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MULTIPLE SOLUTIONS FOR A KIRCHHOFF-TYPE FRACTIONAL COUPLED PROBLEM WITH P-LAPLACIAN

Yi Wang[1], Lixin Tian[1,2,†] and Minjie Dong[3]

Abstract In this paper, we look at a class of two-parameters coupled Kirchhoff-type fractional differential equations. Two differentiated methods are used to prove the existence of two solutions to the equation. The fundamental difference between the two methods is that the first provides asymptotic conditions for the non-linear terms on the right-hand side of the equation, while the second provides algebraic conditions; both methods combine substantial A-R conditions.

Keywords Kirchhoff-type fractional equation, p-Laplacian operator, variational methods, critical point theory.

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1. Introduction

The following coupled Kirchhoff-type fractional differential problem with p-Laplacian is investigated in this paper.

$$\begin{cases} \quad \mathcal{A}(u(t))\Big(\,{}_tD_T^{\alpha}(\frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t)\,{}_0D_t^{\alpha}u(t))) + \ell_1(t)\phi_p(u(t))\Big) \\ = \lambda f_u(t,u(t),v(t)) + \mu g_u(t,u(t),v(t)), \\ \quad \mathcal{B}(v(t))\Big(\,{}_tD_T^{\beta}(\frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)\,{}_0D_t^{\beta}v(t))) + \ell_2(t)\phi_p(v(t))\Big) \\ = \lambda f_v(t,u(t),v(t)) + \mu g_v(t,u(t),v(t)), \\ u(0) = u(T) = 0, \quad v(0) = v(T) = 0, \end{cases} \tag{1.1}$$

where

$$\mathcal{A}(u(t)) = \Big(\hat{a} + \hat{b}\int_0^T \hbar_1(t)|{}_0D_t^{\alpha}u(t)|^p + \ell_1(t)|u(t)|^p dt\Big)^{p-1},$$

$$\mathcal{B}(v(t)) = \Big(\hat{c} + \hat{d}\int_0^T \hbar_2(t)|{}_0D_t^{\beta}v(t)|^p + \ell_2(t)|v(t)|^p dt\Big)^{p-1},$$

†The corresponding author.
Email: wangyistum@163.com(Y. Wang), tianlx@ujs.edu.cn(L. Tian), dongminjiepoppy@126.com(M. Dong)
[1]School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China
[2]School of Mathematical Sciences, Jiangsu University, Zhenjiang 212013, China
[3]School of Physical and Mathematical Sciences, Nanjing Tech University, Nanjing 211816, China

$\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d} > 0$ and $p > 1$ are constants, ${}_0D_t^{\alpha}$, ${}_0D_t^{\beta}$ and ${}_tD_T^{\alpha}$, ${}_tD_T^{\beta}$ are the left and right Riemann-Liouville fractional derivatives of order α, $\beta \in (\frac{1}{p}, 1]$, respectively. $\phi_p(s) = |s|^{p-2}s(s \neq 0)$, $\phi_p(0) = 0$, $\hbar_1$, $\hbar_2 \in L^{\infty}([0,T], R)$ with $\hbar_1^0 = \operatorname{ess\,inf}_{[0,T]} \hbar_1(t) > 0$, $\hbar_1^1 = \operatorname{ess\,sup}_{[0,T]} \hbar_1(t)$, $\hbar_2^0 = \operatorname{ess\,inf}_{[0,T]} \hbar_2(t) > 0$, $\hbar_2^1 = \operatorname{ess\,sup}_{[0,T]} \hbar_2(t)$, ℓ_1, $\ell_2 \in C([0,T], R)$ with $\ell_1^0 = \operatorname{ess\,inf}_{[0,T]} \ell_1(t) > 0$, $\ell_1^1 = \operatorname{ess\,sup}_{[0,T]} \ell_1(t)$, $\ell_2^0 = \operatorname{ess\,inf}_{[0,T]} \ell_2(t) > 0$, $\ell_2^1 = \operatorname{ess\,sup}_{[0,T]} \ell_2(t)$, $f(t,u(t),v(t))$, $g(t,u(t),v(t)) : [0,T] \times R^2$ are C^1 functions, f_s and g_s denote the paitial derivatives of f and g with respect to s, λ, μ are two positive real parameters.

The variational method and the critical point theorem have yielded substantial results in the study of the existence and multiplicity of solutions to fractional order differential equations in recent years. Of these, two main types of fractional order differential equations have been extensively studied. One type is called the fractional advection dispersion equation, which was developed thanks to the modelling of contaminant transport in groundwater by Fix and Roop in [4]. We can turn to the literature for additional investigations of this type of problems, such as [6, 15, 17, 18, 24]. In [6], Jiao and Zhou originally demonstrated that critical point theory is a useful tool for investigating the existence of solutions to the fractional advection dispersion equation below.

$$\begin{cases} \dfrac{d}{dt}\Big(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\Big) + \nabla F(t,u(t)) = 0, \quad a.e.\ t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \le \beta < 1$, respectively, $F : [0,T] \times R^N \to R$ is a given function and $\nabla F(t,x)$ is the gradient of F at x. A differential equation with mixed fractional order derivatives is another type; see, for example, [5, 8, 9, 13, 26].

Unlike the two types of fractional order differential equations listed above, the Kirchhoff-type fractional order differential equations discussed in this study have received less attention to my knowledge (see [2, 3, 10, 12, 19] and references therein). In these articles, just certain asymptotic conditions for the non-linear terms on the right-hand side of the equation are given to derive the existence and multiplicity results for the equation's solutions, ignoring the geometric conditions for the non-linear terms, which are addressed in this paper. For instance, Chen et al. deduced in [3] that the following equation has at least one non-trivial weak solution when $f(t,x)$ is (p^2-1)-superlinear or (p^2-1)-sublinear in x at infinity and possesses infinitely many nontrivial weak solutions when $f(t,x)$ is (p^2-1)-sublinear in x at infinity.

$$\begin{cases} \Big(a + b\displaystyle\int_0^T |{}_0D_t^{\alpha}u(t)|^p dt\Big)^{p-1} {}_tD_T^{\alpha}\phi_p({}_0D_t^{\alpha}u(t)) = f(t,u(t)), \quad t \in (0,T), \\ u(0) = u(T) = 0, \end{cases} \tag{1.3}$$

where a, $b > 0$ and $p > 1$ are constants, ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives of order $\alpha \in (\frac{1}{p}, 1]$, respectively, and $\phi_p : R \to R$ is p-Laplacian defined by $\phi_p(s) = |s|^{p-2}s(s \neq 0)$, $\phi_p(0) = 0$, and $f \in C([0,T] \times R, R)$. In contrast to equation (1.3), we investigate a new complex equation (1.1) and explore the case of two equations with the non-linear term including two perturbation parameters, as well as the nonlocal terms $\mathcal{A}(u(t))$, $\mathcal{B}(v(t))$ on the left-hand side of the equation. The variational structure of equation (1.1) becomes more complex and difficult to investigate as a result of these factors. As a result, our work complements and improves Chen et al. [3] and Kang et al. [10]. Specifically, when $\hat{a} = \hat{c} = 1$, $\hat{b} = \hat{d} = \mu = 0$, $\ell_1(t) = \ell_2(t) = \widetilde{\ell}$ are constant functions then equation (1.1) is equivalent to the following equation

$$\begin{cases} {}_tD_T^{\alpha}\Big(\dfrac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t)_0D_t^{\alpha}u(t))\Big) + \widetilde{\ell}\phi_p(u(t)) = \lambda f_u(t,u(t),v(t)), \\ {}_tD_T^{\beta}\Big(\dfrac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^{\beta}v(t))\Big) + \widetilde{\ell}\phi_p(v(t)) = \lambda f_v(t,u(t),v(t)), \\ u(0) = u(T) = 0, \quad v(0) = v(T) = 0. \end{cases} \tag{1.4}$$

In [13], using a three-critical point theorem, Li et al. discovered that there are at least three weak solutions for the type of problem mentioned above.

The idea behind this article is as follows. In Section2, some important definitions and lemmas are given in this section. In Section 3, we conclude that there are at least two different weak solutions to the equation (1.1) by combining Lemma 2.5 and Lemma 2.6, rather than using the traditional mountain pass theorem as in [3, 6, 10, 19, 26], and then Theorem 2.1 is used to show that the problem has at least two non-trivial solutions based on geometric considerations. Finally, we bring the paper to a close in Section 4.

2. Preliminaries and lemmas

In this section we give some important definitions and lemmas to facilitate the discussion in subsequent articles, and we also establish the problem (1.1) function space and variational structure.

Definition 2.1 ([11, 20]). Let function $u(t)$ be defined on $[a, b]$. The left and right Riemann-Liouville fractional derivatives with order $0 < \hat{\gamma} \le 1$, respectively are defined by

$$ {}_aD_t^{\hat{\gamma}}u(t) = \frac{d}{dt}\,{}_aD_t^{\hat{\gamma}-1}u(t) = \frac{1}{\Gamma(1-\hat{\gamma})}\frac{d}{dt}\int_a^t (t-s)^{-\hat{\gamma}}u(s)ds,$$
$$ {}_tD_b^{\hat{\gamma}}u(t) = -\frac{d}{dt}\,{}_tD_b^{\hat{\gamma}-1}u(t) = -\frac{1}{\Gamma(1-\hat{\gamma})}\frac{d}{dt}\int_t^b (s-t)^{-\hat{\gamma}}u(s)ds.$$

Definition 2.2 ([11, 20]). Let $0 < \hat{\gamma} \le 1$, and $u \in AC([a, b])$. Then, the left and right Caputo fractional derivatives with order $\hat{\gamma}$, respectively are defined by

$$ {}_a^cD_t^{\hat{\gamma}}u(t) = \ {}_aD_t^{\hat{\gamma}-1}u'(t) = \frac{1}{\Gamma(1-\hat{\gamma})}\int_a^t (t-s)^{-\hat{\gamma}}u'(s)ds,$$
$$ {}_t^cD_b^{\hat{\gamma}}u(t) = -\ {}_tD_b^{\hat{\gamma}-1}u'(t) = -\frac{1}{\Gamma(1-\hat{\gamma})}\int_t^b (s-t)^{-\hat{\gamma}}u'(s)ds.$$

Lemma 2.1 ([11, 22]). *Let $\hat{\gamma} > 0$, $p \ge 1$, $q \ge 1$, $\frac{1}{p} + \frac{1}{q} \le 1 + \hat{\gamma}$ or $p \ne 1$, $q \ne 1$, $\frac{1}{p} + \frac{1}{q} = 1 + \hat{\gamma}$. If $u \in L^p([a, b], R)$, $v \in L^q([a, b], R)$, then*

$$\int_a^b ({}_aD_t^{-\hat{\gamma}}u(t))v(t)dt = \int_a^b ({}_tD_b^{-\hat{\gamma}}v(t))u(t)dt.$$

Let $C_0^\infty([0,T],R^N)$ be the set of all functions $u \in C^\infty([0,T],R^N)$ with $u(0) = u(T) = 0$ and the norm $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$, $\|u\|_{L^p} = (\int_0^T |u(t)|^p dt)^{\frac{1}{p}}$.

In the following we establish the function space and variational structure of problem (1.1).

Definition 2.3. Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $\hat{E}_0^\alpha = \overline{C_0^\infty([0,T],R)}$ with the weighted norm

$$\|u\|_{\hat{E}_0^\alpha} = \Big(\int_0^T \hbar_1(t)|_0D_t^\alpha u(t)|^p dt + \int_0^T \ell_1(t)|u(t)|^p dt\Big)^{\frac{1}{p}}. \tag{2.1}$$

Remark 2.1 ([24]). For $u \in \hat{E}_0^\alpha$, we have u, ${}_0D_t^\alpha u \in L^p([0,T],R)$. It is well know that the space $\hat{E}_0^\alpha$ is a reflexive and separable Banach space.

