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ON THE INVARIANCE OF GENERALIZED QUASIARITHMETIC MEANS*

Qian Zhang[1] and Lin Li[2,†]

Abstract The generalized quasiarithmetic mean is generated by two functions and one probability measure, and includes quasiarithmetic, Cauchy and Bajraktarević meas. In this paper, we investigate the invariance of the arithmetic mean with respect to generalized quasiarithmetic means and get some solutions of it under high-order differentiability assumptions.

Keywords Bajraktarević mean, generalized quasiarithmetic means, invariance equation, functional equation.

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1. Introduction

Throughout this paper, let $I \subseteq \mathbb{R}$ be a nonempty open interval. In the sequel, the classes of continuous strictly monotone and continuous positive real-valued functions defined on I will be denoted by $\mathcal{CM}(I)$ and $\mathcal{CP}(I)$, respectively.

The *weighted quasi-arithmetic mean* $A_{\varphi,\lambda} : I^2 \to I$ is defined as

$$A_{\varphi;\lambda}(x,y) := \varphi^{-1}\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right), \quad x, y \in I,$$

where $\lambda \in (0,1)$ and $\varphi \in \mathcal{CM}(I)$.

Let $t, s \in \mathbb{R}_+$, the *weighted two-variable Bajraktarević mean* $B_{f,g} : I^2 \to I$ [2, 15, 17] is defined by

$$B_{f,g;t,s}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{tf(x) + sf(y)}{tg(x) + sg(y)}\right), \qquad x, y \in I,$$

where $f, g : I \to \mathbb{R}$ are two continuous functions such that $g \in \mathcal{CP}(I)$ and the ratio function $f/g \in \mathcal{CM}(I)$. Letting $t = s$ and $\alpha = \frac{f}{g}$, the above Bajraktarević mean can been rewritten by

$$B_g^{\alpha}(x,y) = \alpha^{-1}\left(\frac{g(x)}{g(x)+g(y)}\alpha(x) + \frac{g(y)}{g(x)+g(y)}\alpha(y)\right),$$

which is called *quasi-arithmetic means with weight function.*

†The corresponding author.
Email: qianmo2008@126.com(Q. Zhang), matlinl@zjxu.edu.cn(L. Li)
[1]School of Mathematics and Physics, Southwest University of Science and Technology, Qinglong Avenue, 621010 Mianyang, China
[2]Faculty of Mathematics, Jiaxing University, Guangqiong Road, 314001 Jiaxing, China
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The research on the equality of Bajraktarević means has experienced a long history. As early as 1958, Bajraktarević [2] solved the equality of n-variable quasi-arithmetic means with weight function for a fixed $n \geq 3$, and presented the necessary and sufficient conditions under twice differentiable assumption. Aczél etc [1] obtained the same result without differentiability conditions when the equality holds for all $n \geq 2$, $n \in \mathbb{N}$. The case of fixed $n = 2$ is much more difficult and allows considerably more solutions. Losonczi [8] found 32 new families of solutions under six-times differentiable supposition. More new characterizations of the equality of two-variable Bajraktarević means were obtained by Losonczi etc [11] under the same regularity assumptions. Recently, Páles etc [18] obtained the same conclusion under only first-order differentiability. Meanwhile, Grünwald etc [4] considered the equality problem of generalized Bajraktarević means.

The invariance equation of Bajraktarević means has been investigated extensively. Domsta etc [3] first considered the invariance of arithmetic mean with respect to a special Bajraktarević mean. Further, Jarczyk considered a general class of Bajraktarević means [6] and the invariance of arithmetic mean with weight function was also given [7]. Later, some special invariance problems of Bajraktarević means were considered by Matkowski [13,14]. Páles etc [17] solved the invariance equation for weighted nonsymmetric Bajraktarević means, while Grünwald etc [5] discussed the invariance of the arithmetic mean with respect to generalized Bajraktarević means.

Recall that $f, g : I \to \mathbb{R}$ are two continuous functions on the interval I with $g \in \mathcal{CP}(I)$, $f/g \in \mathcal{CM}(I)$, and μ is a probability measure on the Borel subsets of $[0, 1]$. The two-variable *generalized quasiarithmetic means* [9, 19] $M_{f,g;\mu} : I^2 \to I$ is defined by

$$M_{f,g;\mu}(x, y) := \left(\frac{f}{g}\right)^{-1} \left(\frac{\int_0^1 f(tx + (1-t)y)d\mu(t)}{\int_0^1 g(tx + (1-t)y)d\mu(t)}\right), \quad x, y \in I.$$

Clearly, this mean is a common generalization of Bajraktarević and Cauchy means. Equalities and inequalities of two-variable functional means generated by the same and different measures have been investigated by Losonczi etc [9–11, 19]. Moreover, Páles etc [16] studied the local and global comparison problem of generalized Bajraktarević means.

We say that two pairs of functions $(f, g) : I \to \mathbb{R}^2$ and $(h, k) : I \to \mathbb{R}^2$ are *equivalent* if there exist constants a, b, c, d with $ad \neq cd$ such that

$$h = af + bg, \quad k = cf + dg,$$

and it can be written by $(f, g) \sim (h, k)$.

In this paper, we will consider the invariance of the arithmetic mean with respect to generalized quasiarithmetic means. Our aim is solving the following functional equation

$$\left(\frac{f}{g}\right)^{-1} \left(\frac{\int_0^1 f(tx+(1-t)y)d\mu(t)}{\int_0^1 g(tx+(1-t)y)d\mu(t)}\right) + \left(\frac{h}{k}\right)^{-1} \left(\frac{\int_0^1 h(tx+(1-t)y)d\mu(t)}{\int_0^1 k(tx+(1-t)y)d\mu(t)}\right) = x + y, \tag{1.1}$$

where $x, y \in I$, $f, g, h, k : I \to \mathbb{R}$ are four continuous functions with $g, k \in \mathcal{CP}(I)$, $f/g, h/k \in \mathcal{CM}(I)$, and μ is a probability measure over the Borel sets of $[0, 1]$.

