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MONOTONE ITERATIVE POSITIVE SOLUTIONS FOR A FRACTIONAL DIFFERENTIAL SYSTEM WITH COUPLED HADAMARD TYPE FRACTIONAL INTEGRAL CONDITIONS*

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Abstract In this paper, we study via the monotone iterative technique positive solutions for a class of Hadamard type fractional-order differential systems with coupled Hadamard type fractional-order integral boundary value conditions on an infinite interval. Schemes are constructed to approximate extremal positive solutions of the coupled differential system. Examples are given to illustrate the theory.

Keywords Monotone iterative technique, fractional differential system, Hadamard type fractional integral, infinite interval.

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1. Introduction

Fractional differential equations arise in diffusion processes, engineering mechanics, chaos, biomathematics, fractional dynamic system and are a natural generalization of integer-order differential equations so improve modeling accuracy; see [2,4,5,10–16,19,20,24]. Usually authors discuss three fractional derivatives: Caputo type, Riemann-Liouville type, and Hadamard type. The Hadamard type fractional derivative and integral was introduced in [7] in 1892 and contains the logarithmic function in its definition and arises in fracture analysis and image processing; see [2,3,5,13,18,20] and the references therein.

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Usually to establish existence results for Hadamard type fractional differential equations researchers use fixed point theorems [1, 3, 6, 8, 9, 20, 28, 31]. For example, using the Banach contraction fixed point theorem and the Leray-Schauder alternative, the authors in [1] established the existence of solutions for the following differential system with uncoupled Hadamard type integral boundary conditions:

$$\begin{cases} {}^H D^{\alpha} u(t) = f(t, u(t), v(t)),\ 1 < t < e,\ 1 < \alpha \le 2, \\ {}^H D^{\beta} v(t) = g(t, v(t), u(t)),\ 1 < t < e,\ 1 < \beta \le 2, \\ u(1) = 0,\ u(e) = {}^H I^r u(\sigma_1) = \dfrac{1}{\Gamma(r)} \displaystyle\int_1^{\sigma_1} (\log \sigma_1 - \log s)^{r-1} u(s) \frac{\mathrm{d}s}{s}, \\ v(1) = 0,\ v(e) = {}^H I^r u(\sigma_2) = \dfrac{1}{\Gamma(r)} \displaystyle\int_1^{\sigma_2} (\log \sigma_2 - \log s)^{r-1} v(s) \frac{\mathrm{d}s}{s}, \end{cases} \tag{1.1}$$

where $r > 0, 1 < \sigma_1, \sigma_2 < e, {}^H D^{\alpha}$ and ${}^H D^{\beta}$ denote Hadamard type fractional order derivatives, and ${}^H I^r$ denotes a Hadamard type fractional order integral, $f, g : [1, e] \times \mathbb{R} \times \mathbb{R}$ are given continuous functions. Using the fixed point index the authors in [29] established the existence of solutions for the following system with uncoupled multi-point boundary value problems:

$$\begin{cases} {}^H D^q u(t) + f_1(t, u(t), v(t)) = 0,\ 1 < t \le e, 2 < q \le 3, \\ {}^H D^q v(t) + f_2(t, u(t), v(t)) = 0,\ 1 < t \le e, 2 < q \le 3, \\ u(1) = \delta u(1) = 0,\ u(e) = \displaystyle\sum_{i=1}^{m-1} a_i u(\xi_i), \\ v(1) = \delta v(1) = 0,\ v(e) = \displaystyle\sum_{j=1}^{n-1} b_j v(\eta_j), \end{cases} \tag{1.2}$$

where ${}^H D^q$ denotes the q-order Hadamard type fractional derivative and δ represents the delta derivative, i.e., $\delta u(1) = (tdu/dt)|_{t=1}, v(1) = (tdv/dt)|_{t=1}, f_i \in C([1, e] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+), \mathbb{R}_+ = [0, +\infty), i = 1, 2$. The real constants $a_i, b_j, \xi_i, \eta_j (i = 1, 2, \ldots, m-1, j = 1, 2, \ldots, n-1, m, n > 2)$ satisfy the following: $a_i, b_j > 0, \xi_i, \eta_j \in (1, e)$ with $\sum_{i=1}^{m-1} a_i (\log \xi_i)^{q-1} \in [0, 1)$ and $\sum_{j=1}^{n-1} b_j (\log \eta_j)^{q-1} \in [0, 1)$.

For results on Hadamard type fractional differential equations on the infinite interval we refer the reader to [5, 13, 14, 17, 18, 21, 22, 25, 26]. In [18] the authors established the existence of positive solutions and constructed two explicit monotone iterative sequences which converge to the extremal positive solutions of

$$\begin{cases} {}^H D^{\alpha} u(t) + f(t, u(t), {}^H I^{\gamma} u(t), {}^H D^{\alpha-1} u(t)) = 0,\ 1 < a \le 2, t \in (1, +\infty), \\ u(1) = 0,\ {}^H D^{a-1} u(+\infty) = \displaystyle\sum_{i=1}^{m} \lambda_i {}^H I^{\alpha_i} u(\eta). \end{cases} \tag{1.3}$$

where ${}^H D^{\alpha}$ is a Hadamard type fractional derivative of order α and ${}^H I^{(\cdot)}$ is a Hadamard type fractional order integral, $r, \beta_i, \lambda_i \ge 0 (i = 1, 2, \cdots, m)$ are preset constants and $\alpha, \eta, \beta_i, \lambda_i$ satisfy $\sum_{i=1}^{m} \frac{\lambda_i (\log \eta)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)} < 1$. Motivated by results of literature [1] and [18], the authors in [33] use the monotone iterative technique to investigate the existence of extreme positive solutions of the fractional differential coupled system on an infinite interval

$$\begin{cases} D^{\alpha}u(t)+\varphi(t,u(t),v(t),D^{\beta-1}v(t))=0,\ 2<\alpha\leq 3,\\ D^{\beta}v(t)+\psi(t,u(t),v(t),D^{\alpha-1}u(t))=0,\ 2<\beta\leq 3,\\ u(1)=u'(1)=0,\ D^{\alpha-1}u(+\infty)=\displaystyle\int_1^{+\infty}h(t)v(t)\mathrm{d}t,\\ v(1)=v'(1)=0,\ D^{\beta-1}v(+\infty)=\displaystyle\int_1^{+\infty}g(t)u(t)\mathrm{d}t, \end{cases} \tag{1.4}$$

where D^{α}, D^{β} are Riemann-Liouville fractional derivatives, and the nonlinear terms φ, ψ include coupled unknown functions and the lower-order fractional derivative of unknown functions. Recently the authors in [23] apply fixed point theorems to establish the existence of multiple positive solutions of the Hadamard type fractional differential system with coupled integral boundary conditions:

$$\begin{cases} {}^HD^{p}x(t)+a(t)f(t,x(t),y(t))=0,\ 1<p\leq 2, t\in[1,+\infty),\\ {}^HD^{q}y(t)+b(t)g(t,x(t),y(t))=0,\ 1<q\leq 2, t\in[1,+\infty),\\ x(1)=0,\ {}^HD^{p-1}x(+\infty)=\displaystyle\sum_{i=1}^{m}\lambda_i{}^HI^{\alpha_i}y(\eta_)),\\ y(1)=0,\ {}^HD^{q-1}y(+\infty)=\displaystyle\sum_{j=1}^{n}\sigma_j{}^HI^{\beta_j}x(\xi), \end{cases} \tag{1.5}$$

where ${}^HD^{\phi}$ are Hadamard fractional derivatives of $\phi\in\{p,q\}$,$f,g\in C([1,\infty)\times\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+)$, I^{v} are Hadamard fractional integrals of $v\in\{\alpha_i,\beta_i\}, \lambda_i,\sigma_j>0, i=1,2,\ldots,m, j=1,\ldots,n$. We note that the integral boundary conditions involve coupled unknown functions, but the nonlinearity terms f,g do not include the lower-order fractional derivative of unknown functions.

It is of interest to note that coupled systems involving lower-order Hadamard type fractional derivatives of unknown functions and coupled integral boundary conditions are rarely considered. Motivated by the above we consider the following Hadamard type fractional differential system:

$$\begin{cases} {}^HD^{p}x(t)+f_1(t,x(t),y(t),{}^HD^{p-1}x(t),{}^HD^{q-1}y(t))=0,\\ 1<p\leq 2, 1<q\leq 2, t\in\mathbb{R}_+,\\ {}^HD^{q}y(t)+f_2(t,x(t),y(t),{}^HD^{p-1}x(t),{}^HD^{q-1}y(t))=0,\\ 1<p\leq 2, 1<q\leq 2, t\in\mathbb{R}_+,\\ x(1)=0, {}^HD^{p-1}x(\infty)=\displaystyle\sum_{i=1}^{m}\lambda_i{}^HI^{\alpha_i}y(\eta), \eta\in\mathbb{R}_+,\\ y(1)=0, {}^HD^{q-1}y(\infty)=\displaystyle\sum_{j=1}^{n}\sigma_j{}^HI^{\beta_j}x(\xi), \xi\in\mathbb{R}_+, \end{cases} \tag{1.6}$$

where $\mathbb{R}_+=[1,+\infty),^H D^{\phi}$ are Hadamard fractional derivatives of $\phi\in\{p,q\}$,$f_1,f_2\in C([1,+\infty)\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+)$, I^{ψ} are Hadamard fractional integrals of $\psi\in\{\alpha_i,\beta_i\}$. $\lambda_i,\sigma_j>0, i=1,2,\ldots,m, j=1,\ldots,n$. Our aim in this paper is to obtain in Section 3 two pairs of explicit monotone iterative schemes to approximate the extremal positive solutions. The idea is to extend iterative methods to a system via the definition of a partial order in product spaces, which is quite different from [25–27, 30, 32]. Finally examples are given to illustrate our results.

2. Preliminaries

First we list some definitions and results concerning Hadamard type fractional fractional derivatives and integrals.

Definition 2.1(see [10]). The Hadamard type fractional derivative of order q is given by

$$^{H}D^{q}g(t)=\frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^{n}\int_{1}^{t}(\log t-\log s)^{n-q-1}g(s)\frac{\mathrm{d}s}{s}, n-1<q<n,$$

where $g:[1,\infty)\to\mathbb{R}$ is a integrable function, $[q]$ denotes the integer part of the real number q, $n=[q]+1$ and $\log(\cdot)=\log_{e}(\cdot)$.

Definition 2.2(see [10]). The Hadamard type fractional integral of order q is given by

$$^{H}I^{q}g(t)=\frac{1}{\Gamma(q)}\int_{1}^{t}(\log t-\log s)^{q-1}g(s)\frac{\mathrm{d}s}{s}, q>0,$$

where $g:[1,\infty)\to\mathbb{R}$ is a integrable function, and $\log(\cdot)=\log_{e}(\cdot)$.