Lemma 2.2 ([7]). *Let $0 < \alpha \leq 1$, $1 < p < \infty$. For any $u \in \hat{E}_0^\alpha$, we have*

$$\|u\|_{L^p} \leq \widetilde{\Lambda}_1 \|u\|_{\hat{E}_0^\alpha}, \tag{2.2}$$

moreover, if $\alpha > \frac{1}{p}$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \widetilde{\Lambda}_1^* \|u\|_{\hat{E}_0^\alpha}, \tag{2.3}$$

where $\widetilde{\Lambda}_1 = \frac{T^\alpha}{\Gamma(\alpha+1)(\min\{\hbar_1^0,\ell_1^0\})^{\frac{1}{p}}}$, $\widetilde{\Lambda}_1^* = \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}(\min\{\hbar_1^0,\ell_1^0\})^{\frac{1}{p}}}$.

Analogous to the definition and properties of $\hat{E}_0^\alpha$, we define the fractional derivative space $\hat{E}_0^\beta = \overline{C_0^\infty([0,T],R)}$ with the weighted norm

$$\|v\|_{\hat{E}_0^\beta} = \Big(\int_0^T \hbar_2(t)|_0D_t^\beta v(t)|^p dt + \int_0^T \ell_2(t)|v(t)|^p dt\Big)^{\frac{1}{p}}. \tag{2.4}$$

Similar with (2.2) and (2.3), we have

$$\|v\|_{L^p} \leq \widetilde{\Lambda}_2 \|v\|_{\hat{E}_0^\beta}, \tag{2.5}$$

moreover, if $\beta > \frac{1}{p}$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|v\|_\infty \leq \widetilde{\Lambda}_2^* \|v\|_{\hat{E}_0^\beta}, \tag{2.6}$$

where $\widetilde{\Lambda}_2 = \frac{T^\beta}{\Gamma(\beta+1)(\min\{\hbar_2^0,\ell_2^0\})^{\frac{1}{p}}}$, $\widetilde{\Lambda}_2^* = \frac{T^{\beta-\frac{1}{p}}}{\Gamma(\beta)((\beta-1)q+1)^{\frac{1}{q}}(\min\{\hbar_2^0,\ell_2^0\})^{\frac{1}{p}}}$.

Now, for any $u \in \hat{E}_0^\alpha$, $v \in \hat{E}_0^\beta$, we define a space $\hat{E}_\alpha^\beta = \hat{E}_0^\alpha \times \hat{E}_0^\beta$ with the norm

$$\|(u,v)\|_{\hat{E}_\alpha^\beta} = (\|u\|_{\hat{E}_0^\alpha}^p + \|v\|_{\hat{E}_0^\beta}^p)^{\frac{1}{p}}, \ \forall (u,v) \in \hat{E}_\alpha^\beta.$$

Define $\|(u,v)\|_\infty = \max_{t\in[0,T]} |u(t)| + \max_{t\in[0,T]} |v(t)|$. When α, $\beta > \frac{1}{p}$, we have

$$\begin{aligned}\|(u,v)\|_\infty \leq& \widetilde{\Lambda}_1^* \|u\|_{\hat{E}_0^\alpha} + \widetilde{\Lambda}_2^* \|v\|_{\hat{E}_0^\beta} \leq \widetilde{\Lambda}^* (\|u\|_{\hat{E}_0^\alpha} + \|v\|_{\hat{E}_0^\beta}) \\ \leq& 2^{\frac{p-1}{p}} \widetilde{\Lambda}^* \|(u,v)\|_{\hat{E}_\alpha^\beta},\end{aligned} \tag{2.7}$$

where $\widetilde{\Lambda}^* = \max\{\widetilde{\Lambda}_1^*, \widetilde{\Lambda}_2^*\}$.

Lemma 2.3 ([14]). *For $\alpha, \beta \leq 1$, and $1 < p < \infty$. The fractional derivative space $\hat{E}_{\alpha}^{\beta}$ is a reflexive separable Banach space.*

Lemma 2.4 ([7]). *Let $\frac{1}{p} < \alpha, \beta \leq 1$, $1 < p < \infty$. Assume that the sequence $\{u_k\}$ converges weakly to u in $\hat{E}_0^{\alpha}$ or $\hat{E}_0^{\beta}$, i.e., $u_k \rightharpoonup u$, then $u_k \to u$ in C([0,T],R).*

Definition 2.4. The function $(u, v) \in \hat{E}_{\alpha}^{\beta}$ is a weak solution of problem (1.1) if the identity

$$\begin{aligned} &\Big(\hat{a} + \hat{b}\|u\|_{\hat{E}_0^{\alpha}}^p\Big)^{p-1} \int_0^T \frac{1}{\hbar_1(t)^{p-2}} \phi_p(\hbar_1(t)_0 D_t^{\alpha} u(t))_0 D_t^{\alpha} x(t) \\ &+ \ell_1(t)|u(t)|^{p-2} u(t) x(t) dt \\ &+ \Big(\hat{c} + \hat{d}\|v\|_{\hat{E}_0^{\beta}}^p\Big)^{p-1} \int_0^T \frac{1}{\hbar_2(t)^{p-2}} \phi_p(\hbar_2(t)_0 D_t^{\beta} v(t))_0 D_t^{\beta} y(t) \\ &+ \ell_2(t)|v(t)|^{p-2} v(t) y(t) dt \\ &- \lambda \int_0^T f_u(t, u(t), v(t)) x(t) + f_v(t, u(t), v(t)) y(t) dt \\ &- \mu \int_0^T g_u(t, u(t), v(t)) x(t) + g_v(t, u(t), v(t)) y(t) dt = 0 \end{aligned} \tag{2.8}$$

holds for any $(x, y) \in \hat{E}_{\alpha}^{\beta}$.

Define the functional $\hat{\varphi}_{\lambda}$, $\hat{\Phi}$, $\hat{\Psi} : \hat{E}_{\alpha}^{\beta} \to R$ as follow:

$$\hat{\varphi}_{\lambda}(u, v) = \hat{\Phi}(u, v) - \lambda \hat{\Psi}(u, v), \tag{2.9}$$

$$\begin{aligned} \hat{\Phi}(u, v) =& \frac{1}{\hat{b}p^2}\Big(\hat{a} + \hat{b}\int_0^T (\hbar_1(t)|_0 D_t^{\alpha} u(t)|^p + \ell_1(t)|u(t)|^p) dt\Big)^p - \frac{\hat{a}^p}{\hat{b}p^2} \\ &+ \frac{1}{\hat{d}p^2}\Big(\hat{c} + \hat{d}\int_0^T (\hbar_2(t)|_0 D_t^{\beta} v(t)|^p + \ell_2(t)|v(t)|^p) dt\Big)^p - \frac{\hat{c}^p}{\hat{d}p^2} \\ =& \frac{1}{\hat{b}p^2}(\hat{a} + \hat{b}\|u\|_{\hat{E}_0^{\alpha}}^p)^p - \frac{\hat{a}^p}{\hat{b}p^2} + \frac{1}{\hat{d}p^2}(\hat{c} + \hat{d}\|v\|_{\hat{E}_0^{\beta}}^p)^p - \frac{\hat{c}^p}{\hat{d}p^2}, \end{aligned} \tag{2.10}$$

$$\hat{\Psi}(u, v) = \int_0^T f(t, u(t), v(t)) dt + \frac{\mu}{\lambda}\int_0^T g(t, u(t), v(t)) dt. \tag{2.11}$$

Clearly, $\hat{\varphi}_{\lambda}$, $\hat{\Phi}$, $\hat{\Psi} \in C^1(\hat{E}_{\alpha}^{\beta}, R)$ and for all $(u, v) \in \hat{E}_{\alpha}^{\beta}$, we have

$$\langle \hat{\varphi}_{\lambda}'(u, v), (x, y)\rangle = \langle \hat{\Phi}'(u, v), (x, y)\rangle - \lambda\langle \hat{\Psi}'(u, v), (x, y)\rangle, \tag{2.12}$$

$$\begin{aligned} \langle \hat{\Phi}'(u, v), (x, y)\rangle =& \Big(\hat{a} + \hat{b}\|u\|_{\hat{E}_0^{\alpha}}^p\Big)^{p-1} \int_0^T \frac{1}{\hbar_1(t)^{p-2}} \phi_p(\hbar_1(t)_0 D_t^{\alpha} u(t))_0 D_t^{\alpha} x(t) \\ &+ \ell_1(t)|u(t)|^{p-2} u(t) x(t) dt \\ &+ \Big(\hat{c} + \hat{d}\|v\|_{\hat{E}_0^{\beta}}^p\Big)^{p-1} \int_0^T \frac{1}{\hbar_2(t)^{p-2}} \phi_p(\hbar_2(t)_0 D_t^{\beta} v(t))_0 D_t^{\beta} y(t) \\ &+ \ell_2(t)|v(t)|^{p-2} v(t) y(t) dt, \end{aligned} \tag{2.13}$$

$$\begin{aligned}\langle\hat{\Psi}'(u,v),(x,y)\rangle=&\int_0^T f_u(t,u(t),v(t))x(t)+f_v(t,u(t),v(t))y(t)dt\\&+\frac{\mu}{\lambda}\int_0^T g_u(t,u(t),v(t))x(t)+g_v(t,u(t),v(t))y(t)dt.\end{aligned}\tag{2.14}$$

Therefore, the critical point of functional $\hat{\varphi}_\lambda$ is a weak solution of (1.1).

Definition 2.5 ([21]). Let E be a real Banach space and $\psi\in C^1(E,R)$. We say that ψ satisfies the Palais-Smale condition(denoted by P.S.condition), if any sequence $\{u_k\}\subset E$ for which $\{\psi(u_k)\}$ is bounded and $\psi'(u_k)\to 0$ as $k\to 0$ possesses a convergent subsequence.

Lemma 2.5 ([16]). *Let E is a real Banach space and $\psi\in C^1(E,R)$, and ψ satisfies the P.S.condition. Assume that there exist u_0, $u_1\in E$ and a bounded open neighborhood Ω of u_0 such that $u_1\in E\setminus\overline{\Omega}$ and*

$$\max\{\psi(u_0),\psi(u_1)\}<\inf_{v\in\partial\Omega}\psi(v).$$

Let

$$\sigma=\inf_{j\in\Gamma}\max_{s\in[0,1]}\psi(j(s)),$$

where $\Gamma=\{j|j\in C([0,1],E):j(0)=u_0,\ j(1)=u_1\}$. Obviously, σ is a critical value of ψ, i.e. there exists $u^*\in E$ such that $\psi'(u^*)=0$, $\psi(u^*)=\sigma$ with $\sigma>\max\{\psi(u_0),\psi(u_1)\}$.

Lemma 2.6 (Theorem 38.A, [25]). *For the functional $\psi:B\subseteq E\to R$ with B not empty, $\widetilde{\alpha}=\min_{u\in B}\psi(u)$ admits a solution in the following hold:*

(i) E is a real reflexive Banach space;

(ii) B is bounded and weak sequentially closed, i.e., for each $\{u_k\}\subset B$ such that $u_k\rightharpoonup u$ as $k\to\infty$, we have $u\in B$;

(iii) ψ is weakly sequentially lower semi-continous in B, i.e., for each $\{u_k\}\subset B$ such that $u_k\rightharpoonup u$ as $k\to\infty$, we have $\psi(u)\le\liminf_{k\to\infty}\psi(u_k)$.

Theorem 2.1 (Theorem 1.3, [1]). *Let E be a real Banach space. $\hat{\Phi}$, $\hat{\Psi}:E\to R$ be two continously Gâteaux differentiable functionals such that $\inf_E\hat{\Phi}=\hat{\Phi}(0)=\hat{\Psi}(0)=0$. Assume that there exist $r\in R$ and $\tilde{u}\in E$, with $0<\hat{\Phi}(\tilde{u})<r$, such that*

(i) $\frac{\sup_{\hat{\Phi}(u)\le r}\hat{\Psi}(u)}{r}<\frac{\hat{\Psi}(\tilde{u})}{\hat{\Phi}(\tilde{u})}$,

(ii) for each $\lambda\in\Lambda:=]\frac{\hat{\Phi}(\tilde{u})}{\hat{\Psi}(\tilde{u})},\frac{r}{\sup_{\hat{\Phi}(u)\le r}\hat{\Psi}(u)}[$, the functional $\hat{\varphi}_\lambda=\hat{\Phi}-\lambda\hat{\Psi}$ is unbounded from below and satisfies P.S.condition.

