Journal of Applied Analysis and Computation
Volume 13, Number 3, June 2023, 1597–1612
Website:http://www.jaac-online.com
DOI:10.11948/20220404

MULTIPLE SOLUTIONS FOR NONHOMOGENEOUS QUASILINEAR SCHRÖDINGER–POISSON SYSTEM*

Lanxin Huang[1] and Jiabao Su[1,†]

Abstract We consider the nonhomogeneous quasilinear Schrödinger–Poisson system

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi|u|^{p-2}u = |u|^{q-2}u + h(x) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^p & \text{in } \mathbb{R}^3, \end{cases}$$

where $1 < p < 3$, $p < q < p^* = \frac{3p}{3-p}$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\lambda > 0$ and $h \neq 0$. Under suitable assumptions on h, the Ekeland's variational principle and the mountain pass theorem are applied to establish the existence of multiple solutions for this system. To the best of our knowledge, this paper is one of the first contributions to the study of the nonhomogeneous quasilinear Schrödinger–Poisson system.

Keywords Nonhomogeneous quasilinear Schrödinger–Poisson system, variational methods, multiple solutions.

MSC(2010) 35J10, 35J50, 35J60, 35J92.

1. Introduction and main results

This article is concerned with the nonhomogeneous quasilinear Schrödinger–Poisson system

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi|u|^{p-2}u = |u|^{q-2}u + h(x) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^p & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $1 < p < 3$, $p < q < p^* = \frac{3p}{3-p}$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\lambda > 0$ is a parameter. The function h satisfies the following assumption. From here on we use $\tau' = \frac{\tau}{\tau-1}$ to denote the Hölder conjugate of $\tau > 1$.

(h) h is a nonzero radial function and for $(p^*)' \leqslant s \leqslant p'$,
  (i) $h \in L^s(\mathbb{R}^3)$ with the L^s–norm denoted by $|h|_s$;
  (ii) $(x, \nabla h) \in L^s(\mathbb{R}^3)$ where the gradient ∇h is in the weak sense.

† The corresponding author.
Email: 812419761@qq.com(L. Huang), sujb@cnu.edu.cn.(J. Su)
[1] School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
* The authors were supported by National Natural Science Foundation of China (12271373, 12171326) and KZ202010028048.

When $p=2$ and $h=0$, the system (1.1) reduces to the classical Schrödinger–Poisson system

$$\begin{cases}-\Delta u+u+\lambda\phi u=|u|^{q-2}u & \text{in } \mathbb{R}^3,\\ -\Delta\phi=u^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{1.2}$$

where $\lambda>0$ and $q\in(2,6)$. The system (1.2), also known as the nonlinear Schrödinger–Maxwell equation, can be used to describe the interaction of a charged particle with an electromagnetic field in quantum mechanics. For more details about the mathematical and physical backgrounds we refer the reader to [5,6] and the references therein. In the last decades, many scholars have studied the existence of nontrivial solutions for the system (1.2) with different ranges of q and the similar system involving a general nonlinear term, see [2–4,9,10,22,26,30] and the references therein. More recently, Du etc [12] discussed the following quasilinear Schrödinger–Poisson system

$$\begin{cases}-\Delta_p u+|u|^{p-2}u+\lambda\phi|u|^{p-2}u=|u|^{q-2}u & \text{in } \mathbb{R}^3,\\ -\Delta\phi=|u|^p & \text{in } \mathbb{R}^3,\end{cases} \tag{1.3}$$

where $1<p<3$, $p<q<p^*=\frac{3p}{3-p}$, $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\lambda>0$. Applying the mountain pass theorem, it was proved in [12] that the system (1.3) has a nontrivial solution for $p<q<p^*$. In another paper, Du etc [11] considered the system which is different from (1.3) in the sense that it required that $\frac{4}{3}<p<\frac{12}{5}$ and there is no p involved in the poisson term. Observing from the literature mentioned above, Du etc [11,12] are the first to build the variational framework and establish the existence of nontrivial solutions to the system (1.3) for $p\neq 2$.

When $p=2$ and $h\in L^2(\mathbb{R}^3)$ is a nonzero function, the system (1.1) becomes the following nonhomogeneous Schrödinger–Poisson system

$$\begin{cases}-\Delta u+u+\lambda\phi u=|u|^{q-2}u+h(x) & \text{in } \mathbb{R}^3,\\ -\Delta\phi=u^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{1.4}$$

where $\lambda>0$ and $q\in(2,6)$. Salvatore [23] obtained multiple radial solutions to the system (1.4) for $q\in(4,6)$ and h being radial with small L^2–norm. Subsequently, Jiang etc [17] studied the system (1.4) with $q\in(2,6)$ and $h\in C^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$ is a nonnegative radial function satisfying $(x,\nabla h)\in L^2(\mathbb{R}^3)$. By using the Ekeland's variational principle and the mountain pass theorem, the authors in [17] proved that system (1.4) has at least two radial solutions for $q\in(2,6)$ with the small L^2–norm $|h|_2$ of h. However, for the case that $q\in(2,3]$, it was required in [17] that $\lambda>0$ to be small enough. For related works of the system (1.4) containing the general nonlinearity and similar systems including certain potentials, we refer to [8,14,18,27,29,31,32] and the references therein.

After an accurate bibliographic review, we realised that it is still open whether the system (1.1) has multiple solutions for $1<p<3$ and $h\neq 0$. Strongly inspired by this fact, the aim of this paper is to establish the existence of multiple solutions to the system (1.1).

Now, we state our main results of this paper.

Theorem 1.1. *Suppose that* (h) *and* $\frac{2p(p+1)}{p+2}<q<p^*$ *hold. Then there exists* $\Lambda>0$ *such that the system* (1.1) *admits two solutions for any* $\lambda>0$ *provided* $|h|_s<\Lambda$.

Theorem 1.2. *Suppose that* (h)(i) *and* $p < q < p^*$ *hold. Then there exist* $\Lambda > 0$ *and* $\lambda_* > 0$ *such that the system* (1.1) *admits two solutions for any* $\lambda \in (0, \lambda_*)$ *provided* $|h|_s < \Lambda$.

Remark 1.1. As far as we know, our results are up to date. With the help of the result of [12], we generalize the case of $p = 2$ in [17, 23] to the quasilinear case of $1 < p < 3$ for system (1.1). Furthermore, the solvability of the system (1.1) can be considered for a large class of radial functions h satisfying (h). In this sense the existence results in [17] may be extended to the case that h and $(x, \nabla h)$ belonging to $L^s(\mathbb{R}^3)$ with $\frac{6}{5} \leqslant s \leqslant 2$.

The paper is organized as follows. In Section 2, we introduce the variational framework of (1.1) following [12] and establish some preliminary results. In Section 3, we obtain the existence of the negative energy solution for system (1.1). In Section 4, we study the existence of the positive energy solutions and complete the proof of Theorems 1.1 and 1.2.

2. Preliminaries

Before giving the variational framework of (1.1), we introduce the following notations.

