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ALTERNATING PROJECTION METHOD FOR SOLVING DOUBLY STOCHASTIC INVERSE SINGULAR VALUE PROBLEMS WITH PRESCRIBED ENTRIES

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Abstract Doubly stochastic inverse singular value problem with prescribed entries aims to construct a doubly stochastic matrix from the prescribed singular value and prescribed entries. In this paper, the doubly stochastic inverse singular value problem is considered as the problem of finding a point in the intersection of a compact set and a closed convex set. We present a numerical procedure which is based on an alternating projection process, for solving the problem. The method is iterative in nature. And each subproblem in the alternating projection method can be solved easily. Convergence properties of the algorithm are investigated and numerical results are presented to illustrate the effectiveness of our method.

Keywords Doubly stochastic matrix, inverse singular value problem, prescribed entries, alternating projection method.

MSC(2010) 65F18, 65F15, 49J52, 90C30.

1. Introduction

An inverse singular value problem(ISVP) concerns the construction of a structured matrix with prescribed singular value. There have been extensive studies and practical applications of ISVP, such as in some quadratic groups, structural health monitoring, code division multiple access system, transient circuit simulation (see [5, 17, 23, 26, 28]). In fact, ISVP is a natural extension of inverse eigenvalue problem (IEP) [11]. As noted in [11], if T is a matrix given by

$$T = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}, \tag{1.1}$$

then the ISVP for A is solvable if and only if the IEP for T is solvable. However, the IEP for T might be difficult due to the extra block diagonal zeros. Therefore, methods for solving the ISVP directly have attracted considerable attention in recent years, see [3, 4, 8, 10, 30] and references therein. There are different structured ISVPs such as the affine ISVP, the ISVP for Toeplitz-related matrices, the ISVP

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for nonnegative matrices and so on. In this paper, we consider the doubly stochastic inverse singular value problem (DISVP). Let e be the $n \times 1$ vector of all ones, i.e., $e = (1, 1, \ldots, 1)^T$. A matrix $A \in \mathbb{R}^{n\times n}$ with nonnegative entries is called row stochastic if $Ae = e$. Especially the nonnegative matrix is called doubly stochastic if $Ae = e$ and $e^T A = e^T$ (A has row and column sums equal to one). Doubly stochastic matrices are very important in many applications including probability and statistics, large linear semiconductor circuit networks, economics and operation research, graph theory and graph-based clustering, etc., see [1,2,7,13,15,19,20,29,32,33] and the references therein. The DISVP aims to construct a doubly stochastic matrix from the given singular values. A related problem is the doubly stochastic inverse singular value problem with prescribed diagonal entries(DISVP-PE). The problem can be delineated as follows:

DISVP − PE: Given a list of n nonnegative scalars $\sigma_1, \sigma_2, \ldots, \sigma_n$ and the prescribed positive numbers $g_1, g_2, \ldots, g_n$, find an $n \times n$ doubly stochastic matrix $H = (h_{ij})$ such that it has the singular $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ and

$$h_{ii} = g_i, \quad \forall i = 1, \ldots, n. \tag{1.2}$$

The prescribed entries are used to characterize the underlying structure. Finding necessary and sufficient conditions of the DISVP-PE is a challenging area. Here, we are interested in generally applicable numerical methods for solving DISVP-PE. We consider the DISVP-PE as the problem of finding a point in the intersection of two closed sets. Alternating projection method [6,9] is applied to solve the problem. In this paper, we will show that alternating projection method is also very effective for DISVP-PE.

Throughout this paper, we use the following notations. For an $n \times n$ matrix A, A^T stands for the transpose of a matrix A. We write $A \geq 0$ iff A is a nonnegative matrix (i.e., all the entries are nonnegative) and $A > 0$ iff A is a positive matrix (i.e, all the entries are positive). Let $\mathbb{R}^{n\times n}$ be the set of all real matrices of order n. Denote $\|\cdot\|_F$ to be the Frobenius matrix norm and I_n to be the identity matrix of n dimension. Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite-dimensional vector spaces equipped with a scalar inner product $\langle\cdot\rangle$ and its induced norm $\|\cdot\|$. Let $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ be a linear operator such that $\mathcal{A}(x) \in \mathcal{Y}$ for all $x \in \mathcal{X}$ and the adjoint of $\mathcal{A}$ is denoted by $\mathcal{A}^*$.