Then, for each $\lambda\in\Lambda$, the functional $\hat{\varphi}_\lambda=\hat{\Phi}-\lambda\hat{\Psi}$ has at least two distinct critical points in E.

3. Main Results

In this section we study multiple solutions of problem (1.1) by Lemmas 2.5, 2.6 and Theorem 2.1.

Theorem 3.1. *Let* $\frac{1}{p} < \alpha, \beta \leq 1$, $1 < p < \infty$, $f(t,0,0) = g(t,0,0) = 0$. *Assume that*
(H_1) *For any* $t \in [0,T]$, $(u,v) \in \hat{E}_{\alpha}^{\beta}$ *there exist nonnegative constants* $\widetilde{L}$, $\widetilde{\theta}_1$, $\widetilde{\theta}_2$ *with* $\max\{\widetilde{\theta}_1, \widetilde{\theta}_2\} < \frac{1}{p^2}$ *such that*

$$0 < f(t,u(t),v(t)) \leq \widetilde{\theta}_1\Big(f_u(t,u(t),v(t))u(t) + f_v(t,u(t),v(t))v(t)\Big), |(u,v)| \geq \widetilde{L},$$
$$0 < g(t,u(t),v(t)) \leq \widetilde{\theta}_2\Big(g_u(t,u(t),v(t))u(t) + g_v(t,u(t),v(t))v(t)\Big), |(u,v)| \geq \widetilde{L}.$$

(H_2) *For any* $t \in [0,T]$, $(u,v) \in \hat{E}_{\alpha}^{\beta}$ *such that*

$$\lim_{|u(t)|\to 0} \frac{f(t,u(t),v(t))}{|u(t)|^p} = 0, \qquad \lim_{|v(t)|\to 0} \frac{g(t,u(t),v(t))}{|v(t)|^p} = 0.$$

(H_3) *For any* $t \in [0,T]$, $(u,v) \in \hat{E}_{\alpha}^{\beta}$, *there exists a constant* $\widetilde{\gamma} > p^2$ *such that*

$$\lim_{|u(t)|\to \infty} \frac{f(t,u(t),v(t))}{|u(t)|^{\widetilde{\gamma}}} = \infty, \qquad \lim_{|v(t)|\to \infty} \frac{g(t,u(t),v(t))}{|v(t)|^{\widetilde{\gamma}}} = \infty.$$

Then, problem (1.1) admits at least two weak solutions.

Proof. We complete the proof by three steps:

Step I. We show that $\hat{\varphi}_\lambda : \hat{E}_{\alpha}^{\beta} \to R$ satisfies the P.S. condition.

First, assume that $\{(u_k,v_k)\}_{k=1}^{\infty} \subset \hat{E}_{\alpha}^{\beta}$ is a sequence such that

$$|\hat{\varphi}_\lambda(u_k,v_k)| \leq \widetilde{C}, \quad \hat{\varphi}'_\lambda(u_k,v_k) \to 0, \text{ as } k \to \infty, \tag{3.1}$$

where $\widetilde{C} > 0$ is a constant. We prove that $\{(u_k,v_k)\}_{k=1}^{\infty}$ is a bounded sequence in $\hat{E}_{\alpha}^{\beta}$. From the continuity of $f(t,u(t),v(t)) - \widetilde{\theta}_1 f_u(t,u(t),v(t))u(t) - \widetilde{\theta}_1 f_v(t,u(t),v(t))v(t)$, $g(t,u(t),v(t)) - \widetilde{\theta}_2 g_u(t,u(t),v(t))u(t) - \widetilde{\theta}_2 g_v(t,u(t),v(t))v(t)$ for any $t \in [0,T]$, $|(u,v)| \leq \widetilde{L}$, and (H_1), there exist constants $\widetilde{c}_1$, $\widetilde{c}_2$ such that

$$f(t,u(t),v(t)) \leq \widetilde{\theta}_1\Big(f_u(t,u(t),v(t))u(t) + f_v(t,u(t),v(t))v(t)\Big) + \widetilde{c}_1, \forall |(u,v)| \in R^2,$$
$$g(t,u(t),v(t)) \leq \widetilde{\theta}_2\Big(g_u(t,u(t),v(t))u(t) + g_v(t,u(t),v(t))v(t)\Big) + \widetilde{c}_2, \forall |(u,v)| \in R^2,$$

which together with (2.9), (2.10), (2.11) and (3.1), we have

$$\begin{aligned}
\widetilde{C} \geq & \hat{\varphi}_\lambda(u_k,v_k) \\
= & \frac{1}{\hat{b}p^2}\Big(\hat{a} + \hat{b}\|u_k\|^p_{\hat{E}_0^\alpha}\Big)^p - \frac{\hat{a}^p}{\hat{b}p^2} + \frac{1}{\hat{d}p^2}\Big(\hat{c} + \hat{d}\|v_k\|^p_{\hat{E}_0^\beta}\Big)^p - \frac{\hat{c}^p}{\hat{d}p^2} \\
& - \lambda \int_0^T f(t,u_k(t),v_k(t))dt - \mu \int_0^T g(t,u_k(t),v_k(t))dt \\
\geq & \frac{1}{\hat{b}p^2}\Big(\hat{a} + \hat{b}\|u_k\|^p_{\hat{E}_0^\alpha}\Big)^p - \frac{\hat{a}^p}{\hat{b}p^2} + \frac{1}{\hat{d}p^2}\Big(\hat{c} + \hat{d}\|v_k\|^p_{\hat{E}_0^\beta}\Big)^p - \frac{\hat{c}^p}{\hat{d}p^2} \\
& - \max\{\widetilde{\theta}_1, \widetilde{\theta}_2\}\Big(\lambda \int_0^T f_{u_k}(t,u_k(t),v_k(t))u_k(t) + f_{v_k}(t,u_k(t),v_k(t))v_k(t)dt \\
& + \mu \int_0^T g_{u_k}(t,u_k(t),v_k(t))u_k(t) + g_{v_k}(t,u_k(t),v_k(t))v_k(t)dt\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\lambda\widetilde{c}_1T-\mu\widetilde{c}_2T\\
=&\frac{1}{\hat{b}p^2}\Big(\hat{a}+\hat{b}\|u_k\|^p_{\hat{E}_0^\alpha}\Big)^p-\frac{\hat{a}^p}{\hat{b}p^2}+\frac{1}{\hat{d}p^2}\Big(\hat{c}+\hat{d}\|v_k\|^p_{\hat{E}_0^\beta}\Big)^p-\frac{\hat{c}^p}{\hat{d}p^2}\\
&-\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\}\Big((\hat{a}+\hat{b}\|u_k\|^p_{\hat{E}_0^\alpha})^{p-1}\|u_k\|^p_{\hat{E}_0^\alpha}+(\hat{c}+\hat{d}\|v_k\|^p_{\hat{E}_0^\beta})^{p-1}\|v_k\|^p_{\hat{E}_0^\beta}\\
&-\langle\varphi'_\lambda(u_k,v_k),(u_k,v_k)\rangle\Big)-\lambda\widetilde{c}_1T-\mu\widetilde{c}_2T\\
=&\Big(\hat{a}+\hat{b}\|u_k\|^p_{\hat{E}_0^\alpha}\Big)^{p-1}\Big(\frac{1}{\hat{b}p^2}(\hat{a}+\hat{b}\|u_k\|^p_{\hat{E}_0^\alpha})-\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\}\|u_k\|^p_{\hat{E}_0^\alpha}\Big)\\
&+\Big(\hat{c}+\hat{d}\|v_k\|^p_{\hat{E}_0^\beta}\Big)^{p-1}\Big(\frac{1}{\hat{d}p^2}(\hat{c}+\hat{d}\|v_k\|^p_{\hat{E}_0^\beta})-\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\}\|v_k\|^p_{\hat{E}_0^\beta}\Big)\\
&+\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\}\langle\hat{\varphi}'_\lambda(u_k,v_k),(u_k,v_k)\rangle-\frac{\hat{a}^p}{\hat{b}p^2}-\frac{\hat{c}^p}{\hat{d}p^2}-\lambda\widetilde{c}_1T-\mu\widetilde{c}_2T\\
\geq&\Big(\hat{a}+\hat{b}\|u_k\|^p_{\hat{E}_0^\alpha}\Big)^{p-1}\Big((\frac{1}{p^2}-\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\})\|u_k\|^p_{\hat{E}_0^\alpha}+\frac{\hat{a}}{\hat{b}p^2}\Big)\\
&+\Big(\hat{c}+\hat{d}\|v_k\|^p_{\hat{E}_0^\beta}\Big)^{p-1}\Big((\frac{1}{p^2}-\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\})\|v_k\|^p_{\hat{E}_0^\beta}+\frac{\hat{c}}{\hat{d}p^2}\Big)\\
&-\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\}\|\hat{\varphi}'_\lambda(u_k,v_k)\|\cdot\|(u_k,v_k)\|_{\hat{E}_\alpha^\beta}-\frac{\hat{a}^p}{\hat{b}p^2}-\frac{\hat{c}^p}{\hat{d}p^2}-\lambda\widetilde{c}_1T-\mu\widetilde{c}_2T\\
\geq&\min\{\hat{a}^{p-1},\hat{c}^{p-1}\}(\frac{1}{p^2}-\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\})\|(u_k,v_k)\|^p_{\hat{E}_\alpha^\beta}\\
&-\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\}\|\hat{\varphi}'_\lambda(u_k,v_k)\|\cdot\|(u_k,v_k)\|_{\hat{E}_\alpha^\beta}-\frac{\hat{a}^p}{\hat{b}p^2}-\frac{\hat{c}^p}{\hat{d}p^2}-\lambda\widetilde{c}_1T-\mu\widetilde{c}_2T.
\end{aligned}
$$

Since $\hat{\varphi}'_\lambda(u_k,v_k)\to 0$, there exist $\widetilde{N}_0\in N$, for $k>\widetilde{N}_0$ such that

$$
\begin{aligned}
\widetilde{C}\geq&\min\{\hat{a}^{p-1},\hat{c}^{p-1}\}(\frac{1}{p^2}-\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\})\|(u_k,v_k)\|^p_{\hat{E}_\alpha^\beta}\\
&-\|(u_k,v_k)\|_{\hat{E}_\alpha^\beta}-\frac{\hat{a}^p}{\hat{b}p^2}-\frac{\hat{c}^p}{\hat{d}p^2}-\lambda\widetilde{c}_1T-\mu\widetilde{c}_2T.
\end{aligned}
$$

Then, it follows from $\max\{\widetilde{\theta}_1,\widetilde{\theta}_2\}<\frac{1}{p^2}$ that $\{(u_k,v_k)\}$ is bounded in $\hat{E}_\alpha^\beta$.

