an infinite-dimensional Hamiltonian system on the phase space $\mathcal{P} = H_0^1([0, \pi]) \times L^2([0, \pi])$ with coordinates $u$ and $v = u_t$, where $H_0^1([0, \pi])$ and $L^2([0, \pi])$ are the usual Sobolev spaces. The Hamiltonian for (1.1) is then

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \frac{1}{8}\int_0^{\pi} u^8 dx, \tag{1.2}$$

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*The author was supported by National Natural Science Foundation of China (11971299).

where $A = -d^2/dx^2 + m$ and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $L^2$. The Hamiltonian equations of motions are

$$u_t = \frac{\partial H}{\partial v} = v, \quad v_t = -\frac{\partial H}{\partial u} = -Au - u^7. \tag{1.3}$$

The quasi-periodic solutions of (1.1) to be constructed are of small amplitude. Thus, in first approximation the high order term $u^7$ may be considered as a small perturbation of the linear equation $u_{tt} - u_{xx} + mu = 0$. The latter is of course well understood and has a plenty of quasi-periodic solutions. To be more precise, let

$$\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin jx, \quad \lambda_j = \sqrt{j^2 + m}, \quad j = 1, 2, \cdots$$

be the basic modes and frequencies of the linear system $u_{tt} - u_{xx} + mu = 0$ with Dirichlet boundary conditions. Then every solution is the superposition of their harmonic oscillations and of the form

$$u(t, x) = \sum_{j \geq 1} q_j(t)\phi_j(x), \quad q_j(t) = I_j \cos(\lambda_j t + \varphi_j^0).$$

Their combined motions are periodic, quasi-periodic or almost-periodic, respectively, depending on whether one, finitely many or infinitely many modes are excited. In particular, for every choice

$$J = \{n_1 < n_2 < n_3\} \subset \mathbb{N}^+$$

of 3-modes there is an invariant 6-dimensional linear subspace $E_J$ that is foliated into rotational tori with frequencies $\lambda_{n_1}, \lambda_{n_2}, \lambda_{n_3}$ :

$$E_J = \{(u, v) = (q_1\phi_{n_1} + q_2\phi_{n_2} + q_3\phi_{n_3}, p_1\phi_{n_1} + p_2\phi_{n_2} + p_3\phi_{n_3})\} = \bigcup_{I \in \overline{\mathbb{P}^3}} \mathcal{T}_J(I),$$

where $\mathbb{P}^3 = \{I \in \mathbb{R}^3 : I_l > 0, l = 1, 2, 3\}$ is the positive quadrant in $\mathbb{R}^3$ and

$$\mathcal{T}_J(I) = \{(u, v) : q_l^2 + \lambda_{n_l}^{-2} p_l^2 = I_l, \ l = 1, 2, 3\},$$

using the above representation of $u$ and $v$. Upon restoring the nonlinearity $u^7$, $E_J$ with their quasi-periodic solutions will not persist in their entirety due to the modes and the strong perturbing effect of $u^7$ for large amplitudes. However, there does persist a Cantor subfamily of rotational 3-torus which are only slightly deformed. More exactly, we have the following theorem:

**Theorem 1.1.** *Considering 1D nonlinear wave equation* (1.1)*, assume* $0 < m \leq \frac{1}{9}$ *and the index set* $J = \{n_1 < n_2 < n_3\}$ *satisfies*

$$n_{l+1} \geq \frac{n_l^3}{m}, \quad l = 1, 2. \tag{1.4}$$

*Then there is a set* $\mathcal{C}^*$ *in* $\mathbb{P}^3$ *with positive Lebesgue measure, a family of* 3*-tori*

$$\mathcal{T}_J[\mathcal{C}^*] = \bigcup_{I \in \mathcal{C}^*} \mathcal{T}_J(I) \subset E_J$$

*over $\mathcal{C}^*$, a Lipschitz continuous embedding*

$$\Phi : \mathcal{T}_J[\mathcal{C}^*] \hookrightarrow \mathcal{P},$$

*which is a higher order perturbation of the inclusion map $\Phi_0 : E_J \hookrightarrow \mathcal{P}$ restricted to $\mathcal{T}_J[\mathcal{C}^*]$, such that the restriction of $\Phi$ to each $\mathcal{T}_J(I)$ in the family is an embedding of a rotational invariant 3-torus for the nonlinear wave equation* (1.1).

We prove the above theorem by KAM theory. Historically, KAM theory for partial differential equations was originated by Kuksin [14–16] and Wayne [20]. In order to use KAM theory, some parameters are needed. For nonlinear wave equation, one way of introducing parameters is to consider parameterized potentials, see [1, 8, 9, 16, 20] for examples. However, for a prescribed potential, owing to the absence of exterior parameters, one needs to find out some suitable Birkhoff normal form, and then extract parameters by amplitude-frequency modulation. In this aspect, [17] is the pioneer work, where nonlinear Schrödinger equation with constant potential is studied by Kuksin and Pöschel, and Birkhoff normal form of order four is used to extract parameters. For wave equation with cubic nonlinearity $u^3$, see Pöschel [18] for $m > 0$, Yuan [22] for $-1 < m < 0$, Yuan [23] for the completely resonant case $m = 0$ and Yuan [24] for a prescribed non-constant potential. For wave equation with quintic nonlinearity, 2-dimensional and $b$-dimensional invariant tori are obtained in [12] and [13] respectively. For wave equation with higher order nonlinearity, only 2-dimensional invariant tori is obtained in [11]. For derivative nonlinear wave equations, see [3, 4] by Berti, Biasco and Procesi. See for example [2, 5–7, 10] for the recent results for wave equations. In the following we lay out an outline of the present paper, and meanwhile point out the main difficulties compared with [11–13].

In Section 2, the Hamiltonian is written in infinitely many coordinates, and then put into a partial Birkhoff normal form. In order to obtain the partial Birkhoff normal form, we have to estimate the lower bound of the divisor

$$|\lambda_{j_1} + \cdots + \lambda_{j_8}|$$

for $(j_1, \cdots, j_8)$ with at most 2 components not in $J$. Here the index set $J$ contains 3 indices instead of 2 indices in [11], while in [12] and [13], we need to estimate the lower bound of the divisor

$$|\lambda_{j_1} + \cdots + \lambda_{j_6}|$$

for $(j_1, \cdots, j_6)$ with at most 2 components not in some index set. This is the key difficulty. We solve it by choosing suitable index set $J$ (see (1.4)) and exploring the properties of the function $f(t) = \sqrt{t^2 + m} - t$ (see Lemma 4.1-4.3 in Section 4).

In Section 3, we prove Theorem 1.1 by applying the KAM theorem in Appendix. To this end, we need to check the assumptions of the KAM theorem. One difficulty is to check the nondegeneracy condition $|\omega^{-1}|^{\mathcal{L}}_{\omega(\Pi)} \le L$ in (5.1). We observe that the Lipschitz semi-norm of $\omega^{-1}$ can be controlled by $3||(\frac{\partial \omega}{\partial \xi})^{-1}||_{\ell^1 \to \ell^1}$, where $|\cdot|_1$ is the $\ell^1$-norm for a vector and $||\cdot||_{\ell^1\to\ell^1}$ is the operator norm from $\ell^1$ to $\ell^1$ for a matrix. However, due to the fact that the tangential frequency $\omega(\xi)$ is a vector in $\mathbb{R}^3$ instead of in $\mathbb{R}^2$ in [11], we can not formulate $(\frac{\partial \omega}{\partial \xi})^{-1}$. Another difficulty is to estimate the thrown measure. In order to check the assumption (5.6) in the KAM theorem, we have to estimate the measure of $\zeta \in \Theta$ for $\langle k, \omega(\zeta)\rangle + \Omega_i(\zeta) - \Omega_j(\zeta)$ small. The main idea is to prove either $\langle k,\omega(\zeta)\rangle+\Omega_i(\zeta)-\Omega_j(\zeta)$ or $\frac{\partial^3}{\partial\zeta^3}\Big(\langle k,\omega(\zeta)\rangle+\Omega_i(\zeta)-\Omega_j(\zeta)\Big)$ is larger than some positive constant.

In Section 4, some technical lemmas are given. In Appendix, we copy a KAM theorem from [11], [12] and [13], which is a combination of Theorem A and Theorem D in [19] with some modifications.

## 2. The Hamiltonian and partial Birkhoff normal form

To rewrite (1.2) as a Hamiltonian in infinitely many coordinates we make the ansatz

$$u=\mathcal{S}q=\sum_{j\geq 1}\frac{q_j}{\sqrt{\lambda_j}}\phi_j,\quad v=\sum_{j\geq 1}\sqrt{\lambda_j}p_j\phi_j.$$

The coordinates are taken from some Hilbert space $\ell^{a,s}$ of all real valued sequences $w=(w_1,w_2,\cdots)$ with finite norm

$$||w||_{a,s}^2=\sum_{j\geq 1}|w_j|^2j^{2s}e^{2aj}.$$

Below we will assume that $a\geq 0$ and $s>\frac{1}{2}$. We then obtain the Hamiltonian

$$H=\Lambda+G=\frac{1}{2}\sum_{j\geq 1}\lambda_j(p_j^2+q_j^2)+\frac{1}{8}\int_0^\pi(\mathcal{S}q)^8dx \tag{2.1}$$

with equations of motions

$$\dot{q}_j=\frac{\partial H}{\partial p_j}=\lambda_jp_j,\quad \dot{p}_j=-\frac{\partial H}{\partial q_j}=-\lambda_jq_j-\frac{\partial G}{\partial q_j}. \tag{2.2}$$

These are the Hamiltonian equations of motions with respect to the standard symplectic structure $\sum_j dq_j\wedge dp_j$ on $\ell^{a,s}\times\ell^{a,s}$. The same as Lemma 3 in [18], we know the gradient $G_q$ is real analytic as a map from some neighborhood of the origin in $\ell^{a,s}$ into $\ell^{a,s+1}$ with

$$||G_q||_{a,s+1}=O(||q||_{a,s}^7). \tag{2.3}$$

Thus the associated Hamiltonian vector field

$$X_G=\sum_{j\geq 1}\left(\frac{\partial G}{\partial p_j}\frac{\partial}{\partial q_j}-\frac{\partial G}{\partial q_j}\frac{\partial}{\partial p_j}\right)$$

defines a real analytic map from some neighborhood of the origin in $\ell^{a,s}\times\ell^{a,s}$ into $\ell^{a,s+1}\times\ell^{a,s+1}$.

Note that

$$G(q)=\frac{1}{8}\int_0^\pi(\mathcal{S}q)^8dx=\frac{1}{8}\sum_{j_1,\cdots,j_8\geq 1}G_{j_1\cdots j_8}q_{j_1}\cdots q_{j_8} \tag{2.4}$$

with

$$G_{j_1\cdots j_8}=\frac{1}{\sqrt{\lambda_{j_1}\cdots\lambda_{j_8}}}\int_0^\pi\phi_{j_1}\cdots\phi_{j_8}dx. \tag{2.5}$$

It is not difficult to verify that $G_{j_1\cdots j_8} = 0$ unless $j_1 \pm \cdots \pm j_8 = 0$ for some combination of plus and minus signs. Thus, only a codimension one set of coefficients is actually different from zero, and the sum extends only over $j_1 \pm \cdots \pm j_8 = 0$. For simplicity, we denote $G_{ijkl} = G_{iijjkkll}$, $G_{n_l} = G_{\underbrace{n_l \cdots n_l}_{8}}$. In particular, we have

$$
\begin{aligned}
G_{ijkl} = G_{iijjkkll} =& \frac{16}{\pi^4 \lambda_i \lambda_j \lambda_k \lambda_l} \int_0^\pi \sin^2 ix \sin^2 jx \sin^2 kx \sin^2 lx dx \\
=& \frac{1}{8\pi^3 \lambda_i \lambda_j \lambda_k \lambda_l} (8 + 4\delta_{ij} + 4\delta_{ik} + 4\delta_{il} + 4\delta_{jk} + 4\delta_{jl} + 4\delta_{kl} \\
& - 2\delta_{i+j,k} - 2\delta_{i+k,j} - 2\delta_{j+k,i} - 2\delta_{i+j,l} - 2\delta_{i+l,j} - 2\delta_{l+j,i} \\
& - 2\delta_{k+j,l} - 2\delta_{k+l,j} - 2\delta_{j+l,i} - 2\delta_{i+k,l} - 2\delta_{i+l,k} - 2\delta_{k+l,i} \\
& + \delta_{i+j+k,l} + \delta_{i+j+l,k} + \delta_{j+k+l,i} + \delta_{i+k+l,j} \\
& + \delta_{i+j,k+l} + \delta_{i+k,j+l} + \delta_{i+l,j+k})
\end{aligned} \tag{2.6}
$$

by elementary calculation. In the rest of this section we transform the Hamiltonian (2.1) into some partial Birkhoff normal form of order 14 so that it happens, in a sufficiently small neighborhood of the origin, as a small perturbation of some nonlinear integrable system.

For the rest of this paper we introduce complex coordinates

$$
z_j = \frac{1}{\sqrt{2}}(q_j + \mathbf{i}p_j), \quad \bar{z}_j = \frac{1}{\sqrt{2}}(q_j - \mathbf{i}p_j), \quad j \geq 1.
$$

Inserting them into (2.1), we obtain a real analytic Hamiltonian

$$
\begin{aligned}
H &= \Lambda + G \\
&= \sum_{j\geq 1} \lambda_j |z_j|^2 + \frac{1}{8}\int_0^\pi (\sum_{j\geq 1} \frac{z_j + \bar{z}_j}{\sqrt{2\lambda_j}} \phi_j)^8 dx \\
&= \sum_{j\geq 1} \lambda_j |z_j|^2 + \frac{1}{128} \sum_{j_1,\cdots,j_8 \geq 1} G_{j_1\cdots j_8} (z_{j_1} + \bar{z}_{j_1}) \cdots (z_{j_8} + \bar{z}_{j_8})
\end{aligned} \tag{2.7}
$$

on the now complex Hilbert space $\ell^{a,s}$ with symplectic structure $\mathbf{i}\sum_{j\geq 1} dz_j \wedge d\bar{z}_j$. Real analytic means that $H$ is a function of $z$ and $\bar{z}$, real analytic in the real and imaginary part of $z$. Conveniently introducing $z_{-j} := \bar{z}_j$ for $j \geq 1$, then $H$ in (2.7) is written as

$$
H = \Lambda + G \tag{2.8}
$$

with

$$
\Lambda = \sum_{j\geq 1} \lambda_j z_j z_{-j}, \tag{2.9}
$$

$$
G = \frac{1}{128} \sum_{j_1,\cdots,j_8 \in \mathbb{Z}_*} G_{j_1\cdots j_8} z_{j_1} \cdots z_{j_8}, \tag{2.10}
$$

where $G_{j_1\cdots j_8} := G_{|j_1|\cdots|j_8|}$ for $j_1, \cdots, j_8 \in \mathbb{Z}_* := \mathbb{Z} \setminus \{0\}$.