For $1 \leqslant s < \infty$, $L^s(\mathbb{R}^3)$ denotes the Lebesgue space with the usual norm

$$|u|_s = \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{\frac{1}{s}}.$$

Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

It is well known that the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is continuous. Let $S > 0$ be the embedding constant, i.e,

$$|u|_6^2 \leqslant S^{-1}\|u\|_D^2, \qquad \forall\, u \in D^{1,2}(\mathbb{R}^3). \tag{2.1}$$

Let $W^{1,p}(\mathbb{R}^3)$ denote the Sobolev space endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^p + |u|^p dx\right)^{\frac{1}{p}}.$$

It follows from the classical Sobolev embedding theorems that $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^\ell(\mathbb{R}^3)$ are continuous for all $\ell \in [p, p^*]$. Thus for each $\ell \in [p, p^*]$ there exists $S_\ell > 0$ such that

$$|u|_\ell \leqslant S_\ell \|u\|, \qquad \forall\, u \in W^{1,p}(\mathbb{R}^3). \tag{2.2}$$

We will work on the space of radial functions

$$W_r^{1,p}(\mathbb{R}^3) := \left\{u \in W^{1,p}(\mathbb{R}^3) :\ u(x) = u(|x|)\right\}.$$

It holds that the embedding $W_r^{1,p}(\mathbb{R}^3) \hookrightarrow L^\ell(\mathbb{R}^3)$ is compact for any $p < \ell < p^*$ (see [20, Theorem II.1] or [24, Theorem 1]). We use C to denote various positive constants.

For every $u \in W^{1,p}(\mathbb{R}^3)$, the linear functional $\mathcal{T}_u : D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ is defined as

$$\mathcal{T}_u(v) = \int_{\mathbb{R}^3} |u|^p v dx.$$

By the Hölder inequality and (2.1)–(2.2), one concludes

$$|\mathcal{T}_u(v)| \leqslant \left(\int_{\mathbb{R}^3} |u|^{\frac{6p}{5}} dx\right)^{\frac{5}{6}} \left(\int_{\mathbb{R}^3} |v|^6 dx\right)^{\frac{1}{6}} \leqslant C\|u\|^p \|v\|_D.$$

Then, it follows that $\mathcal{T}_u$ is continuous on $D^{1,2}(\mathbb{R}^3)$. By the Lax–Milgram theorem, we know that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u = |u|^p \quad \text{ in } \mathbb{R}^3.$$

According to [19, Theorem 6.21], ϕ_u has the following explicit expression

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|} dy \geqslant 0.$$

Moreover, ϕ_u has the following properties.

Lemma 2.1 (Proposition 2.1, [12]). *Let $u \in W^{1,p}(\mathbb{R}^3)$.*

(i) *For all $t > 0$, $\phi_{tu} = t^p \phi_u$, and $\phi_{u_t}(x) = t^{kp-2}\phi_u(tx)$ with $u_t(x) = t^k u(tx)$.*

(ii) *$\|\phi_u\|_D \leqslant A\|u\|^p$ with $A > 0$ is a constant.*

(iii) *If $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{p-2} u_n \varphi dx \to \int_{\mathbb{R}^3} \phi_u |u|^{p-2} u \varphi dx, \qquad \forall\, \varphi \in W^{1,p}(\mathbb{R}^3).$$

(iv) *If $u \in W_r^{1,p}(\mathbb{R}^3)$, then $\phi_u \in D_r^{1,2}(\mathbb{R}^3) := \{\phi \in D^{1,2}(\mathbb{R}^3) : \phi(x) = \phi(|x|)\}$.*

Notice that, the forth conclusion comes from a fact that the convolution of two radial functions is still radial.

Now, we establish the variational framework of (1.1). For $h \in L^s(\mathbb{R}^3)$ with $(p^*)' \leqslant s \leqslant p'$, arguing as in [5,6], by Lemma 2.1 and the implicit function theorem, the functional

$$I_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^3} (|\nabla u|^p + |u|^p) dx + \frac{\lambda}{2p}\int_{\mathbb{R}^3} \phi_u |u|^p dx - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q dx - \int_{\mathbb{R}^3} h(x) u dx$$

is a well–defined C^1 functional on $W^{1,p}(\mathbb{R}^3)$ with derivative

$$\begin{aligned}\langle I'_\lambda(u), v\rangle = &\int_{\mathbb{R}^3} (|\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2} uv) dx + \lambda \int_{\mathbb{R}^3} \phi_u |u|^{p-2} uv dx \\ &- \int_{\mathbb{R}^3} |u|^{q-2} uv dx - \int_{\mathbb{R}^3} h(x) v dx, \qquad \forall\, u, v \in W^{1,p}(\mathbb{R}^3).\end{aligned}$$

Note that $(u, \phi_u) \in W^{1,p}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u \in W^{1,p}(\mathbb{R}^3)$ is a critical point of I_λ, see [12] for details. Furthermore, by the principle of symmetric criticality, the critical points of I_λ on $W_r^{1,p}(\mathbb{R}^3)$ are the critical points of I_λ on $W^{1,p}(\mathbb{R}^3)$. Therefore, find a weak solution to the system (1.1) is equivalent to finding a critical point of the functional I_λ on $W_r^{1,p}(\mathbb{R}^3)$. In order to prove our results, we give the following lemmas.

Lemma 2.2. *Assume that* (h)(i) *holds and* $p < q < p^*$. *For* $\lambda > 0$, *there exist* $\Lambda > 0$, $\rho > 0$ *and* $\alpha > 0$ *such that if* $|h|_s < \Lambda$, *then* $I_\lambda(u) \geqslant \alpha$ *for* $u \in W_r^{1,p}(\mathbb{R}^3)$ *with* $\|u\| = \rho$.

Proof. For $u \in W_r^{1,p}(\mathbb{R}^3)$, it follows from the Hölder inequality and (2.2) that

$$\int_{\mathbb{R}^3} hudx \leqslant |h|_s|u|_{s'} \leqslant S_{s'}|h|_s\|u\|, \tag{2.3}$$

where $p \leqslant s' \leqslant p^*$. Since $\lambda > 0$ and $\phi_u \geqslant 0$, we have

$$I_\lambda(u) \geqslant \frac{1}{p}\|u\|^p - \frac{1}{q}|u|_q^q - |h|_s|u|_{s'} \geqslant \|u\| \left(\frac{1}{p}\|u\|^{p-1} - \frac{S_q^q}{q}\|u\|^{q-1} - S_{s'}|h|_s\right).$$

The function

$$f(t) = \frac{1}{p}t^{p-1} - \frac{S_q^q}{q}t^{q-1}$$

is strictly increasing in a right neighborhood of 0 since $p < q$, is continuous and satisfies $f(0) = 0$. Therefore, there exists $\varepsilon_1 \leqslant \varepsilon$ such that for all $t \in (0, \varepsilon_1)$ there results $f(t) > 0$. Now, fixing any $\rho \in (0, \varepsilon_1)$, we obtain $I_\lambda(u) \geqslant \rho\,(f(\rho) - S_{s'}|h|_s)$ if $\|u\| = \rho$. Taking

$$\Lambda = \frac{f(\rho)}{2S_{s'}} \quad \text{and} \quad \alpha = \frac{\rho f(\rho)}{2},$$

we deduce that if $|h|_s < \Lambda$, then $I_\lambda(u) \geqslant \alpha > 0$ for $\|u\| = \rho$. □