In the following, we prove that the sequence $\{(u_k,v_k)\}_{k=1}^\infty$ has a convergent subsequence in $\hat{E}_\alpha^\beta$. Since $\hat{E}_\alpha^\beta$ is a reflexive Banach space, there exists a weakly convergent subsequence with $(u_{k_i},v_{k_i})\rightharpoonup(u,v)$ in $\hat{E}_\alpha^\beta$. For the sake of discussion, we note $\{(u_{k_i},v_{k_i})\}$ as $\{(u_k,v_k)\}$, thus $(u_k,v_k)\rightharpoonup(u,v)$ in $\hat{E}_\alpha^\beta$, then we have

$$
\begin{aligned}
&\langle\hat{\varphi}'_\lambda(u_k,v_k)-\hat{\varphi}'_\lambda(u,v),(u_k,v_k)-(u,v)\rangle\\
=&\langle\hat{\varphi}'_\lambda(u_k,v_k),(u_k,v_k)-(u,v)\rangle-\langle\hat{\varphi}'_\lambda(u,v),(u_k,v_k)-(u,v)\rangle\\
\leq&\|\hat{\varphi}'_\lambda(u_k,v_k)\|\cdot\|(u_k,v_k)-(u,v)\|_{\hat{E}_\alpha^\beta}-\langle\hat{\varphi}'_\lambda(u,v),(u_k,v_k)-(u,v)\rangle\\
&\to 0,\quad\text{as } k\to\infty.
\end{aligned}\tag{3.2}
$$

Additionally, from (2.7) and Lemma 2.4, we get that the $\{(u_k,v_k)\}$ is bounded in $C([0,T],R)^2$ and $\|(u_k,v_k)-(u,v)\|_\infty\to 0$, hence

$$
\int_0^T\Big(f_{u_k}(t,u_k(t),v_k(t))-f_u(t,u(t),v(t))\Big)(u_k(t)-u(t))
$$

$$+\Big(f_{v_k}(t,u_k(t),v_k(t))-f_v(t,u(t),v(t))\Big)(v_k(t)-v(t))dt\to 0,\ \text{as } k\to\infty, \quad (3.3)$$

and

$$\int_0^T \Big(g_{u_k}(t,u_k(t),v_k(t))-g_u(t,u(t),v(t))\Big)(u_k(t)-u(t))$$
$$+\Big(g_{v_k}(t,u_k(t),v_k(t))-g_v(t,u(t),v(t))\Big)(v_k(t)-v(t))dt\to 0,\ \text{as } k\to\infty. \quad (3.4)$$

Then, by straightforward calculation from (2.12), (2.13) and (2.14) we have

$$\begin{aligned}
&\langle \hat{\varphi}'_\lambda(u_k,v_k)-\hat{\varphi}'_\lambda(u,v),(u_k,v_k)-(u,v)\rangle\\
&+\lambda\int_0^T \Big(f_{u_k}(t,u_k(t),v_k(t))-f_u(t,u(t),v(t))\Big)(u_k(t)-u(t))\\
&+\Big(f_{v_k}(t,u_k(t),v_k(t))-f_v(t,u(t),v(t))\Big)(v_k(t)-v(t))dt\\
&+\mu\int_0^T \Big(g_{u_k}(t,u_k(t),v_k(t))-g_u(t,u(t),v(t))\Big)(u_k(t)-u(t))\\
&+\Big(g_{v_k}(t,u_k(t),v_k(t))-g_v(t,u(t),v(t))\Big)(v_k(t)-v(t))dt\\
=&\Big(\hat{a}+\hat{b}\|u_k\|^p_{\hat{E}_0^\alpha}\Big)^{p-1}\int_0^T \frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t)_0D_t^\alpha u_k(t))({}_0D_t^\alpha u_k(t)-\ {}_0D_t^\alpha u(t))\\
&+\ell_1(t)|u_k(t)|^{p-2}u_k(t)(u_k(t)-u(t))dt\\
&+\Big(\hat{c}+\hat{d}\|v_k\|^p_{\hat{E}_0^\beta}\Big)^{p-1}\int_0^T \frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^\beta v_k(t))({}_0D_t^\beta v_k(t)-\ {}_0D_t^\beta v(t))\\
&+\ell_2(t)|v_k(t)|^{p-2}v_k(t)(v_k(t)-v(t))dt\\
&-\Big(\hat{a}+\hat{b}\|u\|^p_{\hat{E}_0^\alpha}\Big)^{p-1}\int_0^T \frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t)_0D_t^\alpha u(t))({}_0D_t^\alpha u_k(t)-\ {}_0D_t^\alpha u(t))\\
&+\ell_1(t)|u(t)|^{p-2}u(t)(u_k(t)-u(t))dt\\
&-\Big(\hat{c}+\hat{d}\|v\|^p_{\hat{E}_0^\beta}\Big)^{p-1}\int_0^T \frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^\beta v(t))({}_0D_t^\beta v_k(t)-\ {}_0D_t^\beta v(t))\\
&+\ell_2(t)|v(t)|^{p-2}v(t)(v_k(t)-v(t))dt\\
=&\Big(\hat{a}+\hat{b}\|u_k\|^p_{\hat{E}_0^\alpha}\Big)^{p-1}\int_0^T \Big(\frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t)_0D_t^\alpha u_k(t))\\
&-\frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t)_0D_t^\alpha u(t))\Big)({}_0D_t^\alpha u_k(t)-\ {}_0D_t^\alpha u(t))\\
&+\Big(\ell_1(t)|u_k(t)|^{p-2}u_k(t)-\ell_1(t)|u(t)|^{p-2}u(t)\Big)(u_k(t)-u(t))dt\\
&+\Big(\hat{c}+\hat{d}\|v_k\|^p_{\hat{E}_0^\beta}\Big)^{p-1}\int_0^T \Big(\frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^\beta v_k(t))\\
&-\frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^\beta v(t))\Big)({}_0D_t^\beta v_k(t)-\ {}_0D_t^\beta v(t))\\
&+\Big(\ell_2(t)|v_k(t)|^{p-2}v_k(t)-\ell_2(t)|v(t)|^{p-2}v(t)\Big)(v_k(t)-v(t))dt\\
&+\Big((\hat{a}+\hat{b}\|u_k\|^p_{\hat{E}_0^\alpha})^{p-1}-(\hat{a}+\hat{b}\|u\|^p_{\hat{E}_0^\alpha})^{p-1}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^T \frac{1}{\hbar_1(t)^{p-2}} \phi_p(\hbar_1(t)_0 D_t^\alpha u(t))({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) \\
&+ \ell_1(t)|u(t)|^{p-2}u(t)(u_k(t) - u(t))dt \\
&+ \Big((\hat{c} + \hat{d}\|v_k\|^p_{\hat{E}_0^\beta})^{p-1} - (\hat{c} + \hat{d}\|v\|^p_{\hat{E}_0^\beta})^{p-1}\Big) \\
&\times \int_0^T \frac{1}{\hbar_2(t)^{p-2}} \phi_p(\hbar_2(t)_0 D_t^\beta v(t))({}_0D_t^\beta v_k(t) - {}_0D_t^\beta v(t)) \\
&+ \ell_2(t)|v(t)|^{p-2}v(t)(v_k(t) - v(t))dt, \qquad (3.5)
\end{aligned}
$$

considering that a weak convergence of $\{(u_k(t), v_k(t))\}$, we have

$$
\begin{aligned}
&\Big((\hat{a} + \hat{b}\|u_k\|^p_{\hat{E}_0^\alpha})^{p-1} - (\hat{a} + \hat{b}\|u\|^p_{\hat{E}_0^\alpha})^{p-1}\Big) \\
&\times \int_0^T \frac{1}{\hbar_1(t)^{p-2}} \phi_p(\hbar_1(t)_0 D_t^\alpha u(t))({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) \\
&+ \ell_1(t)|u(t)|^{p-2}u(t)(u_k(t) - u(t))dt \\
=&\Big((\hat{a} + \hat{b}\|u_k\|^p_{\hat{E}_0^\alpha})^{p-1} - (\hat{a} + \hat{b}\|u\|^p_{\hat{E}_0^\alpha})^{p-1}\Big)(\varphi'_{\lambda 1}(u(t)), u_k(t) - u(t)) \qquad (3.6) \\
&\to 0, \quad \text{as } k \to \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big((\hat{c} + \hat{d}\|v_k\|^p_{\hat{E}_0^\beta})^{p-1} - (\hat{c} + \hat{d}\|v\|^p_{\hat{E}_0^\beta})^{p-1}\Big) \\
&\times \int_0^T \frac{1}{\hbar_2(t)^{p-2}} \phi_p(\hbar_2(t)_0 D_t^\beta v(t))({}_0D_t^\beta v_k(t) - {}_0D_t^\beta v(t)) \\
&+ \ell_2(t)|v(t)|^{p-2}v(t)(v_k(t) - v(t))dt \\
=&\Big((\hat{c} + \hat{d}\|v_k\|^p_{\hat{E}_0^\beta})^{p-1} - (\hat{c} + \hat{d}\|v\|^p_{\hat{E}_0^\beta})^{p-1}\Big)(\varphi'_{\lambda 2}(v(t)), v_k(t) - v(t)) \qquad (3.7) \\
&\to 0, \quad \text{as } k \to \infty,
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\varphi}_{\lambda 1}(u(t)) &= \frac{1}{p}\Big(\int_0^T \hbar_1(t)|{}_0D_t^\alpha u(t)|^p dt + \int_0^T \ell_1(t)|u(t)|^p dt\Big), \\
\hat{\varphi}_{\lambda 2}(v(t)) &= \frac{1}{p}\Big(\int_0^T \hbar_2(t)|{}_0D_t^\beta v(t)|^p dt + \int_0^T \ell_2(t)|v(t)|^p dt\Big),
\end{aligned}
$$

by (3.2) to (3.7), we obtain

$$
\begin{aligned}
&\int_0^T \Big(\frac{1}{\hbar_1(t)^{p-2}} \phi_p(\hbar_1(t)_0 D_t^\alpha u_k(t)) \\
&- \frac{1}{\hbar_1(t)^{p-2}} \phi_p(\hbar_1(t)_0 D_t^\alpha u(t))\Big)({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) \\
&+ \Big(\ell_1(t)|u_k(t)|^{p-2}u_k(t) - \ell_1(t)|u(t)|^{p-2}u(t)\Big)(u_k(t) - u(t))dt \to 0, \qquad (3.8) \\
&\int_0^T \Big(\frac{1}{\hbar_2(t)^{p-2}} \phi_p(\hbar_2(t)_0 D_t^\beta v_k(t))
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^{\beta}v(t))\Big)({}_0D_t^{\beta}v_k(t) - {}_0D_t^{\beta}v(t)) \\
&+ \Big(\ell_2(t)|v_k(t)|^{p-2}v_k(t) - \ell_2(t)|v(t)|^{p-2}v(t)\Big)(v_k(t) - v(t))dt \to 0. \qquad (3.9)
\end{aligned}
$$

Based on a notorious inequality in [23], for any $\widetilde{s}_1$, $\widetilde{s}_2 \in R^N$, there exists a constant $\widetilde{a} > 0$ such that

$$
\langle|\widetilde{s}_1|^{p-2}\widetilde{s}_1 - |\widetilde{s}_2|^{p-2}\widetilde{s}_2, \widetilde{s}_1 - \widetilde{s}_2\rangle \geq \begin{cases} \widetilde{a}|\widetilde{s}_1 - \widetilde{s}_2|^p, & p \geq 2, \\ \widetilde{a}\dfrac{|\widetilde{s}_1 - \widetilde{s}_2|^2}{(|\widetilde{s}_1| + |\widetilde{s}_2|)^{2-p}}, & 1 < p \leq 2, \end{cases} \qquad (3.10)
$$

then by (3.10), there exist constants $\widetilde{d}_1$, $\widetilde{d}_2$, $\widetilde{d}_3$, $\widetilde{d}_4 > 0$ such that