2. Auxiliary results

Given a Borel probability measure μ on the interval $[0,1]$, we define the kth moment and kth centralized moment of μ by

$$\hat{\mu}_k := \int_0^1 t^k d\mu(t) \quad \text{and} \quad \mu_k := \int_0^1 (t-\hat{\mu}_1)^k d\mu(t), \quad k \in \mathbb{N} \cup \{0\},$$

respectively. Clearly, $\hat{\mu}_0 = \mu_0 = 1$, $\mu_1 = 0$ and $\mu_{2k} \geq 0$ for $k \in \mathbb{N}$. Moreover, $\mu_{2k} = 0$ holds if and only if μ is the Dirac measure $\delta_{\hat{\mu}_1}$.

In order to describe the regularity conditions related to the two unknown functions f, g generating the mean $M_{f,g;\mu}$, we introduce some notations. The class $\mathcal{C}_0(I)$ consists of all those pairs (f,g) of continuous functions $f, g : I \to \mathbb{R}$ such that $g \in \mathcal{CP}(I)$ and $f/g \in \mathcal{CM}(I)$. For $n \in \mathbb{N}$, we say that the pair (f,g) is in the class $\mathcal{C}_n(I)$ if f, g are $n-$times continuously differentiable functions such that $g \in \mathcal{CP}(I)$ and the function $f'g - fg'$ does not vanish anywhere on I. Obviously, this latter condition implies that f/g is strictly monotone, i.e., $f/g \in \mathcal{CM}(I)$.

For $(f,g) \in \mathcal{C}_2(I)$, we also introduce the notation

$$\Phi_{f,g} := \frac{W^{2,0}_{f,g}}{W^{1,0}_{f,g}}, \quad \Psi_{f,g} := -\frac{W^{2,1}_{f,g}}{W^{1,0}_{f,g}},$$

where the $(i,j)-$order Wronskian operator $W^{i,j}$ is defined in terms of ith and jth derivatives by

$$W^{i,j}_{f,g} := \begin{vmatrix} f^{(i)} & f^{(j)} \\ g^{(i)} & g^{(j)} \end{vmatrix}.$$

Lemma 2.1 (Lemma 5, [17]). *Let $(f,g) \in \mathcal{C}_2(I)$. Then f, g are solutions of the second-order differential equation*

$$y'' = \Phi_{f,g} y' + \Psi_{f,g} y.$$

In what follows, we need some auxiliary results from [11] about explicit formulae for the high-order directional derivatives of $M_{f,g;\mu}$ at the diagonal points of the Cartesian product $I \times I$. Given a pair $(f,g) \in \mathcal{C}_0(I)$ and a fixed element $x \in I$, define the function $m_x := m_{x;f,g;\mu}$ in a neighborhood of origin by

$$m_x(u) = m_{x;f,g;\mu}(u) := M_{f,g;\mu}(x + (1-\hat{\mu}_1)u, x - \hat{\mu}_1 u), \tag{2.1}$$

where $\hat{\mu}_1$ denotes the first moment of the measure μ.

Lemma 2.2 (Proposition 6, [11]). *Let $n \in \mathbb{N}, (f,g) \in \mathcal{C}_n(I)$, and μ be a Borel probability measure on $[0,1]$. Then, for fixed $x \in I$, the function m_x defined by (2.1) is $n-$times continuously differentiable at the origin and*

$$\sum_{i=0}^{n} \binom{n}{i} \mu_i \begin{vmatrix} f^{(i)}(x) & (f \circ m_x)^{(n-i)}(0) \\ g^{(i)}(x) & (g \circ m_x)^{(n-i)}(0) \end{vmatrix} = 0.$$

Furthermore, $m_x(0) = x$ and in the cases $n = 1,2,3,4,5,$ we have

$$m_x'(0) = 0,$$

$$\begin{aligned}
m''_x(0) &= \mu_2\Phi_{f,g}(x),\\
m'''_x(0) &= \mu_3(\Phi'_{f,g}+\Phi^2_{f,g}+\Psi_{f,g})(x),\\
m^{(4)}(0) &= -3\mu_2^2(\Phi^3_{f,g}+2\Phi_{f,g}\Psi_{f,g})(x)+\mu_4(\Phi''_{f,g}+3\Phi'_{f,g}\Phi_{f,g}+\Phi^3_{f,g}\\
&\quad +2\Phi_{f,g}\Psi_{f,g}+2\Psi'_{f,g})(x),\\
m^{(5)}(0) &= -10\mu_2\mu_3(\Phi^2_{f,g}\Phi'_{f,g}+\Phi^4_{f,g}+(\Phi'_{f,g}+3\Phi^2_{f,g})\Psi_{f,g}+\Phi_{f,g}\Psi'_{f,g}+\Psi^2_{f,g})(x)\\
&\quad +\mu_5(\Phi'''_{f,g}+4\Phi''_{f,g}\Phi_{f,g}+3\Phi'^2_{f,g}+6\Phi'_{f,g}\Phi^2_{f,g}+\Phi^4_{f,g}\\
&\quad +(4\Phi'_{f,g}+3\Phi^2_{f,g})\Psi_{f,g}+5\Phi_{f,g}\Psi'_{f,g}+\Psi^2_{f,g}+3\Psi''_{f,g})(x).
\end{aligned}$$

According to Lemma 2.2, we get the following results.

Lemma 2.3. *Let μ be a Borel probability measure on $[0,1]$, $(f,g),(h,k)\in\mathcal{C}_1(I)$. If equation (1.1) holds, then*

$$\hat{\mu}_1=\frac{1}{2}.$$

Proof. Let $\Delta(I):=\{(x,x)|x\in I\}$ be the diagonal of I^2, and $U\subseteq I^2$ be an open set containing a dense subset D of $\Delta(I)$ such that equation (1.1) holds at every point of U. Let $x\in I$ be fixed satisfying $(x,x)\in D$. Define

$$U_x:=\{u\in\mathbb{R}|(x+(1-\hat{\mu}_1)u,x-\hat{\mu}_1u)\in U\}.$$

Then U_x is a neighbourhood of 0, and equation (1.1) holds on U implies that, for any $u\in U_x$,

$$m_{x;f,g;\mu}(u)+m_{x;h,k;\mu}(u)=2x+(1-2\hat{\mu}_1)u. \tag{2.2}$$

Differentiating equation (2.2) with respect to u, then substituting $u=0$, we get

$$m'_{x;f,g;\mu}(0)+m'_{x;h,k;\mu}(0)=1-2\hat{\mu}_1.$$

Applying the first-order formula of Lemma 2.2, the above equation implies that $\hat{\mu}_1=\frac{1}{2}$. □

Lemma 2.4. *Let μ be a Borel probability measure on $[0,1]$, $(f,g),(h,k)\in\mathcal{C}_2(I)$. If equation (1.1) holds, then*

$$\mu_2(\Phi_{f,g}+\Phi_{h,k})=0. \tag{2.3}$$

Proof. Differentiating equation (2.2) twice with respect to u, then substituting $u=0$, we get

$$m''_{x;f,g;\mu}(0)+m''_{x;h,k;\mu}(0)=0.$$

By the second-order formula of Lemma 2.2, we obtain (2.3). □

Firstly, we consider the case $\mu_2=0$.