Define the normal form set

$$
\mathcal{N} = \{(j_1, \cdots, j_8) \in \mathbb{Z}_*^8 : \text{There exists a } 8 - \text{permutation } \tau
$$

such that $j_{\tau(1)} = -j_{\tau(2)}, \cdots,\ j_{\tau(7)} = -j_{\tau(8)}\}$.

Define the following index sets

$\Delta_l = \{(j_1, \cdots, j_8) \in \mathbb{Z}_*^8$ : There are exactly $l$ components not in$\{\pm n_1, \pm n_2, \pm n_3\}\}$

for $l = 0, 1, 2$ and

$\Delta_3 = \{(j_1, \cdots, j_8) \in \mathbb{Z}_*^8$ : There are at least 3 components not in$\{\pm n_1, \pm n_2, \pm n_3\}\}$.

Split $G$ in (2.10) into three parts:

$$G = \bar{G} + \tilde{G} + \hat{G}, \tag{2.11}$$

where $\bar{G}$ is the normal form part of $G$ with $(j_1, \cdots, j_8) \in (\Delta_0 \cup \Delta_1 \cup \Delta_2) \cap \mathcal{N}$:

$$\begin{aligned}
\bar{G} &= \frac{1}{128} \sum_{(j_1,\cdots,j_8)\in(\Delta_0\cup\Delta_1\cup\Delta_2)\cap\mathcal{N}} G_{j_1\cdots j_8} z_{j_1} \cdots z_{j_8} \qquad (2.12)\\
&= \frac{35}{64}\sum_{l=1}^{3} G_{n_l}|z_{n_l}|^8 + \frac{35}{4}\sum_{\substack{l,l'\\ l\neq l'}} G_{n_l n_l n_l n_{l'}}|z_{n_l}|^6|z_{n_{l'}}|^2 + \frac{315}{16}\sum_{\substack{l,l'\\ l\neq l'}} G_{n_l n_l n_{l'} n_{l'}}|z_{n_l}|^4|z_{n_{l'}}|^4\\
&\quad + \frac{315}{4} \sum_{\substack{l,l',l''\\ l\neq l', l\neq l'', l'\neq l''}} G_{n_l n_l n_{l'} n_{l''}}|z_{n_l}|^4|z_{n_{l'}}|^2|z_{n_{l''}}|^2\\
&\quad + \frac{35}{4}\sum_{l=1}^{b} G_{n_l n_l n_l j}|z_{n_l}|^6|z_j|^2 + \frac{315}{4}\sum_{\substack{l,l'\\ l\neq l'}} G_{n_l n_l n_{l'} j}|z_{n_l}|^4|z_{n_{l'}}|^2|z_j|^2\\
&\quad + \frac{315}{2} \sum_{\substack{l,l',l''\\ l\neq l', l\neq l'', l'\neq l''}} G_{n_l n_{l'} n_{l''} j}|z_{n_l}|^2|z_{n_{l'}}|^2|z_{n_{l''}}|^2|z_j|^2,
\end{aligned}$$

$\tilde{G}$ is the non-normal form part of $G$ with $(j_1, \cdots, j_8) \in (\Delta_0 \cup \Delta_1 \cup \Delta_2) \setminus \mathcal{N}$:

$$\tilde{G} = \frac{1}{128} \sum_{(j_1,\cdots,j_8)\in(\Delta_0\cup\Delta_1\cup\Delta_2)\setminus\mathcal{N}} G_{j_1\cdots j_8} z_{j_1} \cdots z_{j_8}, \tag{2.13}$$

and $\hat{G}$ is the part of $G$ with $(j_1, \cdots, j_8) \in \Delta_3$:

$$\hat{G} = \frac{1}{128} \sum_{(j_1,\cdots,j_8)\in\Delta_3} G_{j_1\cdots j_8} z_{j_1} \cdots z_{j_8}. \tag{2.14}$$

We will eliminate $\tilde{G}$ by a symplectic coordinate transformation $X_F^1$, which is the time-1-map of the flow of a Hamiltonian vector filed $X_F$ given by a Hamiltonian

$$F = \sum_{(j_1,\cdots,j_8)\in\mathbb{Z}_*^8} F_{j_1\cdots j_8} z_{j_1} \cdots z_{j_8} \tag{2.15}$$

with coefficients

$$\mathbf{i}F_{j_1\cdots j_8} = \begin{cases} \dfrac{G_{j_1\cdots j_8}}{128(\lambda_{j_1} + \cdots + \lambda_{j_8})}, & \text{for } (j_1, \cdots, j_8) \in (\Delta_0 \cup \Delta_1 \cup \Delta_2) \setminus \mathcal{N},\\ 0, & \text{otherwise.} \end{cases} \tag{2.16}$$

Here $\lambda_j := \mathrm{sgn} j \cdot \lambda_{|j|}$ for $j \in \mathbb{Z}_*$. Then formally we have

$$\{\Lambda, F\} + \tilde{G} = 0, \tag{2.17}$$

where $\{\cdot,\cdot\}$ is Poisson bracket with respect to the symplectic structure

$$\mathbf{i} \sum_{j\geq 1} dz_j \wedge dz_{-j}.$$

Thus expanding at $t = 0$ and using Taylor's formula we formally get

$$\begin{aligned} H \circ X_F^1 &= H \circ X_F^t|_{t=1} \\ &= \Lambda + \{\Lambda, F\} + \int_0^1 (1-t)\{\{\Lambda, F\}, F\} \circ X_F^t dt + G + \int_0^1 \{G, F\} \circ X_F^t dt \\ &= \Lambda + \bar{G} + \hat{G} + \int_0^1 \{\bar{G} + t\tilde{G} + \hat{G}, F\} \circ X_F^t dt. \end{aligned} \tag{2.18}$$

Now we need to show the correctness of the definition (2.16) and establish the regularity of the vector field $X_F$. To this end, we prove that the divisors $\lambda_{j_1} + \cdots + \lambda_{j_8}$ are away from zero:

**Lemma 2.1.** *Suppose $0 < m \leq \frac{1}{9}$ and the indices $n_1, n_2, n_3 \in \mathbb{N}^+$ satisfy*

$$n_{l+1} \geq \frac{6n_l^3}{m}, \quad l = 1, 2. \tag{2.19}$$

*Then for $(j_1, \cdots, j_8) \in (\Delta_0 \cup \Delta_1 \cup \Delta_2) \setminus \mathcal{N}$, we have*

$$\left|\lambda_{j_1} + \cdots + \lambda_{j_8}\right| \geq \frac{m^2}{480 n_3^3}. \tag{2.20}$$

**Proof.** This lemma is equivalent to prove that, for $j_1, \cdots, j_8 \in \mathbb{N}^+$ and $\sigma_1, \cdots, \sigma_8 \in \{1, -1\}$, if $(\sigma_1 j_1, \cdots, \sigma_8 j_8) \in (\Delta_0 \cup \Delta_1 \cup \Delta_2) \setminus \mathcal{N}$, then we have

$$\Big|\sum_{l=1}^{8} \sigma_l \lambda_{j_l}\Big| \geq \frac{m^2}{480 n_3^3}. \tag{2.21}$$

We firstly consider the case $\sum_{l=1}^{8} \sigma_l j_l \neq 0$. In view of $0 < m \leq \frac{1}{9}$ and

$$\lambda_j = j + (\sqrt{j^2+m} - j) = j + \frac{m}{\sqrt{j^2+m} + j}, \quad j \in \mathbb{N}^+, \tag{2.22}$$

we have

$$\Big|\sum_{l=1}^{8} \sigma_l \lambda_{j_l}\Big| \geq \Big|\sum_{l=1}^{8} \sigma_l j_l\Big| - m \sum_{l=1}^{8} \frac{1}{\sqrt{j_l^2+m} + j_l} \geq 1 - 4m \geq \frac{1}{4}, \tag{2.23}$$

which is larger than $\frac{m^2}{480 n_3^3}$. Therefore, in the following, we assume

$$\sum_{l=1}^{8} \sigma_l j_l = 0. \tag{2.24}$$

Introduce the function

$$f(t)=\sqrt{t^2+m}-t=\frac{m}{\sqrt{t^2+m}+t}, \tag{2.25}$$

which is positive, monotone decreasing and convex for $t\geq 0$. Thus, by (2.24) and (2.25), we have

$$\sum_{l=1}^{8}\sigma_l\lambda_{j_l}=\sum_{l=1}^{8}\sigma_l(\lambda_{j_l}-j_l)=\sum_{l=1}^{8}\sigma_l f(j_l). \tag{2.26}$$

We secondly consider the case $\sigma_k j_k+\sigma_l j_l=0$ for some $1\leq k,l\leq 8$. Without loss of generality, assuming $k=7$, $l=8$, then we have $\sum_{l=1}^{6}\sigma_l j_l=0$. Then by using Lemma 2.1 in [13], we get

$$|\sum_{l=1}^{8}\sigma_l\lambda_{j_l}|=|\sum_{l=1}^{6}\sigma_l\lambda_{j_l}|\geq\frac{m^2}{20n_3^3}, \tag{2.27}$$

which is larger than $\frac{m^2}{480n_3^3}$. Therefore, in the following, we assume

$$\sigma_k j_k+\sigma_l j_l\neq 0,\quad \forall\ 1\leq k,l\leq 8. \tag{2.28}$$

From (2.19), we get

$$n_{l+1}\geq\frac{6n_l^3}{m}\geq\frac{6n_l}{m}\geq 54n_l,\quad l=1,2. \tag{2.29}$$

Now our aim is to prove (2.20) for $(\sigma_1 j_1,\cdots,\sigma_8 j_8)\in(\Delta_0\cup\Delta_1\cup\Delta_2)\setminus\mathcal{N}$ with (2.24) and (2.28). It is obvious $(\Delta_0\cup\Delta_1\cup\Delta_2)\setminus\mathcal{N}=(\Delta_0\setminus\mathcal{N})\cup\Delta_1\cup(\Delta_2\setminus\mathcal{N})$ and no element in $\Delta_0\setminus\mathcal{N}$ fulfills (2.24). In the remaining proof, we consider $\Delta_1$ and $\Delta_2\setminus\mathcal{N}$ respectively.

For $(\sigma_1 j_1,\cdots,\sigma_8 j_8)\in\Delta_1$, denote the unique index different with $n_1,n_2,n_3$ in $\{j_1,\cdots j_8\}$ as $a$, the maximum and minimum indices in $\{j_1,\cdots,j_8\}\setminus\{a\}$ as $n_\nu,n_{\nu'}$ respectively.

Case 1: $n_{\nu'}=n_\nu$. By (2.24) and (2.28), we have $a=7n_\nu$. Thus in view of (2.26), we get

$$|\sum_{l=1}^{8}\sigma_l\lambda_{j_l}|=7f(n_\nu)-f(a)>6f(n_\nu). \tag{2.30}$$

Case 2: $n_\nu>n_{\nu'}$. By the definition of $n_\nu$, all indices in $\{j_1,\cdots,j_8\}\setminus\{a,n_\nu\}$ are smaller than $n_{\nu-1}$. By (2.24) and (2.29),

$$a\geq n_\nu-6n_{\nu-1}\geq(54-6)n_{\nu-1}\geq 48n_{\nu'}. \tag{2.31}$$

Subcase 2.1: There is only one $n_{\nu'}$ in $\{j_1,\cdots,j_8\}\setminus\{a\}$. Recall the definition of $n_{\nu'}$, we conclude that all indices in $\{j_1,\cdots,j_8\}\setminus\{n_{\nu'},a\}$ are bigger than $n_{\nu'+1}\geq 54n_{\nu'}$. Therefore, using Lemma 4.1, (2.26) and (2.31), we have

$$\begin{aligned}|\sum_{l=1}^{8}\sigma_l\lambda_{j_l}|&\geq f(n_{\nu'})-6f(n_{\nu'+1})-f(a)\\&\geq f(n_{\nu'})-6f(54n_{\nu'})-f(48n_{\nu'})\end{aligned}$$

$$\geq f(n_{\nu'}) - \frac{\sqrt{10}+3}{54} f(n_{\nu'}) - \frac{\sqrt{10}+3}{288} f(n_{\nu'})$$
$$\geq \frac{1}{2} f(n_{\nu'}). \tag{2.32}$$

Subcase 2.2: There are at least two $n_{\nu'}$ in $\{j_1, \cdots, j_8\} \setminus \{a\}$. Due to (2.28), we remark that these two $n_{\nu'}$ have the same signs. Moreover, all indices in $\{j_1, \cdots, j_8\} \setminus \{n_{\nu'}, a\}$ are bigger than $n_{\nu'+1} \geq 54 n_{\nu'}$. Therefore, using Lemma 4.1, (2.26) and (2.31), we have

$$\begin{aligned}
\big|\sum_{l=1}^{8} \sigma_l \lambda_{j_l}\big| &\geq 2f(n_{\nu'}) - 5f(n_{\nu'+1}) - f(a) \\
&\geq 2f(n_{\nu'}) - 5f(54 n_{\nu'}) - f(48 n_{\nu'}) \\
&\geq 2f(n_{\nu'}) - \frac{5(\sqrt{10}+3)}{324} f(n_{\nu'}) - \frac{\sqrt{10}+3}{288} f(n_{\nu'}) \\
&\geq f(n_{\nu'}).
\end{aligned} \tag{2.33}$$

It is easy to check that the right hands of (2.56), (2.58), (2.59) are larger that $\frac{m^2}{480 n_3^3}$. Hence (2.53) holds true for $(\sigma_1 j_1, \cdots, \sigma_8 j_8) \in \Delta_1$.