$$
\begin{aligned}
&\int_0^T \Big(\frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t)_0D_t^{\alpha}u_k(t)) - \frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t)_0D_t^{\alpha}u(t))\Big) \\
&\times ({}_0D_t^{\alpha}u_k(t) - {}_0D_t^{\alpha}u(t))dt \\
&\geq \begin{cases} \widetilde{d}_1\int_0^T \frac{1}{\hbar_1(t)^{p-1}}|\hbar_1(t)_0D_t^{\alpha}u_k(t) - \hbar_1(t)_0D_t^{\alpha}u(t)|^p dt, & p \geq 2, \\ \widetilde{d}_1\int_0^T \frac{1}{\hbar_1(t)^{p-1}}\frac{|\hbar_1(t)_0D_t^{\alpha}u_k(t) - \hbar_1(t)_0D_t^{\alpha}u(t)|^2}{(|\hbar_1(t)_0D_t^{\alpha}u_k(t)| + |\hbar_1(t)_0D_t^{\alpha}u(t)|)^{2-p}}dt, & 1 < p \leq 2, \end{cases} \\
&\int_0^T \Big(\ell_1(t)|u_k(t)|^{p-2}u_k(t) - \ell_1(t)|u(t)|^{p-2}u(t)\Big)(u_k(t) - u(t))dt \\
&\geq \begin{cases} \widetilde{d}_2\int_0^T \ell_1(t)|u_k(t) - u(t)|^p dt, & p \geq 2, \\ \widetilde{d}_2\int_0^T \ell_1(t)\frac{|u_k(t)-u(t)|^2}{(|u_k(t)|+|u(t)|)^{2-p}}dt, & 1 < p \leq 2, \end{cases} \\
&\int_0^T \Big(\frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^{\beta}v_k(t)) - \frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^{\beta}v(t))\Big) \\
&\times ({}_0D_t^{\beta}v_k(t) - {}_0D_t^{\beta}v(t))dt \\
&\geq \begin{cases} \widetilde{d}_3\int_0^T \frac{1}{\hbar_2(t)^{p-1}}|\hbar_2(t)_0D_t^{\beta}v_k(t) - \hbar_2(t)_0D_t^{\beta}v(t)|^p dt, & p \geq 2, \\ \widetilde{d}_3\int_0^T \frac{1}{\hbar_2(t)^{p-1}}\frac{|\hbar_2(t)_0D_t^{\beta}v_k(t) - \hbar_2(t)_0D_t^{\beta}v(t)|^2}{(|\hbar_2(t)_0D_t^{\beta}v_k(t)| + |\hbar_2(t)_0D_t^{\beta}v(t)|)^{2-p}}dt, & 1 < p \leq 2, \end{cases} \\
&\int_0^T \Big(\ell_2(t)|v_k(t)|^{p-2}v_k(t) - \ell_2(t)|v(t)|^{p-2}v(t)\Big)(v_k(t) - v(t))dt \\
&\geq \begin{cases} \widetilde{d}_4\int_0^T \ell_2(t)|v_k(t) - v(t)|^p dt, & p \geq 2, \\ \widetilde{d}_4\int_0^T \ell_2(t)\frac{|v_k(t)-v(t)|^2}{(|v_k(t)|+|v(t)|)^{2-p}}dt, & 1 < p \leq 2. \end{cases}
\end{aligned}
$$

When $1 < p \leq 2$, one has

$$
\begin{aligned}
&\int_0^T \hbar_1(t)|{}_0D_t^{\alpha}u_k(t) - {}_0D_t^{\alpha}u(t)|^p dt \\
\leq &\Big(\int_0^T \frac{1}{\hbar_1(t)^{p-1}}\frac{|\hbar_1(t)_0D_t^{\alpha}u_k(t) - \hbar_1(t)_0D_t^{\alpha}u(t)|^2}{(|\hbar_1(t)_0D_t^{\alpha}u_k(t)| + |\hbar_1(t)_0D_t^{\alpha}u(t)|)^{2-p}}dt\Big)^{\frac{p}{2}} \\
&\times \Big(\int_0^T \frac{1}{\hbar_1(t)^{p-1}}(|\hbar_1(t)_0D_t^{\alpha}u_k(t)| + |\hbar_1(t)_0D_t^{\alpha}u(t)|)^p dt\Big)^{\frac{2-p}{2}} \\
\leq &\Big(\int_0^T \frac{1}{\hbar_1(t)^{p-1}}\frac{|\hbar_1(t)_0D_t^{\alpha}u_k(t) - \hbar_1(t)_0D_t^{\alpha}u(t)|^2}{(|\hbar_1(t)_0D_t^{\alpha}u_k(t)| + |\hbar_1(t)_0D_t^{\alpha}u(t)|)^{2-p}}dt\Big)^{\frac{p}{2}}
\end{aligned}
$$

$$\times\, 2^{(p-1)\frac{2-p}{2}}(\hbar_1^1)^{\frac{2-p}{2}}\Big(\int_0^T |{}_0D_t^\alpha u_k(t)|^p + |{}_0D_t^\alpha u(t)|^p dt\Big)^{\frac{2-p}{2}},$$

therefore we infer that

$$\begin{aligned}&\int_0^T \Big(\frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t){}_0D_t^\alpha u_k(t)) - \frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t){}_0D_t^\alpha u(t))\Big)\\&\times ({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t))dt\\ \geq& 2^{(1-p)\frac{2-p}{p}}(\hbar_1^1)^{\frac{p-2}{p}}\widetilde{d}_1(\|{}_0D_t^\alpha u_k(t)\|_{L^p}^p + \|{}_0D_t^\alpha u(t)\|_{L^p}^p)^{\frac{p-2}{p}}\\&\times (\int_0^T \hbar_1(t)|{}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)|^p dt)^{\frac{2}{p}}.\end{aligned} \tag{3.11}$$

A similar calculation yields

$$\begin{aligned}&\int_0^T \ell_1(t)|u_k(t)-u(t)|^p dt\\ \leq&\Big(\int_0^T \ell_1(t)\frac{|u_k(t)-u(t)|^2}{(|u_k(t)|+|u(t)|)^{2-p}}dt\Big)^{\frac{p}{2}}\cdot\Big(\int_0^T \ell_1(t)(|u_k(t)|+|u(t)|)^p dt\Big)^{\frac{2-p}{2}}\\ \leq&\Big(\int_0^T \ell_1(t)\frac{|u_k(t)-u(t)|^2}{(|u_k(t)|+|u(t)|)^{2-p}}dt\Big)^{\frac{p}{2}}\\&\times 2^{(p-1)\frac{2-p}{2}}(\ell_1^1)^{\frac{2-p}{2}}\Big(\int_0^T |u_k(t)|^p + |u(t)|^p dt\Big)^{\frac{2-p}{2}},\end{aligned}$$

therefore we infer that

$$\begin{aligned}&\int_0^T \Big(\ell_1(t)|u_k(t)|^{p-2}u_k(t) - \ell_1(t)|u(t)|^{p-2}u(t)\Big)(u_k(t)-u(t))dt\\ \geq& 2^{(1-p)\frac{2-p}{p}}(\ell_1^1)^{\frac{p-2}{p}}\widetilde{d}_2\Big(\|u_k(t)\|_{L^p}^p + \|u(t)\|_{L^p}^p\Big)^{\frac{p-2}{p}}\\&\times\Big(\int_0^T \ell_1(t)|u_k(t)-u(t)|^p dt\Big)^{\frac{2}{p}}.\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), it follows that

$$\begin{aligned}&\int_0^T \Big(\frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t){}_0D_t^\alpha u_k(t))\\&-\frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t){}_0D_t^\alpha u(t))\Big)({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t))\\&+\Big(\ell_1(t)|u_k(t)|^{p-2}u_k(t) - \ell_1(t)|u(t)|^{p-2}u(t)\Big)(u_k(t)-u(t))dt\\ \geq&\widetilde{K}_1\Big((\int_0^T \hbar_1(t)|{}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)|^p dt)^{\frac{2}{p}} + (\int_0^T \ell_1(t)|u_k(t)-u(t)|^p dt)^{\frac{2}{p}}\Big)\\ \geq& 2^{\frac{p-2}{p}}\widetilde{K}_1\Big(\int_0^T \hbar_1(t)|{}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)|^p dt + \int_0^T \ell_1(t)|u_k(t)-u(t)|^p dt\Big)^{\frac{2}{p}}\\ =&2^{\frac{p-2}{p}}\widetilde{K}_1\|u_k(t)-u(t)\|_{\hat{E}_0^\alpha}^2,\end{aligned} \tag{3.13}$$

where

$$\widetilde{K}_1 = 2^{(1-p)\frac{2-p}{p}}\min\{(\hbar_1^1)^{\frac{p-2}{p}}\widetilde{d}_1(\|{}_0D_t^\alpha u_k(t)\|_{L^p}^p + \|{}_0D_t^\alpha u(t)\|_{L^p}^p)^{\frac{p-2}{p}},$$

$$(\ell_1^1)^{\frac{p-2}{p}}\widetilde{d}_2(\|u_k(t)\|_{L^p}^p+\|u(t)\|_{L^p}^p)^{\frac{p-2}{p}}\}.$$

When $P\geq 2$, one has

$$\begin{aligned}&\int_0^T\Big(\frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t)_0D_t^\alpha u_k(t))\\&-\frac{1}{\hbar_1(t)^{p-2}}\phi_p(\hbar_1(t)_0D_t^\alpha u(t))\Big)({}_0D_t^\alpha u_k(t)-{}_0D_t^\alpha u(t))\\&+\Big(\ell_1(t)|u_k(t)|^{p-2}u_k(t)-\ell_1(t)|u(t)|^{p-2}u(t)\Big)(u_k(t)-u(t))dt\\\geq&\int_0^T\widetilde{d}_1\frac{1}{\hbar_1(t)^{p-1}}|\hbar_1(t)_0D_t^\alpha u_k(t)-\hbar_1(t)_0D_t^\alpha u(t)|^pdt+\int_0^T\widetilde{d}_2\ell_1(t)|u_k(t)-u(t)|^pdt\\\geq&\min\{\widetilde{d}_1,\widetilde{d}_2\}\|u_k(t)-u(t)\|_{\hat{E}_0^\alpha}^p.\end{aligned}\tag{3.14}$$

Likewise, when $1<p<2$, one has

$$\begin{aligned}&\int_0^T\Big(\frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^\beta v_k(t))\\&-\frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^\beta v(t))\Big)({}_0D_t^\beta v_k(t)-{}_0D_t^\beta v(t))\\&+\Big(\ell_2(t)|v_k(t)|^{p-2}v_k(t)-\ell_2(t)|v(t)|^{p-2}v(t)\Big)(v_k(t)-v(t))dt\\\geq&\widetilde{K}_2\Big((\int_0^T\hbar_2(t)|{}_0D_t^\beta v_k(t)-{}_0D_t^\beta v(t)|^pdt)^{\frac{2}{p}}+(\int_0^T\ell_2(t)|v_k(t)-v(t)|^pdt)^{\frac{2}{p}}\Big)\\\geq&2^{\frac{p-2}{p}}\widetilde{K}_2\Big(\int_0^T\hbar_2(t)|{}_0D_t^\beta v_k(t)-{}_0D_t^\beta v(t)|^pdt+\int_0^T\ell_2(t)|v_k(t)-v(t)|^pdt\Big)^{\frac{2}{p}}\\=&2^{\frac{p-2}{p}}\widetilde{K}_2\|v_k(t)-v(t)\|_{\hat{E}_0^\beta}^2,\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}\widetilde{K}_2=&2^{(1-p)\frac{2-p}{p}}\min\{(\hbar_2^1)^{\frac{p-2}{p}}\widetilde{d}_3(\|{}_0D_t^\beta v_k(t)\|_{L^p}^p+\|{}_0D_t^\beta v(t)\|_{L^p}^p)^{\frac{p-2}{p}},\\&(\ell_2^1)^{\frac{p-2}{p}}\widetilde{d}_4(\|v_k(t)\|_{L^p}^p+\|v(t)\|_{L^p}^p)^{\frac{p-2}{p}}\}.\end{aligned}$$

When $P\geq 2$, one has

$$\begin{aligned}&\int_0^T\Big(\frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^\beta v_k(t))\\&-\frac{1}{\hbar_2(t)^{p-2}}\phi_p(\hbar_2(t)_0D_t^\beta v(t))\Big)({}_0D_t^\beta v_k(t)-{}_0D_t^\beta v(t))\\&+\Big(\ell_2(t)|v_k(t)|^{p-2}v_k(t)-\ell_2(t)|v(t)|^{p-2}v(t)\Big)(v_k(t)-v(t))dt\\\geq&\int_0^T\widetilde{d}_3\frac{1}{\hbar_2(t)^{p-1}}|\hbar_2(t)_0D_t^\beta v_k(t)-\hbar_2(t)_0D_t^\beta v(t)|^pdt+\int_0^T\widetilde{d}_4\ell_2(t)|v_k(t)-v(t)|^pdt\\\geq&\min\{\widetilde{d}_3,\widetilde{d}_4\}\|v_k(t)-v(t)\|_{\hat{E}_0^\beta}^p.\end{aligned}\tag{3.16}$$

Therefore, it follows from (3.8) and (3.9) with (3.13) to (3.16) that

$$\|u_k(t)-u(t)\|_{\hat{E}_0^\alpha}\to 0,\quad \|v_k(t)-v(t)\|_{\hat{E}_0^\beta}\to 0,\quad \text{as } k\to\infty.$$

This show that $\|(u_k(t), v_k(t)) - (u(t), v(t))\|_{\hat{E}^\beta_\alpha} \to 0$. Hence, $\hat{\varphi}_\lambda$ satisfies the P.S. condition and the first step of the proof is completed.