Lemma 2.1. *Let $h_i\in C[1,\infty)$ with $0<\int_{1}^{\infty}h_i(s)\frac{ds}{s}<\infty, i=1,2$ and $\Omega=\Gamma(p)\Gamma(q)-\Lambda_1\Lambda_2>0$ with Λ_1 and Λ_2 given below. Then the following coupled Hadamard type fractional differential system*

$$\begin{cases} ^{H}D^{p}x(t)+h_1(t)=0, 1<p\leq 2, t\in\mathbb{R}_{+},\\ ^{H}D^{q}y(t)+h_2(t)=0, 1<q\leq 2, t\in\mathbb{R}_{+},\\ x(1)=0, {}^{H}D^{p-1}x(\infty)=\displaystyle\sum_{i=1}^{m}\lambda_i{}^{H}I^{\alpha_i}y(\eta),\\ y(1)=0, {}^{H}D^{q-1}y(\infty)=\displaystyle\sum_{j=1}^{n}\sigma_j{}^{H}I^{\beta_j}x(\xi), \end{cases}\tag{2.1}$$

is equivalent to the integral system

$$\begin{cases} x(t)=\displaystyle\int_{1}^{+\infty}G_1(t,s)h_1(s)\frac{ds}{s}+\int_{1}^{+\infty}G_2(t,s)h_2(s)\frac{ds}{s},\\ y(t)=\displaystyle\int_{1}^{+\infty}G_3(t,s)h_2(s)\frac{ds}{s}+\int_{1}^{+\infty}G_4(t,s)h_1(s)\frac{ds}{s}, \end{cases}\tag{2.2}$$

where the Green's functions $G_k(t,s), k=1,2,3,4$ are given by

$$G_1(t,s)=g_p(t,s)+\frac{\Lambda_1(\log t)^{p-1}}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j g_{\beta_j}^{p}(\xi,s)}{\Gamma(p+\beta_j)},$$

$$G_2(t,s)=\frac{\Gamma(q)(\log t)^{p-1}}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i g_{\alpha_i}^{q}(\eta,s)}{\Gamma(q+\alpha_i)},$$

$$G_3(t,s) = g_q(t,s) + \frac{\Lambda_2(\log t)^{q-1}}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i g_{\alpha_i}^{q}(\eta,s)}{\Gamma(q+\alpha_i)},$$
$$G_4(t,s) = \frac{\Gamma(p)(\log t)^{q-1}}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j g_{\beta_j}^{p}(\xi,s)}{\Gamma(p+\beta_j)},$$

with

$$\Lambda_1 = \sum_{i=1}^{m}\frac{\lambda_i\Gamma(q)(\log\eta)^{q+\alpha_i-1}}{\Gamma(q+\alpha_i)},\ \Lambda_2 = \sum_{j=1}^{n}\frac{\sigma_j\Gamma(p)(\log\xi)^{p+\beta_j-1}}{\Gamma(p+\beta_j)},$$

and

$$g_\phi(t,s) = \frac{1}{\Gamma(\phi)}\begin{cases}(\log t)^{\phi-1} - (\log t - \log s)^{\phi-1}, 1 \le s \le t < +\infty,\\ (\log t)^{\phi-1}, 1 \le t \le s < +\infty,\end{cases} \tag{2.3}$$
$$g_\psi^\phi(\rho,s) = \begin{cases}(\log\rho)^{\phi+\psi-1} - (\log\rho - \log s)^{\phi+\psi-1}, 1 \le s \le \rho < +\infty,\\ (\log\rho)^{\phi+\psi-1}, 1 \le \rho \le s < +\infty.\end{cases} \tag{2.4}$$

Proof. Apply Lemmas 2.5 and Lemma 2.6 in [23], and we can deduce the above results by direct observation. □

Lemma 2.2. *Let $h_i \in C(\mathbb{R}_+)$ with $0 < \int_1^\infty h_i(s)\frac{ds}{s} < \infty, i = 1,2$ and $\Omega = \Gamma(p)\Gamma(q) - \Lambda_1\Lambda_2 > 0$ with Λ_1 and Λ_2 given in Lemma 2.1. Then the following expression can be obtained from the integral equations* (2.2)

$$\begin{cases}{}^{H}D^{p-1}x(t) = \int_1^{+\infty} G_1^*(t,s)h_1(s)\frac{ds}{s} + \int_1^{+\infty} G_2^*(t,s)h_2(s)\frac{ds}{s},\\ {}^{H}D^{q-1}y(t) = \int_1^{+\infty} G_3^*(t,s)h_2(s)\frac{ds}{s} + \int_1^{+\infty} G_4^*(t,s)h_1(s)\frac{ds}{s},\end{cases} \tag{2.5}$$

where the Green's functions $G_k^*(t,s), 1,2,3,4$ are defined by

$$G_1^*(t,s) = G_0(t,s) + \frac{\Lambda_1\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j g_{\beta_j}^{p}(\xi,s)}{\Gamma(p+\beta_j)},$$
$$G_2^*(t,s) = \frac{\Gamma(p)\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i g_{\alpha_i}^{q}(\eta,s)}{\Gamma(q+\alpha_i)},$$
$$G_3^*(t,s) = G_0(t,s) + \frac{\Lambda_2\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i g_{\alpha_i}^{q}(\eta,s)}{\Gamma(q+\alpha_i)},$$
$$G_4^*(t,s) = \frac{\Gamma(p)\Gamma(q)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j g_{\beta_j}^{p}(\xi,s)}{\Gamma(p+\beta_j)},$$

and

$$G_0(t,s) = \begin{cases}0, 1 \le s \le t < +\infty,\\ 1, 1 \le t \le s < +\infty.\end{cases} \tag{2.6}$$

Proof. Using Lemma 2.5 of of [23], we can obtain

$${}^{H}D^{p-1}x(t) = -\int_1^t h_1(s)\frac{\mathrm{d}s}{s} + c_1\Gamma(p),\ {}^{H}D^{q-1}y(t) = -\int_1^t h_2(s)\frac{\mathrm{d}s}{s} + k_1\Gamma(q),$$

where

$$
\begin{aligned}
c_1 =&\frac{\Gamma(q)}{\Omega}\int_1^\infty h_1(s)\frac{\mathrm{d}s}{s}-\frac{\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^\eta(\log\eta-\log s)^{q+\alpha_i-1}h_2(s)\frac{\mathrm{d}s}{s}\\
&+\frac{\Lambda_1}{\Omega}\int_1^\infty h_2(s)\frac{\mathrm{d}s}{s}-\frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j}{\Gamma(p+\beta_j)}\int_1^\xi(\log\xi-\log s)^{p+\beta_j-1}h_1(s)\frac{\mathrm{d}s}{s},
\end{aligned}
$$

and

$$
\begin{aligned}
k_1 =&\frac{\Gamma(p)}{\Omega}\int_1^\infty h_2(s)\frac{\mathrm{d}s}{s}-\frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j}{\Gamma(p+\beta_j)}\int_1^\xi(\log\xi-\log s)^{p+\beta_j-1}h_1(s)\frac{\mathrm{d}s}{s}\\
&+\frac{\Lambda_2}{\Omega}\int_1^\infty h_1(s)\frac{\mathrm{d}s}{s}-\frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^\eta(\log\eta-\log s)^{q+\alpha_i-1}h_2(s)\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

Since $\Omega=\Gamma(p)\Gamma(q)-\Lambda_1\Lambda_2$ we have

$$
\begin{aligned}
&{}^HD^{p-1}x(t)\\
=&-\int_1^t h_1(s)\frac{\mathrm{d}s}{s}+\Gamma(p)\Big[\frac{\Gamma(q)}{\Omega}\int_1^\infty h_1(s)\frac{\mathrm{d}s}{s}-\frac{\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\\
&\times\int_1^\eta(\log\eta-\log s)^{q+\alpha_i-1}h_2(s)\frac{\mathrm{d}s}{s}+\frac{\Lambda_1}{\Omega}\int_1^\infty h_2(s)\frac{\mathrm{d}s}{s}\\
&-\frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j}{\Gamma(p+\beta_j)}\int_1^\xi(\log\xi-\log s)^{p+\beta_j-1}h_1(s)\frac{\mathrm{d}s}{s}\Big]\\
&+\int_1^\infty h_1(s)\frac{\mathrm{d}s}{s}-\int_1^\infty h_1(s)\frac{\mathrm{d}s}{s}\\
=&\int_1^\infty G_0(t,s)h_1(s)\frac{\mathrm{d}s}{s}+\frac{\Lambda_1\Lambda_2}{\Omega}\int_1^\infty h_1(s)\frac{\mathrm{d}s}{s}-\frac{\Gamma(p)\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j}{\Gamma(p+\beta_j)}\\
&\times\int_1^\xi(\log\xi-\log s)^{p+\beta_j-1}h_1(s)\frac{\mathrm{d}s}{s}+\frac{\Gamma(p)\Lambda_1}{\Omega}\int_1^\infty h_2(s)\frac{\mathrm{d}s}{s}\\
&-\frac{\Gamma(p)\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^\eta(\log\eta-\log s)^{q+\alpha_i-1}h_2(s)\frac{\mathrm{d}s}{s}\\
=&\int_1^\infty G_0(t,s)h_1(s)\frac{\mathrm{d}s}{s}+\frac{\Gamma(p)\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j}{\Gamma(p+\beta_j)}\int_1^\infty(\log\xi)^{p+\beta_j-1}h_1(s)\frac{\mathrm{d}s}{s}\\
&-\frac{\Gamma(p)\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j}{\Gamma(p+\beta_j)}\int_1^\xi(\log\xi-\log s)^{p+\beta_j-1}h_1(s)\frac{\mathrm{d}s}{s}\\
&+\frac{\Gamma(p)\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^\infty(\log\eta)^{q+\alpha_i-1}h_2(s)\frac{\mathrm{d}s}{s}\\
&-\frac{\Gamma(p)\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^\eta(\log\eta-\log s)^{q+\alpha_i-1}h_2(s)\frac{\mathrm{d}s}{s}\\
=&\int_1^\infty G_0(t,s)h_1(s)\frac{\mathrm{d}s}{s}+\frac{\Gamma(p)\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j}{\Gamma(p+\beta_j)}\int_1^\infty g^p_{\beta_j}(\xi,s)h_1(s)\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{\Gamma(p)\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^{\infty} g_{\alpha_i}^{q}(\eta,s)h_2(s)\frac{\mathrm{d}s}{s}\\
&= \int_1^{\infty} G_1^*(t,s)h_1(s)\frac{\mathrm{d}s}{s} + \int_1^{\infty} G_2^*(t,s)h_2(s)\frac{\mathrm{d}s}{s},
\end{aligned}
$$

which shows that the first expression is satisfied in (2.5). In an analogous way, we have

$$
\begin{aligned}
&{}^{H}D^{q-1}y(t)\\
&= -\int_1^{t} h_2(s)\frac{\mathrm{d}s}{s} + \Gamma(q)\Big[\frac{\Gamma(p)}{\Omega}\int_1^{\infty} h_2(s)\frac{\mathrm{d}s}{s} - \frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\lambda_j}{\Gamma(p+\beta_j)}\\
&\quad \times \int_1^{\xi}(\log\xi-\log s)^{p+\beta_j-1}h_1(s)\frac{\mathrm{d}s}{s} + \frac{\Lambda_2}{\Omega}\int_1^{\infty} h_1(s)\frac{\mathrm{d}s}{s} - \frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\\
&\quad \times \int_1^{\eta}(\log\eta-\log s)^{q+\alpha_i-1}h_2(s)\frac{\mathrm{d}s}{s}\Big] + \int_1^{\infty} h_2(s)\frac{\mathrm{d}s}{s} - \int_1^{\infty} h_2(s)\frac{\mathrm{d}s}{s}\\
&= \int_1^{\infty} G_0(t,s)h_2(s)\frac{\mathrm{d}s}{s} + \frac{\Lambda_1\Lambda_2}{\Omega}\int_1^{\infty} h_2(s)\frac{\mathrm{d}s}{s}\\
&\quad - \frac{\Gamma(q)\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(q+\alpha_i)}\int_1^{\eta}(\log\eta-\log s)^{q+\alpha_i-1}h_2(s)\frac{\mathrm{d}s}{s}\\
&\quad + \frac{\Gamma(q)\Lambda_2}{\Omega}\int_1^{\infty} h_1(s)\frac{\mathrm{d}s}{s} - \frac{\Gamma(p)\Gamma(q)}{\Omega}\sum_{j=1}^{n}\frac{\lambda_j}{\Gamma(p+\beta_j)}\int_1^{\xi}(\log\xi-\log s)^{p+\beta_j-1}h_1(s)\frac{\mathrm{d}s}{s}\\
&= \int_1^{\infty} G_3^*(t,s)h_2(s)\frac{\mathrm{d}s}{s} + \int_1^{\infty} G_4^*(t,s)h_1(s)\frac{\mathrm{d}s}{s},
\end{aligned}
$$

which shows that the second expression is also satisfied in (2.5), so we are finished.