Theorem 2.1. *Let μ be a Borel probability measure on $[0,1]$ with $\mu_2=0$. Then the invariance equation (1.1) holds for every $(f,g),(h,k)\in\mathcal{C}_0(I)$.*

Proof. If $\mu_2=0$, then $\mu=\delta_{\hat{\mu}_1}=\delta_{1/2}$, and

$$M_{f,g;\mu}(x,y)=\left(\frac{f}{g}\right)^{-1}\left(\frac{f\left(\frac{x+y}{2}\right)}{g\left(\frac{x+y}{2}\right)}\right)=\frac{x+y}{2}.$$

So the invariance equation (1.1) holds for arbitrary functions f,g,h,k. □

Next, for the case $\mu_2\neq 0$, equation (2.3) leads to

$$\Phi_{f,g}=-\Phi_{h,k}=:\Phi. \tag{2.4}$$

Lemma 2.5. *Let μ be a Borel probability measure on $[0,1]$ with $\mu_2 \neq 0$, and $(f,g),(h,k) \in \mathcal{C}_3(I)$. If equation (1.1) holds, then*

$$\mu_3(\Psi_{f,g} + \Psi_{h,k} + 2\Phi^2) = 0. \tag{2.5}$$

Proof. Differentiating equation (2.2) three times with respect to u, then substituting $u = 0$, we get

$$m'''_{x;f,g;\mu}(0) + m'''_{x;h,k;\mu}(0) = 0.$$

According to the third-order formula of Lemma 2.2, we obtain

$$\mu_3(\Phi'_{f,g} + \Phi^2_{f,g} + \Psi_{f,g}) + \mu_3(\Phi'_{h,k} + \Phi^2_{h,k} + \Psi_{h,k}) = 0,$$

which implies (2.5) by (2.4). □

Lemma 2.6. *Let μ be a Borel probability measure on $[0,1]$ with $\mu_2 \neq 0$, and $(f,g),(h,k) \in \mathcal{C}_4(I)$. If equation (1.1) holds, then*

$$3\mu_4\Phi\Phi' + (\mu_4 - 3\mu_2^2)\Phi(\Psi_{f,g} - \Psi_{h,k}) + \mu_4(\Psi'_{f,g} + \Psi'_{h,k}) = 0. \tag{2.6}$$

Proof. Differentiating equation (2.2) four times with respect to u, then substituting $u = 0$, we get

$$m^{(4)}_{x;f,g;\mu}(0) + m^{(4)}_{x;h,k;\mu}(0) = 0.$$

The fourth-order formula of Lemma 2.2 implies that

$$\begin{aligned} &-3\mu_2^2(\Phi^3_{f,g} + 2\Phi_{f,g}\Psi_{f,g}) + \mu_4(\Phi''_{f,g} + 3\Phi'_{f,g}\Phi_{f,g} + \Phi^3_{f,g} + 2\Phi_{f,g}\Psi_{f,g} + 2\Psi'_{f,g}) \\ &-3\mu_2^2(\Phi^3_{h,k} + 2\Phi_{h,k}\Psi_{h,k}) + \mu_4(\Phi''_{h,k} + 3\Phi'_{h,k}\Phi_{h,k} + \Phi^3_{h,k} + 2\Phi_{h,k}\Psi_{h,k} + 2\Psi'_{h,k}) = 0. \end{aligned} \tag{2.7}$$

Combining (2.3) and (2.7), we get (2.6). □

Furthermore, if we assume $\mu_3 \neq 0$, combining Lemmas 2.5-2.6, we get the following result.

Lemma 2.7. *Let μ be a Borel probability measure on $[0,1]$ with $\mu_3 \neq 0$, and $(f,g),(h,k) \in \mathcal{C}_4(I)$. If equation (1.1) holds, then*

$$\mu_4\Phi\Phi' - (\mu_4 - 3\mu_2^2)\Phi(\Psi_{f,g} - \Psi_{h,k}) = 0. \tag{2.8}$$

Proof. The condition $\mu_3 \neq 0$ implies that $\mu_2 \neq 0$ is also valid. Using Lemma 2.5, we have

$$\Psi_{f,g} + \Psi_{h,k} = -2\Phi^2. \tag{2.9}$$

Then, substituting the above equation into (2.6), we obtain (2.8). □

3. Results for some special denominator functions

Usually, it is very difficult to solve equation (1.1) by applying Lemma 2.6 directly. Fortunately, inspired by the idea of Jarczyk [6], it can be done under additional conditions imposed on the generators g and k. Actually, assuming $k = g$ satisfies the equation of the harmonic oscillator

$$y'' = py, \tag{3.1}$$

for some $p \in \mathbb{R}$. Then, we introduce the sine and cosine type functions $S_p, C_p : \mathbb{R} \to \mathbb{R}$ by

$$S_p(x) := \begin{cases} \sin(\sqrt{-p}x), & p < 0, \\ x, & p = 0, \\ \sinh(\sqrt{p}x), & p > 0. \end{cases} \quad C_p(x) := \begin{cases} \cos(\sqrt{-p}x), & p < 0, \\ 1, & p = 0, \\ \cosh(\sqrt{p}x), & p > 0. \end{cases}$$

Due to basic results on the second-order linear differential equations, the functions S_p and C_p given above form a fundamental system of solutions for the differential equation (3.1).