For $(\sigma_1 j_1, \cdots, \sigma_8 j_8) \in \Delta_2 \backslash \mathcal{N}$, denote $a, a'$ the two indices different with $n_1, n_2, n_3$ in $\{j_1, \cdots, j_8\}$. Without loss of generality, we assume $a \leq a'$. Denote $n_\nu$ and $n_{\nu'}$ the maximum and minimum index in $\{j_1, \cdots, j_8\} \setminus \{a, a'\}$ respectively.

Case 1: $n_\nu = n_{\nu'}$.

Subcase 1.1: $a' - a = 6 n_\nu$. Then in view of (2.26), we get

$$\big|\sum_{l=1}^{8} \sigma_l \lambda_{j_l}\big| = 6 f(n_\nu) + f(a) - f(a') > 6 f(n_\nu). \tag{2.34}$$

Subcase 1.2: $a' + a = 6 n_\nu$ and $a \leq \frac{n_\nu}{6}$. Then using (2.26), Lemma 4.1 and the fact that $f$ is convex, we get

$$\begin{aligned}
\big|\sum_{l=1}^{8} \sigma_l \lambda_{j_l}\big| &= f(a) + f(a') - 6 f(n_\nu) \\
&\geq f(\frac{n_\nu}{6}) + f(\frac{35 n_\nu}{6}) - 6 f(n_\nu) \\
&\geq \frac{36}{\sqrt{10}+3} f(n_\nu) + \frac{6}{35} f(n_\nu) - 6 f(n_\nu) \\
&\geq \frac{1}{480} f(n_\nu).
\end{aligned} \tag{2.35}$$

Subcase 1.3: $a' + a = 6 n_\nu$ and $\frac{n_\nu}{6} < a < \frac{n_\nu}{2}$. By Taylor's formula, we have, for $j \geq 1$,

$$\lambda_j = \sqrt{j^2 + m} = j + \frac{m}{2j} - \frac{m^2}{8\sqrt{j^2 + \theta m}^3}, \tag{2.36}$$

where $0 < \theta < 1$ depends on $j$. Thus, we have

$$\big|\sum_{l=1}^{8} \sigma_l \lambda_{j_l}\big| = |6 \lambda_{n_\nu} - \lambda_a - \lambda_{a'}|$$

$$= \Big|\frac{m}{2}\Big(\frac{6}{n_\nu} - \frac{1}{a} - \frac{1}{a'}\Big) - \frac{m^2}{8}\Big(\frac{6}{\sqrt{n_\nu^2 + \theta_1 m}^3} - \frac{1}{\sqrt{a^2 + \theta_2 m}^3} - \frac{1}{\sqrt{(a')^2 + \theta_3 m}^3}\Big)\Big|, \tag{2.37}$$

where $0 < \theta_1, \theta_2, \theta_3 < 1$. Since $n_\nu$, $a$, $a'$ are integers and $a + a' = 6n_\nu$, we know $|aa' - n_\nu^2| \geq 1$. Otherwise, we have $aa' - n_\nu^2 = 0$, and further $a = (3 - \sqrt{8})n_\nu$, $a' = (3 + \sqrt{8})n_\nu$, which is impossible. Thus,

$$\Big|\frac{6}{n_\nu} - \frac{1}{a} - \frac{1}{a'}\Big| = \frac{|6aa' - n_\nu(a + a')|}{n_\nu aa'} = \frac{6|aa' - n_\nu^2|}{n_\nu aa'} \geq \frac{6}{n_\nu aa'}. \tag{2.38}$$

On the other hand,

$$\begin{aligned}\Big|\frac{6}{\sqrt{n_\nu^2 + \theta_1 m}^3} - \frac{1}{\sqrt{a^2 + \theta_2 m}^3} - \frac{1}{\sqrt{(a')^2 + \theta_3 m}^3}\Big| &\leq \frac{1}{a^3} + \frac{1}{(a')^3} - \frac{6}{\sqrt{n_\nu^2 + m}^3} \\ &\leq \frac{1}{a^3} - \frac{5}{\sqrt{n_\nu^2 + m}^3}.\end{aligned} \tag{2.39}$$

Thus, from (2.37)-(2.39), we get

$$\begin{aligned}\Big|\sum_{l=1}^{8} \sigma_l \lambda_{j_l}\Big| &\geq \frac{3m}{n_\nu aa'} - \frac{m^2}{8}\Big(\frac{1}{a^3} - \frac{5}{\sqrt{n_\nu^2 + m}^3}\Big) \\ &= \frac{3m}{a}\Big(\frac{1}{n_\nu a'} - \frac{m}{24a^2}\Big) + \frac{5m^2}{8\sqrt{n_\nu^2 + m}^3} \\ &\geq \frac{3m}{a}\Big(\frac{1}{n_\nu(6n_\nu)} - \frac{1/9}{24(n_\nu/6)^2}\Big) + \frac{5m^2}{8\sqrt{n_\nu^2 + m}^3} \\ &= \frac{5m^2}{8\sqrt{n_\nu^2 + m}^3}.\end{aligned} \tag{2.40}$$

Subcase 1.4: $a' + a = 6n_\nu$ and $a \geq \frac{n_\nu}{2}$. In view of $a + a' = 6n_\nu$ and our assumption $a \leq a'$, we know $a' \geq 3n_\nu$. Thus using (2.26) and Lemma 4.1, we get

$$\begin{aligned}\Big|\sum_{l=1}^{8} \sigma_l \lambda_{j_l}\Big| &= 6f(n_\nu) - f(a) - f(a') \\ &\geq 6f(n_\nu) - f(\frac{n_\nu}{2}) - f(3n_\nu) \\ &\geq 6f(n_\nu) - 2f(n_\nu) - \frac{\sqrt{10} + 3}{18} f(n_\nu) \\ &> f(n_\nu).\end{aligned} \tag{2.41}$$

Case 2: $n_\nu > n_{\nu'}$.

Subcase 2.1: $a$ and $n_{\nu'}$ have the same signs. By the definition of $n_{\nu'}$, all indices in $\{j_1, \cdots, j_8\} \setminus \{n_{\nu'}, a, a'\}$ are not less than $n_{\nu'+1}$. Therefore using (2.26), (2.29) and Lemma 4.1, we get

$$\Big|\sum_{l=1}^{8} \sigma_l \lambda_{j_l}\Big| \geq f(n_{\nu'}) + f(a) - f(a') - 5f(n_{\nu'+1})$$

$$\begin{aligned}
&\geq f(n_{\nu'}) - 5f(n_{\nu'+1})\\
&\geq f(n_{\nu'}) - 5f(54n_{\nu'})\\
&\geq f(n_{\nu'}) - \frac{5(\sqrt{10}+3)}{324} f(n_{\nu'})\\
&> \frac{1}{2} f(n_{\nu'}). \qquad (2.42)
\end{aligned}$$

Subcase 2.2: $a$ and $n_{\nu'}$ have different signs, and there is only one $n_{\nu'}$. In view of (2.24), we have $a - n_{\nu'} \pm a' \pm n_\nu \pm n_{j_1} \pm \cdots \pm n_{j_4} = 0$. As $a \leq a'$ and $n_\nu \geq 54n_{\nu-1}$, we conclude that $a' \geq n_{\nu-1}$ and has different sign with $n_\nu$. If $\nu \geq \nu' + 2$, then $a' \geq n_{\nu'+1}$; otherwise $\nu = \nu' + 1$, then there are 3 $n_\nu$'s and thus $a' \geq n_\nu = n_{\nu'+1}$. Now all indices in $\{j_1, \cdots, j_8\} \backslash \{n_{\nu'}, a\}$ are not less than $n_{\nu'+1}$. Using (2.19), (2.26), Lemma 4.2 and the fact that $a'$ has different sign with $n_\nu$, we have

$$\begin{aligned}
|\sum_{l=1}^{8} \sigma_l f(\lambda_{j_l})| &= |f(a) - f(n_{\nu'}) \pm f(a') \pm f(n_\nu) \pm f(n_{j_1}) \pm \cdots \pm f(n_{j_4})|\\
&\geq |f(a) - f(n_{\nu'})| - 3f(n_{\nu'+1})\\
&\geq \frac{m}{2}\frac{1}{(n_{\nu'}+1)^2 + m} - \frac{3m}{2n_{\nu'+1}}\\
&\geq \frac{2m}{17n_{\nu'}^2} - \frac{3m^2}{12n_{\nu'}^3}\\
&\geq \frac{m}{20n_{\nu'}^3}, \qquad (2.43)
\end{aligned}$$

where in the last inequality we use $0 < m \leq \frac{1}{9}$.

Subcase 2.3: $a$ and $n_{\nu'}$ have different signs, and there are $l$ $n_{\nu'}$s for $2 \leq l \leq 4$. In view of (2.24), we have $a - ln_{\nu'} \pm a' \pm n_\nu \pm n_{j_1} \pm \cdots \pm n_{j_{5-l}} = 0$. As $a \leq a'$ and $n_\nu \geq 54n_{\nu-1}$, we conclude that $a' \geq n_{\nu-1}$ and has different sign with $n_\nu$. If $\nu \geq \nu' + 2$, then $a' \geq n_{\nu'+1}$; otherwise $\nu = \nu' + 1$, then there are 3 $n_\nu$'s and thus $a' \geq n_\nu = n_{\nu'+1}$. Now all indices in $\{j_1, \cdots, j_8\} \setminus \{n_{\nu'}, a\}$ are not less than $n_{\nu'+1}$. Using (2.19), (2.26), Lemma 4.3 and the fact that $a'$ has different sign with $n_\nu$, we have

$$\begin{aligned}
|\sum_{l=1}^{8} \sigma_l f(\lambda_{j_l})| &\geq |f(a) - lf(n_{\nu'})| - 3f(n_{\nu'+1})\\
&\geq \frac{m^2}{2n_{\nu'}^3} - (3 - \frac{l}{2})\frac{m}{n_{\nu'+1}}\\
&\geq \frac{m^2}{2n_{\nu'}^3} - (3 - \frac{l}{2})\frac{m^2}{6n_{\nu'}^3}\\
&> \frac{m^2}{6n_{\nu'}^3}. \qquad (2.44)
\end{aligned}$$

Subcase 2.4: $a$ and $n_{\nu'}$ have different signs, and there are 5 $n_{\nu'}$. In view of (2.24), we have $a - 5n_{\nu'} \pm a' \pm n_\nu = 0$. As $a \leq a'$ and $n_{\nu'} \leq \frac{n_\nu}{54}$, we conclude that $a' \geq \frac{1}{2}(n_\nu - 5n_{\nu'}) \geq \frac{9n_\nu}{20}$ and has different sign with $n_\nu$. Using (2.19), (2.26), Lemma 4.3 and the fact that $a'$ has different sign with $n_\nu$, we have

$$\begin{aligned}|\sum_{l=1}^{8}\sigma_l f(\lambda_{j_l})| &\geq |f(a)-5f(n_{\nu'})| - f(\frac{9n_\nu}{20})\\ &\geq \frac{m^2}{2n_{\nu'}^3}-\frac{10m}{9n_\nu}\\ &\geq \frac{m^2}{2n_{\nu'}^3}-\frac{5m^2}{27n_{\nu'}^3}\\ &\geq \frac{25m^2}{54n_{\nu'}^3}.\end{aligned}\tag{2.45}$$

It is easy to check that the right hands of (2.34), (2.35), (2.40), (2.41), (2.42), (2.43), (2.44), (2.45) are larger than $\frac{m^2}{480n_3^3}$. Hence (2.53) holds true for $(\sigma_1 j_1,\cdots,\sigma_8 j_8) \in \Delta_2\setminus\mathcal{N}$. This completes the proof of this lemma. □

In view of (2.5) and the above lemma, in the same way as [18], the regularity of the vector field $X_F$ could be easily established:

$$X_F\in\mathcal{A}(\ell_b^{a,s},\ell_b^{a,s+1}),\tag{2.46}$$

where $\mathcal{A}(\ell_b^{a,s},\ell_b^{a,s+1})$ denotes the class of all real analytic maps from some neighborhood of the origin in $\ell_b^{a,s}$ into $\ell_b^{a,s+1}$, and $\ell_b^{a,s}$ denotes the Hilbert space of all bi-infinite sequences with finite norm $||q||_{a,s}^2=|q_0|^2+\sum_j|q_j|^2|j|^{2s}e^{2|j|a}$. Therefore, in view of (2.18), we obtain the following theorem:

**Theorem 2.1.** *Suppose* $0<m\leq\frac{1}{9}$ *and the indices* $n_1,n_2,n_3\in\mathbb{N}^+$ *satisfy* (2.19). *Then by the symplectic change of coordinates* $\Gamma_1:=X_F^1$, *which is real analytic in some neighborhood of the origin in* $\ell_b^{a,p}$, *the Hamiltonian* $H=\Lambda+G$ *in* (2.8) *is taken into*

$$H\circ\Gamma_1=\Lambda+\bar{G}+\hat{G}+K,\tag{2.47}$$

*where* $\Lambda$ *is in (2.9),* $\bar{G}$ *is in (2.12),* $\hat{G}$ *is in (2.14), and*

$$K=\int_0^1\{\bar{G}+t\tilde{G}+\hat{G},F\}\circ X_F^t dt.\tag{2.48}$$

*Moreover,* $X_{\bar{G}}$, $X_{\hat{G}}$, $X_K\in\mathcal{A}(\ell_b^{a,p},\ell_b^{a,p+1})$.