The rest of this paper is organized as follows. In Section 2, we reformulate the DISVP-PE as the problem of finding a point in the intersection of two closed sets. Meanwhile, the alternating projection method is proposed to solve the DISVP-PE. Section 3 is devoted to analyzing the convergence of the algorithm. Numerical tests for the alternating projection method are presented in Section 4.

2. Reformulation

For the given singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$, without loss of generality, we assume that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n. \tag{2.1}$$

Let

$$\Sigma = \mathrm{diag}(\sigma), \qquad \sigma := (\sigma_1, \sigma_2, \ldots, \sigma_n). \tag{2.2}$$

The set of all $n \times n$ real matrices with the singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$ can be defined by

$$\mathcal{M} = \{U\Sigma V | U, V \in \mathcal{O}(n)\}, \tag{2.3}$$

where $\mathcal{O}(n) = \{Q \in \mathbb{R}^{n\times n} | Q^TQ = I_n\}$.

Given the prescribed positive diagonal values $g = \{g_1, g_2, \ldots, g_n\}$, the set

$$\mathcal{K} = \{A \in \mathbb{R}^{n\times n} | Ae = e, e^TA = e^T, A \geq 0, A_{ii} = g_i\} \tag{2.4}$$

contains the doubly stochastic matrices with the prescribed entries at the desired locations.

The DISVP-PE can be reformulated as the following problem:

$$\text{Find} \quad H \in \mathcal{M} \cap \mathcal{K}. \tag{2.5}$$

Alternating projection method for the DISVP-PE is to project iteratively onto each of the closed sets $\mathcal{M}$ and $\mathcal{K}$.

Consider a Euclidean space $\mathbb{E}$ with Frobenius norm $\|H\| = \text{trace}(H^TH)$. Given a nonempty closed set $\mathcal{C} \in \mathbb{E}$, for any $M \in \mathcal{C}$

$$\|M - H\| \leq \|T - H\| \quad \text{for all} \quad T \in \mathcal{C}, \tag{2.6}$$

then M is called a projection of H onto $\mathcal{C}$, written as $M = P_{\mathcal{C}}(H)$.

Alternating projection method applied to the DISVP-PE can be described as follows.

Algorithm 1 Alternating projection method

Initialization: Given initial matrix $H_0 \in \mathbb{R}^{n\times n}$ and $k := 0$. Iterate the following steps.
Step 1. $M_{k+1} = P_{\mathcal{M}}(H_k)$.
Step 2. $H_{k+1} = P_{\mathcal{K}}(M_{k+1})$
Step 3. Replace k by $k+1$ and go to Step 1.

The key fact of the algorithm is that it is often easier to project onto the individual set $\mathcal{M}$ or $\mathcal{K}$ than it is to project onto the intersection $\mathcal{M} \cap \mathcal{K}$. Moreover, the following distance reduction property always holds for alternating projection between two closed sets.

Theorem 2.1 (Orsi [22])**.** *Let $\mathcal{M}$ and $\mathcal{K}$ be closed (nonempty) sets in a finite dimensional Hilbert space $\mathbb{H}$. For any initial value $y_0 \in \mathcal{K}$, if $x_1 = P_{\mathcal{M}}(y_0)$, $y_1 = P_{\mathcal{K}}(x_1)$, $x_2 = P_{\mathcal{M}}(y_1)$, then*

$$\|x_2 - y_1\| \leq \|x_1 - y_1\| \leq \|x_1 - y_0\|. \tag{2.7}$$

Projections play an important part in the algorithm. We next show that it is indeed possible to calculate projections onto these sets.

2.1. Solving the first projection

In this subsection, we will show how to calculate the first projection $M_{k+1} = P_{\mathcal{M}}(H_k)$, where $\mathcal{M} = \{U\Sigma V | U, V \in \mathcal{O}(n)\}$. For this purpose, we state the following important result of Hoffman Wielandt.