Lemma 2.3. *Assume that* (h)(i) *holds. Then any bounded sequence* $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^3)$ *satisfying* $I'_\lambda(u_n) \to 0$ *has a strongly convergent subsequence.*

Proof. Since $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^3)$ is bounded, we can deduce that there exists $u \in W_r^{1,p}(\mathbb{R}^3)$ such that, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_r^{1,p}(\mathbb{R}^3), \\ u_n \to u & \text{in } L^q(\mathbb{R}^3), \quad p < q < p^*, \\ u_n(x) \to u(x) & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{2.4}$$

We will claim that $u_n \to u$ strongly in $W_r^{1,p}(\mathbb{R}^3)$, namely,

$$\|u_n - u\| \to 0, \quad n \to \infty. \tag{2.5}$$

Indeed, by the Hölder inequality, (2.2) and (2.4), as $n \to \infty$,

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx\right| \\ &\leqslant \left(|u_n|_q^{q-1} + |u|_q^{q-1}\right)|u_n - u|_q \\ &\leqslant S_q^{q-1}\left(\|u_n\|^{q-1} + \|u\|^{q-1}\right)|u_n - u|_q \\ &\to 0. \end{aligned} \tag{2.6}$$

Combining Lemma 2.1(ii), the Hölder inequality, (2.1) and (2.4), we conclude that, as $n \to \infty$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} (\phi_{u_n}|u_n|^{p-2}u_n - \phi_u|u|^{p-2}u)(u_n-u)dx\right| \\
&\leqslant |\phi_{u_n}|_6 \left(\int_{\mathbb{R}^3} \left||u_n|^{p-2}u_n(u_n-u)\right|^{\frac{6}{5}} dx\right)^{\frac{5}{6}} + |\phi_u|_6 \left(\int_{\mathbb{R}^3} \left||u|^{p-2}u(u_n-u)\right|^{\frac{6}{5}} dx\right)^{\frac{5}{6}} \\
&\leqslant |\phi_{u_n}|_6 \left(\int_{\mathbb{R}^3} |u_n|^{\frac{6(p-1)}{5}}|u_n-u|^{\frac{6}{5}} dx\right)^{\frac{5}{6}} + |\phi_u|_6 \left(\int_{\mathbb{R}^3} |u|^{\frac{6(p-1)}{5}}|u_n-u|^{\frac{6}{5}} dx\right)^{\frac{5}{6}} \quad (2.7)\\
&\leqslant S^{-\frac{1}{2}}\|\phi_{u_n}\|_D|u_n|_p^{p-1}|u_n-u|_{\frac{6p}{6-p}} + S^{-\frac{1}{2}}\|\phi_u\|_D|u|_p^{p-1}|u_n-u|_{\frac{6p}{6-p}} \\
&\leqslant C\left(\|u_n\|^p|u_n|_p^{p-1} + \|u\|^p|u|_p^{p-1}\right)|u_n-u|_{\frac{6p}{6-p}} \to 0.
\end{aligned}$$

By $I'_\lambda(u_n) \to 0$ and (2.4) we have

$$\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \to 0, \quad n \to \infty. \tag{2.8}$$

It follows from (2.6)–(2.8) that

$$\begin{aligned}
&\int_{\mathbb{R}^3} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u) \\
&+ (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)dx = o(1).
\end{aligned} \tag{2.9}$$

Now, let us recall the following elementary inequality (see [25, p240]). There exists $c_p > 0$ such that for all $\xi, \eta \in \mathbb{R}^N$,

$$\begin{cases}
\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi-\eta\right) \geqslant c_p|\xi-\eta|^p & \text{for } p \geqslant 2,\\
\left(|\xi|+|\eta|\right)^{2-p}\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi-\eta\right) \geqslant c_p|\xi-\eta|^2 & \text{for } 1<p<2.
\end{cases} \tag{2.10}$$

For $2 \leqslant p < 3$, by (2.10) we get

$$\begin{aligned}
&\int_{\mathbb{R}^3} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)dx \geqslant C\int_{\mathbb{R}^3} |\nabla u_n - \nabla u|^p dx,\\
&\int_{\mathbb{R}^3} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)dx \geqslant C\int_{\mathbb{R}^3} |u_n-u|^p dx.
\end{aligned} \tag{2.11}$$

For $1 < p < 2$, from the boundedness of $\{u_n\}$, the Hölder inequality and (2.10), we get

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla(u_n-u)|^p dx \\
&\leqslant C\int_{\mathbb{R}^3} \left[\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla(u_n-u)\right]^{\frac{p}{2}} \left(|\nabla u_n| + |\nabla u|\right)^{\frac{p(2-p)}{2}} dx \\
&\leqslant C\left(\int_{\mathbb{R}^3} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n-u)dx\right)^{\frac{p}{2}}\left(\int_{\mathbb{R}^3} (|\nabla u_n|^p + |\nabla u|^p)\, dx\right)^{\frac{2-p}{2}} \\
&\leqslant C\left(\int_{\mathbb{R}^3} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n-u)dx\right)^{\frac{p}{2}}.
\end{aligned} \tag{2.12}$$

In the same way we obtain

$$\int_{\mathbb{R}^3} |u_n - u|^p dx \leqslant C \left(\int_{\mathbb{R}^3} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx \right)^{\frac{p}{2}}. \tag{2.13}$$

Therefore, it follows from (2.9), (2.11)–(2.13) that (2.5) holds. □

3. A solution with negative energy

In this section we will find a solution of (1.1) with negative energy for $p < q < p^*$ and h satisfying (h)(i) and small $|h|_s$. With the aid of Ekeland's variational principle [21] (see also [13]), this solution is obtained by searching for the local minimum of the energy functional I_λ. Now, we give the following Ekeland's variational principle so that it is convenience for readers to understand the proof of our result.

Theorem 3.1 (Theorem 4.1, [21]). *Let M be a complete metric space with metric d and let $\Phi : M \to (-\infty, +\infty]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\epsilon > 0$ be given and $u \in M$ be such that*

$$\Phi(u) \leqslant \inf_M \Phi + \epsilon.$$

Then there exists $v \in M$ such that

$$\Phi(v) \leqslant \Phi(u), \quad d(u,v) \leqslant 1$$

and, for each $w \neq v$ in M,

$$\Phi(w) > \Phi(v) - \epsilon d(v,w).$$

Lemma 3.1. *Assume that* (h)(i) *holds and $p < q < p^*$. Then for $\lambda > 0$,*

$$c_* = \inf_{u \in \bar{B}_\rho} I_\lambda(u) < 0,$$

where $\bar{B}_\rho = \{u \in W_r^{1,p}(\mathbb{R}^3) : \|u\| \leqslant \rho\}$ and ρ was given by Lemma 2.2.