By simple calculation we have

$$K=\{\bar{G}+\hat{G}+\frac{1}{2}\tilde{G},F\}+\int_0^1\Big\{\{(1-t)(\bar{G}+\hat{G})+\frac{1}{2}(1-t^2)\tilde{G},F\},F\Big\}\circ X_F^t dt,\tag{2.49}$$

where the first term is order 14 and the second term is at least order 20. In order to obtain a partial Birkhoff normal form of order 14, we need another real analytic, symplectic coordinate change. To this end, define the normal form set

$$\begin{aligned}\mathcal{N}'=&\{(j_1,\cdots,j_{14})\in\mathbb{Z}_*^{14}:\text{There exists a 14-permutation }\tau\text{ such that}\\ &j_{\tau(1)}=-j_{\tau(2)},\ j_{\tau(3)}=-j_{\tau(4)},\cdots,\ j_{\tau(13)}=-j_{\tau(14)}\},\end{aligned}$$

and the following index sets

$$\Delta_l'=\{(j_1,\cdots,j_{14})\in\mathbb{Z}_*^{14}:\text{There are exactly } l \text{ components not in}\{\pm n_1,\pm n_2,\pm n_3\}\}$$

for $l = 0, 1$, and

$\Delta_2' = \{(j_1, \cdots, j_{14}) \in \mathbb{Z}_*^{14} :$ There are at least 2 components not in $\{\pm n_1, \pm n_2, \pm n_3\}\}$.

Split the first term of $K$ in (2.49) into three parts:

$$\{\bar{G} + \hat{G} + \frac{1}{2}\tilde{G}, F\} = \bar{K} + \tilde{K} + \hat{K}, \tag{2.50}$$

where $\bar{K}$ is the normal form part with $(j_1, \cdots, j_{14}) \in (\Delta_0' \cup \Delta_1') \cap \mathcal{N}'$, $\tilde{K}$ is the non-normal form part with $(j_1, \cdots, j_{14}) \in (\Delta_0' \cup \Delta_1') \setminus \mathcal{N}'$, and $\hat{K}$ is the part with $(j_1, \cdots, j_{14}) \in \Delta_2'$. The same procedure as eliminating $\tilde{G}$, we will eliminate $\tilde{K}$ by another symplectic coordinate transformation. Similarly, a lemma about the divisors $\lambda_{j_1} + \cdots + \lambda_{j_{14}}$ is needed:

**Lemma 2.2.** *Suppose $0 < m \leq \frac{1}{9}$ and the indices $n_1, n_2, n_3 \in \mathbb{N}^+$ satisfy*

$$n_{l+1} \geq \frac{6n_l^3}{m}, \quad l = 1, 2. \tag{2.51}$$

*Then for $(j_1, \cdots, j_{14}) \in (\Delta_0' \cup \Delta_1') \setminus \mathcal{N}'$, we have*

$$\left|\lambda_{j_1} + \cdots + \lambda_{j_{14}}\right| \geq \frac{m^2}{4n_3^3}. \tag{2.52}$$

**Proof.** This lemma is equivalent to prove that, for $j_1, \cdots, j_{14} \in \mathbb{N}^+$ and $\sigma_1, \cdots, \sigma_{14} \in \{1, -1\}$, if $(\sigma_1 j_1, \cdots, \sigma_{14} j_{14}) \in (\Delta_0' \cup \Delta_1') \setminus \mathcal{N}'$, then we have

$$\Big|\sum_{l=1}^{14} \sigma_l \lambda_{j_l}\Big| \geq \frac{m^2}{4n_3^3}. \tag{2.53}$$

Similarly to (2.24) and (2.28) in Lemma 2.1, we may assume

$$\sum_{l=1}^{14} \sigma_l j_l = 0 \tag{2.54}$$

and

$$\sigma_k j_k + \sigma_l j_l \neq 0, \quad 1 \leq k, l \leq 14. \tag{2.55}$$

Now our aim is to prove (2.52) for $(\sigma_1 j_1, \cdots, \sigma_{14} j_{14}) \in (\Delta_0' \cup \Delta_1') \setminus \mathcal{N}'$ with (2.54) and (2.55). It is obvious $(\Delta_0' \cup \Delta_1') \setminus \mathcal{N}' = (\Delta_0' \setminus \mathcal{N}') \cup \Delta_1'$ and no element in $\Delta_0' \setminus \mathcal{N}'$ fulfills (2.54). In the remaining proof, we consider $\Delta_1'$.

For $(j_1, \cdots, j_{14}) \in \Delta_1'$, denote the unique index different with $n_1, n_2, n_3$ in $\{j_1, \cdots, j_{14}\}$ as $a$, the maximum and minimum indices in $\{j_1, \cdots, j_{14}\} \setminus \{a\}$ as $n_\nu, n_{\nu'}$ respectively.

Case 1: $n_{\nu'} = n_\nu$. By (2.54) and (2.55), we have $a = 13n_\nu$. Thus in view of (2.26), we get

$$\Big|\sum_{l=1}^{14} \sigma_l \lambda_{j_l}\Big| = 13f(n_\nu) - f(a) > 12f(n_\nu). \tag{2.56}$$

Case 2: $n_\nu > n_{\nu'}$. By the definition of $n_\nu$, all indices in $\{j_1, \cdots, j_{14}\} \setminus \{a, n_\nu\}$ are smaller than $n_{\nu-1}$. By (2.51) and (2.54),

$$a \geq n_\nu - 12n_{\nu-1} \geq 42n_{\nu-1} \geq 27n_{\nu'}. \tag{2.57}$$

Subcase 2.1: There is only one $n_{\nu'}$ in $\{j_1, \cdots, j_{14}\} \setminus \{a\}$. Recall the definition of $n_{\nu'}$, we conclude that all indices in $\{j_1, \cdots, j_{14}\} \setminus \{n_{\nu'}, a\}$ are bigger than $n_{\nu'+1} \geq 54n_{\nu'}$. Therefore, using Lemma 4.1, (2.26) and (2.57), we have

$$\begin{aligned}
\Big|\sum_{l=1}^{14} \sigma_l \lambda_{j_l}\Big| &\geq f(n_{\nu'}) - 12f(n_{\nu'+1}) - f(a) \\
&\geq f(n_{\nu'}) - 12f(54n_{\nu'}) - f(27n_{\nu'}) \\
&\geq f(n_{\nu'}) - \frac{\sqrt{10}+3}{27} f(n_{\nu'}) - \frac{\sqrt{10}+3}{162} f(n_{\nu'}) \\
&\geq \frac{1}{2} f(n_{\nu'}).
\end{aligned} \tag{2.58}$$

Subcase 2.2: There are at least two $n_{\nu'}$ in $\{j_1, \cdots, j_{14}\} \setminus \{a\}$. Due to (2.55), we remark that these two $n_{\nu'}$ have the same signs. Moreover, all indices in $\{j_1, \cdots, j_{14}\} \setminus \{n_{\nu'}, a\}$ are bigger than $n_{\nu'+1} \geq 54n_{\nu'}$. Therefore, using Lemma 4.1, (2.26) and (2.57), we have

$$\begin{aligned}
\Big|\sum_{l=1}^{14} \sigma_l \lambda_{j_l}\Big| &\geq 2f(n_{\nu'}) - 11f(n_{\nu'+1}) - f(a) \\
&\geq 2f(n_{\nu'}) - 11f(54n_{\nu'}) - f(27n_{\nu'}) \\
&\geq 2f(n_{\nu'}) - \frac{11(\sqrt{10}+3)}{324} f(n_{\nu'}) - \frac{\sqrt{10}+3}{162} f(n_{\nu'}) \\
&\geq f(n_{\nu'}).
\end{aligned} \tag{2.59}$$

It is easy to check that the right hands of (2.56), (2.58), (2.59) are larger that $\frac{m^2}{4n_3^3}$. Hence (2.53) holds true for $(\sigma_1 j_1, \cdots, \sigma_{14} j_{14}) \in \Delta_1'$. $\square$

Therefore, we can obtain the following result:

**Theorem 2.2.** *Suppose $0 < m \leq \frac{1}{9}$ and the indices $n_1, n_2, n_3 \in \mathbb{N}^+$ satisfy*

$$n_{l+1} \geq \frac{6n_l^3}{m}, \quad l = 1, 2. \tag{2.60}$$

*Then by another symplectic change of coordinates $\Gamma_2$, which is the time-1-map of the flow of a Hamiltonian vector field $X_{\tilde{F}}$ and is real analytic in some neighborhood of the origin in $\ell_b^{a,s}$, the Hamiltonian $H \circ \Gamma_1$ in (2.47) is taken into*

$$(H \circ \Gamma_1) \circ \Gamma_2 = \Lambda + \bar{G} + \hat{G} + \bar{K} + \hat{K} + T, \tag{2.61}$$

*where $\bar{K}$ is of the form*

$$\bar{K} = \sum_{\substack{l_1, l_2, l_3 \in \mathbb{N} \\ l_1 + l_2 + l_3 = 7}} K_{l_1 l_2 l_3} |z_{n_1}|^{2l_1} |z_{n_2}|^{2l_2} |z_{n_3}|^{2l_3} \tag{2.62}$$

*with coefficients $K_{l_1 l_2 l_3}$ real and depending only on $n_1$, $n_2, n_3$ and $m$, and*

$$|\hat{K}| = O(\|z\|_{a,s}^{12} \|\hat{z}\|_{a,s}^2), \tag{2.63}$$

$$|T| = O(\|z\|_{a,s}^{20}), \tag{2.64}$$

*$\hat{z} = (z_j)_{j \in \mathbb{N}^+ \setminus \{n_1, n_2, n_3\}}$. Moreover, $X_{\bar{K}}$, $X_{\hat{K}}$, $X_T \in \mathcal{A}(\ell_b^{a,s}, \ell_b^{a,s+1})$.*

# 3. Proof of Theorem 1.1

By the last section there exist two real analytic, symplectic changes of coordinates $\Gamma_1, \Gamma_2$ which takes $H$ into $H \circ \Gamma_1 \circ \Gamma_2 = \Lambda + \bar{G} + \hat{G} + \bar{K} + \hat{K} + T$, where $\Lambda$ is in (2.9), $\bar{G}$ is in (2.12), $\hat{G}$ is in (2.14), $\bar{K}$ is in (2.62), $\hat{K}$ is in (2.63), $T$ is in (2.64). Let $I = (|z_j|^2 : j \in \mathbb{N}^+)$, then we have

$$\Lambda = \sum_{j\geq 1} \lambda_j I_j, \tag{3.1}$$

$$\begin{aligned}\bar{G} =& \frac{35}{8\pi^3}\Big(\sum_{l=1}^{3} \frac{35 I_{n_l}^4}{64\lambda_{n_l}^4} + \sum_{\substack{l,l'\\ l\neq l'}} \frac{5 I_{n_l}^3 I_{n_{l'}}}{\lambda_{n_l}^3 \lambda_{n_{l'}}} + \sum_{\substack{l,l'\\ l\neq l'}} \frac{81 I_{n_l}^2 I_{n_{l'}}^2}{8\lambda_{n_l}^2 \lambda_{n_{l'}}^2} + \sum_{\substack{l,l',l''\\ l\neq l', l\neq l'', l'\neq l''}} \frac{27 I_{n_l}^2 I_{n_{l'}} I_{n_{l''}}}{\lambda_{n_l}^2 \lambda_{n_{l'}} \lambda_{n_{l''}}}\Big)\\ &+ \frac{35}{4}\Big(\sum_{l=1}^{b} G_{n_l n_l n_l j} I_{n_l}^3 + 9 \sum_{\substack{l,l'\\ l\neq l'}} G_{n_l n_l n_{l'} j} I_{n_l}^2 I_{n_{l'}}\\ &+ 18 \sum_{\substack{l,l',l''\\ l\neq l', l\neq l'', l'\neq l''}} G_{n_l n_{l'} n_{l''} j} I_{n_l} I_{n_{l'}} I_{n_{l''}}\Big) I_j,\end{aligned} \tag{3.2}$$

$$\bar{K} = \sum_{\substack{l_1,l_2,l_3\in\mathbb{N}\\ l_1+l_2+l_3=7}} K_{l_1 l_2 l_3} I_{n_1}^{l_1} I_{n_2}^{l_2} I_{n_3}^{l_3}. \tag{3.3}$$

Moreover, we know

$$|\hat{G}| = O(\|z\|_{a,s}^5 \|\hat{z}\|_{a,s}^3), \quad |\hat{K}| = O(\|z\|_{a,s}^{12} \|\hat{z}\|_{a,s}^2), \quad |T| = O(\|z\|_{a,s}^{20}). \tag{3.4}$$

*Step 1: New coordinates.* We introduce symplectic polar and real coordinates $(x, y, u, v)$ by setting

$$\begin{cases} z_{n_l} = \sqrt{\xi_l^{\frac{1}{3}} + y_l}\, e^{-\mathbf{i}x_l}, & l = 1,2,3\\ z_j = \dfrac{1}{\sqrt{2}}(u_j + \mathbf{i}v_j), & j \neq n_1, n_2, n_3, \end{cases} \tag{3.5}$$

$$\mathbf{i}\sum_{j\geq 1} dz_j \wedge d\bar{z}_j = \sum_{i=1}^{3} dx_i \wedge dy_i + \sum_{j\neq n_1,n_2,n_3} du_j \wedge dv_j$$

and

$$\begin{cases} I_{n_l} = \xi_l^{\frac{1}{3}} + y_l, & l = 1,2,3,\\ I_j = \dfrac{1}{2}(u_j^2 + v_j^2), & j \neq n_1, n_2, n_3. \end{cases}$$