Theorem 3.1. *Let μ be a Borel probability measure on $[0,1]$ with $\mu_2 \neq 0$, and $(f,g),(h,k) \in \mathcal{C}_4(I)$. Assume $k = g$ satisfying equation (3.1). If equation (1.1) holds, then there exists $r \in \mathbb{R}$ such that*

$$f = g \int \frac{1}{g^2} exp(\int rg^\beta), \quad h = g \int \frac{1}{g^2} exp(\int -rg^\beta), \tag{3.2}$$

where $\beta := \frac{2(\mu_4 - 3\mu_2^2)}{3\mu_4}$.

Proof. Since equation (3.1) holds, by the definitions of $\Phi_{f,g}$ and $\Psi_{f,g}$ we get

$$\Psi_{f,g} = -\frac{\begin{vmatrix} f'' & f' \\ g'' & g' \end{vmatrix}}{\begin{vmatrix} f' & f \\ g' & g \end{vmatrix}} = -\frac{\begin{vmatrix} f'' & f' \\ pg & g' \end{vmatrix}}{\begin{vmatrix} f' & f \\ g' & g \end{vmatrix}} = -\frac{g'}{g}\Phi_{f,g} + p.$$

Similarly, we have

$$\Psi_{h,k} = -\frac{g'}{g}\Phi_{h,k} + p,$$

by the fact $k = g$ and (3.1).

Using (2.4) and the above two equalities, we have

$$\Psi_{f,g} + \Psi_{h,k} = 2p, \quad \Psi_{f,g} - \Psi_{h,k} = -2 \cdot \frac{g'}{g}\Phi,$$

and thus $\Psi'_{f,g} + \Psi'_{h,k} = 0$. Consequently, (2.6) becomes

$$3\mu_4 \Phi\Phi' - 2(\mu_4 - 3\mu_2^2)\frac{g'}{g}\Phi^2 = 0,$$

which implies that there exists some $r \in \mathbb{R}$ such that

$$\Phi = rg^{\frac{2(\mu_4 - 3\mu_2^2)}{3\mu_4}}.$$

Hence, we obtain

$$\Phi_{f,g} = \frac{(W^{1,0}_{f,g})'}{W^{1,0}_{f,g}} = rg^\beta, \quad \Phi_{h,k} = \frac{(W^{1,0}_{h,k})'}{W^{1,0}_{h,k}} = -rg^\beta,$$

where $\beta = \frac{2(\mu_4 - 3\mu_2^2)}{3\mu_4}$.

Since $W_{f,g}^{1,0} = \left(\frac{f}{g}\right)' \cdot g^2$ and $W_{h,k}^{1,0} = \left(\frac{h}{k}\right)' \cdot k^2$, integrating the above equations, we get (3.2). $\square$

In what follows, we will restrict to the basic solutions to the equation of harmonic oscillator, i.e. consider the functions

(H1) $g(x) = x$ for $x \in I \subset (0, \infty)$;

(H2) $g(x) = e^x$ for $x \in I$;

(H3) $g(x) = \cos x$ for $x \in I \subset (0, \pi/2)$.

Remark 3.1. Let $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$, $f, g, h, k : I \to \mathbb{R}$ be four continuous functions with $g, k \in \mathcal{CP}(I)$, $f/g, h/k \in \mathcal{CM}(I)$, and μ be a probability measure over the Borel sets of $[0, 1]$. Then the pairs $(f, g), (h, k)$ satisfy equation (1.1) if and only if the pairs $(f_{a,b}, g_{a,b}), (h_{a,b}, k_{a,b})$, where $I_{a,b} := \{a \in \mathbb{R} : ax + b \in I\}$ and $f_{a,b}, g_{a,b}, h_{a,b}, k_{a,b} : I_{a,b} \to \mathbb{R}$ are defined by

$$f_{a,b}(x) = f(ax + b), \quad g_{a,b}(x) = g(ax + b),$$

and

$$h_{a,b}(x) = h(ax + b), \quad k_{a,b}(x) = k(ax + b),$$

satisfy the equation

$$\begin{aligned} &\left(\frac{f_{a,b}}{g_{a,b}}\right)^{-1} \left(\frac{\int_0^1 f_{a,b}(tx + (1-t)y) d\mu(t)}{\int_0^1 g_{a,b}(tx + (1-t)y) d\mu(t)}\right) \\ &+ \left(\frac{h_{a,b}}{k_{a,b}}\right)^{-1} \left(\frac{\int_0^1 h_{a,b}(tx + (1-t)y) d\mu(t)}{\int_0^1 k_{a,b}(tx + (1-t)y) d\mu(t)}\right) = x + y. \end{aligned}$$

According to Remark 3.1, for the differential equation (3.1) we only need to consider the fundamental system S_p and C_p. We may omit the number p in the functions S_p, C_p and the proof of functions $x \mapsto e^{-x}, x \mapsto \sin x = \cos(\pi/2 - x)$. Substituting $k = g = x, e^x$ or $\cos x$ into (3.2), after a simple calculation, we get the following result directly.

Theorem 3.2. *Let μ be a Borel probability measure on $[0, 1]$ with $\mu_2 \neq 0$, and $(f, g), (h, k) \in \mathcal{C}_4(I)$. Assuming that $k = g$ satisfies one of conditions (H1)-(H3). If equation (1.1) holds, letting $\beta = \frac{2(\mu_4 - 3\mu_2^2)}{3\mu_4}$, then*

(i) for case (H1), if $\beta \neq -1$, there exist $b_1, b_2 \in \mathbb{R}_+, r \in \mathbb{R}$ such that

$$f(x) = x \int b_1 x^{-2} exp(\frac{r}{\beta + 1} x^{\beta+1}), \quad h(x) = x \int b_2 x^{-2} exp(-\frac{r}{\beta + 1} x^{\beta+1}),$$

where $x \in I$ and if $\beta = -1$, there exist $b_1, b_2 \in \mathbb{R}_+, c_1, c_2, r \in \mathbb{R}$ such that

$$f(x) = \begin{cases} b_1 x^r + c_1 x, & r \neq 1, \\ b_1 x \ln x + c_1 x, & r = 1, \end{cases} \qquad h(x) = \begin{cases} b_2 x^{-r} + c_2 x, & r \neq -1, \\ b_2 x \ln x + c_2 x, & r = -1, \end{cases}$$

where $x \in I$.