For brevity, we introduce the following symbols and results:

$$
\begin{aligned}
M_1 &= \frac{1}{\Gamma(p)} + \frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j(\log\xi)^{p+\beta_j-1}}{\Gamma(p+\beta_j)} = \frac{\Gamma(q)}{\Omega},\\
M_2 &= \frac{\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i(\log\eta)^{q+\alpha_i-1}}{\Gamma(q+\alpha_i)} = \frac{\Lambda_1}{\Omega},\\
M_3 &= \frac{1}{\Gamma(q)} + \frac{\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i(\log\eta)^{q+\alpha_i-1}}{\Gamma(q+\alpha_i)} = \frac{\Gamma(p)}{\Omega},\\
M_4 &= \frac{\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j(\log\xi)^{p+\beta_j-1}}{\Gamma(p+\beta_j)} = \frac{\Lambda_2}{\Omega},\\
N_1 &= 1 + \frac{\Lambda_1\Gamma(p)}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j(\log\xi)^{p+\beta_j-1}}{\Gamma(p+\beta_j)} = \frac{\Gamma(p)\Gamma(q)}{\Omega},\\
N_2 &= \frac{\Gamma(p)\Gamma(q)}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i(\log\eta)^{q+\alpha_i-1}}{\Gamma(q+\alpha_i)} = \frac{\Gamma(p)\Lambda_1}{\Omega},\\
N_3 &= 1 + \frac{\Gamma(q)\Lambda_2}{\Omega}\sum_{i=1}^{m}\frac{\lambda_i(\log\eta)^{q+\alpha_i-1}}{\Gamma(q+\alpha_i)} = \frac{\Gamma(p)\Gamma(q)}{\Omega},
\end{aligned}
$$

$$N_4 = \frac{\Gamma(p)\Gamma(q)}{\Omega} \sum_{j=1}^{n} \frac{\sigma_j (\log \xi)^{p+\beta_j-1}}{\Gamma(p+\beta_j)} = \frac{\Gamma(q)\Lambda_2}{\Omega}.$$

□

Lemma 2.3 (see [23]). *The Green functions $G_k(t,s), k = 1,2,3,4$ defined in* (2.2) *has the following properties:*
(A1) $G_k(t,s)$ are continuous and $G_k(t,s) \geq 0$ for all $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+, k = 1,2,3,4$;
(A2) $\dfrac{G_k(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} \leq M_k$ for all $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+, k = 1,2,3,4$.

Lemma 2.4. *The Green functions $G_k(t,s)$ and $G_k^*(t,s), k = 1,2,3,4$ defined in* (2.2) *and* (2.5) *have the following properties:*
(B1) $G_k(t,s) \leq M_k(\log t)^{p-1}, k = 1,2; G_k(t,s) \leq M_k(\log t)^{q-1}, k = 3,4$ for $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$;
(B2) $0 \leq G_k^*(t,s) \leq N_k, k = 1,2,3,4$ for $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Proof. From (2.3) and (2.4), it is easy to see that

$$g_p(t,s) \leq \frac{(\log t)^{p-1}}{\Gamma(p)}, \; g_{\beta_j}^p(\xi,s) \leq (\log \xi)^{p+\beta_j-1},$$
$$g_{\alpha_i}^q(\eta,s) \leq (\log \eta)^{q+\alpha_i-1}, (t,s) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

and then

$$G_1(t,s) \leq [\frac{1}{\Gamma(p)} + \frac{\Lambda_1}{\Omega}\sum_{j=1}^{n}\frac{\sigma_j(\log\xi)^{p+\beta_j-1}}{\Gamma(p+\beta_j)}](\log t)^{p-1} = M_1(\log t)^{p-1}, (t,s) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

$$G_2(t,s) \leq \frac{\Gamma(q)}{\Omega} \sum_{i=1}^{m} \frac{\lambda_i(\log\eta)^{q+\alpha_i-1}}{\Gamma(q+\alpha_i)}(\log t)^{p-1} = M_2(\log t)^{p-1}, (t,s) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

In a analogous way, we can obtain $G_k(t,s) \leq M_k(\log t)^{q-1}$ for $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+, k = 3,4$, so property (B1) holds. From the Green functions $G_k^*(t,s), k = 1,2,3,4$ in Lemma 2.2, it is easy to observe that property (B2) holds.

Define two spaces of continuous functions on $\mathbb{R}_+$:

$$X = \Big\{x \in C(\mathbb{R}_+), {}^H D^{p-1}x \in C(\mathbb{R}_+) : \sup_{t\in\mathbb{R}_+} \frac{|x(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}} < +\infty,$$
$$\sup_{t\in\mathbb{R}_+} |{}^H D^{p-1}x(t)| < +\infty\Big\},$$
$$Y = \Big\{y \in C(\mathbb{R}_+), {}^H D^{q-1}y \in C(\mathbb{R}_+) : \sup_{t\in\mathbb{R}_+} \frac{|y(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}} < +\infty,$$
$$\sup_{t\in\mathbb{R}_+} |{}^H D^{q-1}y(t)| < +\infty\Big\}$$

equipped with the norms

$$\|x\|_X = \max\Big\{\sup_{t\in\mathbb{R}_+} \frac{|x(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}}, \sup_{t\in\mathbb{R}_+} |{}^H D^{p-1}x(t)|\Big\},$$
$$\|y\|_Y = \max\Big\{\sup_{t\in\mathbb{R}_+} \frac{|y(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}}, \sup_{t\in\mathbb{R}_+} |{}^H D^{q-1}y(t)|\Big\}.$$

□

Lemma 2.5 (see [21]). *$(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces.*

Moreover, it is easy to see the product space $(X \times Y, \|\cdot\|_{X\times Y})$ is also a Banach space with the norm

$$\|\cdot\|_{X\times Y} = \max\{\|x\|_X, \|y\|_Y\}.$$

Lemma 2.6 (see [22]). *Let $U \subset X$ be a bounded set. Then U is relatively compact in X if the following hold:*

(i) For any $x \in U$, $\dfrac{x(t)}{1+(\log t)^{p-1}+(\log t)^{q-1}}$ and ${}^H D^{p-1}x(t)$ are equicontinuous on any compact interval of $\mathbb{R}_+$;

(ii) For any $\varepsilon > 0$, there is a constant $C = C(\varepsilon) > 0$ such that

$$\left|\frac{x(t_1)}{1+(\log t_1)^{p-1}+(\log t_2)^{q-1}} - \frac{x(t_2)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}\right| < \varepsilon$$

and $|{}^H D^{p-1}x(t_1) - {}^H D^{p-1}x(t_2)| < \varepsilon$ for any $t_1, t_2 \geq C$ and $x \in U$.

3. Main results

Now we define the cone $P \subset X \times Y$ as

$$P = \{(x,y) \in X \times Y | x(t) \geq 0, y(t) \geq 0, {}^H D^{p-1}x(t) \geq 0, {}^H D^{q-1}y(t) \geq 0, t \in \mathbb{R}_+\},$$

and the operator $\mathcal{F} : P \to X \times Y$ as $\mathcal{F}(x,y)(t) = (\mathcal{F}_1(x,y)(t), \mathcal{F}_2(x,y)(t))$ for all $t \in \mathbb{R}_+$, where the operators $\mathcal{F}_1 : P \to X \times Y$ and $\mathcal{F}_2 : P \to X \times Y$ are given by

$$\begin{pmatrix} \mathcal{F}_1(x,y)(t) \\ \mathcal{F}_2(x,y)(t) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_1^{+\infty} G_1(t,s) f_{1(x,y)}(s) \frac{\mathrm{d}s}{s} + \int_1^{+\infty} G_2(t,s) f_{2(x,y)}(s) \frac{\mathrm{d}s}{s} \\ \displaystyle\int_1^{+\infty} G_3(t,s) f_{2(x,y)}(s) \frac{\mathrm{d}s}{s} + \int_1^{+\infty} G_4(t,s) f_{1(x,y)}(s) \frac{\mathrm{d}s}{s} \end{pmatrix}, \tag{3.1}$$

for $x, y \in P, t \in \mathbb{R}_+$, where

$$\begin{cases} f_{1(x,y)}(s) = f_1(s, x(s), y(s), {}^H D^{p-1}x(s), {}^H D^{q-1}y(s)), \\ f_{2(x,y)}(s) = f_2(s, x(s), y(s), {}^H D^{p-1}x(s), {}^H D^{q-1}y(s)). \end{cases}$$

From Lemma 2.2 and (3.1), for $x, y \in P, t \in \mathbb{R}_+$, we have

$$\begin{pmatrix} {}^H D^{\alpha-1}\mathcal{F}_1(x,y)(t) \\ {}^H D^{\beta-1}\mathcal{F}_2(x,y)(t) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_1^{+\infty} G_1^*(t,s) f_{1(x,y)}(s) \frac{\mathrm{d}s}{s} + \int_1^{+\infty} G_2^*(t,s) f_{2(x,y)}(s) \frac{\mathrm{d}s}{s} \\ \displaystyle\int_1^{+\infty} G_3^*(t,s) f_{2(x,y)}(s) \frac{\mathrm{d}s}{s} + \int_1^{+\infty} G_4^*(t,s) f_{1(x,y)}(s) \frac{\mathrm{d}s}{s} \end{pmatrix}. \tag{3.2}$$

From Lemma 2.1 it is clear that (x,y) is a pair of positive solutions for the fractional differential system (1.6) if and only if $(x,y) \in P$ is a pair of positive fixed points of the operator $\mathcal{F}$. We consider the existence of the fixed points of the operator $\mathcal{F}$.