Proof. By (h)(i), we can choose a function $\varphi \in W_r^{1,p}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} h(x)\varphi(x)dx > 0$. It follows from $h \in L^s(\mathbb{R}^3)$ that $|h|^{s-2}h \in L^{s'}(\mathbb{R}^3)$. Then there exists a radial sequence $\{h_n\} \subset C_0^\infty(\mathbb{R}^3)$ such that $h_n \to |h|^{s-2}h$ strongly in $L^{s'}(\mathbb{R}^3)$ since $C_0^\infty(\mathbb{R}^3)$ is dense in $L^{s'}(\mathbb{R}^3)$ and h is radial. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\left|h_{n_0} - |h|^{s-2}h\right|_{s'} \leqslant \frac{1}{2}|h|_s^{s-1}.$$

Using the Hölder inequality, we get

$$\int_{\mathbb{R}^3} h(x)h_{n_0}(x)dx \geqslant -|h|_s \left(\left|h_{n_0} - |h|^{s-2}h\right|_{s'}\right) + |h|_s^s > 0.$$

Evidently, $h_{n_0} \in W_r^{1,p}(\mathbb{R}^3)$. Then $\int_{\mathbb{R}^3} h(x)\varphi(x)dx > 0$ holds by taking $\varphi(x) = h_{n_0}(x)$. For $t > 0$ small enough, we deduce that

$$I_\lambda(t\varphi) = \frac{t^p}{p}\|\varphi\|^p + \frac{t^{2p}}{2p}\lambda \int_{\mathbb{R}^3} \phi_\varphi |\varphi|^p dx - \frac{t^q}{q}\int_{\mathbb{R}^3} |\varphi|^q dx - t\int_{\mathbb{R}^3} h\varphi dx < 0.$$

This shows that

$$c_* = \inf_{u \in \bar{B}_\rho} I_\lambda(u) < 0,$$

where $\bar{B}_\rho = \{u \in W_r^{1,p}(\mathbb{R}^3) : \|u\| \leqslant \rho\}$ and ρ was given by Lemma 2.2. □

Theorem 3.2. *Assume that* (h)(i) *holds and* $p < q < p^*$. *Then* I_λ *has a critical point* $u_* \in W_r^{1,p}(\mathbb{R}^3)$ *with* $I_\lambda(u_*) < 0$ *for* $\lambda > 0$ *provided* $|h|_s < \Lambda$, *where* Λ *was given in Lemma 2.2.*

Proof. By Lemma 3.1, I_λ satisfies all assumptions in Theorem 3.1 (Ekeland's variational principle), then we know that there is a sequence $\{u_n\} \subset \bar{B}_\rho$ satisfying

$$c_* \leqslant I_\lambda(u_n) \leqslant c_* + \frac{1}{n}, \tag{3.1}$$

$$I_\lambda(w) \geqslant I_\lambda(u_n) - \frac{1}{n}\|w - u_n\| \quad \text{for all } w \in \bar{B}_\rho. \tag{3.2}$$

Now, we will prove that $\{u_n\}$ is a bounded $(\mathrm{PS})_{c_*}$ sequence of I_λ.

Firstly, we claim that $\|u_n\| < \rho$ for all $n \in \mathbb{N}$ large. Otherwise, we may assume that $\|u_n\| = \rho$, up to a subsequence. Hence, by Lemma 2.2, there exist $\Lambda > 0$, $\rho > 0$ and $\alpha > 0$ such that if $|h|_s < \Lambda$, then $I_\lambda(u_n) \geqslant \alpha$ for $\|u_n\| = \rho$. Taking the limit as $n \to \infty$ and by using (3.1), we get $0 > c_* \geqslant \alpha > 0$, which is a contradiction. Then, we may assume that $\|u_n\| < \rho$ for all $n \in \mathbb{N}$.

Next, we show that $I'_\lambda(u_n) \to 0$ in $[W_r^{1,p}(\mathbb{R}^3)]^*$. Indeed, for any $z \in W_r^{1,p}(\mathbb{R}^3)$ with $\|z\| = 1$, we choose sufficiently small $\delta > 0$ such that $\|u_n + tz\| < \rho$ for all $|t| < \delta$. By using (3.2), we deduce that

$$\frac{I_\lambda(u_n + tz) - I_\lambda(u_n)}{t} \geqslant -\frac{1}{n}.$$

Taking the limit as $t \to 0$, we get $\langle I'_\lambda(u_n), z\rangle \geqslant -\frac{1}{n}$. Similarly, replacing z with $-z$ in the above arguments, we get $\langle I'_\lambda(u_n), z\rangle \leqslant \frac{1}{n}$. Then, for any $z \in W_r^{1,p}(\mathbb{R}^3)$ with $\|z\| = 1$, we conclude that $\langle I'_\lambda(u_n), z\rangle \to 0$ as $n \to \infty$. This shows at once that $\{u_n\}$ is a bounded $(\mathrm{PS})_{c_*}$ sequence of I_λ. Therefore, by Lemma 2.3, there exists $u_* \in W_r^{1,p}(\mathbb{R}^3)$ such that $I_\lambda(u_*) = c_* < 0$ and $I'_\lambda(u_*) = 0$. We finish the proof. □

4. A solution with positive energy

In this section we will find a solution of (1.1) with positive energy for $p < q < p^*$. The section is divided into two subsections. In Subsection 4.1, we consider the case $\frac{2p(p+1)}{p+2} < q < p^*$ for any $\lambda > 0$. In Subsection 4.2, we consider the case $p < q < p^*$ for $\lambda > 0$ small. Now, we give the following well–known mountain pass theorem [7] (see also [1]), which will be used in the proof of the rest paper.

Theorem 4.1 (Theorem 2.2, [7])**.** *Let* Φ *be a* C^1 *function on a Banach space* E. *Suppose*

(M_1) *there exist a neighborhood* U *of* 0 *in* E *and a constant* α *such that* $\Phi(u) \geqslant \alpha$ *for every* u *in the boundary of* U,

(M_2) $\Phi(0) < \alpha$ *and* $\Phi(v) < \alpha$ *for some* $v \notin U$.

Set

$$c = \inf_{P\in\Gamma} \max_{w\in P} \Phi(w) \geqslant \alpha,$$

where Γ denotes the class of continuous paths joining 0 to v. Then, there is a sequence $\{u_n\}$ in E such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ in E^*.

4.1. The case $\frac{2p(p+1)}{p+2} < q < p^*$

In this subsection, we will prove that system (1.1) has a positive energy solution for $\frac{2p(p+1)}{p+2} < q < p^*$ and h satisfying (h) and small $|h|_s$. Then we give some lemmas and theorems.