(ii) for case (H2), if $\beta \neq 0$, there exist $b_1, b_2 \in \mathbb{R}_+, r \in \mathbb{R}$ such that

$$f(x) = e^x \int b_1 e^{-2x} exp(\frac{r}{\beta} e^{\beta x}), \quad h(x) = e^x \int b_2 e^{-2x} exp(-\frac{r}{\beta} e^{\beta x}), \quad x \in I,$$

and if $\beta = 0$, there exist $b_1, b_2 \in \mathbb{R}_+, c_1, c_2, r \in \mathbb{R}$ such that

$$f(x) = \begin{cases} (\frac{b_1}{r-2} e^{(r-2)x} + c_1) e^x, & r \neq 2, \\ (b_1 x + c_1) e^x, & r = 2, \end{cases} \quad x \in I,$$

$$h(x) = \begin{cases} (\frac{b_2}{-r-2} e^{(-r-2)x} + c_2) e^x, & r \neq -2, \\ (b_2 x + c_2) e^x, & r = -2, \end{cases} \quad x \in I.$$

(iii) for case (H3), there exists $r \in \mathbb{R}$ such that

$$f(x) = \cos x \int \frac{1}{\cos^2 x} \exp(\int r \cos^\beta x), \quad x \in I,$$
$$h(x) = \cos x \int \frac{1}{\cos^2 x} \exp(\int -r \cos^\beta x), \quad x \in I.$$

4. Results for the case $\mu_3 \neq 0$

In this section, for the case $\mu_3 \neq 0$, we will get some results of equation (1.1) under the assumption that $\mu_5 \neq 10\mu_3\mu_2$ as follows.

Theorem 4.1. *Let μ be a Borel probability measure on $[0, 1]$ with $\mu_3 \neq 0$, $\mu_4 = 3\mu_2^2$, and $\mu_5 \neq 10\mu_2\mu_3$, $(f, g), (h, k) \in \mathcal{C}_5(I)$. If equation (1.1) holds, then one of the following alternatives holds:*

(i) When $\mu_5 = 2\mu_2\mu_3$, there exist $p, q \in \mathbb{R}$ such that

$$(f, g) \sim (e^{\frac{p}{2}x} \cdot S_{\frac{p^2}{4}}, e^{\frac{p}{2}x} \cdot C_{\frac{p^2}{4}}), \quad (h, k) \sim (e^{-\frac{p}{2}x} \cdot S_{-\frac{7p^2}{4}}, e^{-\frac{p}{2}x} \cdot C_{-\frac{7p^2}{4}}).$$

(ii) When $\mu_5 \neq 2\mu_2\mu_3$, there exist $p, q \in \mathbb{R}$ such that f, g and h, k are solutions of the following differential equations

$$y'' = py' + \frac{2p^2 q e^{p\kappa x}}{1 - q e^{p\kappa x}} y \quad \text{and} \quad y'' = -py' + \frac{-2p^2}{1 - q e^{p\kappa x}} y, \tag{4.1}$$

respectively, where

$$\kappa = -\frac{2(\mu_5 - 10\mu_3\mu_2)}{\mu_5 - 2\mu_3\mu_2}.$$

Proof. Differentiating equation (2.2) five times with respect to u, then substituting $u = 0$, we get

$$m^{(5)}_{x;f,g;\mu}(0) + m^{(5)}_{x;h,k;\mu}(0) = 0.$$

The fifth-order formula of Lemma 2.2 implies that

$$- 10\mu_3\mu_2(2\Phi^4 + \Phi'(\Psi_{f,g} - \Psi_{h,k}) + 3\Phi^2(\Psi_{f,g} + \Psi_{h,k}) + \Phi(\Psi'_{f,g} - \Psi'_{h,k})$$
$$+ (\Psi^2_{f,g} + \Psi^2_{h,k})) + \mu_5(8\Phi''\Phi + 6\Phi'^2 + 2\Phi^4 + 4\Phi'(\Psi_{f,g} - \Psi_{h,k}) + 3\Phi^2(\Psi_{f,g}$$

$$+\Psi_{h,k}) + 5\Phi(\Psi'_{f,g} - \Psi'_{h,k}) + (\Psi^2_{f,g} + \Psi^2_{h,k}) + 3(\Psi''_{f,g} + \Psi''_{h,k})) = 0. \tag{4.2}$$

Since $\mu_4 = 3\mu_2^2$, using Lemma 2.7 and $\mu_4 \neq 0$, we get $\Phi\Phi' = 0$. It follows that there exists $p \in \mathbb{R}$ such that

$$\Phi(x) = p,\ \Psi_{f,g}(x) + \Psi_{h,k}(x) = -2p^2,\ x \in I,$$

and thus the equation (4.2) becomes

$$5(\mu_5 - 2\mu_3\mu_2)p\Psi'_{f,g} + (\mu_5 - 10\mu_3\mu_2)(\Psi^2_{f,g} + 2p^2\Psi_{f,g}) = 0. \tag{4.3}$$

Since $\mu_5 \neq 10\mu_2\mu_3$, in order to solve the above differential equation, we will discuss two cases.

(i) For the case $\mu_5 = 2\mu_2\mu_3$, since $\Psi_{f,g} + \Psi_{h,k} = -2p^2$ we get

$$\Psi_{f,g} = 0, \quad \Psi_{h,k} = -2p^2,$$

or

$$\Psi_{f,g} = -2p^2, \quad \Psi_{h,k} = 0.$$

Due to the symmetry of (f,g) and (h,k) in (1.1) and the arbitrarily chosen of p, it suffices to consider one of the above two cases, that is, f,g and h,k are respectively solutions of the following differential equations

$$y'' = py' \quad \text{and} \quad y'' = -py' - 2p^2y, \tag{4.4}$$

by Lemma 2.1 and the fact that $\Phi_{f,g} = -\Phi_{h,k} = p$.