Throughout this paper we assume that f_1, f_2 satisfy the following hypotheses:

(H1) $f_1, f_2 \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}_+)$ and $\Omega = \Gamma(p)\Gamma(q) - \Lambda_1\Lambda_2 > 0$.

(H2) The nonnegative functions $a_i(t), b_i(t) \in L[1,+\infty)(i=0,1,2,3,4)$ and the nonnegative constants $0 \leq \varsigma_k, \tau_k < 1(k=1,2,3,4)$ satisfy

$$|f_1(t,x,y,w,z)| \leq a_0(t) + a_1(t)|x|^{\varsigma_1} + a_2(t)|y|^{\varsigma_2} + a_3(t)|w|^{\varsigma_3} + a_4(t)|z|^{\varsigma_4},$$
$$x,y,w,z \in \mathbb{R}, \forall t \in \mathbb{R}_+,$$

with

$$\int_1^{+\infty} a_0(t)\frac{\mathrm{d}t}{t} = a_0^* < +\infty,$$
$$\int_1^{+\infty} a_1(t)[1+(\log t)^{p-1}+(\log t)^{q-1}]^{\varsigma_1}\frac{\mathrm{d}t}{t} = a_1^* < +\infty,$$
$$\int_1^{+\infty} a_2(t)[1+(\log t)^{p-1}+(\log t)^{q-1}]^{\varsigma_2}\frac{\mathrm{d}t}{t} = a_2^* < +\infty,$$
$$\int_1^{+\infty} a_3(t)\mathrm{d}t = a_3^* < +\infty, \int_1^{+\infty} a_4(t)\frac{\mathrm{d}t}{t} = a_4^* < +\infty,$$

and

$$|f_2(t,x,y,w,z)| \leq b_0(t) + b_1(t)|x|^{\tau_1} + b_2(t)|y|^{\tau_2} + b_3(t)|w|^{\tau_3} + b_4(t)|z|^{\tau_3},$$
$$x,y,w,z \in \mathbb{R}, \forall t \in \mathbb{R}_+,$$

with

$$\int_1^{+\infty} b_0(t)\frac{\mathrm{d}t}{t} = b_0^* < +\infty,$$
$$\int_1^{+\infty} b_1(t)[1+(\log t)^{p-1}+(\log t)^{q-1}]^{\tau_1}\frac{\mathrm{d}t}{t} = b_1^* < +\infty,$$
$$\int_1^{+\infty} b_2(t)[1+(\log t)^{p-1}+(\log t)^{q-1}]^{\tau_2}\frac{\mathrm{d}t}{t} = b_2^* < +\infty,$$
$$\int_1^{+\infty} b_3(t)\frac{\mathrm{d}t}{t} = b_3^* < +\infty, \int_1^{+\infty} b_4(t)\frac{\mathrm{d}t}{t} = b_3^* < +\infty.$$

(H3) $f_1(t,x,y,w,z)$ and $f_2(t,x,y,w,z)$ are increasing with respect to the variables x,y,w,z, and $f_1(t,0,0,0,0) \not\equiv 0, f_2(t,0,0,0,0) \not\equiv 0, \forall t \in \mathbb{R}_+$.

Lemma 3.1. *If hypotheses (H1) and (H2) are satisfied, then*

$$\int_1^{+\infty} |f_{1(x,y)}(s)|\frac{ds}{s} \leq a_0^* + \sum_{k=1}^{4} a_k^* ||(x,y)||_{X\times Y}^{\varsigma_k}, \forall (x,y) \in X \times Y,$$

and

$$\int_1^{+\infty} |f_{2(x,y)}(s)|\frac{ds}{s} \leq b_0^* + \sum_{k=1}^{4} b_k^* ||(x,y)||_{X\times Y}^{\tau_k}, \forall (x,y) \in X \times Y.$$

Proof. For all $(x,y) \in X \times Y$, by hypotheses (H1) and (H2) we have

$$\int_1^{+\infty} |f_{1(x,y)}(s)|\frac{\mathrm{d}s}{s}$$

$$
\begin{aligned}
&\le \int_1^{+\infty} \Big(a_0(s) + a_1(s)|x(s)|^{\varsigma_1} + a_2(s)|y(s)|^{\varsigma_2} + a_3(s)|{}^H D^{p-1} x(s))|^{\varsigma_3} \\
&\quad + a_4(s)|{}^H D^{q-1} y(s))|^{\varsigma_4} \Big) \frac{\mathrm{d}s}{s} \\
&\le a_0^* + \int_1^{+\infty} a_1(s))[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\varsigma_1} \frac{|x(s)|^{\varsigma_1}}{[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\varsigma_1}} \frac{\mathrm{d}s}{s} \\
&\quad + \int_1^{+\infty} a_2(s))[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\varsigma_2} \frac{|y(s)|^{\varsigma_2}}{[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\varsigma_2}} \frac{\mathrm{d}s}{s} \\
&\quad + \int_1^{+\infty} a_3(s)|{}^H D^{p-1} x(s)|^{\varsigma_3} \frac{\mathrm{d}s}{s} + \int_1^{+\infty} a_4(s)|{}^H D^{q-1} y(s)|^{\varsigma_4} \frac{\mathrm{d}s}{s} \\
&\le a_0^* + a_1^*||x||_X^{\varsigma_1} + a_2^*||y||_Y^{\varsigma_2} + a_3^*||x||_X^{\varsigma_3} + a_4^*||y||_Y^{\varsigma_4} \\
&\le a_0^* + \sum_{k=1}^4 a_k^* ||(x,y)||_{X\times Y}^{\varsigma_k}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_1^{+\infty} |f_{2(x,y)}(s)| \frac{\mathrm{d}s}{s} \\
&\le \int_1^{+\infty} \Big(b_0(s) + b_1(s)|x(s)|^{\tau_1} + b_2(s)|y(s)|^{\tau_2} + b_3(s)|{}^H D^{p-1} x(s))|^{\tau_3} \\
&\quad + b_4(s)|{}^H D^{q-1} y(s))|^{\tau_4} \Big) \frac{\mathrm{d}s}{s} \\
&\le b_0^* + \int_1^{+\infty} b_1(s))[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\tau_1} \frac{|x(s)|^{\tau_1}}{[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\tau_1}} \frac{\mathrm{d}s}{s} \\
&\quad + \int_1^{+\infty} b_2(s))[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\tau_2} \frac{|y(s)|^{\tau_2}}{[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\tau_2}} \frac{\mathrm{d}s}{s} \\
&\quad + \int_1^{+\infty} b_3(s)|{}^H D^{p-1} x(s)|^{\tau_3} \frac{\mathrm{d}s}{s} + \int_1^{+\infty} b_4(s)|{}^H D^{q-1} y(s)|^{\tau_4} \frac{\mathrm{d}s}{s} \\
&\le b_0^* + b_1^*||x||_X^{\tau_1} + b_2^*||y||_Y^{\tau_2} + b_3^*||x||_X^{\tau_3} + b_4^*||y||_y^{\tau_4} \\
&\le b_0^* + \sum_{k=1}^4 b_k^* ||(x,y)||_{X\times Y}^{\tau_k}.
\end{aligned}
$$

□

Lemma 3.2. *If hypotheses (H1) and (H2) are satisfied, then the operator* $\digamma : P \to P$ *is continuous and completely continuous.*

Proof. Since $G_k(t,s) \ge 0, G_k^*(t,s) \ge 0, k = 1,2,3,4$ and $f_1, f_2 \ge 0$, we obtain $\digamma_1(x,y)(t) \ge 0, \digamma_2(x,y)(t) \ge 0, {}^H D^{p-1}\digamma_1(x,y)(t) \ge 0, {}^H D^{q-1}\digamma_2(x,y)(t) \ge 0$ for any $(x,y) \in P,\ t \in \mathbb{R}_+$, so $\digamma : P \to P$.

Let $U = \{(x,y)|(x,y) \in P, ||(x,y)||_{X\times Y} \le \Delta\}$ for some $\Delta > 0$. For all $(x,y) \in U$, from Lemma 3.1, Lemma 2.3 and (3.1), we have

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}_+} \frac{|\digamma_1(x,y)(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}} \\
&\le \sup_{t\in\mathbb{R}_+} \Big| \int_1^{+\infty} \frac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} f_{1(x,y)}(s) \frac{\mathrm{d}s}{s} \Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\sup_{t\in J}\Big|\int_1^{+\infty}\frac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}f_{2(x,y)}(s)\frac{\mathrm{d}s}{s}\Big|\\
\le& M_1\int_1^{+\infty}|f_{1(x,y)}(s)|\frac{\mathrm{d}s}{s}+M_2\int_1^{+\infty}|f_{2(x,y)}(s)|\frac{\mathrm{d}s}{s}\\
\le&(M_1+M_2)\Big[a_0^*+\sum_{k=1}^{4}a_k^*||(x,y)||_{X\times Y}^{\varsigma_k}+b_0^*+\sum_{k=1}^{4}b_k^*||(x,y)||_{X\times Y}^{\tau_k}\Big]\\
\le&(M_1+M_2)\Big[a_0^*+b_0^*+\sum_{k=1}^{4}(a_k^*\Delta^{\varsigma_k}+b_k^*\Delta^{\tau_k})\Big],
\end{aligned}\tag{3.3}
$$

and from Lemma 2.3, Lemma 3.1 and (3.2) we have

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}_+}|{}^HD^{p-1}F_1(x,y)(t)|\\
\le&\sup_{t\in\mathbb{R}_+}\Big|\int_1^{\infty}G_1^*(t,s)f_{1(x,y)}(s)\frac{\mathrm{d}s}{s}\Big|+\sup_{t\in\mathbb{R}_+}\Big|\int_1^{\infty}G_2^*(t,s)f_{2(x,y)}(s)\frac{\mathrm{d}s}{s}\Big|\\
\le&N_1\int_1^{+\infty}|f_{1(x,y)}(s)|\frac{\mathrm{d}s}{s}+N_2\int_1^{+\infty}|f_{2(x,y)}(s)|\frac{\mathrm{d}s}{s}\\
\le&(N_1+N_2)\Big[a_0^*+\sum_{k=1}^{4}a_k^*||(x,y)||_{X\times Y}^{\varsigma_k}+b_0^*+\sum_{k=1}^{4}b_k^*||(x,y)||_{X\times Y}^{\tau_k}\Big]\\
\le&(N_1+N_2)\Big[a_0^*+b_0^*+\sum_{k=1}^{4}(a_k^*\Delta^{\varsigma_k}+b_k^*\Delta^{\tau_k})\Big].
\end{aligned}\tag{3.4}
$$

Then for all $(x,y)\in U$ we have

$$
\begin{aligned}
&||F_1(x,y)||_X\\
=&\max\Big\{\sup_{t\in\mathbb{R}_+}\frac{|F_1(x,y)(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}},\sup_{t\in\mathbb{R}_+}|{}^HD^{p-1}F_1(x,y)(t)|\Big\}\\
\le&\max\{M_1+M_2,N_1+N_2\}\Big[a_0^*+b_0^*+\sum_{k=1}^{4}(a_k^*\Delta^{\varsigma_k}+b_k^*\Delta^{\tau_k})\Big],
\end{aligned}\tag{3.5}
$$

and similarly

$$
||F_2(x,y)||_Y\le\max\{M_3+M_4,N_3+N_4\}\Big[a_0^*+b_0^*+\sum_{k=1}^{4}(a_k^*\Delta^{\varsigma_k}+b_k^*\Delta^{\tau_k})\Big],
$$

so

$$
\begin{aligned}
&||F(x,y)||_{X\times Y}\\
=&\max\Big\{\|F_1(x,y)\|_X,\|F_2(x,y)\|_Y\Big\}\\
\le&\max\Big\{M_1+M_2,N_1+N_2,M_3+M_4,N_3+N_4\Big\}\Big[a_0^*+b_0^*+\sum_{k=1}^{3}(a_k^*\Delta^{\varsigma_k}+b_k^*\Delta^{\tau_k})\Big],
\end{aligned}\tag{3.6}
$$

i.e. FU is uniformly bounded.