StepII. We will show that there exists an $\widetilde{r} > 0$ such that the functional $\hat{\varphi}_\lambda$ has a local minimum $(u_0^*, v_0^*) \in \Theta_{\widetilde{r}} : \{(u, v) \in \hat{E}^\beta_\alpha : \|(u, v)\|_{\hat{E}^\beta_\alpha} < \widetilde{r}\}$.

Apparently, $\overline{\Theta}_{\widetilde{r}}$ is a bound and real reflexive Banach space. In addition, we can show that $\overline{\Theta}_{\widetilde{r}}$ is weak sequentially closed. In fact, let $\{(u_k, v_k)\} \subseteq \overline{\Theta}_{\widetilde{r}}$ such that $(u_k, v_k) \rightharpoonup (u, v)$, by the Mazur theorem [16], there exists a sequence of convex combinations

$$\widetilde{u}_k = \sum_{j=1}^{k} \varepsilon_{k_j} u_j, \quad \sum_{j=1}^{k} \varepsilon_{k_j} = 1, \quad \varepsilon_{k_j} > 0, \quad j \in N,$$
$$\widetilde{v}_k = \sum_{j=1}^{k} \varsigma_{k_j} v_j, \quad \sum_{j=1}^{k} \varsigma_{k_j} = 1, \quad \varsigma_{k_j} > 0, \quad j \in N,$$

such that $\widetilde{u}_k \to u$ in $\hat{E}^\alpha_0$, $\widetilde{v}_k \to v$ in $\hat{E}^\beta_0$, i.e., $(\widetilde{u}_k, \widetilde{v}_k) \to (u, v)$. Since $\overline{\Theta}_{\widetilde{r}}$ is a closed convex set, so $(\widetilde{u}_k, \widetilde{v}_k) \subseteq \overline{\Theta}_{\widetilde{r}}$ and $(u, v) \in \overline{\Theta}_{\widetilde{r}}$. Next, we show that $\hat{\varphi}_\lambda$ is weakly sequentially lower semi-continous in $\overline{\Theta}_{\widetilde{r}}$. Assuming $(u_k, v_k) \rightharpoonup (u, v)$ in $\hat{E}^\beta_\alpha$, from the Lemma 2.4, then (u_k, v_k) uniformly converges (u, v) in $C([0, T], R)^2$. So $\hat{\Psi}(u, v)$ is weakly sequentially continuous. Apparently $\hat{\Phi}(u, v)$ is a convex and continuous function, we have that $\hat{\Phi}(u, v)$ is weakly sequentially lower semi-continous. Then $\hat{\varphi}_\lambda$ is weakly sequentially lower semi-continous in $\overline{\Theta}_{\widetilde{r}}$. i.e.

$$\begin{aligned}\liminf_{k\to\infty} \hat{\varphi}_\lambda(u_k, v_k) =& \liminf_{k\to\infty} \frac{1}{\hat{b}p^2}(\hat{a} + \hat{b}\|u_k\|^p_{\hat{E}^\alpha_0})^p - \frac{\hat{a}^p}{\hat{b}p^2} + \frac{1}{\hat{d}p^2}(\hat{c} + \hat{d}\|v_k\|^p_{\hat{E}^\beta_0})^p - \frac{\hat{c}^p}{\hat{d}p^2} \\ &- \lambda \int_0^T f(t, u_k(t), v_k(t))dt - \mu \int_0^T g(t, u_k(t), v_k(t))dt \\ \geq& \hat{\varphi}_\lambda(u, v).\end{aligned}$$

Therefore, it follows from Lemma 2.6 that $\hat{\varphi}_\lambda$ has a local minimum. Without loss of generality, we assume $\hat{\varphi}_\lambda(u_0^*, v_0^*) = \min_{(u,v)\in\overline{\Theta}_{\widetilde{r}}} \varphi_\lambda(u, v)$. Next we will demonstrate that

$$\hat{\varphi}_\lambda(u_0^*, v_0^*) < \inf_{(u,v)\in\partial\Theta_{\widetilde{r}}} \hat{\varphi}_\lambda(u, v). \tag{3.17}$$

If (3.17) holds, the proof of Step II is completed.

There we choose $\widetilde{r} = \widetilde{r}_0 > 0$, from (H_2), there exist $0 < \widetilde{\varepsilon} < \min\{1, \frac{1}{\lambda}, \frac{1}{\mu}\}$ and $\widetilde{\delta} > 0$, for any $t \in [0, T]$, $(u, v) \in \hat{E}^\beta_\alpha$, with $|u(t)| \leq \widetilde{\delta}$, $|v(t)| \leq \widetilde{\delta}$, such that

$$f(t, u(t), v(t)) \leq \frac{\widetilde{\varepsilon}\hat{a}^{p-1}}{p\widetilde{\Lambda}_1^p}|u(t)|^p, \quad g(t, u(t), v(t)) \leq \frac{\widetilde{\varepsilon}\hat{c}^{p-1}}{p\widetilde{\Lambda}_2^p}|v(t)|^p. \tag{3.18}$$

So, for any $(u, v) \in \partial\Theta_{\widetilde{r}_0}$, by (2.2), (2.5), (2.9), (2.10), (2.11) and (3.18), we have

$$\begin{aligned}\hat{\varphi}_\lambda(u, v) =& \frac{1}{\hat{b}p^2}(\hat{a} + \hat{b}\|u\|^p_{\hat{E}^\alpha_0})^p - \frac{\hat{a}^p}{\hat{b}p^2} + \frac{1}{\hat{d}p^2}(\hat{c} + \hat{d}\|v\|^p_{\hat{E}^\beta_0})^p - \frac{\hat{c}^p}{\hat{d}p^2} \\ &- \lambda \int_0^T f(t, u(t), v(t))dt - \mu \int_0^T g(t, u(t), v(t))dt\end{aligned}$$

$$\begin{aligned}
\geq & \frac{\hat{a}^{p-1}}{p}\|u(t)\|^p_{\hat{E}_0^\alpha} + \frac{\hat{c}^{p-1}}{p}\|v(t)\|^p_{\hat{E}_0^\beta} - \lambda\frac{\widetilde{\varepsilon}\hat{a}^{p-1}}{p\widetilde{\Lambda}_1^p}\int_0^T |u(t)|^p dt \\
& - \mu\frac{\widetilde{\varepsilon}\hat{c}^{p-1}}{p\widetilde{\Lambda}_2^p}\int_0^T |v(t)|^p dt \\
\geq & \frac{\hat{a}^{p-1}}{p}\|u(t)\|^p_{\hat{E}_0^\alpha} + \frac{\hat{c}^{p-1}}{p}\|v(t)\|^p_{\hat{E}_0^\beta} - \lambda\frac{\widetilde{\varepsilon}\hat{a}^{p-1}}{p}\|u(t)\|^p_{\hat{E}_0^\alpha} \\
& - \mu\frac{\widetilde{\varepsilon}\hat{c}^{p-1}}{p}\|v(t)\|^p_{\hat{E}_0^\beta} \\
\geq & \frac{\hat{a}^{p-1}}{p}(1-\lambda\widetilde{\varepsilon})\|u(t)\|^p_{\hat{E}_0^\alpha} + \frac{\hat{c}^{p-1}}{p}(1-\mu\widetilde{\varepsilon})\|v(t)\|^p_{\hat{E}_0^\beta} \\
\geq & \min\{\hat{a}^{p-1}(1-\lambda\widetilde{\varepsilon}), \hat{c}^{p-1}(1-\mu\widetilde{\varepsilon})\}\frac{\widetilde{r}_0^p}{p}.
\end{aligned}$$

Noticing $\hat{\varphi}_\lambda(0,0) = 0$, then $\hat{\varphi}_\lambda(u_0^*, v_0^*) \leq \hat{\varphi}_\lambda(0,0) < \hat{\varphi}_\lambda(u,v)$ for any $(u,v) \in \partial\Theta_{\widetilde{r}_0}$. So (3.17) holds and $(u_0^*, v_0^*) \in \Theta_{\widetilde{r}_0}$.

StepIII. We will demonstrate that there exist an (u_1^*, v_1^*) with $\|(u_1^*, v_1^*)\|_{\hat{E}_\alpha^\beta} > \widetilde{r}_0$ such that $\hat{\varphi}_\lambda(u_1^*, v_1^*) < \inf_{(u,v)\in\partial\Theta_{\widetilde{r}_0}} \hat{\varphi}_\lambda(u,v)$.

Considering that (H3), there exist nonnegative constants $\widetilde{d}_1^*$, $\widetilde{d}_2^*$, $\widetilde{d}_3^*$ and $\widetilde{d}_4^*$ such that for any $t \in [0,T]$, $(u,v) \in \hat{E}_\alpha^\beta$,

$$f(t,u(t),v(t)) \geq \widetilde{d}_1^*|u(t)|^{\widetilde{\gamma}} - \widetilde{d}_2^*, \qquad g(t,u(t),v(t)) \geq \widetilde{d}_3^*|v(t)|^{\widetilde{\gamma}} - \widetilde{d}_4^*. \tag{3.19}$$

So, for any $(u,v) \in \hat{E}_\alpha^\beta \setminus (0,0)$, since $\widetilde{\gamma} > p^2$, by (3.19), we have

$$\begin{aligned}
\hat{\varphi}_\lambda(\hat{\xi}u, \hat{\xi}v) = & \frac{1}{\hat{b}p^2}(\hat{a} + \hat{b}\hat{\xi}^p\|u\|^p_{\hat{E}_0^\alpha})^p - \frac{\hat{a}^p}{\hat{b}p^2} + \frac{1}{\hat{d}p^2}(\hat{c} + \hat{d}\hat{\xi}^p\|v\|^p_{\hat{E}_0^\beta})^p - \frac{\hat{c}^p}{\hat{d}p^2} \\
& - \lambda\int_0^T f(t,\hat{\xi}u(t),\hat{\xi}v(t))dt - \mu\int_0^T g(t,\hat{\xi}u(t),\hat{\xi}v(t))dt \\
\leq & \frac{1}{\hat{b}p^2}(\hat{a} + \hat{b}\hat{\xi}^p\|u\|^p_{\hat{E}_0^\alpha})^p + \frac{1}{\hat{d}p^2}(\hat{c} + \hat{d}\hat{\xi}^p\|v\|^p_{\hat{E}_0^\beta})^p \\
& - \lambda\widetilde{d}_1^*\int_0^T |\hat{\xi}u(t)|^{\widetilde{\gamma}}dt + \lambda\widetilde{d}_2^*T - \mu\widetilde{d}_3^*\int_0^T |\hat{\xi}v(t)|^{\widetilde{\gamma}}dt + \mu\widetilde{d}_4^*T \\
= & \frac{1}{\hat{b}p^2}(\hat{a} + \hat{b}\hat{\xi}^p\|u\|^p_{\hat{E}_0^\alpha})^p + \frac{1}{\hat{d}p^2}(\hat{c} + \hat{d}\hat{\xi}^p\|v\|^p_{\hat{E}_0^\beta})^p \\
& - \lambda\widetilde{d}_1^*\hat{\xi}^{\widetilde{\gamma}}\|u(t)\|^{\widetilde{\gamma}}_{L^{\widetilde{\gamma}}} - \mu\widetilde{d}_3^*\hat{\xi}^{\widetilde{\gamma}}\|v(t)\|^{\widetilde{\gamma}}_{L^{\widetilde{\gamma}}} + \lambda\widetilde{d}_2^*T + \mu\widetilde{d}_4^*T \\
\to & -\infty, \quad \text{as} \quad \hat{\xi} \to \infty.
\end{aligned} \tag{3.20}$$

Then the (3.20) implies that there exists ξ_0^* sufficiently large with $\|(\xi_0^*u, \xi_0^*v)\|_{\hat{E}_\alpha^\beta} > \widetilde{r}_0$, $\hat{\varphi}_\lambda(\xi_0^*u, \xi_0^*v) \leq 0$. Therefore we take $(u_1^*, v_1^*) = (\xi_0^*u, \xi_0^*v)$ with $\|(u_1^*, v_1^*)\|_{\hat{E}_\alpha^\beta} > \widetilde{r}_0$ and $\hat{\varphi}_\lambda(u_1^*, v_1^*) \leq 0$ which implied that $\hat{\varphi}_\lambda(u_1^*, v_1^*) < \inf_{(u,v)\in\partial\Theta_{\widetilde{r}_0}} \hat{\varphi}_\lambda(u,v)$. Then StepIII is proved.