Up to a constant depending only on $\xi$, the normal form $\Lambda + \bar{G} + \bar{K}$ becomes

$$\langle \omega(\xi), y\rangle + \frac{1}{2}\langle \Omega(\xi), u^2 + v^2\rangle + Q$$

with tangential frequencies

$$\omega_l(\xi) = \lambda_{n_l} + \frac{1225\xi_l}{128\pi^3\lambda_{n_l}^4} + \frac{525}{4\pi^3}\sum_{l'\neq l} \frac{\xi_l^{\frac{2}{3}} \xi_{l'}^{\frac{1}{3}}}{\lambda_{n_l}^3 \lambda_{n_{l'}}} + \frac{525}{8\pi^3}\sum_{l'\neq l} \frac{\xi_{l'}}{\lambda_{n_l}\lambda_{n_{l'}}^3} \tag{3.6}$$

$$+\frac{2835}{16\pi^3}\sum_{l'\neq l}\frac{\xi_l^{\frac{1}{3}}\xi_{l'}^{\frac{2}{3}}}{\lambda_{n_l}^2\lambda_{n_{l'}}^2}+\frac{945}{4\pi^3}\sum_{\substack{l',l''\neq l,\\ l'\neq l''}}\frac{\xi_l^{\frac{1}{3}}\xi_{l'}^{\frac{1}{3}}\xi_{l''}^{\frac{1}{3}}}{\lambda_{n_l}^2\lambda_{n_{l'}}\lambda_{n_{l''}}}+\frac{945}{4\pi^3}\sum_{\substack{l',l''\neq l\\ l'\neq l''}}\frac{\xi_{l'}^{\frac{2}{3}}\xi_{l''}^{\frac{1}{3}}}{\lambda_{n_l}\lambda_{n_{l'}}^2\lambda_{n_{l''}}}$$

$$+7K_{0\cdots 070\cdots 0}\xi_l^2+6\sum_{\substack{j_l=6\\ j_{l'}=1}}K_{j_1\cdots j_b}\xi_l^{\frac{5}{3}}\xi_{l'}^{\frac{1}{3}}+5\sum_{\substack{j_l=5\\ j_{l'}+j_{l''}=2}}K_{j_1\cdots j_b}\xi_l^{\frac{5}{3}}\xi_{l'}^{\frac{j_{l'}}{3}}\xi_{l''}^{\frac{j_{l''}}{3}}$$

$$+4\sum_{\substack{j_l=4\\ j_{l'}+j_{l''}+j_{l'''}=3}}K_{j_1\cdots j_b}\xi_l^{\frac{4}{3}}\xi_{l'}^{\frac{j_{l'}}{3}}\xi_{l''}^{\frac{j_{l''}}{3}}\xi_{l'''}^{\frac{j_{l'''}}{3}}$$

$$+3\sum_{\substack{j_l=3\\ j_{l'}+j_{l''}+j_{l'''}+j_{l^{(4)}}=4}}K_{j_1\cdots j_b}\xi_l\xi_{l'}^{\frac{j_{l'}}{3}}\xi_{l''}^{\frac{j_{l''}}{3}}\xi_{l'''}^{\frac{j_{l'''}}{3}}\xi_{l^{(4)}}^{\frac{j_{l^{(4)}}}{3}}$$

$$+2\sum_{\substack{j_l=2\\ j_{l'}+j_{l''}+j_{l'''}+j_{l^{(4)}}+j_{l^{(5)}}=5}}K_{j_1\cdots j_b}\xi_l^{\frac{2}{3}}\xi_{l'}^{\frac{j_{l'}}{3}}\xi_{l''}^{\frac{j_{l''}}{3}}\xi_{l'''}^{\frac{j_{l'''}}{3}}\xi_{l^{(4)}}^{\frac{j_{l^{(4)}}}{3}}\xi_{l^{(5)}}^{\frac{j_{l^{(5)}}}{3}}$$

$$+\sum_{\substack{j_l=1\\ j_{l'}+j_{l''}+j_{l'''}+j_{l^{(4)}}+j_{l^{(5)}}+j_{l^{(6)}}=6}}K_{j_1\cdots j_b}\xi_l^{\frac{1}{3}}\xi_{l'}^{\frac{j_{l'}}{3}}\xi_{l''}^{\frac{j_{l''}}{3}}\xi_{l'''}^{\frac{j_{l'''}}{3}}\xi_{l^{(4)}}^{\frac{j_{l^{(4)}}}{3}}\xi_{l^{(5)}}^{\frac{j_{l^{(5)}}}{3}}\xi_{l^{(6)}}^{\frac{j_{l^{(6)}}}{3}},$$

normal frequencies

$$\begin{aligned}\Omega_j(\xi)=&\lambda_j+\frac{35}{4}\Big(\sum_{l=1}^{b}G_{n_ln_ln_lj}\xi_l+9\sum_{\substack{l,l'\\ l\neq l'}}G_{n_ln_ln_{l'}j}\xi_l^{\frac{2}{3}}\xi_{l'}^{\frac{1}{3}}\\ &+18\sum_{\substack{l,l',l''\\ l\neq l',l\neq l'',l'\neq l''}}G_{n_ln_{l'}n_{l''}j}\xi_l^{\frac{1}{3}}\xi_{l'}^{\frac{1}{3}}\xi_{l''}^{\frac{1}{3}}\Big),\quad j\neq n_1,n_2,n_3\end{aligned}\tag{3.7}$$

and remainder

$$Q=O(|y|^2)+O(|y|\cdot\|u^2+v^2\|).\tag{3.8}$$

The total Hamiltonian $H=N+P$ with

$$N=\langle\omega(\xi),y\rangle+\frac{1}{2}\langle\Omega(\xi),u^2+v^2\rangle,\tag{3.9}$$

$$P=Q+\hat{G}+\hat{K}+T.\tag{3.10}$$

Now let $r>0$ and consider the phase space domain

$$D(2,r):\quad |\mathrm{Im}x|<2,\quad |y|<r^2,\quad \|u\|_{a,s}+\|v\|_{a,s}<r\tag{3.11}$$

and the parameter domain

$$\Pi=\{\xi=(\xi_1,\xi_2,\xi_3):r^{\frac{6}{5}}\leq\xi_1,\xi_2,\xi_3\leq\frac{6}{5}r^{\frac{6}{5}}\}.\tag{3.12}$$

*Step 2: Checking assumption A of Theorem 5.1.* Let

$$\hat{\omega}=T\Lambda_1(\omega-\omega_0),\tag{3.13}$$

where

$$T = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \tag{3.14}$$

$\Lambda_1 = \mathrm{diag}(\frac{128\pi^3}{35}\lambda_{n_l} : 1 \le l \le 3)$ and $\omega_0 = (\lambda_{n_l})_{1\le l\le 3}$. Denote

$$\eta = \Lambda_2^{-1}\xi, \tag{3.15}$$

where $\Lambda_2 = \mathrm{diag}(\lambda_{n_l}^3 : 1 \le l \le 3)$. From (3.13) and (3.15), we get

$$\frac{\partial \omega}{\partial \xi} = \Lambda_1^{-1} T^{-1} \frac{\partial \hat{\omega}}{\partial \eta} \Lambda_2^{-1}. \tag{3.16}$$

Calculate directly, we have

$$\begin{aligned} \hat{\omega}_l =& 445\eta_l + 35(\eta_{l'} + \eta_{l''}) + 1032\eta_l^{\frac{2}{3}}(\eta_{l'}^{\frac{1}{3}} + \eta_{l''}^{\frac{1}{3}}) + 696\eta_l^{\frac{1}{3}}(\eta_{l'}^{\frac{2}{3}} + \eta_{l''}^{\frac{2}{3}}) \\ &+ 264(\eta_{l'}^{\frac{1}{3}}\eta_{l''}^{\frac{2}{3}} + \eta_{l'}^{\frac{2}{3}}\eta_{l''}^{\frac{1}{3}}) + 1728\eta_l^{\frac{1}{3}}\eta_{l'}^{\frac{1}{3}}\eta_{l''}^{\frac{1}{3}}, \\ & l = 1,2,3,\ 1 \le l', l'' \le 3,\ l', l'' \ne l, l' \ne l'' \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \hat{\omega}_i}{\partial \eta_j} &= 445 + 688\eta_j^{-\frac{1}{3}}(\eta_{l'}^{\frac{1}{3}} + \eta_{l''}^{\frac{1}{3}}) + 232\eta_j^{-\frac{2}{3}}(\eta_{l'}^{\frac{2}{3}} + \eta_{l''}^{\frac{2}{3}}) \\ &\quad + 576\eta_j^{-\frac{2}{3}}\eta_{l'}^{\frac{1}{3}}\eta_{l''}^{\frac{1}{3}} + O(r^{\frac{4}{3}}), \quad i = j,\ 1 \le l', l'' \le 3,\ l', l'' \ne j, l' \ne l'', \qquad (3.17) \\ \frac{\partial \hat{\omega}_i}{\partial \eta_j} &= 35 + 344\eta_i^{\frac{2}{3}}\eta_j^{-\frac{2}{3}} + 464\eta_i^{\frac{1}{3}}\eta_j^{-\frac{1}{3}} + 88(\eta_j^{-\frac{2}{3}}\eta_{l'}^{\frac{2}{3}} + 2\eta_j^{-\frac{1}{3}}\eta_{l'}^{\frac{1}{3}}) \\ &\quad + 576\eta_i^{\frac{1}{3}}\eta_j^{-\frac{2}{3}}\eta_{l'}^{\frac{1}{3}} + O(r^{\frac{4}{3}}), \quad i \ne j,\ 1 \le l' \le 3,\ l' \ne i, j. \qquad (3.18) \end{aligned}$$

In order to estimate $|\omega|_{\Pi}^{\mathcal{L}}$ and $|\omega^{-1}|_{\omega(\Pi)}^{\mathcal{L}}$, we estimate $||\frac{\partial \omega}{\partial \xi}||_{\ell^1\to\ell^1}$ and $||(\frac{\partial \omega}{\partial \xi})^{-1}||_{\ell^1\to\ell^1}$ in the following. From (3.17) and (3.18), we have

$$\begin{aligned} ||\frac{\partial \hat{\omega}}{\partial \eta}||_{\ell^1\to\ell^1} &= \max_{1\le i\le 3}\sum_{j=1}^{3}|\frac{\partial \hat{\omega}_i}{\partial \eta_j}| \\ &\le 515 + 576\max_{1\le i\le b}\eta_i^{-\frac{2}{3}}\sum_{l,l'}\eta_l^{\frac{1}{3}}\eta_{l'}^{\frac{1}{3}} + 344\max_{1\le i\le 3}\eta_i^{-\frac{2}{3}}\sum_{l}\eta_l^{\frac{2}{3}} \\ &\quad + 688\max_{1\le j\le b}\eta_j^{-\frac{1}{3}}\sum_{l}\eta_l^{\frac{1}{3}} + O(r^{\frac{4}{3}}) \\ &\le \frac{6000\lambda_{n_3}}{\lambda_{n_1}} \end{aligned} \tag{3.19}$$

and

$$|\frac{\partial \hat{\omega}_j}{\partial \eta_j}| - |\sum_{i\ne j}\frac{\partial \hat{\omega}_j}{\partial \eta_j}| \ge 410 + O(r^{\frac{4}{3}}) \ge 409 > 0, \quad j = 1,2,3. \tag{3.20}$$

Then applying Lemma 4.4 to $(\frac{\partial\hat{\omega}}{\partial\eta})$, we have

$$||(\frac{\partial\hat{\omega}}{\partial\eta})^{-1}||_{\ell^1\to\ell^1} \le \frac{1}{409}. \tag{3.21}$$

By direct calculation, we have

$$T^{-1} = \frac{1}{4}\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \tag{3.22}$$

$$||T||_{\ell^1\to\ell^1} = \max_{1\le i\le 3}\sum_{j=1}^{3}|T_{ij}| = 3 \tag{3.23}$$

and

$$||T^{-1}||_{\ell^1\to\ell^1} = \max_{1\le i\le 3}\sum_{j=1}^{3}|T_{ij}^{-1}| = 1. \tag{3.24}$$

From (3.16) (3.19) (3.21) (3.23) and (3.24), we can obtain the estimate of the Lipschitz semi-norm of $\omega$ and $\omega^{-1}$:

$$\begin{aligned} |\omega|_{\Pi}^{\mathcal{L}} &\le |\frac{\partial\omega}{\partial\xi_1}|_{\Pi} + \cdots + |\frac{\partial\omega}{\partial\xi_3}|_{\Pi} \\ &\le \max\{|\frac{\partial\omega_1}{\partial\xi_1}|_{\Pi}, |\frac{\partial\omega_2}{\partial\xi_1}|_{\Pi}, |\frac{\partial\omega_3}{\partial\xi_1}|_{\Pi}\} + \cdots + \max\{|\frac{\partial\omega_1}{\partial\xi_3}|_{\Pi}, |\frac{\partial\omega_2}{\partial\xi_3}|_{\Pi}, |\frac{\partial\omega_3}{\partial\xi_3}|_{\Pi}\} \\ &\le 3||\frac{\partial\omega}{\partial\xi}||_{\ell^1\to\ell^1} \\ &= 3||\Lambda_1^{-1}T^{-1}\frac{\partial\hat{\omega}}{\partial\eta}\Lambda_2^{-1}||_{\ell^1\to\ell^1} \\ &\le 3||\Lambda_1^{-1}||_{\ell^1\to\ell^1}||T^{-1}||_{\ell^1\to\ell^1}||\frac{\partial\hat{\omega}}{\partial\eta}||_{\ell^1\to\ell^1}||\Lambda_2^{-1}||_{\ell^1\to\ell^1} \\ &\le 3\cdot\frac{35}{128\pi^3\lambda_{n_1}}\cdot 1\cdot\frac{6000\lambda_{n_3}}{\lambda_{n_1}}\cdot\frac{1}{\lambda_{n_1}^3} \\ &= \frac{39375\lambda_{n_3}}{\pi^3\lambda_{n_1}^5}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} |\omega^{-1}|_{\omega(\Pi)}^{\mathcal{L}} &\le 3||(\frac{\partial\omega}{\partial\xi})^{-1}||_{\ell^1\to\ell^1} \\ &= 3||\Lambda_2(\frac{\partial\hat{\omega}}{\partial\eta})^{-1}T\Lambda_1||_{\ell^1\to\ell^1} \\ &\le 3||\Lambda_2||_{\ell^1\to\ell^1}||(\frac{\partial\hat{\omega}}{\partial\eta})^{-1}||_{\ell^1\to\ell^1}||T||_{\ell^1\to\ell^1}||\Lambda_1||_{\ell^1\to\ell^1} \\ &\le 3\cdot\lambda_{n_3}^3\cdot\frac{1}{409}\cdot 3\cdot\frac{128\pi^3\lambda_{n_3}}{35} \\ &< \pi^3\lambda_{n_3}^4. \end{aligned} \tag{3.26}$$