Let $y = \tilde{y}e^{\frac{p}{2}x}$, then $y' = \tilde{y}'e^{\frac{p}{2}x} + \frac{p}{2}\tilde{y}e^{\frac{p}{2}x}, y'' = \tilde{y}''e^{\frac{p}{2}x} + p\tilde{y}'e^{\frac{p}{2}x} + \frac{p^2}{4}\tilde{y}e^{\frac{p}{2}x}$. Substituting them into the first equation of (4.4), we get that

$$\tilde{y}'' = \frac{p^2}{4}\tilde{y}.$$

Hence, $(f \cdot e^{-\frac{p}{2}x}, g \cdot e^{-\frac{p}{2}x}) \sim (S_{\frac{p^2}{4}}, C_{\frac{p^2}{4}})$, which implies $(f,g) \sim (e^{\frac{p}{2}x} \cdot S_{\frac{p^2}{4}}, e^{\frac{p}{2}x} \cdot C_{\frac{p^2}{4}})$. A completely analogous argument shows that $(h,k) \sim (e^{-\frac{p}{2}x} \cdot S_{-\frac{7p^2}{4}}, e^{-\frac{p}{2}x} \cdot C_{-\frac{7p^2}{4}})$.

(ii) For the case $\mu_5 \neq 2\mu_2\mu_3$, it follows from (4.3) that there exists $q \in \mathbb{R}$ such that

$$\Psi_{f,g}(x) = \frac{2p^2qe^{p\kappa x}}{1 - qe^{p\kappa x}}, \quad \Psi_{h,k}(x) = \frac{-2p^2}{1 - qe^{p\kappa x}},$$

where $p \in \mathbb{R}$ and $\kappa := -\frac{2(\mu_5 - 10\mu_3\mu_2)}{5(\mu_5 - 2\mu_3\mu_2)}$. Using Lemma 2.1, we obtain (4.1). $\square$

Theorem 4.2. *Let μ be a Borel probability measure on $[0,1]$ with $\mu_3 \neq 0$, $\mu_4 \neq 3\mu_2^2$ and $\mu_5 \neq 10\mu_2\mu_3$, $(f,g),(h,k) \in \mathcal{C}_5(I)$. If equation (1.1) holds, then one of the following alternatives holds:*

(i) The pairs $(f,g),(h,k)$ satisfy

$$(f,g) \sim (1,x),\ (h,k) \sim (1,x). \tag{4.5}$$

(ii) There exists $c \in \mathbb{R}$ such that f, g and h, k are solutions of the following differential equations

$$y'' = \Phi(x)y' + (\alpha\Phi'(x) - \Phi^2(x))y \text{ and } y'' = -\Phi(x)y' + (-\alpha\Phi'(x) - \Phi^2(x))y,$$

respectively, where α is defined by (4.8) and the function Φ satisfies either

$$\Phi(x) = \pm\frac{1}{\sqrt{\frac{r}{q}x + c}}, \quad x \in I$$

for case

$$\mu_5 = \frac{10\mu_2\mu_3\mu_4}{\mu_4 + 12\mu_2^2},$$

or

$$\Phi' = \sqrt{c\Phi^{-\frac{2q}{p}} + \frac{r}{2p+q}\Phi^4}$$

for case

$$\mu_5 \neq \frac{10\mu_2\mu_3\mu_4}{\mu_4 + 12\mu_2^2},$$

where p, q, r defined by (4.11) are determined by μ.

Proof. Since $\mu_4 \neq 3\mu_2^2$, making use of Lemma 2.7 and equation (2.8), one of the following alternatives holds:

Case A $\Phi(x) \equiv 0,\ x \in I$,

Case B $\Psi_{f,g}(x) - \Psi_{h,k}(x) = \frac{\mu_4}{\mu_4 - 3\mu_2^2}\Phi'(x),\ x \in I$,

Case C There exist two subintervals $I_1, I_2 \subset I$ such that

$$\Phi(x) = 0,\ \Psi_{f,g}(x) - \Psi_{h,k}(x) \neq \frac{\mu_4}{\mu_4 - 3\mu_2^2}\Phi'(x), x \in I_1,$$

and

$$\Phi(x) \neq 0,\ \Psi_{f,g}(x) - \Psi_{h,k}(x) = \frac{\mu_4}{\mu_4 - 3\mu_2^2}\Phi'(x), x \in I_2.$$

Firstly, we will show that **Case C** is invalid. In fact, substituting $\Phi(x) = 0,\ x \in I_1$ into (4.2), we get

$$(\mu_5 - 10\mu_2\mu_3)\Psi_{f,g}^2(x) = 0, \quad x \in I_1,$$

which leads to $\Psi_{f,g}(x) = \Psi_{h,k}(x) = 0,\ x \in I_1$, a contradiction to the first inequality of **Case C**. Hence, we only need to consider **Case A** and **Case B**.

(i) If

$$\Phi(x) \equiv 0$$

holds for all $x \in I$, then $\Psi_{f,g} + \Psi_{h,k} = 0$ and (4.2) becomes

$$(\mu_5 - 10\mu_2\mu_3)\Psi_{f,g}^2 = 0,$$

which implies $\Psi_{f,g} = 0$ since $\mu_5 \neq 10\mu_2\mu_3$. Therefore, $(f, g), (h, k)$ are the solutions of equation $y'' = 0$.

(ii) If

$$\Psi_{f,g}(x) - \Psi_{h,k}(x) = \frac{\mu_4}{\mu_4 - 3\mu_2^2}\Phi'(x),\ x \in I, \tag{4.6}$$

combining (2.9) we get

$$\Psi_{f,g} = \alpha\Phi' - \Phi^2, \quad \Psi_{h,k} = -\alpha\Phi' - \Phi^2, \tag{4.7}$$

where

$$\alpha := \frac{\mu_4}{2(\mu_4 - 3\mu_2^2)}. \tag{4.8}$$

We further obtain

$$\Psi_{f,g}^2 + \Psi_{h,k}^2 = 2\Phi^4 + \frac{1}{2}\left(\frac{\mu_4}{\mu_4 - 3\mu_2^2}\right)^2 \Phi'^2, \quad \Psi''_{f,g} + \Psi''_{h,k} = -4\Phi'^2 - 4\Phi\Phi''. \tag{4.9}$$

Substituting (2.9), (4.6) and (4.9) into (4.2), we get

$$p\Phi\Phi'' + q\Phi'^2 - r\Phi^4 = 0, \tag{4.10}$$

where

$$\begin{aligned}
p &:= \frac{\mu_4}{\mu_4 - 3\mu_2^2}(5\mu_5 - 10\mu_2\mu_3) - 4\mu_5,\\
q &:= \frac{\mu_4}{\mu_4 - 3\mu_2^2}(4\mu_5 - 10\mu_2\mu_3) + \frac{1}{2}\left(\frac{\mu_4}{\mu_4 - 3\mu_2^2}\right)^2 (\mu_5 - 10\mu_2\mu_3) - 6\mu_5,\\
r &:= 2(\mu_5 - 10\mu_2\mu_3).
\end{aligned} \tag{4.11}$$

Next, we will discuss two cases for μ_5 as in (ii) of Theorem 4.2.