Lemma 4.1. *Assume that* (h)(i) *holds and $\frac{2p(p+1)}{p+2} < q < p^*$. For $\lambda > 0$, there exists $\Lambda > 0$ such that if $|h|_s < \Lambda$, then I_λ satisfies the assumptions (M_1)–(M_2) in Theorem 4.1.*

Proof. By Lemma 2.2, there exist $B_\rho = \{u \in W_r^{1,p}(\mathbb{R}^3) : \|u\| < \rho\}$ and constants $\Lambda > 0$, $\alpha > 0$ such that if $|h|_s < \Lambda$, then $I_\lambda(u) \geqslant \alpha$ with $\|u\| = \rho$. It is clear that (M_1) in Theorem 4.1 is true. Now, we need to verify (M_2) in Theorem 4.1. For any fixed $u \in W_r^{1,p}(\mathbb{R}^3)\backslash\{0\}$, we define $u_t(x) = t^{\frac{p+2}{p}} u(tx)$. By Lemma 2.1(i), we estimate that

$$\begin{aligned} I_\lambda(u_t) =& \frac{t^{\beta_1}}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{t^{\beta_2}}{p} \int_{\mathbb{R}^3} |u|^p dx + \frac{\lambda t^{\beta_1}}{2p} \int_{\mathbb{R}^3} \phi_u |u|^p dx \\ &- \frac{t^{\beta_3}}{q} \int_{\mathbb{R}^3} |u|^q dx - t^{\beta_4} \int_{\mathbb{R}^3} h\left(\frac{x}{t}\right) u dx, \end{aligned}$$

where

$$\beta_1 = 2p-1, \quad \beta_2 = p-1, \quad \beta_3 = \frac{(p+2)q-3p}{p}, \quad \beta_4 = \frac{2-2p}{p}. \tag{4.1}$$

In view of $\frac{2p(p+1)}{p+2} < q$ and $1 < p < 3$, we have

$$\beta_3 > \beta_1 > \beta_2 > 0 > \beta_4.$$

Therefore, there exists $t_0 > 0$ such that $\|u_{t_0}\| > \rho$ and $I_\lambda(u_{t_0}) < 0 < \alpha$. From this and $I_\lambda(0) = 0 < \alpha$ we immediately prove that (M_2) is true. $\square$

By Theorem 4.1 and Lemma 4.1, we define the mountain pass level

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0,$$

where $\Gamma_\lambda = \{\gamma \in C([0,1], W_r^{1,p}(\mathbb{R}^3)) : \gamma(0) = 0 \text{ and } I_\lambda(\gamma(1)) < 0\}$, then there is a sequence $\{u_n\}$ in $W_r^{1,p}(\mathbb{R}^3)$ such that $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$ in $[W_r^{1,p}(\mathbb{R}^3)]^*$. Combining with this and employing a scaling technique introduced by Jeanjean [15] (see also [12]), we will construct a special bounded $(\mathrm{PS})_{c_\lambda}$ sequence of I_λ in the next lemma.

Lemma 4.2. *Assume that* (h) *holds and $\frac{2p(p+1)}{p+2} < q < p^*$. There exists a bounded sequence $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^3)$ satisfying $I_\lambda(u_n) \to c_\lambda$, $I'_\lambda(u_n) \to 0$, $J_\lambda(u_n) \to 0$, where the functional $J_\lambda : W_r^{1,p}(\mathbb{R}^3) \to \mathbb{R}$ defined with the numbers β_i given in (4.1) by*

$$J_\lambda(u) = \frac{\beta_1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{\beta_2}{p} \int_{\mathbb{R}^3} |u|^p dx + \frac{\lambda\beta_1}{2p} \int_{\mathbb{R}^3} \phi_u |u|^p dx - \frac{\beta_3}{q} \int_{\mathbb{R}^3} |u|^q dx$$

$$- \beta_4 \int_{\mathbb{R}^3} hudx + \int_{\mathbb{R}^3} (x, \nabla h(x))udx.$$

Proof. Define the map $\Psi : \mathbb{R} \times W_r^{1,p}(\mathbb{R}^3) \to W_r^{1,p}(\mathbb{R}^3)$ for $\sigma \in \mathbb{R}$, $v \in W_r^{1,p}(\mathbb{R}^3)$ and $x \in \mathbb{R}^3$ by

$$\Psi(\sigma, v)(x) = e^{\frac{p+2}{p}\sigma} v(e^{\sigma} x).$$

By Lemma 2.1(i), the functional $I_\lambda \circ \Psi$ is computed as

$$\begin{aligned} I_\lambda(\Psi(\sigma, v)) =& \frac{e^{\beta_1 \sigma}}{p} \int_{\mathbb{R}^3} |\nabla v|^p dx + \frac{e^{\beta_2 \sigma}}{p} \int_{\mathbb{R}^3} |v|^p dx + \frac{\lambda e^{\beta_1 \sigma}}{2p} \int_{\mathbb{R}^3} \phi_v |v|^p dx \\ & - \frac{e^{\beta_3 \sigma}}{q} \int_{\mathbb{R}^3} |v|^q dx - e^{\beta_4 \sigma} \int_{\mathbb{R}^3} h\left(\frac{x}{e^{\sigma}}\right) vdx. \end{aligned}$$

It is standard to verify that $I_\lambda \circ \Psi$ is continuously Fréchet–differentiable on $\mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)$. Together with $I_\lambda(\Psi(0,0)) = 0$, we set the family of paths

$$\bar{\Gamma}_\lambda = \{\bar{\gamma} \in C([0,1], \mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)) : \bar{\gamma}(0) = (0,0) \text{ and } (I_\lambda \circ \Psi)(\bar{\gamma}(1)) < 0\}.$$

As $\Gamma_\lambda = \{\Psi \circ \bar{\gamma} : \bar{\gamma} \in \bar{\Gamma}_\lambda\}$, the mountain pass levels of I_λ and $I_\lambda \circ \Psi$ coincide:

$$c_\lambda = \inf_{\bar{\gamma} \in \bar{\Gamma}_\lambda} \sup_{t \in [0,1]} (I_\lambda \circ \Psi)(\bar{\gamma}(t)).$$

Let $\bar{\gamma} = (0, \gamma)$. For every $\epsilon \in (0, \frac{c_\lambda}{2})$, there exists $\gamma \in \Gamma_\lambda$ such that

$$\sup (I_\lambda \circ \Psi)(0, \gamma) \leqslant c_\lambda + \epsilon.$$

Then, by [28, Theorem 2.8], there exists $(\sigma, v) \in \mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)$ such that

$$\begin{cases} c_\lambda - 2\epsilon \leqslant (I_\lambda \circ \Psi)(\sigma, v) \leqslant c_\lambda + 2\epsilon, \\ \operatorname{dist}\{(\sigma, v), (0, \gamma)\} \leqslant 2\sqrt{\epsilon}, \ \text{ where } \operatorname{dist}\{(\sigma, v), (\varsigma, w)\} = (|\sigma - \varsigma|^2 + \|v - w\|^2)^{\frac{1}{2}}, \\ (I_\lambda \circ \Psi)'(\sigma, v) \to 0 \text{ in } [\mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)]^*. \end{cases}$$

Therefore, there exists a sequence $\{(\sigma_n, v_n)\} \subset \mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)$ such that as $n \to \infty$,

$$\sigma_n \to 0, \quad (I_\lambda \circ \Psi)(\sigma_n, v_n) \to c_\lambda, \quad (I_\lambda \circ \Psi)'(\sigma_n, v_n) \to 0.$$