*Step 3: Checking assumption B of Theorem 5.1.* By (3.7) we know $\bar{\Omega}_j=\lambda_j$,

$$\hat{\Omega}_j=\frac{35}{4}(\sum_{l=1}^{b}G_{n_ln_ln_lj}\xi_l+9\sum_{\substack{l,l'\\ l\neq l'}}G_{n_ln_ln_{l'}j}\xi_l^{\frac{2}{3}}\xi_{l'}^{\frac{1}{3}}+18G_{n_1n_2n_3j}\xi_1^{\frac{1}{3}}\xi_2^{\frac{1}{3}}\xi_3^{\frac{1}{3}}),\quad j\neq n_1,n_2,n_3. \tag{3.27}$$

From (2.6), we get

$$G_{n_ln_ln_lj}=\begin{cases}\frac{7}{4\pi^3\lambda_{n_l}^3\lambda_j}, j=2n_l;\\ \frac{19}{8\pi^3\lambda_{n_l}^3\lambda_j}, j=3n_l;\\ \frac{5}{2\pi^3\lambda_{n_l}^3\lambda_j},\ j\neq n_1,n_2,n_3,2n_l,3n_l,\end{cases} \tag{3.28}$$

$$G_{n_ln_ln_{l'}j}=\begin{cases}\frac{5}{4\pi^3\lambda_{n_l}^2\lambda_{n_{l'}}\lambda_j}, j=2n_l;\\ \frac{1}{4\pi^3\lambda_{n_l}^2\lambda_{n_{l'}}\lambda_j}, j=n_l\pm n_{l'}\ l>l';\\ \frac{13}{8\pi^3\lambda_{n_l}^2\lambda_{n_{l'}}\lambda_j}, j=2n_l\pm n_{l'}\ l>l';\\ \frac{3}{2\pi^3\lambda_{n_l}^2\lambda_{n_{l'}}\lambda_j},\ j\neq n_1,n_2,n_3,n_l\pm n_{l'},n_l\pm n_{l'},2n_l,2n_l\pm n_{l'},\ l>l',\end{cases} \tag{3.29}$$

$$G_{n_1n_2n_3j}=\begin{cases}\frac{3}{4\pi^3\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}\lambda_j}, j=n_2\pm n_1,n_3\pm n_2,n_3\pm n_1,\\ \frac{9}{8\pi^3\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}\lambda_j}, j=n_1+n_2+n_3,n_3+n_2-n_1,n_3-n_2-n_1,\\ \frac{1}{\pi^3\lambda_{n_1}\lambda_{n_2}\lambda_{n_3}\lambda_j},\ j\neq n_1,n_2,n_3,n_l\pm n_{1'},n_1+n_2+n_3,\\ \qquad\qquad n_3+n_2-n_1,n_3-n_2-n_1,l>l'.\end{cases}$$

We can easily see that $\hat{\Omega}$ is a Lipschitz map from $\Pi$ to $\ell_\infty^1$, here $\ell_\infty^p$ the space of all complex sequences with finite norm $|w|_p=\sup_j|w_j||j|^p$, and by calculation we can get

$$|\Omega|_{1,\Pi}^{\mathcal{L}}<\sum_{1\leq l,l',l''\leq 3}\frac{10}{\pi^3\lambda_{n_l}\lambda_{l'}\lambda_{n_{l''}}}\leq\frac{270}{\pi^3\lambda_{n_1}^3}. \tag{3.30}$$

In view of (3.25) (3.26) and (3.30), denoting

$$M=\frac{39375\lambda_{n_3}}{\pi^3\lambda_{n_1}^5}+\frac{270}{\pi^3\lambda_{n_1}^3},\quad L=\pi^3\lambda_{n_3}^4, \tag{3.31}$$

then the assumptions (5.1) in Theorem 5.1 are satisfied. Finally, observing that $\lambda_j=\sqrt{j^2+m}=j+\frac{m}{2j}+O(j^{-3})$ and $\frac{1}{\lambda_j}=\frac{1}{j}+O(j^{-3})$. We know the assumption (5.2) in Theorem 5.1 is satisfied with $\kappa=2$.

*Step 4: Checking assumption C and smallness condition* (5.5) *of Theorem 5.1.* Observing (3.10) for the perturbation $P$, it can be easily checked that $P$ is real analytic in the space coordinates and Lipschitz in the parameters, and for each $\xi\in\Pi$ its hamiltonian vector field $X_P$ is an analytic map from $\mathcal{P}^{a,p}$ to $\mathcal{P}^{a,\bar{p}}$ with $\bar{p}=p+1$. In the following we check the smallness condition (5.5). In view of (3.8), we have

$$|Q|=O(r^4). \tag{3.32}$$

In view of (3.4) and $|\xi|=O(r^{\frac{6}{5}})$, we have

$$|\hat{G}|=O((r^{\frac{1}{5}})^5r^3)=O(r^4), \tag{3.33}$$

$$|\hat{K}| = O((r^{\frac{1}{5}})^{12} r^2) = O(r^{\frac{22}{5}}), \tag{3.34}$$

$$|T| = O((r^{\frac{1}{5}})^{20}) = O(r^4). \tag{3.35}$$

From (3.10), (3.33)-(3.35), we know $|P| = O(r^4)$ and thus

$$|X_P|_{r,D(s,r)\times\Pi} = O(r^2). \tag{3.36}$$

Since $X_P$ is real analytic in $\xi$, we have

$$|X_P|^{\mathcal{L}}_{r,D(s,r)\times\Pi} = O(r^2 r^{-\frac{6}{5}}) = O(r^{\frac{4}{5}}). \tag{3.37}$$

We choose

$$\alpha = r^{\frac{15}{8}}\gamma^{-1}, \tag{3.38}$$

where $\gamma$ is taken from the KAM theorem. It¡¯s obvious that when $r$ is small enough,

$$\epsilon := ||X_P||_{r,D(s,r)\times\Pi} + \frac{\alpha}{M}||X_P||^{\mathcal{L}}_{r,D(s,r)\times\Pi} = O(r^2) \leq \gamma\alpha, \tag{3.39}$$

which is just the smallness condition (5.5). Till now there only remains the assumption (5.6) of Theorem 5.1.

*Step 5: Checking assumption* (5.6) *Theorem 5.1.* For convenience, we introduce

$$\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \Theta,$$

as parameters with $\zeta_1 = \frac{\xi_1^{\frac{1}{3}}}{\lambda_{n_1}}, \zeta_2 = \frac{\xi_2^{\frac{1}{3}}}{\lambda_{n_2}}, \zeta_3 = \frac{\xi_3^{\frac{1}{3}}}{\lambda_{n_3}}$, where

$$\Theta = \{\zeta = (\zeta_1, \zeta_2, \zeta_3) : r^{\frac{2}{5}} \leq \lambda_{n_1}\zeta_1, \lambda_{n_2}\zeta_2, \lambda_{n_3}\zeta_3 \leq \sqrt[3]{\frac{6}{5}} r^{\frac{2}{5}}\}.$$

In view of (5.3), denoting

$$\tilde{\mathcal{R}}_{kl}(\alpha) = \{\zeta \in \Theta : |\langle k, \omega(\zeta)\rangle + \langle l, \Omega(\zeta)\rangle| < \alpha\frac{\langle l\rangle}{A_k}\}, \tag{3.40}$$

then to prove the assumption (5.6) is equivalent to prove

$$|\bigcup_{(k,l)\in\mathcal{X}} \tilde{\mathcal{R}}_{kl}(\hat{\alpha})| \leq c r^{\frac{4}{5}} \alpha^{\frac{\kappa}{3(\kappa+1-\chi/4)}} \tag{3.41}$$

where $\hat{\alpha} = \alpha^{\frac{\kappa+1-\chi}{\kappa+1-\chi/4}}, 0 \leq \chi < 1$, and $c$ is a positive constant. We only need to give the proof of the most difficult case that $l$ has two non-zero components of opposite sign. In this case, rewrite $\tilde{\mathcal{R}}_{kl}(\alpha)$ in (3.35) as

$$\tilde{\mathcal{R}}_{kij}(\alpha) = \{\zeta \in \Theta : |\langle k, \omega(\zeta)\rangle + \Omega_i(\zeta) - \Omega_j(\zeta)| < \alpha\frac{|i-j|}{A_k}\}, \tag{3.42}$$

where $k \in \mathbb{Z}^3$ and $i, j \in \mathbb{N}^+ \setminus \{n_1, n_2, n_3\}, i \neq j$. In view of (5.4), it is sufficient to prove

$$|\bigcup_{\substack{0<|k|<K_*\\0<i+j<L_*}} \tilde{\mathcal{R}}_{kl}(\hat{\alpha})| \leq c r^{\frac{4}{5}} \alpha^{\frac{\kappa}{3(\kappa+1-\chi/4)}}, \tag{3.43}$$

where $K_*, L_*$ are defined in the KAM theorem and here they satisfy

$$K_* = 16LM \leq 64 \times 10^4, \tag{3.44}$$
$$L_* = 36(|\omega|_\Pi + 1)LM/\beta < 10^8 \lambda_{n_3}^7. \tag{3.45}$$

From (3.6) (3.7), we have, for $0 < |k| < K_*, 0 < i + j < L_*$,

$$\begin{aligned}
&\langle k, \omega(\zeta)\rangle + \Omega_i(\zeta) - \Omega_j(\zeta) \\
=&k_1\lambda_{n_1} + k_2\lambda_{n_2} + k_3\lambda_{n_3} + \lambda_i - \lambda_j \\
&+ \frac{35}{128\pi^3}\sum_{l=1}^{3}(35\frac{k_l}{\lambda_{n_l}} + 240\sum_{l'\neq l}\frac{k_{l'}}{\lambda_{n_{l'}}} + 32\pi^3\lambda_{n_l}^3(G_{n_l n_l n_l i} - G_{n_l n_l n_l j}))\zeta_l^3 \\
&+ \frac{315}{16\pi^3}\sum_{\substack{l',l'' \\ l'\neq l''}}(9\frac{k_l}{\lambda_{n_l}} + 12\sum_{l\neq l',l''}\frac{k_l}{\lambda_{n_l}} + 4\pi^3\lambda_{n_{l'}}^2\lambda_{n_{l''}}\lambda_{n_{l'}}(G_{n_{l'} n_{l'} n_{l''} i} - G_{n_{l'} n_{l'} n_{l''} j}))\zeta_{l'}^2\zeta_{l''} \\
&+ \frac{315}{4\pi^3}\sum_{\substack{l,l',l'' \\ l\neq l',l'',l'\neq l''}}^{3}(3\frac{k_l}{\lambda_l} + 2(G_{n_l n_{l'} n_{l''} i} - G_{n_l n_{l'} n_{l''} j}))\zeta_l\zeta_{l'}\zeta_{l''} \\
&+ O(r^{\frac{12}{5}}).
\end{aligned}$$

If for every $0 < |k| < K_*, 0 < i + j < L_*$, at least one of the following 4 inequalities holds:

$$\left|k_1\lambda_{n_1} + k_2\lambda_{n_2} + k_3\lambda_{n_3} + \lambda_i - \lambda_j\right| \geq \frac{1}{4\lambda_{n_3}}, \tag{3.46}$$
$$\left|35\frac{k_l}{\lambda_{n_l}} + 240\sum_{l'\neq l}\frac{k_{l'}}{\lambda_{n_{l'}}} + 32\pi^3\lambda_{n_l}^3(G_{n_l n_l n_l i} - G_{n_l n_l n_l j})\right| \geq \frac{1}{4\lambda_{n_3}}, \tag{3.47}$$

then for $r$ small enough, either $\left|\langle k, \omega(\zeta)\rangle + \Omega_i(\zeta) - \Omega_j(\zeta)\right|$ or

$$\left|\frac{\partial^3}{\partial\zeta_l^3}(\langle k, \omega(\zeta)\rangle + \Omega_i(\zeta) - \Omega_j(\zeta))\right|$$

is bigger than $\frac{1}{8\lambda_{n_3}}$. If $\left|\langle k, \omega(\zeta)\rangle + \Omega_i(\zeta) - \Omega_j(\zeta)\right| > \frac{1}{8\lambda_{n_3}}$, then

$$|\tilde{R}_{kij}(\hat{\alpha})| = 0.$$

If $\left|\frac{\partial^3}{\partial\zeta_l^3}(\langle k, \omega(\zeta)\rangle + \Omega_i(\zeta) - \Omega_j(\zeta))\right| > \frac{1}{8\lambda_{n_3}}$ for some $1 \leq l \leq 3$, by using Lemma 4.5 in the next section and noting that $|k|, i, j$ can be bounded by a positive constant depending only on $\lambda_{n_3}$, we get

$$|\tilde{R}_{kij}(\hat{\alpha})| = 2(2 + 3 + 8\lambda_{n_3})(\hat{\alpha}\frac{|i-j|}{A_k})^{\frac{1}{3}}(\mathrm{diam}\Theta)^2 = O(\hat{\alpha}^{\frac{1}{3}}r^{\frac{4}{5}}). \tag{3.48}$$

Since the number of $(k, i, j)$ satisfying $0 < |k| < K_*, 0 < i + j < L_*$ can be bounded by a positive constant depending only on $\lambda_{n_3}$, we finally get

$$\left|\cup_{\substack{0<|k|<K_* \\ 0<i+j<L_*}}\tilde{R}_{kij}(\hat{\alpha})\right| = O(\hat{\alpha}^{\frac{1}{3}}r^{\frac{4}{5}}) = O(\hat{\alpha}^{\frac{1}{3}}r^{\frac{4}{5}}) = O(\alpha^{\frac{\kappa+1-\chi}{\kappa+1-\chi/4}}r^{\frac{4}{5}}), \tag{3.49}$$

which is less than the right hand of (3.41) by the fact $\chi < 1$. Therefore, till now the only remaining task is to prove that at least one of the 4 inequalities in (3.46) (3.47) holds. Supposing this not true, then we have

$$\left|k_1\lambda_{n_1} + k_2\lambda_{n_2} + k_3\lambda_{n_3} + \lambda_i - \lambda_j\right| < \frac{1}{4\lambda_{n_3}}, \tag{3.50}$$

$$\left|35\frac{k_l}{\lambda_{n_l}} + 240\sum_{l'\neq l}\frac{k_{l'}}{\lambda_{n_{l'}}} + 32\pi^3\lambda_{n_l}^3(G_{n_l n_l n_l i} - G_{n_l n_l n_l j})\right| < \frac{1}{4\lambda_{n_3}},\ l = 1,2,3. \tag{3.51}$$

We will discuss (3.50) (3.51) in three different cases in the following. We also mention that $k \in \mathbb{Z}^3, 0 < |k| < K_*$ and $i, j \in \mathbb{N}^+ \setminus \{n_1, n_2, n_3\}$, $i \neq j$, $0 < i+j < L_*$.