Theorem 2.2 (Horn and Johnson [17])**.** *Let $A, B \in \mathbb{C}^{n\times n}$ and suppose that $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ are the nonincreasingly ordered singular values of A and B, respectively. Then*

$$\|A - B\| \geq \sum_{i=1}^{n} (\tau_i - \sigma_i)^2. \tag{2.8}$$

Based on Theorem 2.2, Wu and Lin have obtained the following result.

Theorem 2.3 (Wu and Lin [31]). *Let A be a nonnegative $n \times n$ matrix. Suppose that the singular value decomposition of A is*

$$A = U\Sigma_A V^T, \tag{2.9}$$

where $U, V \in \mathbb{R}^{n\times n}$ are two unitary matrices, and $\Sigma_A = \mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_n)$ with entries in the order $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$. Then $U\Sigma V^T$ is a best approximation in $\mathcal{M}$ to A in the Frobenius norm, where $\mathcal{M} = \{U\Sigma V | U, V \in \mathcal{O}(n)\}$.

The projection of $A \in \mathbb{R}^{n\times n}$ onto $\mathcal{M}$ is given by Theorem 2.3. Suppose that the singular value decomposition of H_k is $H_k = U_k \Sigma_{H_k} V_k^T$, then $M_{k+1} = P_{\mathcal{M}}(H_k) = U_k \Sigma V_k^T$. The reason for this is that the set $\mathcal{M}$ is nonconvex, and hence projections onto this set are not guaranteed to be unique.

2.2. Solving the second projection

As for the second projection $H_{k+1} = P_{\mathcal{K}}(M_{k+1})$, we might reasonably consider the case of exact projections on the closed convex set $\mathcal{K}$. However, projecting onto the set $\mathcal{K}$ may be much harder, so a more realistic analysis allows relaxed projections. The projection $H = P_{\mathcal{K}}(M)$ can be solved by the following quadratic optimization problem

$$\begin{aligned} \min\ & \frac{1}{2}\|H - M\|^2 \\ \text{s.t. }\ & He = e, e^T H = e^T, \\ & H \geq 0, \\ & H_{i,i} = g_i, i = 1, 2, \ldots, n. \end{aligned} \tag{2.10}$$

Let

$$\mathcal{F}(H) = \begin{bmatrix} He \\ \begin{bmatrix} I_{n-1} & 0 \end{bmatrix} H^T e \\ e_1^T H e_1 \\ \vdots \\ e_n^T H e_n \end{bmatrix}, \quad b = \begin{bmatrix} e \\ \begin{bmatrix} I_{n-1} & 0 \end{bmatrix} e \\ g_1 \\ \vdots \\ g_n \end{bmatrix}, \tag{2.11}$$

where e_i denotes the ith column of the n-by-n identity matrix. Then Problem (2.10) is equivalent to

$$\begin{aligned} \min\ & \frac{1}{2}\|H - M\|^2 \\ \text{s.t. }\ & \mathcal{F}(H) = b, \\ & H \in \mathcal{N}, \end{aligned} \tag{2.12}$$

where $\mathcal{N} := \{H : H \in \mathbb{R}^{n\times n}, H \geq 0\}$. Motivated by [24], we adopt a Newton-type method for computing the solution of Problem (2.12). Denote the dual cone of $\mathcal{N}$ by $\mathcal{N}^*$, i.e., $\mathcal{N}^* = \{U : U \in \mathbb{R}^{n\times n}, \langle U, H\rangle \geq 0, \forall H \in \mathcal{N}\}$. The Lagrangian function for Problem (2.12) is given by

$$L(H, y, S) = \frac{1}{2}\|H - M\|^2 + \langle y, b - \mathcal{F}(H)\rangle - \langle S, H\rangle, \tag{2.13}$$

where the dual variable $(y, S) \in \mathbb{R}^{3n-1} \times \mathcal{N}^*$. Furthermore, the Lagrangian dual problem (see [18]) of (2.12) is given as follows