It follows from Lemma 2.5 that the critical value

$$\sigma^* = \inf_{(\widetilde{j}_1,\widetilde{j}_2)\in\widetilde{\Gamma}} \max_{s\in[0,1]} \hat{\varphi}_\lambda(\widetilde{j}_1(s), \widetilde{j}_2(s)),$$

where

$$\begin{aligned}\widetilde{\Gamma} =&\{(\widetilde{j}_1(s),\widetilde{j}_2(s))|(\widetilde{j}_1(s),\widetilde{j}_2(s))\in C([0,1],R)^2:(\widetilde{j}_1(0),\widetilde{j}_2(0))=(u_0^*,v_0^*),\\ &(\widetilde{j}_1(1),\widetilde{j}_2(1))=(u_1^*,v_1^*)\},\end{aligned}$$

then, there exists a critical point $(\widetilde{u}^*,\widetilde{v}^*)\in\hat{E}_\alpha^\beta$, such that $\hat{\varphi}'_\lambda(\widetilde{u}^*,\widetilde{v}^*)=0$. In addition, by Lemma 2.6, we know that (u_0^*,v_0^*) is also a critical point, therefore, $(\widetilde{u}^*,\widetilde{v}^*)$ and (u_0^*,v_0^*) are two different critical points of $\hat{\varphi}_\lambda$ and they are weak solutions of (1.1). Proof of Theorem 3.1 is completed. □

Before we begin the following proof, we give some notations.

$$\begin{aligned}G_1(\alpha,\varrho)=&\frac{1}{(\varrho T)^p}\Big(\int_0^{\varrho T}\hbar_1(t)t^{(1-\alpha)p}dt+\int_{\varrho T}^{(1-\varrho)T}\hbar_1(t)(t^{1-\alpha}-(t-\varrho T)^{1-\alpha})^p dt\\ &+\int_{(1-\varrho)T}^{T}\hbar_1(t)(t^{1-\alpha}-(t-\varrho T)^{1-\alpha}-(t-(1-\varrho)T)^{1-\alpha})^p dt\Big)\\ &+(\frac{\Gamma(2-\alpha)}{\varrho T})^p\int_0^{\varrho T}\ell_1(t)t^p dt+(\Gamma(2-\alpha))^p\int_{\varrho T}^{(1-\varrho)T}\ell_1(t)dt\\ &+(\frac{\Gamma(2-\alpha)}{\varrho T})^p\int_{(1-\varrho)T}^{T}\ell_1(t)(T-t)^p dt,\\ G_2(\beta,\varrho)=&\frac{1}{(\varrho T)^p}\Big(\int_0^{\varrho T}\hbar_2(t)t^{(1-\beta)p}dt+\int_{\varrho T}^{(1-\varrho)T}\hbar_2(t)(t^{1-\beta}-(t-\varrho T)^{1-\beta})^p dt\\ &+\int_{(1-\varrho)T}^{T}\hbar_2(t)(t^{1-\beta}-(t-\varrho T)^{1-\beta}-(t-(1-\varrho)T)^{1-\beta})^p dt\Big)\\ &+(\frac{\Gamma(2-\beta)}{\varrho T})^p\int_0^{\varrho T}\ell_2(t)t^p dt+(\Gamma(2-\beta))^p\int_{\varrho T}^{(1-\varrho)T}\ell_2(t)dt\\ &+(\frac{\Gamma(2-\beta)}{\varrho T})^p\int_{(1-\varrho)T}^{T}\ell_2(t)(T-t)^p dt,\\ \widetilde{M}=&\max\{\frac{1}{\hat{a}^{p-1}}\widetilde{\Lambda}_1^{*p},\frac{1}{\hat{c}^{p-1}}\widetilde{\Lambda}_2^{*p}\},\\ \widetilde{\Omega}(\widetilde{\iota})=&\{(u,v)\in R^2:\frac{1}{p}|u|^p+\frac{1}{p}|v|^p\le\widetilde{\iota}\},\\ \widetilde{g}^0=&\int_0^T\max_{(u,v)\in\widetilde{\Omega}(\widetilde{c})}g(t,u,v)dt,\\ \widetilde{g}_0=&\inf_{[0,T]\times[0,\Gamma(2-\alpha)\varpi_1]\times[0,\Gamma(2-\beta)\varpi_2]}g(t,u,v),\\ \widetilde{G}^*=&\frac{1}{\hat{b}p^2}(\hat{a}+\hat{b}\varpi_1^pG_1(\alpha,\varrho))^p+\frac{1}{\hat{d}p^2}(\hat{c}+\hat{d}\varpi_2^pG_2(\beta,\varrho))^p,\\ \widetilde{\mu}^*=&\min\{\frac{-\lambda\int_0^T\max_{(u,v)\in\widetilde{\Omega}(\widetilde{c})}f(t,u,v)dt+\frac{\widetilde{c}}{\widetilde{M}}}{\widetilde{g}^0},\\ &\frac{-\lambda\int_{\varrho T}^{(1-\varrho)T}f(t,\Gamma(2-\alpha)\varpi_1,\Gamma(2-\beta)\varpi_2)dt+\widetilde{G}^*}{T\widetilde{g}_0}\}.\end{aligned}$$

Theorem 3.2. *Let $\frac{1}{p}<\alpha,\beta\le 1$, $f(t,0,0)=g(t,0,0)=0$. Assume that (H1) and (H3) hold and there exist three positive constants $\widetilde{c}$, ϖ_1, ϖ_2 such that*

(H_1')

$$\frac{\widetilde{c}}{\widetilde{M}} > \frac{1}{\hat{b}p^2}(\hat{a}+\hat{b}\varpi_1^p G_1(\alpha,\varrho))^p + \frac{1}{\hat{d}p^2}(\hat{c}+\hat{d}\varpi_2^p G_2(\beta,\varrho))^p = \widetilde{G}^*;$$

(H_2')

$$\begin{aligned}&\int_{\varrho T}^{(1-\varrho)T} f(t,\Gamma(2-\alpha)\varpi_1,\Gamma(2-\beta)\varpi_2)dt > 0;\\ &\int_0^{\varrho T} f(t,\frac{\Gamma(2-\alpha)\varpi_1}{\varrho T}t,\frac{\Gamma(2-\beta)\varpi_2}{\varrho T}t)dt\\ &+\int_0^{\varrho T} f(T-t,\frac{\Gamma(2-\alpha)\varpi_1}{\varrho T}t,\frac{\Gamma(2-\beta)\varpi_2}{\varrho T}t)dt > 0;\end{aligned}$$

(H_3')

$$\frac{\widetilde{M}\int_0^T \max_{(u,v)\in\widetilde{\Omega}(\widetilde{c})} f(t,u,v)dt}{\widetilde{c}} < \frac{\int_{\varrho T}^{(1-\varrho)T} f(t,\Gamma(2-\alpha)\varpi_1,\Gamma(2-\beta)\varpi_2)dt}{\widetilde{G}^*}.$$

Then, for each

$$\lambda \in (\frac{\widetilde{G}^*}{\int_{\varrho T}^{(1-\varrho)T} f(t,\Gamma(2-\alpha)\varpi_1,\Gamma(2-\beta)\varpi_2)dt}, \frac{\widetilde{c}}{\widetilde{M}\int_0^T \max_{(u,v)\in\widetilde{\Omega}(\widetilde{c})} f(t,u,v)dt}),$$

and $\mu \in (0,\widetilde{\mu}^*)$*, the problem (1.1) has at least two non trivial solutions.*

Proof. It is clear that $\inf_{(u,v)\in\hat{E}_\alpha^\beta}\hat{\Phi}(0,0)=\hat{\Psi}(0,0)=0$. Besides, $\hat{\Phi}$, $\hat{\Psi}:\hat{E}_\alpha^\beta\to R$ are two continously Gâteaux differentiable functionals and $\hat{\Phi}'$, $\hat{\Psi}'$ can be seen in (2.13), (2.14).

Set $r=\frac{\widetilde{c}}{\widetilde{M}}$ and consider the following two functions $\hat{u}_1(t)\in\hat{E}_0^\alpha$, $\hat{v}_1(t)\in\hat{E}_0^\beta$ defined by

$$\hat{u}_1(t) := \begin{cases}\frac{\Gamma(2-\alpha)\varpi_1}{\varrho T}t, & t\in[0,\varrho T],\\ \Gamma(2-\alpha)\varpi_1, & t\in[\varrho T,(1-\varrho)T],\\ \frac{\Gamma(2-\alpha)\varpi_1}{\varrho T}(T-t), & t\in[(1-\varrho)T,T],\end{cases} \tag{3.21}$$

and

$$\hat{v}_1(t) := \begin{cases}\frac{\Gamma(2-\beta)\varpi_2}{\varrho T}t, & t\in[0,\varrho T],\\ \Gamma(2-\beta)\varpi_2, & t\in[\varrho T,(1-\varrho)T],\\ \frac{\Gamma(2-\beta)\varpi_2}{\varrho T}(T-t), & t\in[(1-\varrho)T,T].\end{cases} \tag{3.22}$$

Clearly $\hat{u}_1(0)=\hat{u}_1(T)=\hat{v}_1(0)=\hat{v}_1(T)=0$, $\hat{u}_1(t)$, $\hat{v}_1(t)\in L^p[0,T]$. By the definition 2.1, we have

$$
{}_0D_t^{\alpha}\hat{u}_1(t) := \begin{cases} \dfrac{\varpi_1}{\varrho T} t^{1-\alpha}, & t \in [0, \varrho T], \\ \dfrac{\varpi_1}{\varrho T}(t^{1-\alpha} - (t-\varrho T)^{1-\alpha}), & t \in [\varrho T, (1-\varrho)T], \\ \dfrac{\varpi_1}{\varrho T}(t^{1-\alpha} - (t-\varrho T)^{1-\alpha} - (t-(1-\varrho)T)^{1-\alpha}), & t \in [(1-\varrho)T, T], \end{cases}
$$

and

$$
{}_0D_t^{\beta}\hat{v}_1(t) := \begin{cases} \dfrac{\varpi_2}{\varrho T} t^{1-\beta}, & t \in [0, \varrho T], \\ \dfrac{\varpi_2}{\varrho T}(t^{1-\beta} - (t-\varrho T)^{1-\beta}), & t \in [\varrho T, (1-\varrho)T], \\ \dfrac{\varpi_2}{\varrho T}(t^{1-\beta} - (t-\varrho T)^{1-\beta} - (t-(1-\varrho)T)^{1-\beta}), & t \in [(1-\varrho)T, T]. \end{cases}
$$