Case 1. Both $i$ and $j$ are not in $\{2n_1, 2n_2, 2n_3, 3n_1, 3n_2, 3n_3\}$. Then (3.51) becomes

$$\left| -205\frac{k_l}{\lambda_{n_l}} + 240\sum_{l'=1}^{3}\frac{k_{l'}}{\lambda_{n_{l'}}} + \frac{80}{\lambda_i} - \frac{80}{\lambda_j}\right| < \frac{1}{4\lambda_{n_3}}, \quad l = 1,2,3. \tag{3.52}$$

Eliminating $\frac{80}{\lambda_i} - \frac{80}{\lambda_j}$, we get

$$205\left|\frac{k_l}{\lambda_{n_l}} - \frac{k_{l'}}{\lambda_{n_{l'}}}\right| < \frac{1}{2\lambda_{n_3}}, \quad 1 \leq l, l' \leq 3. \tag{3.53}$$

As $k = (k_1, k_2, k_3) \neq 0$, we may assume $k_l \neq 0$ for some $1 \leq l \leq b$. Without loss of generality, suppose $k_l > 0$. We claim that $k_{l'} > 0$ for all $1 \leq l' \leq 3$. Otherwise, there exits some $1 \leq l' \leq 3, l' \neq l$ such that $k_{l'} \leq 0$, and then we have

$$205\left|\frac{k_l}{\lambda_{n_l}} - \frac{k_{l'}}{\lambda_{n_{l'}}}\right| \geq 205\frac{k_l}{\lambda_{n_l}} > \frac{4}{\lambda_{n_3}}, \tag{3.54}$$

which contradicts with (3.53). Noting that $\lambda_i - \lambda_j$ and $\frac{80}{\lambda_i} - \frac{80}{\lambda_j} = \frac{80(\lambda_j - \lambda_i)}{\lambda_i\lambda_j}$ have different signs, we discuss in the following two cases:

Subcase 1.1: If $\lambda_i - \lambda_j > 0$, then we have

$$\left|k_1\lambda_{n_1} + k_2\lambda_{n_2} + k_3\lambda_{n_3} + \lambda_i - \lambda_j\right| > k_1\lambda_{n_1} + k_2\lambda_{n_2} + k_3\lambda_{n_3} > \frac{1}{4\lambda_{n_3}}, \tag{3.55}$$

which contradicts with (3.50).

Subcase 1.2: If $\lambda_i - \lambda_j < 0$, then we have

$$\left| -205\frac{k_l}{\lambda_{n_l}} + 240\sum_{l'=1}^{3}\frac{k_{l'}}{\lambda_{n_{l'}}} + \frac{80}{\lambda_i} - \frac{80}{\lambda_j}\right| > 35\frac{k_l}{\lambda_{n_l}} + \sum_{l'\neq l}240\frac{k_{l'}}{\lambda_{n_{l'}}} > \frac{1}{4\lambda_{n_3}}, \tag{3.56}$$

which contradicts with (3.52).

Case 2: Both $i$ and $j$ are in $\{2n_1, 2n_2, 2n_3\}$. Then (3.51) becomes

$$\left| -205\frac{k_l}{\lambda_{n_l}} + 240\sum_{l'=1}^{3}\frac{k_{l'}}{\lambda_{n_{l'}}} + \frac{56}{\lambda_i} - \frac{56}{\lambda_j}\right| < \frac{1}{4\lambda_{n_3}}, \quad l = 1,2,3. \tag{3.57}$$

In the same way as Case 1, we can derive contradictions.

Case 3: Both $i$ and $j$ are in $\{3n_1, 3n_2, 3n_3\}$. Then (3.51) becomes

$$\big| -205\frac{k_l}{\lambda_{n_l}} + 240\sum_{l'=1}^{3}\frac{k_{l'}}{\lambda_{n_{l'}}} + \frac{76}{\lambda_i} - \frac{76}{\lambda_j}\big| < \frac{1}{4\lambda_{n_3}}, \quad l = 1,2,3. \tag{3.58}$$

In the same way as Case 1, we can derive contradictions.

Case 4. $i = 2n_{\bar{l}}$ for some $1 \le \bar{l} \le 3$ and $j = 3n_{\bar{l}'}$ for some $1 \le \bar{l}' \le 3$.

Subcase 4.1. $\bar{l} = \bar{l}'$. Then (3.51) becomes

$$\big| -205\frac{k_l}{\lambda_{n_l}} + 240\sum_{l'=1}^{3}\frac{k_{l'}}{\lambda_{n_{l'}}} + \frac{80}{\lambda_{2n_{\bar{l}}}} - \frac{80}{\lambda_{3n_{\bar{l}}}}\big| < \frac{1}{4\lambda_{n_3}}, \quad 1 \le l \le 3, l \ne \bar{l}, \tag{3.59}$$

$$\big| -205\frac{k_l}{\lambda_{n_l}} + 240\sum_{l'=1}^{3}\frac{k_{l'}}{\lambda_{n_{l'}}} + \frac{56}{\lambda_i} - \frac{76}{\lambda_j}\big| < \frac{1}{4\lambda_{n_3}}, \quad 1 \le l \le 3, l = \bar{l}. \tag{3.60}$$

Eliminating $\frac{80}{\lambda_i} - \frac{80}{\lambda_j}$ by (3.59), we get

$$205\big|\frac{k_{l'}}{\lambda_{n_{l'}}} - \frac{k_{l''}}{\lambda_{n_{l''}}}\big| < \frac{2}{\lambda_{n_3}}, \quad l', l'' \ne \bar{l}. \tag{3.61}$$

By (3.59) and (3.60), we get

$$\big|205\big(\frac{k_{l'}}{\lambda_{n_{l'}}} - \frac{k_{\bar{l}}}{\lambda_{n_{\bar{l}}}}\big) + \frac{24}{\lambda_{2n_{\bar{l}}}} - \frac{4}{\lambda_{3n_{\bar{l}}}}\big| < \frac{1}{2\lambda_{n_3}}, \quad l' \ne \bar{l}. \tag{3.62}$$

If $k_{\bar{l}} \ge 0$, then from (3.62), we get

$$205\frac{k_{l'}}{\lambda_{n_{l'}}} \ge 205\frac{k_{\bar{l}}}{\lambda_{n_{\bar{l}}}} - \frac{24}{\lambda_{2n_{\bar{l}}}} + \frac{4}{\lambda_{3n_{\bar{l}}}} - \frac{1}{2\lambda_{n_3}} > 0, \tag{3.63}$$

which means $k_{l'} > 0$, and from (3.61) we get

$$205\frac{k_{l''}}{\lambda_{n_{l''}}} \ge \frac{205k_{l'}}{\lambda_{n_{l'}}} - \frac{1}{2\lambda_{n_3}} \ge \frac{205}{\lambda_{n_{l'}}} - \frac{1}{2\lambda_{n_3}} > 0, \tag{3.64}$$

which means $k_{l''} > 0$. Otherwise if $k_{\bar{l}} < 0$, then from (3.62), we get

$$205\frac{k_{l'}}{\lambda_{n_{l'}}} \le 205\frac{k_{\bar{l}}}{\lambda_{n_{\bar{l}}}} - \frac{24}{\lambda_{2n_{\bar{l}}}} + \frac{4}{\lambda_{3n_{\bar{l}}}} + \frac{1}{2\lambda_{n_3}} < 0, \tag{3.65}$$

which means $k_{l'} < 0$, and from (3.61) we get

$$205\frac{k_{l''}}{\lambda_{n_{l''}}} \le \frac{205k_{l'}}{\lambda_{n_{l'}}} + \frac{1}{2\lambda_{n_3}} \le -\frac{205}{\lambda_{n_{l'}}} + \frac{1}{2\lambda_{n_3}} < 0, \tag{3.66}$$

which means $k_{l''} < 0$. In the same way as Subcase 1.2, we can derive contradictions.

Subcase 4.2. $\bar{l} \ne \bar{l}'$. Then (3.51) becomes

$$\big| -205\frac{k_l}{\lambda_{n_l}} + 240\sum_{l'=1}^{3}\frac{k_{l'}}{\lambda_{n_{l'}}} + \frac{80}{\lambda_{2n_{\bar{l}}}} - \frac{80}{\lambda_{3n_{\bar{l}'}}}\big| < \frac{1}{4\lambda_{n_3}}, \quad 1 \le l \le 3, l \ne \bar{l}, \bar{l}', \tag{3.67}$$

$$|-205\frac{k_{\bar{l}}}{\lambda_{n_{\bar{l}}}}+240\sum_{l'=1}^{3}\frac{k_{l'}}{\lambda_{n_{l'}}}+\frac{56}{\lambda_{2n_{\bar{l}}}}-\frac{80}{\lambda_{3n_{\bar{l}'}}}|<\frac{1}{4\lambda_{n_3}}, \tag{3.68}$$

$$|-205\frac{k_{\bar{l}'}}{\lambda_{n_{\bar{l}'}}}+240\sum_{l'=1}^{3}\frac{k_{l'}}{\lambda_{n_{l'}}}+\frac{80}{\lambda_{2n_{\bar{l}}}}-\frac{76}{\lambda_{3n_{\bar{l}'}}}|<\frac{1}{4\lambda_{n_3}}. \tag{3.69}$$

Eliminating $\frac{80}{\lambda_{3n_{\bar{l}'}}}$ by (3.67) (3.68), we get

$$|205(\frac{k_l}{\lambda_{n_l}}-\frac{k_{\bar{l}}}{\lambda_{n_{\bar{l}}}})-\frac{24}{\lambda_{2n_{\bar{l}}}}|<\frac{1}{2\lambda_{n_3}}. \tag{3.70}$$

Eliminating $\frac{80}{\lambda_{2n_{\bar{l}}}}$ by (3.67) (3.69), we get

$$|205(\frac{k_l}{\lambda_{n_l}}-\frac{k_{\bar{l}'}}{\lambda_{n_{\bar{l}'}}})+\frac{4}{\lambda_{3n_{\bar{l}'}}}|<\frac{1}{2\lambda_{n_3}}. \tag{3.71}$$

By (3.68) and (3.69), we get

$$|205(\frac{k_{\bar{l}'}}{\lambda_{n_{\bar{l}'}}}-\frac{k_{\bar{l}}}{\lambda_{n_{\bar{l}}}})-\frac{24}{\lambda_{2n_{\bar{l}}}}-\frac{4}{\lambda_{3n_{\bar{l}'}}}|<\frac{1}{2\lambda_{n_3}}. \tag{3.72}$$

If $k_{\bar{l}}>0$, then from (3.72), we get

$$205\frac{k_{\bar{l}'}}{\lambda_{n_{\bar{l}'}}}>205\frac{k_{\bar{l}}}{\lambda_{n_{\bar{l}}}}+\frac{24}{\lambda_{2n_{\bar{l}}}}+\frac{4}{\lambda_{3n_{\bar{l}'}}}-\frac{1}{2\lambda_{n_3}}>0, \tag{3.73}$$

which means $k_{\bar{l}'}>0$, and from (3.70), we get

$$205\frac{k_l}{\lambda_{n_l}}>205\frac{k_{\bar{l}}}{\lambda_{n_{\bar{l}}}}+\frac{24}{\lambda_{2n_{\bar{l}}}}-\frac{1}{2\lambda_{n_3}}>0, \tag{3.74}$$

which means $k_l>0$. Otherwise if $k_{\bar{l}}<0$, then from (3.70), we get

$$205\frac{k_{\bar{l}'}}{\lambda_{n_{\bar{l}'}}}<205\frac{k_{\bar{l}}}{\lambda_{n_{\bar{l}}}}+\frac{24}{\lambda_{2n_{\bar{l}}}}+\frac{1}{2\lambda_{n_3}}<0, \tag{3.75}$$

which means $k_{\bar{l}'}<0$, and from (3.71), we get

$$205\frac{k_l}{\lambda_{n_l}}<205\frac{k_{\bar{l}'}}{\lambda_{n_{\bar{l}'}}}-\frac{4}{\lambda_{3n_{\bar{l}'}}}+\frac{1}{2\lambda_{n_3}}<0, \tag{3.76}$$

which means $k_{\bar{l}'}<0$. In the same way as Subcase 1.1 and Subcase 1.2, we can derive contradictions.

Till now, all the assumptions in the KAM theorem have been checked. Taking $\chi=\frac{4}{5}$, in view of (5.7), the measure of the excluding set of parameters is

$$O(r^{\frac{12}{5}}\alpha^{\frac{\kappa}{\kappa+1-\chi/4}})=O(r^{\frac{12}{5}}r^{\frac{75}{56}}),$$

which is of higher order than $O(r^{\frac{18}{5}})$. This means that, when $r$ is small enough, the rotational tori persist for most of $\xi\in\Pi$. Thus Theorem 1.1 follows from Theorem 5.1 in Appendix.

# 4. Technical Lemmas

In this section, the first four lemmas are estimates of the function

$$f(t)=\frac{m}{\sqrt{t^2+m}+t}, \tag{4.1}$$

which are used to perform Birkhoff normal form in Section 2.