$$\begin{aligned}
&\min\{\theta(y,S) := -\min_H L(H,y,S) = \tfrac{1}{2}\|\mathcal{F}^*(y) + M + S\|^2 - \langle y, b\rangle - \tfrac{1}{2}\|M\|^2\}\\
&\text{s.t.}\quad y = (y_1, y_2, \dots, y_{3n-1}) \in \mathbb{R}^{3n-1}, S \in \mathcal{N}^*,
\end{aligned} \tag{2.14}$$

where $\mathcal{F}^*$ is the adjoint of $\mathcal{F}$, defined by

$$\mathcal{F}^*(y) = \begin{bmatrix} y_1 + y_{n+1} + y_{2n} & y_1 + y_{n+2} & \cdots & y_1 + y_{2n-1} & y_1 \\ y_2 + y_{n+1} & y_2 + y_{n+2} + y_{2n+1} & \cdots & y_2 + y_{2n-1} & y_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ y_n + y_{n+1} & y_n + y_{n+2} & \cdots & y_n + y_{2n-1} & y_n + y_{3n-1} \end{bmatrix}. \tag{2.15}$$

From (2.14), we can find that if the optimal solution $(\bar{y}, \bar{S}) \in \arg\min \theta(y, S)$, it should satisfy

$$\bar{S} = P_{\mathcal{N}^*}(-\mathcal{F}^*(\bar{y}) - M) \quad \text{and} \quad \bar{y} \in \arg\min \theta(y, \bar{S}).$$

The Moreau decomposition theorem [21] states that $x = P_{\mathcal{C}}(x) - P_{\mathcal{C}^*}(-x)$ for any nonempty closed convex cone $\mathcal{C}$. So the optimal solution $\bar{y}$ should satisfy the following unconstrained minimization problem

$$\min_y \xi(y) := \frac{1}{2}\|P_{\mathcal{N}}(\mathcal{F}^*(y) + M)\|^2 - \langle y, b\rangle - \frac{1}{2}\|M\|^2. \tag{2.16}$$

On the other hand, we observe that if $H \in \arg\min L(H, y, S)$, then $H = \mathcal{F}^*(y) + M + S = P_{\mathcal{N}}(\mathcal{F}^*(y) + M)$. Once we can compute an optimal solution $\bar{y}$, then we can obtain the optimal solution $H = P_{\mathcal{N}}(\mathcal{F}^*(\bar{y}) + M)$ of Problem (2.10). Next, we will discuss how to solve the optimal problem (2.16). Notice that $\xi(y)$ is a continuously differentiable function and then at solution $\bar{y}$ of (2.16)

$$G(\bar{y}) := \nabla\xi(\bar{y}) = \mathcal{F}[P_{\mathcal{N}}(\mathcal{F}^*(\bar{y}) + M)] - b = 0. \tag{2.17}$$

Furthermore, $\forall A \in \mathbb{R}^{n\times n}$, we have

$$P_{\mathcal{N}}(A) = \begin{bmatrix} (a_{11})_+ & \cdots & (a_{1n})_+ \\ \vdots & \dots & \vdots \\ (a_{n1})_+ & \cdots & (a_{nn})_+ \end{bmatrix}, \tag{2.18}$$

where $(a_{ij})_+ = \max(a_{ij}, 0)$. Combining (2.11), (2.15) and (2.18), we have, for any

$y \in \mathbb{R}^{3n-1}$,

$$G(y) = \begin{bmatrix} (m_{11}+y_1+y_{n+1}+y_{2n})_+ + \sum_{j=2}^{n-1}(m_{1,j}+y_1+y_{n+j})_+ + (m_{1,n}+y_1)_+ \\ \vdots \\ \sum_{j=1}^{n-1}(m_{n,j}+y_n+y_{n+j})_+ + (m_{n,n}+y_n+y_{3n-1})_+ \\ (m_{11}+y_1+y_{n+1}+y_{2n})_+ + \sum_{i=2}^{n}(m_{i,1}+y_i+y_{n+1})_+ \\ \vdots \\ \sum_{i=1}^{n-1}(m_{i,n-1}+y_i+y_{2n-1})_+ + (m_{n,n-1}+y_n+y_{2n-1})_+ \\ (m_{11}+y_1+y_{n+1}+y_{2n})_+ \\ \vdots \\ (m_{n,n}+y_n+y_{3n-1})_+ \end{bmatrix} - \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ g_1 \\ \vdots \\ g_n \end{bmatrix}. \tag{2.19}$$