For every $(\zeta, w) \in \mathbb{R} \times W_r^{1,p}(\mathbb{R}^3)$, there results

$$\langle (I_\lambda \circ \Psi)'(\sigma_n, v_n), (\zeta, w) \rangle = \langle I'_\lambda(\Psi(\sigma_n, v_n)), \Psi(\sigma_n, w) \rangle + J_\lambda(\Psi(\sigma_n, v_n))\zeta.$$

Take $u_n = \Psi(\sigma_n, v_n)$. Then we conclude that

$$I_\lambda(u_n) \to c_\lambda, \quad I'_\lambda(u_n) \to 0, \quad J_\lambda(u_n) \to 0. \tag{4.2}$$

Now, our claim is to prove that $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$. By (4.1) and (4.2), for n large enough, we deduce that

$$c_\lambda + 1 \geqslant I_\lambda(u_n) - \frac{1}{\beta_3} J_\lambda(u_n)$$

$$
\begin{aligned}
&= \frac{1}{p}\left(1-\frac{\beta_1}{\beta_3}\right)\int_{\mathbb{R}^3}|\nabla u_n|^p dx + \frac{1}{p}\left(1-\frac{\beta_2}{\beta_3}\right)\int_{\mathbb{R}^3}|u_n|^p dx \\
&\quad + \frac{\lambda}{2p}\left(1-\frac{\beta_1}{\beta_3}\right)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p dx \\
&\quad - \left(1-\frac{\beta_4}{\beta_3}\right)\int_{\mathbb{R}^3} h u_n dx - \frac{1}{\beta_3}\int_{\mathbb{R}^3}(x,\nabla h)u_n dx \\
&\geqslant \frac{1}{p}\left(1-\frac{\beta_1}{\beta_3}\right)\|u_n\|^p - \left(1-\frac{\beta_4}{\beta_3}\right)\int_{\mathbb{R}^3} h u_n dx - \frac{1}{\beta_3}\int_{\mathbb{R}^3}(x,\nabla h)u_n dx, \quad (4.3)
\end{aligned}
$$

where

$$
1 > \frac{\beta_1}{\beta_3} > \frac{\beta_2}{\beta_3} > 0 > \frac{\beta_4}{\beta_3}. \tag{4.4}
$$

It follows from (4.3) that

$$
c_\lambda + 1 + \left(1-\frac{\beta_4}{\beta_3}\right)\int_{\mathbb{R}^3} h u_n dx + \frac{1}{\beta_3}\int_{\mathbb{R}^3}(x,\nabla h)u_n dx \geqslant \frac{1}{p}\left(1-\frac{\beta_1}{\beta_3}\right)\|u_n\|^p. \tag{4.5}
$$

We deduce from (h)(ii), (2.2) and the Hölder inequality that

$$
\left|\int_{\mathbb{R}^3}(x,\nabla h)u_n dx\right| \leqslant \left(\int_{\mathbb{R}^3}|(x,\nabla h)|^s\, dx\right)^{\frac{1}{s}}\left(\int_{\mathbb{R}^3}|u_n|^{s'} dx\right)^{\frac{1}{s'}} \leqslant C\|u_n\|, \tag{4.6}
$$

where $p \leqslant s' \leqslant p^*$. Therefore by (2.3) and (4.4)–(4.6) that $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$. □

Theorem 4.2. *Assume that* (h) *holds and* $\frac{2p(p+1)}{p+2} < q < p^*$. *Then* I_λ *has a critical point* $u^* \in W_r^{1,p}(\mathbb{R}^3)$ *with* $I_\lambda(u^*) > 0$ *for* $\lambda > 0$ *provided* $|h|_s < \Lambda$, *where* Λ *was given in Lemma 2.2.*

Proof. For $\lambda > 0$, by Theorem 4.1 and Lemmas 4.1–4.2, there exists $\Lambda > 0$ such that if $|h|_s < \Lambda$, then we obtain a bounded $(\mathrm{PS})_{c_\lambda}$ sequence $\{u_n\}$ of I_λ. In view of Lemma 2.3, we know that there exists a critical point $u^* \in W_r^{1,p}(\mathbb{R}^3)$ of I_λ. Moreover, $I_\lambda(u^*) = c_\lambda > 0$. □

The result of Theorem 1.1 follows from Theorems 3.2, 4.2.

4.2. The case $p < q < p^*$

In this subsection, we will prove that if $\lambda > 0$ small, then system (1.1) has a positive energy solution for $p < q < p^*$ and h satisfying (h)(i) and small $|h|_s$. In what follows, we shall use a truncated technique which is due to Jeanjean and Le Coz [16]. Then, we define the cut–off function $\chi \in C^\infty(\mathbb{R}_+, [0,1])$ satisfying $|\chi'|_\infty \leqslant 2$,

$$
\chi(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}], \\ 0, & t \in [1, \infty), \end{cases}
$$

and $\chi(t) \in [0,1]$ for $t \in (\frac{1}{2}, 1)$. Define a penalized functional $I_{\lambda,K} : W_r^{1,p}(\mathbb{R}^3) \to \mathbb{R}$ as

$$
I_{\lambda,K}(u) = \frac{1}{p}\int_{\mathbb{R}^3}(|\nabla u|^p + |u|^p)dx + \frac{\lambda}{2p}L_K(u)\int_{\mathbb{R}^3}\phi_u|u|^p dx - \frac{1}{q}\int_{\mathbb{R}^3}|u|^q dx - \int_{\mathbb{R}^3} h u dx, \tag{4.7}
$$

where $K > 0$ and $L_K(u) = \chi\left(\frac{\|u\|^p}{K^p}\right)$. It is easy to check that $I_{\lambda,K}$ is of class C^1 and

$$\begin{aligned}\langle I'_{\lambda,K}(u), v\rangle =& (1 + a_{\lambda,K}(u)) \int_{\mathbb{R}^3} (|\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv)dx \\ &+ \lambda L_K(u) \int_{\mathbb{R}^3} \phi_u |u|^{p-2}uvdx \\ &- \int_{\mathbb{R}^3} |u|^{q-2}uvdx - \int_{\mathbb{R}^3} hvdx, \quad \forall\, u, v \in W_r^{1,p}(\mathbb{R}^3), \end{aligned} \tag{4.8}$$

where

$$a_{\lambda,K}(u) = \frac{\lambda}{2K^p}\chi'\left(\frac{\|u\|^p}{K^p}\right)\int_{\mathbb{R}^3} \phi_u |u|^p dx. \tag{4.9}$$

Notice that if $\|u\| \leqslant \frac{K}{2}$, then there hold $L_K(u) = 1$ and $I_{\lambda,K}(u) = I_\lambda(u)$. Now, we first verify that the functional $I_{\lambda,K}$ possesses a mountain pass geometry for each $K > 0$.