Next let $I \subset \mathbb{R}_+$ be any compact interval. For all $t_1, t_2 \in I, t_2 > t_1$ and $(x, y) \in U$, we have

$$\begin{aligned}&\Big|\frac{F_1(x,y)(t_2)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{F_1(x,y)(t_1)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\Big|\\ &\le\int_1^{+\infty}\Big|\frac{G_1(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{G_1(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\Big|\big|f_{1(x,y)}(s)\big|\frac{\mathrm{d}s}{s}\\ &\quad+\int_1^{+\infty}\Big|\frac{G_2(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{G_2(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\Big|\big|f_{2(x,y)}(s)\big|\frac{\mathrm{d}s}{s}.\end{aligned} \tag{3.7}$$

Note the functions $G_1(t_2, s)/(1+(\log t_2)^{p-1}+(\log t_2)^{q-1})-G_1(t_1, s)/(1+(\log t_1)^{p-1}+(\log t_1)^{q-1})$ and $G_2(t_2, s)/(1+(\log t_2)^{p-1}+(\log t_2)^{q-1})-G_2(t_1, s)/(1+(\log t_1)^{p-1}+(\log t_1)^{q-1})$ are uniformly continuous for any $(t_1, s), (t_2, s) \in I \times I$. In fact, for all $s \in I$ and $s \le t$, we have

$$\begin{aligned}&\frac{G_1(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{G_1(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\\ =&\frac{g_p(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}+\frac{\Lambda_1(\log t_2)^{p-1}}{\Omega[1+(\log t_2)^{p-1}+(\log t_2)^{q-1}]}\\ &\times\sum_{j=1}^{n}\frac{\sigma_j g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)}-\frac{g_p(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\\ &-\frac{\Lambda_1(\log t_1)^{p-1}}{\Omega[1+(\log t_1)^{p-1}+(\log t_1)^{q-1}]}\sum_{j=1}^{n}\frac{\sigma_j g^p_{\beta_j}(\xi,s)}{\Gamma(p+\beta_j)}\\ =&\frac{(\log t_2)^{p-1}-(\log t_2-\log s)^{p-1}}{\Gamma(p)[1+(\log t_2)^{p-1}+(\log t_2)^{q-1}]}+\frac{\Lambda_1(\log t_2)^{p-1}}{\Omega[1+(\log t_2)^{p-1}+(\log t_2)^{q-1}]}\\ &\times\sum_{j=1}^{n}\frac{\sigma_j[\log\xi)^{p+\beta_j-1}-(\log\xi-\log s)^{p+\beta_j-1}]}{\Gamma(p+\beta_j)}\\ &-\frac{(\log t_1)^{p-1}-(\log t_1-\log s)^{p-1}}{\Gamma(p)[1+(\log t_1)^{p-1}+(\log t_1)^{q-1}]}-\frac{\Lambda_1(\log t_1)^{p-1}}{\Omega[1+(\log t_1)^{p-1}+(\log t_1)^{q-1}]}\\ &\times\sum_{j=1}^{n}\frac{\sigma_j[\log\xi)^{p+\beta_j-1}-(\log\xi-\log s)^{p+\beta_j-1}]}{\Gamma(p+\beta_j)},\end{aligned}$$

so $G_1(t_2, s)/(1 + (\log t_2)^{p-1} + (\log t_2)^{q-1}) - G_1(t_1, s)/(1 + (\log t_1)^{p-1} + (\log t_1)^{q-1})$ is continuous in any compact interval I, so uniformly continuous for any $s \in I$. In a similar way, for all $s \in I$ and $s \le t$, we have

$$\begin{aligned}&\frac{G_2(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{G_2(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\\ =&\frac{\Gamma(q)(\log t_2)^{p-1}}{\Omega[1+(\log t_2)^{p-1}+(\log t_2)^{q-1}]}\sum_{i=1}^{m}\frac{\lambda_i g^q_{\alpha_i}(\eta,s)}{\Gamma(q+\alpha_i)}\\ &-\frac{\Gamma(q)(\log t_1)^{p-1}}{\Omega[1+(\log t_1)^{p-1}+(\log t_1)^{q-1}]}\sum_{i=1}^{m}\frac{\lambda_i g^q_{\alpha_i}(\eta,s)}{\Gamma(q+\alpha_i)}\end{aligned}$$

$$=\frac{\Gamma(q)(\log t_2)^{p-1}}{\Omega[1+(\log t_2)^{p-1}+(\log t_2)^{q-1}]}\sum_{i=1}^{m}\frac{\lambda_i[\log\eta)^{q+\alpha_i-1}-(\log\eta-\log s)^{q+\alpha_i-1}]}{\Gamma(q+\alpha_i)}$$
$$-\frac{\Gamma(q)(\log t_1)^{p-1}}{\Omega[1+(\log t_1)^{p-1}+(\log t_1)^{q-1}]}\sum_{i=1}^{m}\frac{\lambda_i[\log\eta)^{q+\alpha_i-1}-(\log\eta-\log s)^{q+\alpha_i-1}]}{\Gamma(q+\alpha_i)},$$

which is also uniformly continuous for any $s\in I$.

In addition, note

$$\frac{G_1(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{G_1(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}$$
$$=\frac{(\log t_2)^{p-1}}{\Gamma(p)[1+(\log t_2)^{p-1}+(\log t_2)^{q-1}]}$$
$$+\frac{\Lambda_1(\log t_2)^{p-1}}{\Omega[1+(\log t_2)^{p-1}+(\log t_2)^{q-1}]}\sum_{j=1}^{n}\frac{\sigma_j\log\xi)^{p+\beta_j-1}}{\Gamma(p+\beta_j)}$$
$$-\frac{(\log t_1)^{p-1}}{\Gamma(p)[1+(\log t_1)^{p-1}+(\log t_1)^{q-1}]}$$
$$-\frac{\Lambda_1(\log t_1)^{p-1}}{\Omega[1+(\log t_1)^{p-1}+(\log t_1)^{q-1}]}\sum_{j=1}^{n}\frac{\sigma_j\log\xi)^{p+\beta_j-1}}{\Gamma(p+\beta_j)},$$

is independent of s for $s\geq t$, so the function $G_1(t_2,s)/(1+(\log t_2)^{p-1}+(\log t_2)^{q-1})-G_1(t_1,s)/(1+(\log t_1)^{p-1}+(\log t_1)^{q-1})$ is uniformly continuous on $\mathbb{R}_+/I$. In a similar way, for all $s\in\mathbb{R}_+/I$ and $s\geq t$, we have

$$\frac{G_2(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{G_2(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}$$
$$=\frac{\Gamma(q)(\log t_2)^{p-1}}{\Omega[1+(\log t_2)^{p-1}+(\log t_2)^{q-1}]}\sum_{i=1}^{m}\frac{\lambda_i\log\eta)^{q+\alpha_i-1}}{\Gamma(q+\alpha_i)}$$
$$-\frac{\Gamma(q)(\log t_1)^{p-1}}{\Omega[1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\sum_{i=1}^{m}\frac{\lambda_i\log\eta)^{q+\alpha_i-1}}{\Gamma(q+\alpha_i)},$$

which is independent of s so the function $G_2(t_2,s)/(1+(\log t_2)^{p-1}+(\log t_2)^{q-1})-G_2(t_1,s)/(1+(\log t_1)^{p-1}+(\log t_1)^{q-1})$ is uniformly continuous on $\mathbb{R}_+/I$.

Thus, for all $s\in\mathbb{R}_+$ and $t_1,t_2\in I$, we have

$$\begin{aligned}&\forall\epsilon>0,\exists\delta(\epsilon)\quad\text{such that if}\quad|t_1-t_2|<\delta,\ \text{ then}\\&\Big|\frac{G_1(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{G_1(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\Big|<\epsilon,\\&\Big|\frac{G_2(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{G_2(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\Big|<\epsilon.\end{aligned}\tag{3.8}$$

From Lemma 3.1, (3.7) and (3.8), for all $s\in\mathbb{R}_+,\ (x,y)\in U$ and $t_1,t_2\in I$, we have

$$\Big|\frac{F_1(x,y)(t_2)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}}-\frac{F_1(x,y)(t_1)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\Big|$$

$$\leq\Big[a_0^* + \sum_{k=1}^{4} a_k^* \Delta^{\varsigma_k} + b_0^* + \sum_{k=1}^{4} b_k^* \Delta^{\tau_k}\Big]\epsilon,$$

so the function $\mathcal{F}_1(x,y)(t)/(1+(\log t)^{p-1}+(\log t)^{q-1})$ is equicontinuous on I.

Also

$${}^H D^{p-1}\mathcal{F}_1(x,y)(t) = \int_1^{+\infty} G_1^*(t,s) f_{1(x,y)}(s)\frac{\mathrm{d}s}{s} + \int_1^{+\infty} G_2^*(t,s) f_{2(x,y)}(s)\frac{\mathrm{d}s}{s}$$

and from the representations of the Green functions $G_1^*(t,s), G_2^*(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+)$ then ${}^H D^{p-1}\mathcal{F}_1(x,y)(t)$ is equicontinuous on I. In the same way we have $\mathcal{F}_2(x,y)(t)/(1+(\log t)^{p-1}+(\log t)^{q-1})$ and ${}^H D^{q-1}\mathcal{F}_2(x,y)(t)$ are equicontinuous. Thus hypothesis (i) of Lemma 2.6 is satisfied.

Next we show the operator $\mathcal{F}_1, \mathcal{F}_2$ are equiconvergent at $+\infty$. Since

$$\lim_{t\to+\infty} \frac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} = 0, \ \lim_{t\to+\infty} \frac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}} = 0,$$

then for any $\epsilon > 0$, there exists a sufficiently large constant $C = C(\epsilon) > 0$, such that for any $t_1, t_2 \geq C$ and $s \in \mathbb{R}_+$, we have

$$\Big|\frac{G_1(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} - \frac{G_1(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\Big| < \epsilon,$$
$$\Big|\frac{G_2(t_2,s)}{1+(\log t_2)^{p-1}+(\log t_2)^{q-1}} - \frac{G_2(t_1,s)}{1+(\log t_1)^{p-1}+(\log t_1)^{q-1}}\Big| < \epsilon.$$

From Lemma 3.1 and (3.7), we conclude that $\mathcal{F}_1(x,y)(t)/(1+(\log t)^{p-1}+(\log t)^{q-1})$ are equiconvergent at $+\infty$. Furthermore from the representations of the Green functions $G_1^*(t,s), G_2^*(t,s)$ we have that ${}^H D^{p-1}\mathcal{F}_1(x,y)(t)$ is equiconvergent at $+\infty$. Similarly, $\mathcal{F}_2(x,y)(t)/(1+(\log t)^{p-1}+(\log t)^{q-1})$ and ${}^H D^{q-1}\mathcal{F}_2(x,y)(t)$ are equiconvergent at $+\infty$. Thus hypothesis (ii) of Lemma 2.6 is hold.