So that

$$
\begin{aligned}
\|\hat{u}_1(t)\|^p_{\hat{E}_0^{\alpha}} &= \int_0^T \hbar_1(t)|{}_0D_t^{\alpha}\hat{u}_1(t)|^p dt + \ell_1(t)|\hat{u}_1(t)|^p dt \\
&= \int_0^{\varrho T} + \int_{\varrho T}^{(1-\varrho)T} + \int_{(1-\varrho)T}^{T} \hbar_1(t)|{}_0D_t^{\alpha}\hat{u}_1(t)|^p + \ell_1(t)|\hat{u}_1(t)|^p dt \\
&= \varpi_1^p G_1(\alpha, \varrho), \\
\|\hat{v}_1(t)\|^p_{\hat{E}_0^{\beta}} &= \int_0^T \hbar_2(t)|{}_0D_t^{\beta}\hat{v}_1(t)|^p dt + \ell_2(t)|\hat{v}_1(t)|^p dt \\
&= \int_0^{\varrho T} + \int_{\varrho T}^{(1-\varrho)T} + \int_{(1-\varrho)T}^{T} \hbar_2(t)|{}_0D_t^{\beta}\hat{v}_1(t)|^p + \ell_2(t)|\hat{v}_1(t)|^p dt \\
&= \varpi_2^p G_2(\beta, \varrho).
\end{aligned}
$$

Then, by (H_1') we have

$$
\begin{aligned}
0 <& \hat{\Phi}(\hat{u}_1(t), \hat{v}_1(t)) \\
=& \frac{1}{\hat{b}p^2}(\hat{a} + \hat{b}\varpi_1^p G_1(\alpha, \varrho))^p - \frac{\hat{a}^p}{\hat{b}p^2} + \frac{1}{\hat{d}p^2}(\hat{c} + \hat{d}\varpi_2^p G_2(\beta, \varrho))^p - \frac{\hat{c}^p}{\hat{d}p^2} \\
\le& \frac{1}{\hat{b}p^2}(\hat{a} + \hat{b}\varpi_1^p G_1(\alpha, \varrho))^p + \frac{1}{\hat{d}p^2}(\hat{c} + \hat{d}\varpi_2^p G_2(\beta, \varrho))^p < \frac{\widetilde{c}}{\widetilde{\widetilde{M}}} = r.
\end{aligned}
$$

Now, using that

$$
\begin{aligned}
& \{(u,v) \in \hat{E}_{\alpha}^{\beta} : \hat{\Phi}(u,v) \le r\} \\
=& \{(u,v) \in \hat{E}_{\alpha}^{\beta} : \frac{1}{\hat{b}p^2}(\hat{a} + \hat{b}\|u\|^p_{\hat{E}_0^{\alpha}})^p - \frac{\hat{a}^p}{\hat{b}p^2} + \frac{1}{\hat{d}p^2}(\hat{c} + \hat{d}\|v\|^p_{\hat{E}_0^{\beta}})^p - \frac{\hat{c}^p}{\hat{d}p^2} \le r\} \\
\subseteq& \{(u,v) \in \hat{E}_{\alpha}^{\beta} : \frac{\hat{a}^{p-1}}{p}\|u\|^p_{\hat{E}_0^{\alpha}} + \frac{\hat{c}^{p-1}}{p}\|v\|^p_{\hat{E}_0^{\beta}} \le r\} \\
\subseteq& \{(u,v) \in \hat{E}_{\alpha}^{\beta} : \frac{\hat{a}^{p-1}}{p}\frac{1}{\widetilde{\Lambda}_1^{*p}}\|u\|^p_{\infty} + \frac{\hat{c}^{p-1}}{p}\frac{1}{\widetilde{\Lambda}_2^{*p}}\|v\|^p_{\infty} \le r\}
\end{aligned}
$$

$$\subseteq\{(u,v)\in \hat{E}_{\alpha}^{\beta}:\frac{|u|^p}{p}+\frac{|v|^p}{p}\leq \widetilde{M}r=\widetilde{c}\}.$$

Due to $\mu\in(0,\widetilde{\mu}^*)$, we have

$$\mu<\frac{-\lambda\int_0^T \max_{(u,v)\in\widetilde{\Omega}(\widetilde{c})} f(t,u,v)dt+\frac{\widetilde{c}}{\widetilde{M}}}{\widetilde{g}^0},$$

then, the following inequality holds:

$$\frac{\sup_{\hat{\Phi}(u,v)\leq r}\hat{\Psi}(u,v)}{r}\leq \widetilde{M}\frac{\int_0^T \max_{(u,v)\in\widetilde{\Omega}(\widetilde{c})} f(t,u,v)dt+\frac{\mu}{\lambda}\widetilde{g}^0}{\widetilde{c}}<\frac{1}{\lambda}. \tag{3.23}$$

In addition, due to $\mu\in(0,\widetilde{\mu}^*)$, we have

$$\mu<\frac{-\lambda\int_{\varrho T}^{(1-\varrho)T} f(t,\Gamma(2-\alpha)\varpi_1,\Gamma(2-\beta)\varpi_2)dt+\widetilde{G}^*}{T\widetilde{g}_0},$$

then, the following inequality holds:

$$\frac{\hat{\Psi}(\hat{u}_1(t),\hat{v}_1(t))}{\hat{\Phi}(\hat{u}_1(t),\hat{v}_1(t))}\geq\frac{\int_{\varrho T}^{(1-\varrho)T} f(t,\Gamma(2-\alpha)\varpi_1,\Gamma(2-\beta)\varpi_2)dt+\frac{\mu}{\lambda}T\widetilde{g}_0}{\widetilde{G}^*}>\frac{1}{\lambda}. \tag{3.24}$$

By (3.23) and (3.24), the hypothesis (i) of Theorem 2.1 holds. In addition, note that when (H1) holds, StepI in the proof of Theorem 3.1 shows that $\hat{\varphi}_\lambda=\hat{\Phi}-\lambda\hat{\Psi}$ satisfies P.S.condition. When (H3) holds, the (3.20) implies that $\hat{\varphi}_\lambda=\hat{\Phi}-\lambda\hat{\Psi}$ is unbounded from blow on $\hat{E}_{\alpha}^{\beta}$. Then, the hypothesis (ii) of Theorem 2.1 holds. Therefore, all the assumptions of Theorem 2.1 are satisfied.

Hence, Theorem 3.2 implies that for each

$$\lambda\in(\frac{\widetilde{G}^*}{\int_{\varrho T}^{(1-\varrho)T} f(t,\Gamma(2-\alpha)\varpi_1,\Gamma(2-\beta)\varpi_2)dt},\frac{\widetilde{c}}{\widetilde{M}\int_0^T \max_{(u,v)\in\widetilde{\Omega}(\widetilde{c})} f(t,u,v)dt}),$$

and $\mu\in(0,\widetilde{\mu}^*)$, the functional $\hat{\varphi}_\lambda$ has least two non-zero critical point that are non trivial solutions of (1.1). □

4. Conclusions

In this paper we study the fractional-order Kirchhoff-type coupled equation with p-Laplacian operators in two approaches, obtaining results on the existence of two solutions to the equation under important A-R conditions. To the best of my knowledge, there has been relatively little research on such equations. Therefore, we are inclined to continue to explore such problems in the future.

References

[1] G. Bonanno, *Relations between the mountain pass theorem and local minima*, Adv. Nonlinear Anal., 2012, 1(3), 205–220.

[2] G. Chai and W. Liu, *Existence of solutions for the fractional Kirchhoff equations with sign-changing potential*, Bound. Value Probl., 2018. DOI: 10.1186/s13661-018-1046-3.

[3] T. Chen, W. Liu and H. Jin, *Nontrivial solutions of the Kirchhoff-type fractional p-Laplacian Dirichlet problem*, J. Funct. Spaces, 2020. DOI: 10.1155/2020/8453205.

[4] G. Fix and J. Roop, *Least squares finite-element solution of a fractional order two-point boundary value problem*, Comput. Math. Appl., 2004, 48(7–8), 1017–1033.

[5] S. Heidarkhani and A. Salari, *Nontrivial solutions for impulsive fractional differential systems through variational methods*, Math. Meth. Appl. Sci., 2020, 43(10), 6529–6541.

[6] F. Jiao and Y. Zhou, *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Comput. Math. Appl., 2011, 62(3), 1181–1199.

[7] F. Jiao and Y. Zhou, *Existence results for fractional boundary value problem via critical point theory*, Int. J. Bifurcation Chaos, 2012, 22(4), 1250086.

[8] F. Kamache, R. Guefaifia and S. Boulaaras, *Existence of three solutions for perturbed nonlinear fractional p-Laplacian boundary value systems with two control parameters*, J. Pseudo-Differ. Oper. Appl., 2020, 11(4), 1781–1803.

[9] F. Kamache, R. Guefaifia, S. Boulaaras and A. Alharbi, *Existence of weak solutions for a new class of fractional p-Laplacian boundary value systems*, Mathematics, 2020, 8(4), 475.

[10] D. Kang, C. Liu and X. Zhang, *Existence of solutions for Kirchhoff-type fractional Dirichlet problem with p-Laplacian*, Mathematics, 2020. DOI: 10.3390/math8010106.

[11] A. Kilbas, H. Srivastava and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[12] M. Kratou, *Ground state solutions of p-Laplacian singular Kirchhoff problem involving a Riemann-Liouville fractional derivative*, Filomat, 2019, 33(7), 2073–2088.

[13] D. Li, F. Chen and Y. An, *The multiplicity of solutions for a class of nonlinear fractional Dirichlet boundary value problems with p-Laplacian type via variational approach*, Int. J. Nonlinear Sci. Numer. Simul., 2019, 20(3–4), 361–371.

[14] D. Li, F. Chen and Y. An, *Existence and multiplicity of nontrivial solutions for nonlinear fractional differential systems with p-Laplacian via critical point theory*, Math. Meth. Appl. Sci., 2018, 41(8), 3197–3212.

[15] D. Ma, L. Liu and Y.Wu, *Existence of nontrivial solutions for a system of fractional advection-dispersion equations*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., 2019, 113(2), 1041–1057.

[16] J. Mawhin and M.Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, Berlin, 1989.

[17] D. Min and F. Chen, *Three solutions for a class of fractional impulsive advection-dispersion equations with Sturm-Liouville boundary conditions via variational approach*, Math. Meth. Appl. Sci., 2020, 43(15), 9151–9168.

[18] N. Nyamoradi and E. Tayyebi, *Existence of solutions for a class of fractional boundary value equations with impulsive effects via critical point theory*, Mediterr. J. Math., 2018, 15(3), 1–25.

[19] N. Nyamoradi, Y. Zhou, E. Tayyebi, B. Ahmad and A. Alsaedi, *Nontrivial solutions for time fractional nonlinear Schrödinger-Kirchhoff type equations*, Discrete Dyn. Nat. Soc., 2017. DOI: 10.1155/2017/9281049.

[20] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[21] P. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, American Mathematical Society, Providence, 1986.

[22] S. Samko, A. Kilbas and O. Marichev, *Fractional Integrals and Derivatives: Theory and Application*, Gordon and Breach Science Publishers, New York, 1993.

[23] J. Simon, *Régularité de la solution d'un problème aux limites non linéaires*, Ann. Fac. Sci. Toulouse, 1981, 3, 247–274.

[24] Y. Tian and J. Nieto, *The applications of critical-point theory to discontinuous fractional-order differential equations*, Proc. Edinb. Math. Soc., 2017, 60(4), 1021–1051.

[25] E. Zeidler, *Nonlinear Functional Analysis and its Applications, III: Variational Methods and Optimization*, Springer, New York, 1985.

[26] Y. Zhao, C. Luo and H. Chen, *Existence results for non-instantaneous impulsive nonlinear fractional differential equation via variational methods*, Bull. Malays. Math. Sci. Soc., 2020, 43(3), 2151–2169.