When

$$\mu_5 = \frac{10\mu_2\mu_3\mu_4}{\mu_4 + 12\mu_2^2}, \tag{4.12}$$

we first claim that $p = 0, qr \neq 0$, that is,

$$\frac{\mu_4}{\mu_4 - 3\mu_2^2}(5\mu_5 - 10\mu_2\mu_3) - 4\mu_5 = 0, \quad \mu_5 - 10\mu_2\mu_3 \neq 0 \tag{4.13}$$

and

$$\frac{\mu_4}{\mu_4 - 3\mu_2^2}(4\mu_5 - 10\mu_2\mu_3) + \left(\frac{\mu_4}{\mu_4 - 3\mu_2^2}\right)^2 (\mu_5 - 10\mu_2\mu_3) - 6\mu_5 \neq 0 \tag{4.14}$$

hold simultaneously.

In fact, (4.13) is obtained by (4.12) directly. In order to prove (4.14), assume that

$$\mu_5(\mu_4^2 - 24\mu_2^2\mu_4 + 54\mu_2^4) + 10\mu_2\mu_3\mu_4(2\mu_4 - 3\mu_2^2) = 0.$$

By substituting condition (4.12) into the above equation, we get

$$\mu_4^2 - \mu_2^2\mu_4 + 6\mu_2^4 = 0,$$

which is impossible since the real numbers $\mu_2, \mu_4 > 0$. Consequently, (4.13)-(4.14) are proved.

What's more, we claim that $qr > 0$ when (4.12) is valid. In fact, substituting condition (4.12) into equation (4.11) about the expressions of q and r, we get

$$qr = \frac{7200\mu_2^4\mu_3^2\mu_4(6(\mu_2^2 - 1/4\mu_4)^2 + 5/8\mu_4^2)}{(\mu_4 + 12\mu_2^2)(\mu_4 - 3\mu_2^2)^2} > 0.$$

Therefore, equation (4.10) becomes

$$\Phi'^2 - \frac{r}{q}\Phi^4 = 0, \tag{4.15}$$

which can be rewritten by

$$(\Phi' - \sqrt{\frac{r}{q}}\Phi^2) \cdot (\Phi' + \sqrt{\frac{r}{q}}\Phi^2) = 0.$$

Then, we conclude that $\Phi(x) = 0$ for all $x \in I$ or there exits $c \in \mathbb{R}$ such that

$$\Phi(x) = -\frac{1}{\pm\sqrt{\frac{r}{q}}x + c}, \quad x \in I. \tag{4.16}$$

If $\Phi(x) = 0,\ x \in I$, then (4.5) is obviously true. Otherwise, by Lemma 2.1, (4.7) and (4.16), we also get the result.

When

$$\mu_5 \neq \frac{10\mu_2\mu_3\mu_4}{\mu_4 + 12\mu_2^2}$$

holds, that is, $p = \frac{\mu_4}{\mu_4 - 3\mu_2^2}(5\mu_5 - 10\mu_2\mu_3) - 4\mu_5 \neq 0$. Then, equation (4.10) can be rewritten as

$$\Phi\Phi'' + \frac{q}{p}\Phi'^2 - \frac{r}{p}\Phi^4 = 0. \tag{4.17}$$

Now we can reduce the above second-order nonlinear equation to the first-order Bernoulli equation. In fact, let

$$\Phi' = u, \tag{4.18}$$

we have $\Phi'' = u\frac{du}{d\Phi}$, and (4.17) becomes

$$\frac{du}{d\Phi} + \frac{q}{p\Phi}u = \frac{r}{p}\Phi^3 u^{-1}. \tag{4.19}$$

Then, put

$$z = u^2, \tag{4.20}$$

we have $\frac{dz}{d\Phi} = 2u\frac{du}{d\Phi}$ and (4.19) becomes

$$\frac{dz}{d\Phi} + \frac{2q}{p\Phi}z = \frac{2r}{p}\Phi^3,$$

which is a first-order linear non-homogeneous differential equation. Hence, there exists $c \in \mathbb{R}$ such that

$$z = c\Phi^{-\frac{2q}{p}} + \frac{r}{2p+q}\Phi^4.$$

By (4.18) and (4.20), we obtain that

$$\Phi' = \sqrt{c\Phi^{-\frac{2q}{p}} + \frac{r}{2p+q}\Phi^4}.$$

□

Remark 4.1. Note that the case $\mu_5 = 10\mu_2\mu_3$ is not considered in this section. Actually, under this assumption, the left side of equation (4.2) is vanished when $\Phi \equiv 0$ and thus the further discussion is unavailable. Therefore, it remains an open problem in the case that $\mu_5 = 10\mu_2\mu_3$.

5. Examples

In this section, we obtain the relevant conclusions of some special cases of the equation (1.1) from the specialization of measures and derived functions respectively.

First, let $k = g = 1$, the invariance equation (1.1) becomes the invariance of arithmetic mean with respect to Makó-Páles means [12]. By Theorems 2.1 and 3.1, we get

Example 5.1. Let μ be a Borel probability measure on $[0, 1]$, $f, h \in \mathcal{C}_4(I)$. The invariance equation

$$f^{-1}\left(\int_0^1 f(tx+(1-t)y)d\mu(t)\right)+h^{-1}\left(\int_0^1 h(tx+(1-t)y)d\mu(t)\right)=x+y,\ \ x, y \in I$$

holds if and only if $\hat{\mu}_1 = \frac{1}{2}$ and one of the following alternatives is true:

(i) When $\mu_2 = 0$, then $\mu = \delta_{1/2}$ and f, h are arbitrary.