From the above we can apply Lemma 2.6 so $\mathcal{F} : P \to P$ is completely continuous.

Next we prove $\mathcal{F} : P \to P$ is continuous. Let $(x_n, y_n), (x,y) \in P$, such that $(x_n, y_n) \to (x,y) (n \to \infty)$. Then $||(x_n,y_n)||_{X\times Y} < +\infty, ||(x,y)||_{X\times Y} < +\infty$. Similar to (3.3) and (3.4), we obtain

$$\sup_{t\in J} \frac{|\mathcal{F}_1(x_n,y_n)(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}}$$
$$\leq (M_1+M_2)\Big[a_0^* + b_0^* + \sum_{k=1}^{4}\big(a_k^*||(x_n,y_n)||_{X\times Y}^{\varsigma_k} + b_k^*||(x_n,y_n)||_{X\times Y}^{\tau_k}\big)\Big] < +\infty,$$

and

$$\sup_{t\in J} |{}^H D^{p-1}\mathcal{F}_1(x_n,y_n)(t)|$$
$$\leq (N_1+N_2)\Big[a_0^* + b_0^* + \sum_{k=1}^{4}\big(a_k^*||(x_n,y_n)||_{X\times Y}^{\varsigma_k} + b_k^*||(x_n,y_n)||_{X\times Y}^{\tau_k}\big)\Big] < +\infty.$$

Since the functions f_1, f_2 are continuous, we have

$$\lim_{n\to\infty} \frac{\mathcal{F}_1(x_n,y_n)(t)}{1+(\log t)^{p-1}+(\log t)^{q-1}}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\Big[\int_1^{+\infty}\frac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}f_{1(x_n,y_n)}(s)\frac{\mathrm{d}s}{s}\\
&\quad+\int_1^{+\infty}\frac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}f_{2(x_n,y_n)}(s)\frac{\mathrm{d}s}{s}\Big]\\
&=\int_1^{+\infty}\frac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}f_{1(x,y)}(s)\frac{\mathrm{d}s}{s}\\
&\quad+\int_1^{+\infty}\frac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}f_{2(x,y)}(s)\frac{\mathrm{d}s}{s}\\
&=\frac{\digamma_1(x,y)(t)}{1+(\log t)^{p-1}+(\log t)^{q-1}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty}{}^HD^{p-1}\digamma_1(x_n,y_n)(t)\\
&=\lim_{n\to\infty}\Big[\int_1^{+\infty}G_1^*(t,s)f_{1(x_n,y_n)}(s)\frac{\mathrm{d}s}{s}+\int_1^{+\infty}G_2^*(t,s)f_{2(x_n,y_n)}(s)\frac{\mathrm{d}s}{s}\Big]\\
&=\int_1^{\infty}G_1^*(t,s)f_{1(x,y)}(s)\frac{\mathrm{d}s}{s}+\int_1^{+\infty}G_2^*(t,s)f_{2(x,y)}(s)\frac{\mathrm{d}s}{s}\\
&={}^HD^{p-1}\digamma_1(x,y)(t).
\end{aligned}
$$

Then from the Lebesgue dominated convergence theorem

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}_+}\frac{|\digamma_1(x_n,y_n)(t)-\digamma_1(x,y)(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}}\\
&\le\sup_{t\in\mathbb{R}_+}\int_1^{+\infty}\frac{G_1(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}\big|f_{1(x_n,y_n)}(s)-f_{1(x,y)}(s)\big|\frac{\mathrm{d}s}{s}\\
&\quad+\sup_{t\in\mathbb{R}_+}\int_1^{+\infty}\frac{G_2(t,s)}{1+(\log t)^{p-1}+(\log t)^{q-1}}\big|f_{2(x_n,y_n)}(s)-f_{2(x,y)}(s)\big|\frac{\mathrm{d}s}{s}\\
&\le(M_1+M_2)\Big[\int_1^{+\infty}\big|f_{1(x_n,y_n)}(s)-f_{1(x,y)}(s)\big|\frac{\mathrm{d}s}{s}\\
&\quad+\int_1^{+\infty}\big|f_{2(x_n,y_n)}(s)-f_{2(x,y)}(s)\big|\frac{\mathrm{d}s}{s}\Big]\to 0\ \text{ as }\ n\to\infty;
\end{aligned}
$$

note

$$
\begin{aligned}
&\big|f_{1(x_n,y_n)}(s)-f_{1(x,y)}(s)\big|\\
&\le|f_{1(x_n,y_n)}(s)|+|f_{1(x,y)}(s)|\\
&\le 2a_0(s)+a_1(s))[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\varsigma_1}\big(||(x_n,y_n)||_{X\times Y}^{\varsigma_1}+||(x,y)||_{X\times Y}^{\varsigma_1}\big)\\
&\quad+a_2(s))[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\varsigma_2}\big(||(x_n,y_n)||_{X\times Y}^{\varsigma_2}+||(x,y)||_{X\times Y}^{\varsigma_2}\big)\\
&\quad+a_3(s)\big(||(x_n,y_n)||_{X\times Y}^{\varsigma_3}+||(x,y)||_{X\times Y}^{\varsigma_3}\big)\\
&\quad+a_4(s)\big(||(x_n,y_n)||_{X\times Y}^{\varsigma_4}+||(x,y)||_{X\times Y}^{\varsigma_4}\big)
\end{aligned}
$$

and

$$
\big|f_{2(x_n,y_n)}(s)-f_{2(x,y)}(s)\big|
$$

$$
\begin{aligned}
&\leq 2b_0(s)+b_1(s))[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\tau_1}\big(||(x_n,y_n)||^{\tau_1}_{X\times Y}+||(x,y)||^{\tau_1}_{X\times Y}\big)\\
&\quad +b_2(s))[1+(\log s)^{p-1}+(\log s)^{q-1}]^{\tau_2}\big(||(x_n,y_n)||^{\tau_2}_{X\times Y}+||(x,y)||^{\tau_2}_{X\times Y}\big)\\
&\quad +b_3(s)\big(||(x_n,y_n)||^{\tau_3}_{X\times Y}+||(x,y)||^{\tau_3}_{X\times Y}\big)\\
&\quad +b_4(s)\big(||(x_n,y_n)||^{\tau_4}_{X\times Y}+||(x,y)||^{\tau_4}_{X\times Y}\big).
\end{aligned}
$$

Also note from the Lebesgue dominated convergence theorem that

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}_+}|{}^HD^{p-1}\digamma_1(x_n,y_n)(t)-{}^HD^{p-1}\digamma_1(x,y)(t)|\\
\leq &\sup_{t\in\mathbb{R}_+}\int_1^{+\infty}G_1^*(t,s)\big|f_{1(x_n,y_n)}(s)-f_{1(x,y)}(s)\big|\frac{\mathrm{d}s}{s}\\
&+\sup_{t\in\mathbb{R}_+}\int_1^{+\infty}G_2^*(t,s)\big|f_{2(x_n,y_n)}(s)-f_{2(x,y)}(s)\big|\frac{\mathrm{d}s}{s}\\
\leq &(N_1+N_2)\Big[\int_1^{+\infty}\big|f_{1(x_n,y_n)}(s)-f_{1(x,y)}(s)\big|\frac{\mathrm{d}s}{s}\\
&+\int_1^{+\infty}\big|f_{2(x_n,y_n)}(s)-f_{2(x,y)}(s)\big|\frac{\mathrm{d}s}{s}\Big]\to 0\ \text{ as }\ n\to\infty.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|\digamma_1(x_n,y_n)-\digamma_1(x,y)\|_X\\
=&\max\Big\{\sup_{t\in\mathbb{R}_+}\frac{|\digamma_1(x_n,y_n)(t)-\digamma_1(x,y)(t)|}{1+(\log t)^{p-1}+(\log t)^{q-1}},\\
&\sup_{t\in\mathbb{R}_+}|{}^HD^{p-1}\digamma_1(x_n,y_n)(t)-{}^HD^{p-1}\digamma_1(x,y)(t)|\Big\}\to 0, n\to\infty,
\end{aligned}
$$

so $\digamma_1$ is continuous. In a similar way we can show that $\digamma_2$ is continuous. Thus $\digamma$ is continuous.

Consequently $\digamma: P\to P$ is continuous and completely continuous.

Define a partial order on the product space:

$$
\begin{pmatrix}x_1\\ y_1\end{pmatrix}\geq\begin{pmatrix}x_2\\ y_2\end{pmatrix}
$$

if $x_1(t)\geq x_2(t),\ y_1(t)\geq y_2(t),\ {}^HD^{p-1}x_1(t)\geq {}^HD^{p-1}x_2(t),\ {}^HD^{q-1}y_1(t)\geq D^{q-1}y_2(t)$, $t\in\mathbb{R}_+$. □

Theorem 3.3. *Suppose hypotheses (H1),(H2) and (H3) are satisfied. Then system* (1.6) *have two pairs of positive solutions* (x^*,y^*) *and* (w^*,z^*) *satisfying* $0\leq\|(x^*,y^*)\|_{X\times Y}\leq R$ *and* $0\leq\|(w^*,z^*)\|_{X\times Y}\leq R$ *with* $\lim_{n\to\infty}(x_n,y_n)=(x^*,y^*)$ *and* $\lim_{n\to\infty}(w_n,z_n)=(w^*,z^*)$, *where* R *is a given real constant,* (x_n,y_n) *and* (w_n,z_n) *can be defined via the following two pairs of iterative schemes*

$$
\begin{pmatrix}x_n(t)\\ y_n(t)\end{pmatrix}=\begin{pmatrix}\digamma_1(x_{n-1},y_{n-1})(t)\\ \digamma_2(x_{n-1},y_{n-1})(t)\end{pmatrix}, n=1,2,\ldots,\quad \textit{with}\quad \begin{pmatrix}x_0(t)\\ y_0(t)\end{pmatrix}=\begin{pmatrix}R(\log t)^p\\ R(\log t)^q\end{pmatrix} \tag{3.9}
$$

and

$$\begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix} = \begin{pmatrix} \digamma_1(w_{n-1}, z_{n-1})(t) \\ \digamma_2(w_{n-1}, z_{n-1})(t) \end{pmatrix}, n = 1, 2, \ldots, \quad with \quad \begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.10)$$

Furthermore they have the following monotonicity properties:

$$\begin{aligned} \begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix} \leq \begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} w^*(t) \\ z^*(t) \end{pmatrix} \\ \leq \cdots \leq \begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix} \\ \leq \cdots \leq \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \leq \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \leq \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \begin{pmatrix} {}^H D^{p-1} w_0(t) \\ {}^H D^{q-1} z_0(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{p-1} w_1(t) \\ {}^H D^{q-1} z_1(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} {}^H D^{p-1} w_n(t) \\ {}^H D^{q-1} z_n(t) \end{pmatrix} \\ \leq \cdots \leq \begin{pmatrix} {}^H D^{p-1} w^*(t) \\ {}^H D^{q-1} z^*(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} {}^H D^{p-1} x^*(t) \\ {}^H D^{q-1} y^*(t) \end{pmatrix} \\ \leq \cdots \leq \begin{pmatrix} {}^H D^{p-1} x_n(t) \\ {}^H D^{q-1} y_n(t) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} {}^H D^{p-1} x_2(t) \\ {}^H D^{q-1} y_2(t) \end{pmatrix} \\ \leq \begin{pmatrix} {}^H D^{p-1} x_1(t) \\ {}^H D^{q-1} y_1(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{p-1} x_0(t) \\ {}^H D^{q-1} y_0(t) \end{pmatrix}. \end{aligned} \quad (3.12)$$

Proof. Recall $\digamma(P) \subset P$.