**Lemma 4.1.** *For $f(t)$ in (4.1) and $0<m\le\frac{1}{9}$, we have*

$$f(t)\le\lambda f(\lambda t)\le\frac{\sqrt{10}+3}{6}f(t),\quad \lambda\ge 1,\ t\ge 1. \tag{4.2}$$

**Proof.** This is Lemma 4.1 in [13] with $0<m\le\frac{1}{9}$. □

**Lemma 4.2.** *For $f(t)$ in (4.1) and $0<m\le\frac{1}{9}$, we have*

$$|f(a)-f(n)|\ge\frac{m}{2((n+1)^2+m)},\quad a,n\in\mathbb{N}^+, a\ne n. \tag{4.3}$$

**Proof.** This is Lemma 4.2 in [13] with $0<m\le\frac{1}{9}$. □

**Lemma 4.3.** *For $f(t)$ in (4.1) and $0<m\le\frac{1}{9}$, we have*

$$|f(a)-lf(n)|\ge\frac{m^2}{2n^3},\quad a,n\in\mathbb{N}^+,\quad l=2,3,4,5. \tag{4.4}$$

**Proof.** For $l=2,3$, this is Lemma 4.3 and Lemma 4.4 in [13] with $0<m\le\frac{1}{9}$ respectively. We only need to prove it for $l=4$ and $l=5$.

Note that

$$|f(a)-4f(n)|=m\left|\frac{1}{\sqrt{a^2+m}+a}-\frac{1}{\sqrt{(\frac{n}{4})^2+\frac{m}{16}}+\frac{n}{4}}\right|. \tag{4.5}$$

If $n=4k$, then for all $a\in\mathbb{N}^+$,

$$\begin{aligned}|f(a)-4f(n)|&\ge|f(k)-4f(4k)|\\&=\frac{m}{\sqrt{k^2+\frac{m}{16}}+k}-\frac{m}{\sqrt{k^2+m}+k}\\&=\frac{15m^2}{16}\frac{1}{(\sqrt{k^2+\frac{m}{16}}+k)(\sqrt{k^2+m}+k)(\sqrt{k^2+\frac{m}{16}}+\sqrt{k^2+m})}\\&>\frac{m^2}{64k^3}\\&=\frac{m^2}{n^3}.\end{aligned} \tag{4.6}$$

If $n=4k+1$, then for all $a\in\mathbb{N}^+$,

$$\begin{aligned}|f(a)-4f(n)|&\ge|f(k)-4f(4k+1)|\\&=\frac{m}{\sqrt{k^2+m}+k}-\frac{m}{\sqrt{(k+\frac{1}{4})^2+\frac{m}{16}}+(k+\frac{1}{4})}\end{aligned}$$

$$\geq \frac{m}{4(\sqrt{(k+\frac{1}{4})^2+\frac{m}{16}}+k+\frac{1}{4})(\sqrt{k^2+m}+k)}$$
$$\geq \frac{m}{64(k+\frac{1}{4})^2}$$
$$= \frac{m}{4n^2}. \tag{4.7}$$

If $n = 4k+2$, then for all $a \in \mathbb{N}^+$,

$$\begin{aligned}|f(a)-4f(n)| &\geq |f(k+1)-4f(4k+2)| \\ &= \frac{m}{\sqrt{(k+\frac{1}{2})^2+\frac{m}{16}}+(k+\frac{1}{2})} - \frac{m}{\sqrt{(k+1)^2+m}+k+1} \\ &\geq \frac{m}{4(\sqrt{(k+\frac{1}{2})^2+\frac{m}{16}}+k+\frac{1}{2})(\sqrt{(k+1)^2+m}+k+1)} \\ &\geq \frac{m}{64(k+\frac{1}{2})^2} \\ &= \frac{m}{4n^2}.\end{aligned} \tag{4.8}$$

If $n = 4k+3$, then for all $a \in \mathbb{N}^+$,

$$\begin{aligned}|f(a)-4f(n)| &\geq |f(k+1)-4f(4k+3)| \\ &= \frac{m}{\sqrt{(k+\frac{3}{4})^2+\frac{m}{16}}+(k+\frac{3}{4})} - \frac{m}{\sqrt{(k+1)^2+m}+k+1} \\ &\geq \frac{m}{4(\sqrt{(k+\frac{3}{4})^2+\frac{m}{16}}+k+\frac{3}{4})(\sqrt{(k+1)^2+m}+k+1)} \\ &\geq \frac{m}{64(k+\frac{3}{4})^2} \\ &= \frac{m}{4n^2}.\end{aligned} \tag{4.9}$$

Now from (4.6), (4.7), (4.8) and (4.9), we obtain (4.4). For $l = 5$, the proof is similar, we omit it here. □

**Lemma 4.4.** *Suppose that $A = (a_{ij})_{n\times n}$ is a matrix of order $n$ which satisfies*

$$\tau_{ij} = |a_{jj}| - \sum_{i\neq j} |a_{ij}| > 0, \quad j = 1, \cdots, n \tag{4.10}$$

*then*

$$||A^{-1}||_{\ell^1\to\ell^1} \leq (\min_{1\leq j\leq n} \tau_j)^{-1}. \tag{4.11}$$

**Proof.** This is Lemma 4.5 in [11]. □

The next lemma is a special case of Lemma 2.1 in [21], which is used to estimate the measure of parameters in Section 3.

**Lemma 4.5.** *Suppose that $g(x)$ is a 3-th differentiable function on the closure $\bar{I}$ of $I$, where $I \subset R$ is an interval. Let $I_h = \{x : |g(x)| < h, x \in I\}, h > 0$. If on $I$, $|\frac{d^3 g(x)}{d\xi^3}| \geq d \geq 0$, where $d$ is a constant, then $|I_h| \leq 2(2+3+\cdots+m+d^{-1})h^{\frac{1}{3}}$.*

# 5. Appendix: A KAM Theorem

Consider small perturbations of a family of linear integrable hamiltonians

$$N = \sum_{1\le j\le n} \omega_j(\xi) y_j + \frac{1}{2}\sum_{j\ge 1} \Omega_j(\xi)(u_j^2+v_j^2), \quad 1\le n<\infty$$

on a phase space

$$\mathcal{P}^{a,p} = \mathbb{T}^n \times \mathbb{R}^n \times \ell^{a,p} \times \ell^{a,p} \ni (x,y,u,v)$$

with symplectic structure $\sum_{1\le j\le n} \mathrm{d}x_j \wedge \mathrm{d}y_j + \sum_{j\ge 1} \mathrm{d}u_j \wedge \mathrm{d}v_j$, The frequencies $\omega = (\omega_1, \cdots, \omega_n)$ and $\Omega = (\Omega_1, \Omega_2, \cdots)$ depend on $n$-parameters $\xi \in \Pi \subset \mathbb{R}^n$, with $\Pi$ a closed bounded set of positive Lebesgue measure. For each $\xi$ there is an invariant $n$-torus $\mathcal{T}_0 = \mathbb{T}^n \times \{0,0,0\}$ with frequencies $\omega(\xi)$. In its normal space described by the $uv$-coordinates the origin is an elliptic fixed point with characteristic frequencies $\Omega(\xi)$. The aim is to prove the persistence of a large portion of this family linearly stable rotational tori under small perturbations $H = N + P$ of $N$. To this end the following assumptions are made.

*Assumption A: Nondegeneracy.* The map $\xi \mapsto \omega(\xi)$ between $\Pi$ and its image is a homeomorphism which is Lipschitz continuous in both directions. Moreover,

$$\langle l, \Omega(\xi)\rangle \neq 0 \quad \text{on } \Pi$$

for all integer vectors $l \in \mathbb{Z}^\infty$ with $1 \le |l| \le 2$.

*Assumption B: Spectral Asymptotics.* There exists $\delta < 0$ such that

$$\Omega_j(\xi) = j + \cdots + O(j^\delta),$$

where the dots stand for fixed lower order terms in $j$, allowing also negative exponents. More precisely, there exists a fixed, parameter-independent sequence $\bar\Omega$ with $\bar\Omega_j = j + \cdots$ such that $\hat\Omega_j = \Omega_j - \bar\Omega_j$ give rise to a Lipschitz map

$$\hat\Omega : \Pi \to \ell_\infty^{-\delta}$$

with $\ell_\infty^p$ the space of all complex sequences with finite norm $|w|_p = \sup_j |w_j| j^p$.

*Assumption C: Regularity.* The perturbation $P$ is real analytic in the space coordinates and Lipschitz in the parameters, and for each $\xi \in \Pi$ its hamiltonian vector field $X_P = (P_y, -P_x, P_v, -P_u)^T$ defines near $\mathcal{T}_0$ a real analytic map

$$X_P : \mathcal{P}^{a,p} \to \mathcal{P}^{a,\bar p}, \quad \bar p > p.$$

To make this more precise we introduce complex neighbourhoods

$$D(s,r) : |\mathrm{Im} x| < s,\ |y| < r^2,\ ||u||_{a,p} + ||v||_{a,p} < r$$

of $\mathcal{T}_0$ and weighted norms

$$|(x,y,u,v)|_r = |(x,y,u,v)|_{\bar p, r} = |x| + \frac{1}{r^2}|y| + \frac{1}{r}||u||_{a,\bar p} + \frac{1}{r}||v||_{a,\bar p},$$

where $|\cdot|$ is the sup-norm for complex vectors. Then we assume that the hamiltonian vector field $X_P$ is real analytic on $D(s,r)$ for some $s$ and $r$ uniformly in $\xi$ with finite norm $|X_P|_{r,D(s,r)} = \sup_{D(s,r)} |X_P|_r$, and that the same holds for its Lipschitz semi-norm

$$|X_P|_r^{\mathcal{L}} = \sup_{\xi \neq \zeta} \frac{|\Delta_{\xi\zeta} X_P|_r}{|\xi - \zeta|},$$

where $\Delta_{\xi\zeta} X_P = X_P(\cdot, \xi) - X_P(\cdot, \zeta)$, and where the supremum is taken over $\Pi$.

To state the KAM theorem we also assume that

$$|\omega|_{\Pi}^{\mathcal{L}} + |\hat{\Omega}|_{-\delta,\Pi}^{\mathcal{L}} \leq M < \infty, \quad |\omega^{-1}|_{\omega(\Pi)}^{\mathcal{L}} \leq L < \infty, \tag{5.1}$$

where the Lipschitz semi-norms are defined analogously to $|X_P|_r^{\mathcal{L}}$. Let $\kappa > 0$ be the largest exponent such that

$$\frac{\Omega_i - \Omega_j}{i - j} = 1 + O(j^{-\kappa}), \quad i > j, \tag{5.2}$$

uniformly on $\Pi$. Without loss of generality, we can assume that $-\delta \leq \kappa$ by increasing $\delta$ if necessary. Moreover, we introduce

$$\mathcal{R}_{kl}(\alpha) = \{\xi \in \Pi : |\langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle| < \alpha \frac{\langle l \rangle}{A_k}\}, \tag{5.3}$$

where $\langle l \rangle = \max(1, |\sum j l_j|)$, $A_k = 1 + |k|^\tau$, $\tau \geq (n+3)\frac{\delta - 1}{\delta}$. Finally let

$$\mathcal{X} = \{(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty : 0 < |k| < K_*, 0 < |l|_\sigma < L_*\}, \tag{5.4}$$

where $K_* = 16LM$, $\sigma = \min(1, -\delta)$, $|l|_\sigma = \sum |l_j| j^\sigma$, $L_* = 36(|\omega|_\Pi + 1)LM/\beta$ with $\beta$ the largest positive constant such that $|\langle l, \Omega \rangle| \geq \frac{27}{2}\beta\langle l \rangle$ for every $1 \leq |l| \leq 2$.

**Theorem 5.1.** *Suppose $H = N + P$ satisfies assumptions A, B, C, and*

$$\epsilon = |X_P|_{r,D(s,r)\times\Pi} + \frac{\alpha}{M}|X_P|_{r,D(s,r)\times\Pi}^{\mathcal{L}} \leq \gamma\alpha, \tag{5.5}$$

*where $0 < \alpha \leq 1$ is another parameter, and $\gamma$ depends on $n, \tau, s$. Then there exists a Cantor set $\Pi_\alpha \subset \Pi$, a Lipschitz continuous family of torus embeddings $\Phi : \mathbb{T}^n \times \Pi_\alpha \to \mathcal{P}^{a,\bar{p}}$, and a Lipschitz continuous map $\tilde{\omega} : \Pi_\alpha \to \mathbb{R}^n$, such that for each $\xi \in \Pi_\alpha$ the map $\Phi$ restricted to $\mathbb{T}^n \times \{\xi\}$ is a real analytic embedding of a rotational torus with frequencies $\tilde{\omega}(\xi)$ for the hamiltonian $H$ at $\xi$.*

*Each embedding is real analytic on $|Imx| < s/2$, and*

$$|\Phi - \Phi_0|_r + \frac{\alpha}{M}|\Phi - \Phi_0|_r^{\mathcal{L}} \leq \frac{c\epsilon}{\alpha},$$
$$|\tilde{\omega} - \omega| + \frac{\alpha}{M}|\tilde{\omega} - \omega|^{\mathcal{L}} \leq c\epsilon,$$

*uniformly on that domain and $\Pi_\alpha$, where $\Pi_0 : \mathbb{T}^n \times \Pi \to \mathcal{T}_0$ is the trivial embedding, and $c \leq \gamma^{-1}$ depends on the same parameter as $\gamma$.*

*Moreover, denoting $\hat{\alpha} = \alpha^{1-3w}$, $w = \frac{\chi}{4\kappa + 4 - \chi}$ with $\chi$ any fixed number in $0 \leq \chi < \min(\bar{p} - p, 1)$, then if*

$$|\bigcup_{(k,l)\in\mathcal{X}} \mathcal{R}_{kl}(\hat{\alpha})| \leq c_1 \rho^{n-1} \alpha^{\frac{\kappa}{\kappa + 1 - \chi/4}} \tag{5.6}$$

*for all sufficiently small $\alpha$, where $\rho = \mathrm{diam}\Pi$, and the constant $c_1$ depends on $\chi$ and $\bar{p} - p$, then we have*

$$|\Pi \setminus \Pi_\alpha| \leq c_2 \rho^{n-1} \alpha^{\frac{\kappa}{\kappa + 1 - \chi/4}}, \tag{5.7}$$

*where the constant $c_2$ also depends on $\chi$ and $\bar{p} - p$.*

This theorem is copied from [12], which is a combination of Theorem A and Theorem D in [19] with some modifications.

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