According to Redemacher's theorem [25, Chapter 9.J], the Lipschitz continuous function G is Fréchet differentiable almost everywhere. Let $\mathcal{O}$ be the set of differential points of G and $G'(y)$ be the Jacobian of $G(y)$ where it exists. Denote

$$\partial_B G(y) := \{T | T = \lim_{j\to\infty} G'(y^j), y^j \to y, y^j \in \mathcal{O}\}. \tag{2.20}$$

Then Clarke's generalized Jacobian [12] is

$$\partial G(y) := \mathrm{conv}\{\partial_B G(y)\}, \tag{2.21}$$

where "conv" represents the convex hull.

Based on $\partial G(y)$, Qi and Sun [24] proposed the following nonsmooth Newton method for solving Problem $G(y) = 0$:

$$y^{j+1} = y^j - D_j^{-1} G(y^j), \tag{2.22}$$

where $D_j \in \partial G(y^j)$.

By (2.20) and (2.21), for any $D \in \partial G(y)$, D has the form

$$D = \left(\begin{array}{ccc|ccc|ccc} \sum_{i=1}^n d_{1,i} & \cdots & 0 & d_{1,1} & \cdots & d_{1,n-1} & d_{1,1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \sum_{i=1}^n d_{n,i} & d_{n,1} & \cdots & d_{n,n-1} & 0 & \cdots & d_{n,n} \\ \hline d_{1,1} & \cdots & d_{n,1} & \sum_{i=1}^n d_{i,1} & \cdots & 0 & d_{1,1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{1,n-1} & \cdots & d_{n,n-1} & 0 & \cdots & \sum_{i=1}^n d_{i,n-1} & 0 & \cdots & 0 \\ \hline d_{1,1} & \cdots & 0 & d_{1,1} & \cdots & 0 & d_{1,1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 0 & \cdots & d_{n-1,n-1} & & & \\ 0 & \cdots & d_{n,n} & 0 & \cdots & 0 & 0 & \cdots & d_{n,n} \end{array}\right), \tag{2.23}$$

where for $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ and $i \neq j$,

$$\begin{cases} d_{i,j} = 1, & \text{if} \quad m_{i,j} + y_i + y_{n+j} > 0, \\ d_{i,j} \in [0,1], & \text{if} \quad m_{i,j} + y_i + y_{n+j} = 0, \\ d_{i,j} = 0, & \text{if} \quad m_{i,j} + y_i + y_{n+j} < 0, \end{cases}$$

for $i = 1, \ldots, n-1$, $j = n$,

$$\begin{cases} d_{i,n} = 1, & \text{if} \quad m_{i,n} + y_i > 0, \\ d_{i,n} \in [0,1], & \text{if} \quad m_{i,n} + y_i = 0, \\ d_{i,n} = 0, & \text{if} \quad m_{i,n} + y_i < 0, \end{cases}$$

for $i = 1, \ldots, n-1$,

$$\begin{cases} d_{i,i} = 1, & \text{if} \quad m_{i,i} + y_i + y_{n+i} + y_{2n+i-1} > 0, \\ d_{i,i} \in [0,1], & \text{if} \quad m_{i,i} + y_i + y_{n+i} + y_{2n+i-1} = 0, \\ d_{i,i} = 0, & \text{if} \quad m_{i,i} + y_i + y_{n+i} + y_{2n+i-1} < 0, \end{cases}$$

and

$$\begin{cases} d_{n,n} = 1, & \text{if} \quad m_{n,n} + y_n + y_{3n+1} > 0, \\ d_{n,n} \in [0,1], & \text{if} \quad m_{n,n} + y_n + y_{3n+1} = 0, \\ d_{n,n} = 0, & \text{if} \quad m_{n,n} + y_n + y_{3n+1} < 0. \end{cases}$$

In this paper, we only consider these cases:

- $d_{i,j} = 0$, if $m_{i,j} + y_i + y_{n+j} = 0$,
- $d_{i,n} = 0$, if $m_{i,n} + y_i = 0$.