(ii) When $\mu_2 \neq 0$, then there exist $a, b, c, d \in \mathbb{R}$ such that

$$f(x) = \begin{cases} ax + b, & r = 0, \\ ae^{rx} + b, & r \neq 0, \end{cases} \qquad h(x) = \begin{cases} cx + d, & r = 0, \\ ce^{rx} + d, & r \neq 0. \end{cases}$$

Proof. (i) is obtained by Theorem 2.1 directly. Since $k = g = 1$ which satisfies (3.1), using Theorem 3.1 we get our result (ii). □

Remark 5.1. The same conclusion was obtained in Corollary 13 of [12] under the assumption that μ is a symmetric measure with respect to the point 1/2.

Example 5.2. Consider the functional equation

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{f\,(x)+f(y)}{g\,(x)+g(y)}\right)+\left(\frac{h}{k}\right)^{-1}\left(\frac{h\,(x)+h(y)}{k\,(h)+k(y)}\right)=x+y, \tag{5.1}$$

where $f, g, h, k \in \mathcal{C}_4(I)$. Assume that $k = g$ satisfies $y'' = py$ for some $p \in \mathbb{R}$. If equation (5.1) holds, then there exists $r \in \mathbb{R}$ such that

$$f = g\int \frac{1}{g^2}exp(\int rg^{-\frac{4}{3}}), \quad h = k\int \frac{1}{k^2}exp(\int -rk^{-\frac{4}{3}}). \tag{5.2}$$

Furthermore, assume $k = g$ satisfies one of conditions (H1)-(H3), equation (5.1) implies

(i) for case (H1), there exist $b_1, b_2 \in \mathbb{R}_+, r \in \mathbb{R}$ such that

$$f(x) = x\int b_1x^{-2}exp(-3rx^{-\frac{1}{3}}), \quad h(x) = x\int b_2x^{-2}exp(3rx^{-\frac{1}{3}}), \tag{5.3}$$

where $x \in I$,

(ii) for case (H2), there exist $b_1, b_2 \in \mathbb{R}_+, r \in \mathbb{R}$ such that

$$f(x) = e^x\int b_1e^{-2x}exp(-\frac{3r}{4}e^{-\frac{4}{3}x}), \quad h(x) = e^x\int b_2e^{-2x}exp(\frac{3r}{4}e^{-\frac{4}{3}x}), \tag{5.4}$$

where $x \in I$.

(iii) for case (H3), there exists $r \in \mathbb{R}$ such that

$$\begin{aligned} f(x) &= \cos x \int \frac{1}{\cos^2 x} \exp\Big(\int r \cos^{-\frac{4}{3}} x\Big), \\ h(x) &= \cos x \int \frac{1}{\cos^2 x} \exp\Big(\int -r \cos^{-\frac{4}{3}} x\Big), \end{aligned} \tag{5.5}$$

where $x \in I$.

Proof. Actually, the measure μ is given by

$$\mu = \frac{\delta_0 + \delta_1}{2}.$$

Then, we get $\hat{\mu}_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{4}$, $\mu_3 = 0$, $\mu_4 = \frac{1}{16}$, $\mu_5 = 0$, and $\beta = \frac{2(\mu_4 - 3\mu_2^2)}{3\mu_4} = -\frac{4}{3}$. Using Theorems 3.1-3.2 we get (5.2)-(5.5). $\square$

Remark 5.2. Our result (5.2) is the same as Theorem 1 in [6]. Jarczyk proved that (5.1) holds under the assumption of (H1), (H2) or (H3) if and only if $r = 0$ in equations (5.3)-(5.5), that is, equation (5.1) holds if and only if

$$f(x) = x\left(\frac{a}{x} + b\right), \quad h(x) = x\left(\frac{c}{x} + d\right), \quad x \in I$$

in case (H1) and

$$f(x) = e^x(ae^{-2x} + b), \quad h(x) = e^x(e^{-2x} + d), \quad x \in I$$

in case (H2) and

$$f(x) = \cos x(a \tan x + b), \quad h(x) = \cos x(c \tan x + d), \quad x \in I$$

in case (H3).

Example 5.3. Consider the functional equation

$$\left(\frac{f}{g}\right)^{-1}\left(\frac{2f\left(\frac{x+3y}{4}\right) + f(x)}{2g\left(\frac{x+3y}{4}\right) + g(x)}\right) + \left(\frac{h}{k}\right)^{-1}\left(\frac{2h\left(\frac{x+3y}{4}\right) + h(x)}{2k\left(\frac{x+3y}{4}\right) + k(x)}\right) = x + y, \tag{5.6}$$

where $f, g, h, k \in \mathcal{C}_4(I)$. If equation (5.6) holds, then we have either

$$(f, g) \sim (1, x),\ (h, k) \sim (1, x),$$

or f, g and h, k are solutions of the following differential equations

$$y'' = \Phi(x)y' + (-\frac{1}{2}\Phi'(x) - \Phi^2(x))y \quad \text{and} \quad y'' = -\Phi(x)y' + (\frac{1}{2}\Phi'(x) - \Phi^2(x))y,$$

respectively, where there exists $c \in \mathbb{R}$ such that the function Φ satisfies

$$\Phi' = \sqrt{\frac{12}{35}\Phi^4 + c\Phi^{-3}}.$$

Proof. In fact, the measure μ is given by

$$\mu = \frac{2\delta_{1/4} + \delta_1}{3}.$$

Then, we have $\hat{\mu}_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{8}$, $\mu_3 = \frac{1}{32}$, $\mu_4 = \frac{3}{128}$, $\mu_5 = \frac{5}{512}$ and $\alpha = \frac{\mu_4}{2(\mu_4 - 3\mu_3^2)} = -\frac{1}{2}$.

Since $\mu_4 - 3\mu_2^2 \neq 0$ and $\mu_5 - 10\mu_2\mu_3 \neq 0$, after a simple calculation, equation (4.10) becomes

$$10\Phi\Phi'' + 15\Phi'^2 - 12\Phi^4 = 0.$$

Making using of Theorem 4.2, we obtain that there exists $c \in \mathbb{R}$ such that Φ satisfies

$$\Phi' = \sqrt{\frac{12}{35}\Phi^4 + c\Phi^{-3}}.$$

This finishes the proof. □

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