Let

$$R \geq \max\Big\{10\Lambda a_0^*, 10\Lambda b_0^*, (10\Lambda a_k^*)^{1/(1-\varsigma_k)}, (10\Lambda b_k^*)^{1/(1-\tau_k)}, k = 1, 2, 3, \Big\}$$

where

$$\Lambda = \max\Big\{M_1 + M_2, N_1 + N_2, M_3 + M_4, N_3 + N_4\Big\}.$$

Set $U_R = \{(x, y) \in P : ||(x, y)||_{X\times Y} \leq R\}$. For any $(x, y) \in U_R$, similar to (3.3) and (3.4), we have

$$\sup_{t\in\mathbb{R}_+} \frac{|\digamma_1(x, y)(t)|}{1 + (\log t)^{p-1} + (\log t)^{q-1}} \leq \Lambda\Big[a_0^* + b_0^* + \sum_{k=1}^{4}(a_k^* R^{\varsigma_k} + b_k^* R^{\tau_k})\Big] \leq R$$

and

$$\sup_{t\in\mathbb{R}_+} |{}^H D^{p-1} x(t)| \leq \Lambda\Big[a_0^* + b_0^* + \sum_{k=1}^{4}(a_k^* R^{\varsigma_k} + b_k^* R^{\tau_k})\Big] \leq R,$$

which infers that $||\digamma_1(x, y)||_X \leq R$. In a similar way, $||\digamma_2(x, y)||_Y \leq R$ for all $(x, y) \in U_R$. Thus for all $(x, y) \in U_R$ we have

$$||\digamma(x, y)||_{X\times Y} = \Big\{\|\digamma_1(x, y)\|_X, \|\digamma_2(x, y)\|_Y\Big\} \leq R.$$

That is, $\digamma(U_R) \subset U_R$.

From (3.9) and (3.10), we see that $(x_0(t), y_0(t)), (w_0(t), z_0(t)) \in U_R$. Now we define two pairs of (x_n, y_n) and (w_n, z_n) as $(x_n, y_n) = \digamma(x_{n-1}, y_{n-1}), (w_n, z_n) = \digamma(w_{n-1}, z_{n-1})$ for $n=1, 2, \ldots$. Since $\digamma(U_R) \subset U_R$, we can see that $(x_n, y_n), (w_n, z_n) \in \digamma(U_R)$ for $n=1, 2, \ldots$.

From Lemma 2.5, (3.1) and (3.9), for arbitrarily $t \in \mathbb{R}_+$, we have

$$
\begin{aligned}
x_1(t) = \digamma_1(x_0, y_0)(t) &\leq \Lambda\Big[a_0^* + \sum_{k=1}^{4} a_k^* R^{\varsigma_k} + b_0^* + \sum_{k=1}^{4} b_k^* R^{\tau_k}\Big](\log t)^{p-1} \\
&\leq R(\log t)^{p-1} = x_0(t)
\end{aligned}
$$

and

$$
\begin{aligned}
y_1(t) = \digamma_2(x_0, y_0)(t) &\leq \Lambda\Big[b_0^* + \sum_{k=1}^{4} b_k^* R^{\tau_k} + a_0^* + \sum_{k=1}^{4} a_k^* R^{\varsigma_k}\Big](\log t)^{q-1} \\
&\leq R(\log t)^{q-1} = y_0(t),
\end{aligned}
$$

that is

$$
\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} \digamma_1(x_0, y_0)(t) \\ \digamma_2(x_0, y_0)(t) \end{pmatrix} \leq \begin{pmatrix} R(\log t)^{p-1} \\ R(\log t)^{q-1} \end{pmatrix} = \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix}. \tag{3.13}
$$

From (3.13) and Lemma 2.4, we see

$$
\begin{aligned}
{}^H D^{p-1} x_1(t) &= {}^H D^{p-1} \digamma_1(x_0, y_0)(t) \\
&= \int_0^{+\infty} G_1^*(t, s) f_{1(x_0, y_0)}(s) \frac{\mathrm{d}s}{s} + \int_0^{+\infty} G_2^*(t, s) f_{2(x_0, y_0)}(s) \frac{\mathrm{d}s}{s} \\
&\leq \Lambda\Big[a_0^* + \sum_{k=1}^{4} a_k^* R^{\varsigma_k} + b_0^* + \sum_{k=1}^{4} b_k^* R^{\tau_k}\Big] \leq R = {}^H D^{p-1} x_0(t), \\
{}^H D^{q-1} y_1(t) &= {}^H D^{q-1} \digamma_2(x_0, y_0)(t) \\
&= \int_0^{+\infty} G_3^*(t, s) f_{2(x_0, y_0)}(s) \frac{\mathrm{d}s}{s} + \int_0^{+\infty} G_4^*(t, s) f_{1(u_0, v_0)}(s) \frac{\mathrm{d}s}{s} \\
&\leq \Lambda\Big[a_0^* + \sum_{k=1}^{3} a_k^* R^{\varsigma_k} + b_0^* + \sum_{k=1}^{3} b_k^* R^{\tau_k}\Big] \leq R = {}^H D^{q-1} y_0(t),
\end{aligned}
$$

that is

$$
\begin{pmatrix} {}^H D^{p-1} x_1(t) \\ {}^H D^{q-1} y_1(t) \end{pmatrix} = \begin{pmatrix} {}^H D^{p-1} \digamma_1(x_0, y_0)(t) \\ {}^H D^{q-1} \digamma_2(x_0, y_0)(t) \end{pmatrix} \leq \begin{pmatrix} R \\ R \end{pmatrix} = \begin{pmatrix} {}^H D^{p-1} x_0(t) \\ {}^H D^{q-1} y_0(t) \end{pmatrix}. \tag{3.14}
$$

From (3.13) and (3.14) and (H3) for all $t \in \mathbb{R}_+$, we do the second iteration

$$
\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \digamma_1(x_1, y_1)(t) \\ \digamma_2(x_1, y_1)(t) \end{pmatrix} \leq \begin{pmatrix} \digamma_1(x_0, y_0)(t) \\ \digamma_2(x_0, y_0)(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix},
$$

$$
\begin{pmatrix} {}^H D^{p-1} x_2(t) \\ {}^H D^{q-1} y_2(t) \end{pmatrix} = \begin{pmatrix} {}^H D^{p-1} \digamma_1(x_1, y_1)(t) \\ {}^H D^{q-1} \digamma_2(x_1, y_1)(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{p-1} \digamma_1(x_0, y_0)(t) \\ {}^H D^{q-1} \digamma_2(x_0, y_0)(t) \end{pmatrix}
$$

$$= \begin{pmatrix} {}^H D^{p-1} x_1(t) \\ {}^H D^{q-1} y_1(t) \end{pmatrix}.$$

For $t \in \mathbb{R}_+$, recursively, we have

$$\begin{pmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{pmatrix} \le \begin{pmatrix} x_n(t) \\ y_n(t)) \end{pmatrix}, \begin{pmatrix} {}^H D^{p-1} x_{n+1}(t) \\ {}^H D^{q-1} y_{n+1}(t) \end{pmatrix} \le \begin{pmatrix} {}^H D^{p-1} x_n(t) \\ {}^H D^{p-1} x_n(t) \end{pmatrix}. \tag{3.15}$$

Now $\mathcal{F} : P \to P$ completely continuous guarantees a $(x^*, y^*) \in U_R$ and a subsequence S of $\mathbf{N}$ with $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$ in S. This with (3.15) enables us to deduce that $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$. Now the continuity of $\mathcal{F}$ and $(x_{n+1}, y_{n+1}) = \mathcal{F}(x_n, y_n)$ yields $(x^*, y^*) = \mathcal{F}(x^*, y^*)$, i.e. (x^*, y^*) is a pair of fixed point of $\mathcal{F}$.

For $\{(w_n, z_n)\}_{n=0}^{\infty}$, via a similar argument, we have

$$\begin{aligned}
\begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix} &= \begin{pmatrix} \mathcal{F}_1(w_0, z_0)(t) \\ \mathcal{F}_2(w_0, z_0)(t) \end{pmatrix} \\
&= \begin{pmatrix} \int_1^{+\infty} G_1(t,s) f_{1(w_0,z_0)}(s) \frac{\mathrm{d}s}{s} + \int_1^{+\infty} G_2(t,s) f_{2(w_0,z_0)}(s) \mathrm{d}s \\ \int_0^{+\infty} G_3(t,s) f_{2(w_0,z_0)}(s) \frac{\mathrm{d}s}{s} + \int_1^{+\infty} G_4(t,s) f_{2(w_0,z_0)}(s) \frac{\mathrm{d}s}{s} \end{pmatrix} \\
&\ge \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} {}^H D^{p-1} w_1(t) \\ {}^H D^{q-1} z_1(t) \end{pmatrix} &= \begin{pmatrix} \int_1^{+\infty} G_1^*(t,s) f_{1(w_0,z_0)}(s) \frac{\mathrm{d}s}{s} + \int_1^{+\infty} G_2^*(t,s) f_{2(w_0,z_0)}(s) \frac{\mathrm{d}s}{s} \\ \int_1^{+\infty} G_3^*(t,s) f_{2(w_0,z_0)}(s) \frac{\mathrm{d}s}{s} + \int_1^{+\infty} G_4^*(t,s) f_{1(w_0,z_0)}(s) \frac{\mathrm{d}s}{s} \end{pmatrix} \\
&\ge \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} {}^H D^{p-1} w_0(t) \\ {}^H D^{q-1} z_0(t) \end{pmatrix}.
\end{aligned}$$

From (H3) we have

$$\begin{pmatrix} w_2(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(w_1, z_1)(t) \\ \mathcal{F}_2(w_1, z_1)(t) \end{pmatrix} \ge \begin{pmatrix} \mathcal{F}_1(w_0, z_0)(t) \\ \mathcal{F}_2(w_0, z_0)(t) \end{pmatrix} = \begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix},$$

$$\begin{aligned}
\begin{pmatrix} {}^H D^{p-1} w_2(t) \\ {}^H D^{q-1} z_2(t) \end{pmatrix} &= \begin{pmatrix} {}^H D^{p-1} \mathcal{F}_1(w_1, z_1)(t) \\ {}^H D^{q-1} \mathcal{F}_2(w_1, z_1)(t) \end{pmatrix} \ge \begin{pmatrix} {}^H D^{p-1} \mathcal{F}_1(w_0, z_0)(t) \\ {}^H D^{q-1} \mathcal{F}_2(w_0, z_0)(t) \end{pmatrix} \\
&= \begin{pmatrix} {}^H D^{p-1} w_1(t) \\ {}^H D^{q-1} z_1(t) \end{pmatrix}.
\end{aligned}$$

Similar, for $n = 1, 2, \ldots$ and $t \in \mathbb{R}_+$, we have

$$\begin{pmatrix} w_{n+1}(t) \\ z_{n+1}(t) \end{pmatrix} \geq \begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix}, \quad \begin{pmatrix} {}^H D^{p-1} w_{n+1}(t) \\ {}^H D^{q-1} z_{n+1}(t) \end{pmatrix} \geq \begin{pmatrix} {}^H D^{p-1} w_n(t) \\ {}^H D^{q-1} z_n(t) \end{pmatrix}.$$

Using $(w_{n+1}, z_{n+1}) = \mathcal{F}(w_n, z_n)$ and the complete continuity property of the operator $\mathcal{F}$, we see that $(w_n, z_n) \to (w^*, z^*)$ and $\mathcal{F}(w^*, z^*) = (w^*, z^*)$. Thus (w^*, z^*) is also one pair fixed points of $\mathcal{F}$.