Lemma 4.3. *Assume that* (h)(i) *holds and* $p < q < p^*$. *For* $\lambda > 0$ *and* $K > 0$, *there exists* $\Lambda > 0$ *such that if* $|h|_s < \Lambda$, *then* $I_{\lambda,K}$ *satisfies the assumptions* (M_1)–(M_2) *in Theorem 4.1.*

Proof. First of all, by Lemma 2.2, it is easy to see that there exists $\Lambda > 0$ such that if $|h|_s < \Lambda$, then (M_1) in Theorem 4.1 is true. Next, we claim that $I_{\lambda,K}$ satisfies (M_2). Arguing as in the proof of Lemma 3.1, we can choose a function $\varphi_1 \in W_r^{1,p}(\mathbb{R}^3)$ such that $\|\varphi_1\| = 1$ and $\int_{\mathbb{R}^3} h(x)\varphi_1(x)dx > 0$. For each $K > 0$ and $t \geqslant K$, $L_K(t\varphi_1) = 0$. From this it is clear that

$$I_{\lambda,K}(t\varphi_1) = \frac{1}{p}t^p - \frac{1}{q}t^q \int_{\mathbb{R}^3} |\varphi_1|^q dx - t\int_{\mathbb{R}^3} h(x)\varphi_1 dx. \tag{4.10}$$

Indeed, if we choose $t_K > K$ large, then $\|t_K\varphi_1\| > \rho$ and $I_{\lambda,K}(t_K\varphi_1) < 0$. Therefore, it is enough to take $v = t_K\varphi_1$ to complete the proof. □

By Lemma 4.3, $I_{\lambda,K}$ satisfies all assumptions in Theorem 4.1. Now, we define the mountain pass level of $I_{\lambda,K}$ for each $K > 0$ as follows:

$$c_{\lambda,K} = \inf_{\gamma\in\Gamma_{\lambda,K}} \max_{t\in[0,1]} I_{\lambda,K}(\gamma(t)) > 0, \tag{4.11}$$

where $\Gamma_{\lambda,K} := \left\{\gamma \in C([0,1], W_r^{1,p}(\mathbb{R}^3)) : \gamma(0) = 0 \text{ and } I_{\lambda,K}(\gamma(1)) < 0\right\}$. Then there exists a sequence $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^3)$ such that as $n \to \infty$

$$I_{\lambda,K}(u_n) \to c_{\lambda,K}, \qquad I'_{\lambda,K}(u_n) \to 0 \;\text{ in } [W_r^{1,p}(\mathbb{R}^3)]^*. \tag{4.12}$$

Next, we will prove that $\{u_n\}$ is bounded in $W_r^{1,p}(\mathbb{R}^3)$ for large K and small λ.

Lemma 4.4. *There exist* $K > 0$ *and* $\lambda_* > 0$ *such that for all* $\lambda \in (0, \lambda_*)$ *the sequence* $\{u_n\}$ *given by* (4.12) *satisfies*

$$\|u_n\| \leqslant \frac{K}{2}. \tag{4.13}$$

Proof. We first claim that $\{u_n\}$ must be bounded. From (4.7) and (4.8), it is easy to see that

$$\begin{aligned} c_{\lambda,K}+1+\|u_n\| \geqslant & I_{\lambda,K}(u_n)-\frac{1}{q}\langle I'_{\lambda,K}(u_n),u_n\rangle \\ &=\left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p+\left(\frac{\lambda}{2p}-\frac{\lambda}{q}\right)L_K(u_n)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p dx \\ &\quad -\frac{a_{\lambda,K}(u_n)}{q}\|u_n\|^p-\frac{q-1}{q}\int_{\mathbb{R}^3}h(x)u_n dx. \end{aligned}$$

This shows that

$$\begin{aligned} \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p \leqslant c_{\lambda,K}+1+\|u_n\|+\left(\frac{\lambda}{q}-\frac{\lambda}{2p}\right)L_K(u_n)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p dx \\ +\frac{a_{\lambda,K}(u_n)}{q}\|u_n\|^p+\frac{q-1}{q}\int_{\mathbb{R}^3}h(x)u_n dx. \end{aligned} \tag{4.14}$$

From the definition one sees that if $\|u_n\| \geqslant K$, then there result $L_K(u_n)=0$ and $a_{\lambda,K}(u_n)=0$, which and (4.14) yield

$$\left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p \leqslant c_{\lambda,K}+1+\|u_n\|+\frac{q-1}{q}\int_{\mathbb{R}^3}h(x)u_n dx.$$

From this and from (2.3) we obtain that $\{u_n\}$ is bounded. By Lemma 2.1(ii), for all $n\in\mathbb{N}$ we have,

$$\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p dx=-\int_{\mathbb{R}^3}\phi_{u_n}\Delta\phi_{u_n}dx=\|\phi_{u_n}\|_D^2 \leqslant A^2\|u_n\|^{2p}. \tag{4.15}$$

Notice that if $\|u_n\| \geqslant K$ then $\chi'\left(\frac{\|u_n\|^p}{K^p}\right)=0$. It follows from (4.9) and (4.15) that

$$|a_{\lambda,K}(u_n)| \leqslant \frac{\lambda}{2K^p}\left|\chi'\left(\frac{\|u_n\|^p}{K^p}\right)\right|\left|\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p dx\right| \leqslant \lambda A^2K^p. \tag{4.16}$$

Then by using (2.3), (4.15)–(4.16) and the Hölder inequality, an easy computation shows that

$$\begin{cases} \left(\dfrac{\lambda}{q}-\dfrac{\lambda}{2p}\right)L_K(u_n)\displaystyle\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^p dx \leqslant \lambda A^2K^{2p}, \\ \dfrac{a_{\lambda,K}(u_n)}{q}\|u_n\|^p \leqslant \lambda A^2K^{2p}, \\ \dfrac{q-1}{q}\displaystyle\int_{\mathbb{R}^3}h(x)u_n dx \leqslant C\Lambda\|u_n\|. \end{cases} \tag{4.17}$$

Let φ_1 be the function taken in the proof of Lemma 4.3. By (4.10),

$$I_{\lambda,K}(K\varphi_1) \leqslant \frac{K^p}{p}-\frac{K^q}{q}|\varphi_1|_q^q.$$

So there exists $K_1 > 0$ such that $I_{\lambda,K}(K\varphi_1) < 0$ for all $K \geqslant K_1$. Thus by (4.7) and (4.11),

$$\begin{aligned} c_{\lambda,K} &\leqslant \max_{t\in[0,1]} I_{\lambda,K}(tK\varphi_1) \leqslant \max_{t\in[0,1]} \left\{ \frac{1}{p}(tK)^p - \frac{1}{q}(tK)^q |\varphi_1|_q^q \right\} \\ &\quad + \max_{t\in[0,1]} \frac{\lambda}{2p}(tK)^{2p} L_K(tK\varphi_1) \int_{\mathbb{R}^3} \phi_{\varphi_1} |\varphi_1|^p dx \\ &\leqslant C + \lambda A^2 K^{2p}. \end{aligned} \tag{4.18}$$

Now, it follows from (4.14)–(4.18) that, for all $K \geqslant K_1$,

$$\left(\frac{1}{p} - \frac{1}{q}\right) \|u_n\|^p \leqslant C + 1 + 3\lambda A^2 K^{2p} + (1 + C\Lambda)\|u_n\|. \tag{4.19}$$

Summing up, if we choose $\lambda_* = (A^2 K^{2p})^{-1}$, then inequality (4.19) implies (4.13) for any $K \geqslant K_1$ and $\lambda \in (0, \lambda_*)$. The proof is complete. □

Theorem 4.3. *Assume that* (h)(i) *holds and $p < q < p^*$. Then there exists $\lambda_* > 0$ such that I_λ has a critical point $u^* \in W_r^{1,p}(\mathbb{R}^3)$ with $I_\lambda(u^*) > 0$ for any $\lambda \in (0, \lambda_*)$ provided $|h|_s < \Lambda$, where Λ was given in Lemma 2.2.*

Proof. Combining Theorem 4.1 (the mountain pass theorem) and Lemmas 4.3–4.4, for $K > 0$ large enough and $\lambda > 0$ small, we obtain a bounded $(\mathrm{PS})_{c_{\lambda,K}}$ sequence $\{u_n\}$ of $I_{\lambda,K}$ with $\|u_n\| \leqslant \frac{K}{2}$. By the definition of χ and (4.7), we derive that $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^3)$ is a bounded $(\mathrm{PS})_{c_{\lambda,K}}$ sequence for I_λ with $c_{\lambda,K} > 0$. By Lemma 2.3, we can find a critical point u^* of I_λ with $I_\lambda(u^*) = c_{\lambda,K} > 0$. □

The result of Theorem 1.2 follows from Theorems 3.2, 4.3.

Acknowledgements

The authors would like to thank the anonymous reviewers for their careful reading of the manuscript and for many valuable suggestions.

References

[1] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 1973, 14(4), 349–381.

[2] A. Azzollini, *Concentration and compactness in nonlinear Schrödinger–Poisson system with a general nonlinearity*, J. Differential Equations, 2010, 249(7), 1746–1763.

[3] A. Azzollini, P. d'Avenia and A. Pomponio, *On the Schrödinger–Maxwell equations under the effect of a general nonlinear term*, Ann. Inst. H. PoincaršŠ Anal. Non Linéaire, 2010, 27(2), 779–791.

[4] A. Azzollini and A. Pomponio, *Ground state solutions for the nonlinear Schrödinger–Maxwell equations*, J. Math. Anal. Appl., 2008, 345(1), 90–108.

[5] V. Benci and D. Fortunato, *An eigenvalue problem for the Schrödinger–Maxwell equations*, Topol. Methods Nonlinear Anal., 1998, 11(2), 283–293.

[6] V. Benci and D. Fortunato, *Solitary waves of the nonlinear Klein–Gordon equation coupled with the Maxwell equations*, Rev. Math. Phys., 2002, 14(4), 409–420.

[7] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., 1983, 36(4), 437–477.

[8] S. Chen and C. Tang, *Multiple solutions for nonhomogeneous Schrödinger–Maxwell and Klein–Gordon–Maxwell equations on* $\mathbb{R}^3$, NoDEA Nonlinear Differential Equations Appl., 2010, 17(5), 559–574.

[9] T. D'Aprile and D. Mugnai, *Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A, 2004, 134(5), 893–906.

[10] P. d'Avenia, *Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations*, Adv. Nonlinear Stud., 2002, 2(2), 177–192.

[11] Y. Du, J. Su and C. Wang, *The Schrödinger–Poisson system with p–Laplacian*, Appl. Math. Lett., 2021, 120(107286), 1–7.

[12] Y. Du, J. Su and C. Wang, *On a quasilinear Schrödinger–Poisson system*, J. Math. Anal. Appl., 2022, 505(125446), 1–14.

[13] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., 1974, 47, 324–353.

[14] L. Huang, X. Wu and C. Tang, *Multiple positive solutions for nonhomogeneous Schrödinger–Poisson systems with Berestycki-Lions type conditions*, Electron. J. Differential Equations, 2021, 2021(1), 1–14.

[15] L. Jeanjean, *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal., 1997, 28(10), 1633–1659.

[16] L. Jeanjean and S. Le Coz, *An existence and stability result for standing waves of nonlinear Schrödinger equations*, Adv. Differential Equations, 2006, 11(7), 813–840.

[17] Y. Jiang, Z. Wang and H. Zhou, *Multiple solutions for a nonhomogeneous Schrödinger–Maxwell system in* $\mathbb{R}^3$, Nonlinear Anal., 2013. DOI: 10.1016/j.na.2013.01.006.

[18] S. Khoutir and H. Chen, *Multiple nontrivial solutions for a nonhomogeneous Schrödinger–Poisson system in* $\mathbb{R}^3$, Electron. J. Qual. Theory Differ. Equ., 2017. DOI: 10.14232/ejqtde.2017.1.28.

[19] E. H. Lieb and M. Loss, *Analysis*, American Mathematical Society, 2001.

[20] P. L. Lions, *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal., 1982, 49(3), 315–334.

[21] J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, Springer-Verlag, New York, 1989.

[22] D. Ruiz, *The Schrödinger–Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal., 2006, 237(2), 655–674.

[23] A. Salvatore, *Multiple solitary waves for a non-homogeneous Schrödinger–Maxwell system in* $\mathbb{R}^3$, Adv. Nonlinear Stud., 2006, 6(2), 157–169.

[24] J. Su, Z. Wang and M. Willem, *Weighted Sobolev embedding with unbounded and decaying radial potential*, J. Differential Equations, 2007, 238(1), 201–219.

[25] J. Su and Z. Wang, *Sobolev type embedding and quasilinear elliptic equations with radial potentials*, J. Differential Equations, 2011, 250(1), 223–242.

[26] J. Sun and S. Ma, *Ground state solutions for some Schrödinger–Poisson systems with periodic potentials*, J. Differential Equations, 2016, 260(3), 2119–2149.

[27] L. Wang, S. Ma and N. Xu, *Multiple solutions for nonhomogeneous Schrödinger–Poisson equations with sign–changing potential*, Acta Math. Sci. Ser. B, 2017, 37(2), 555–572.

[28] M. Willem, *Minimax Theorems*, Birkhäuser Boston, Inc., Boston, 1996.

[29] Y. Ye, *Multiple positive solutions for nonhomogeneous Schrödinger–Poisson system in* $\mathbb{R}^3$, Lith. Math. J., 2020, 60(2), 276–287.

[30] L. Yin, X. Wu and C. Tang, *Ground state solutions for an asymptotically* 2*–linear Schrödinger–Poisson system*, Appl. Math. Lett., 2019. DOI: 10.1016/j.aml.2018.07.017.

[31] J. Zhang, J. M. doÓ and M. Squassina, *Schrödinger–Poisson systems with a general critical nonlinearity*, Commun. Contemp. Math., 2017, 19(4), 1–16.

[32] Q. Zhang, F. Li and Z. Liang, *Existence of multiple positive solutions to nonhomogeneous Schrödinger–Poisson system*, Appl. Math. Comput., 2015. DOI: 10.1016/j.amc.2015.02.044.