Finally we show that (x^*, y^*) and (w^*, z^*) are two pairs of extreme positive solutions for the system (1.6). Assume that $(\xi(t), \eta(t))$ is a pair of positive solutions for the system (1.6). Then $\mathcal{F}(\xi(t), \eta(t)) = (\xi(t), \eta(t))$ and

$$\begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} R(\log t)^{p-1} \\ R(\log t)^{q-1} \end{pmatrix} = \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix},$$
$$\begin{pmatrix} {}^H D^{p-1} w_0(t) \\ {}^H D^{q-1} z_0(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{p-1} \xi(t) \\ {}^H D^{q-1} \eta(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{p-1} x_0(t) \\ {}^H D^{q-1} y_0(t) \end{pmatrix}.$$

From the monotone property of the operator $\mathcal{F}$, we have

$$\begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(w_0, z_0)(t) \\ \mathcal{F}_2(w_0, z_0)(t) \end{pmatrix} \leq \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} \mathcal{F}_1(x_0, y_0)(t) \\ \mathcal{F}_2(x_0, y_0)(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix},$$
$$\begin{pmatrix} {}^H D^{p-1} w_1(t) \\ {}^H D^{q-1} z_1(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{p-1} \xi(t) \\ {}^H D^{q-1} \eta(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{p-1} x_1(t) \\ {}^H D^{q-1} y_1(t) \end{pmatrix}.$$

Repeating the above process, we have

$$\begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix} \leq \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix}$$
$$\begin{pmatrix} {}^H D^{p-1} w_n(t) \\ {}^H D^{q-1} z_n(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{p-1} \xi(t) \\ {}^H D^{q-1} \eta(t) \end{pmatrix} \leq \begin{pmatrix} {}^H D^{p-1} x_n(t) \\ {}^H D^{q-1} y_n(t) \end{pmatrix}.$$

From $\lim_{n\to\infty} (w_n, z_n) = (w^*, z^*)$ and $\lim_{n\to\infty} (x_n, y_n) = (x^*, y^*)$, the results in (3.11) and (3.12) hold.

Now $f_1(t, 0, 0, 0, 0) \neq 0$ and $f_2(t, 0, 0, 0, 0) \neq 0$ for all $t \in \mathbb{R}_+$, so the system (1.6) has no zero solution. From (3.11) and (3.12), it is clear that (w^*, z^*) and (x^*, y^*) are two pairs of extreme positive solutions for system (1.6), which are given via two pairs of monotone iterative schemes in (3.9) and (3.10). Therefore the proof is completed. □

Example 3.4. Consider the following coupled fractional differential system on an infinite interval

$$\begin{cases} -{}^H D^{1.5}x(t) = f_1(t, x(t), y(t), {}^H D^{p-1}x(t), {}^H D^{q-1}y(t)), \\ -{}^H D^{1.1}y(t) = f_2(t, x(t), y(t), {}^H D^{p-1}x(t), {}^H D^{q-1}y(t)), \\ x(1) = 0, \ {}^H D^{0.5}x(+\infty) = \frac{1}{10} I^{\frac{1}{2}} y(\frac{7}{4}) + \frac{1}{20} I^{\frac{3}{2}} y(\frac{7}{4}), \\ y(1) = 0, \ D^{1.1}y(+\infty) = \frac{1}{8} I^{\frac{1}{3}} x(\frac{1}{3}) + \frac{1}{7} I^{\frac{2}{3}} x(\frac{1}{3}) + \frac{1}{12} I^{\frac{4}{3}} x(\frac{1}{3}). \end{cases} \tag{3.16}$$

where $p = 1.5, q = 1.1$ and

$$\begin{aligned} f_{1(x,y)} &= \frac{2t}{(9+t)^2} + \frac{te^{-t}|x(t)|^{0.1}}{[1 + (\log t)^{0.5} + (\log t)^{0.1}]^{0.1}} + \frac{te^{-2t}|y(t)|^{0.3}}{[1 + (\log t)^{0.5} + (\log t)^{0.1}]^{0.3}} \\ &\quad + te^{-10t}|{}^H D^{0.5}x(t)|^{0.4} + \frac{t|{}^H D^{0.1}y(t)|^{0.1}}{1+t^2}, \\ f_{2(x,y)} &= \frac{t}{20(1+t^2)} + \frac{te^{-3t}|x(t)|^{0.2}}{[1 + (\log t)^{0.5} + (\log t)^{0.1}]^{0.2}} + \frac{te^{-4t}|y(t)|^{0.4}}{[1 + (\log t)^{0.5} + (\log t)^{0.1}]^{0.4}} \\ &\quad + \frac{3t^3|{}^H D^{0.5}x(t)|^{0.2}}{(3+t^3)^2} + \frac{t|{}^H D^{0.1}y(t)|^{0.3}}{10(1+t^2)}, \end{aligned}$$

and $\varsigma_1 = 0.1, \varsigma_2 = 0.3, \varsigma_3 = 0.4, \varsigma_4 = 0.1, \tau_1 = 0.2, \tau_2 = 0.4, \tau_3 = 0.2, \tau_4 = 0.3, \lambda_1 = \frac{1}{10}, \lambda_2 = \frac{1}{20}, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{3}{2}, \eta = \frac{7}{4}, \sigma_1 = \frac{1}{8}, \sigma_2 = \frac{1}{7}, \sigma_3 = \frac{1}{12}, \beta_1 = \frac{1}{3}, \beta_2 = \frac{2}{3}, \beta_3 = \frac{4}{3}, \xi = \frac{1}{3}, \Gamma(1.5) = 0.886227, \Gamma(1.1) = 0.951351, \Lambda_1 = 0.088302, \Lambda_2 = 0.141864, \Omega = \Gamma(1.5)\Gamma(1.1) - \Lambda_1\Lambda_2 > 0$. Thus hypothesis (H1) holds.

Also we have

$$\begin{aligned} |f_{1(x,y)}| &\le \frac{2t}{(9+t)^2} + \frac{te^{-t}|x|^{0.1}}{[1 + (\log t)^{0.5} + (\log t)^{0.1}]^{0.1}} + \frac{te^{-2t}|y|^{0.3}}{[1 + (\log t)^{0.5} + (\log t)^{0.1}]^{0.3}} \\ &\quad + te^{-10t}|w|^{0.4} + \frac{t|z|^{0.1}}{1+t^2} \\ &= a_0(t) + a_1(t)|x|^{0.1} + a_2(t)|y|^{0.3} + a_3(t)|w|^{0.4} + a_4(t)|z|^{0.1}, \\ |f_{2(x,y)}| &\le \frac{t}{20(1+t^2)} + \frac{te^{-3t}|x|^{0.2}}{[1+(\log t)^{0.5}+(\log t)^{0.1}]^{0.2}} + \frac{te^{-4t}|y|^{0.4}}{[1+(\log t)^{0.5}+(\log t)^{0.1}]^{0.4}} \\ &\quad + \frac{3t^3|w|^{0.2}}{(3+t^3)^2} + \frac{t|z|^{0.3}}{10(1+t^2)} \\ &= b_0(t) + b_1(t)|x|^{0.2} + b_2(t)|y|^{0.4} + b_3(t)|w|^{0.2} + b_3(t)|z|^{0.3} \end{aligned}$$

and

$$\begin{aligned} a_0^* &= \int_1^{+\infty} a_0(t)\frac{\mathrm{d}t}{t} = 0.200000, \\ a_1^* &= \int_1^{+\infty} a_1(t)[1 + (\log t)^{0.5} + (\log t)^{0.1}]^{0.1}\frac{\mathrm{d}t}{t} = 0.367879, \\ a_2^* &= \int_1^{+\infty} a_2(t)[1 + (\log t)^{0.5} + (\log t)^{0.1}]^{0.3}\frac{\mathrm{d}t}{t} = 0.183940, \\ a_3^* &= \int_1^{+\infty} a_3(t)\frac{\mathrm{d}t}{t} = 0.036788, \quad a_4^* = \int_1^{+\infty} a_4(t)\frac{\mathrm{d}t}{t} = 0.785398, \end{aligned}$$

$$b_0^* = \int_1^{+\infty} b_0(t)\frac{\mathrm{d}t}{t} = 0.039200,$$
$$b_1^* = \int_1^{+\infty} b_1(t)[1 + (\log t)^{0.5} + (\log t)^{0.1}]^{0.2}\frac{\mathrm{d}t}{t} = 0.122626,$$
$$b_2^* = \int_1^{+\infty} b_2(t)[1 + (\log t)^{0.5} + (\log t)^{0.1}]^{0.4}\frac{\mathrm{d}t}{t} = 0.091970,$$
$$b_3^* = \int_1^{+\infty} b_3(t)\frac{\mathrm{d}t}{t} = 0.250000, \quad b_4^* = \int_1^{+\infty} b_4(t)\frac{\mathrm{d}t}{t} = 0.078540$$

so hypothesis (H2) holds.

It is easy to verify that f_1, f_2 are increasing with respect to the variables x, y, w, z and $f_1(t, 0, 0, 0, 0) \not\equiv 0, f_2(t, 0, 0, 0, 0) \not\equiv 0, \forall t \in \mathbb{R}_+$. Thus hypothesis (H3) holds. From Theorem 3.3, it follows that the fractional differential system (3.16) have two pairs of positive solutions, which can be given via two pairs of monotone iterative schemes in (3.9) and (3.10).

4. Conclusion

In this paper, we apply the monotone iterative technique to study a class of Hadamard type fractional differential systems in an infinite interval, which involves lower-order coupled Hadamard type fractional derivatives of unknown functions and coupled Hadamard type fractional integral boundary conditions. Two pairs of explicit monotone iterative schemes converging to the extremal positive solutions are presented.

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