The details of this method for solving the optimal solution $\bar{y}$ are given in Algorithm 2.

In the following proposition, we present that the matrix $D \in \mathbb{R}^{(3n-1)\times(3n-1)}$ is positive semi-definite.

Proposition 2.1. *$D \in \mathbb{R}^{(3n-1)\times(3n-1)}$ is positive semi-definite.*

Proof. For any $h = (h_1, h_2, \ldots, h_{3n-1})^T \in \mathbb{R}^{3n-1}$, a tedious calculation yields

$$\begin{aligned} h^T D h = & \sum_{i=1}^{n-1} d_{ii}(h_i + h_{n+i} + h_{2n-1+i})^2 + \sum_{j=2}^{n-1} d_{1j}(h_1 + h_{n+j})^2 \\ & + \sum_{i=2}^{n} \sum_{j=1, j\neq i}^{n-1} d_{ij}(h_i + h_{n+j})^2 + \sum_{i=1}^{n-1} d_{in} h_i^2 + d_{nn}(h_n + h_{3n-1})^2. \end{aligned} \tag{2.25}$$

Since $d_{ij} \geq 0$, we have

$$h^T D h \geq 0. \tag{2.26}$$

We thus complete the proof. □

The following theorem contains two important sufficient conditions ensuring the convergency of nonsmooth Newton method [24] for solving Problem (2.16).

Algorithm 2 A semismooth Newton-CG method for solving $G(y)$

Initialization: Given $M = (m_{ij})_{n\times n}$, the prescribed positive values $g = \{g_1, g_2, \ldots, g_n\}$, $y^0 \in \mathbb{R}^{3n-1}$, $\eta \in (0,1)$, $\rho, \delta \in (0, 1/2)$. $l := 0$, iterate the following steps for $l \geq 0$.

Step 1. Let $D^l \in \partial G(y^l)$, which is defined by (2.23), and solve the following linear system to find $\Delta y^l \in \mathbb{R}^{3n-1}$ by the conjugate gradient (CG) method:

$$G(y^l) + D^l \Delta y^l = 0,$$

such that

$$\|G(y^l) + D^l \Delta y^l\| \leq \min\{\eta, \|G(y^l)\|\}\|G(y^l)\|, \tag{2.24}$$

if D^l is nonsingular. If (2.24) is not achieved, or if the condition

$$\langle \Delta y^l, G(y^l)\rangle \leq -\min\{\eta, \|G(y^l)\|\}\|\Delta y^l\|.$$

is not satisfied, or D^l is singular, let

$$\Delta y^l = -G(y^l).$$

Step 2. (Line search) Set $\alpha_l = \rho^{m_l}$, where m_l is the first nonnegative integer m for which

$$\xi(y^l + \rho^m \Delta y^l) \leq \xi(y^l) + \delta \rho^m \langle \Delta y^l, G(y^l)\rangle.$$

Step 3. Set $y^{l+1} = y^l + \alpha_l \Delta y^l$.

Theorem 2.4. *Let $\bar{H} = P_{\mathcal{N}}(\mathcal{F}^*(\bar{y}) + M)$ be the unique solution of Problem (2.12), where $\bar{y} \in \mathbb{R}^{3n-1}$ satisfies $G(\bar{y}) = 0$. If $e_i^T \bar{H} > 0$ and $0 < g + \bar{H} e_i < 1$ for some $1 \leq i \leq n$ or $\bar{H} e_j > 0$ and $0 < g^T + e_j^T \bar{H} < 1$ for some $1 \leq j \leq n$, the sequence y^l generated by nonsmooth Newton method (2.22) converges to $\bar{y}$ quadratically provided that the starting point y^0 is sufficiently close to $\bar{y}$. Here $g = \{g_1, g_2, \ldots, g_n\}$ is the prescribed positive diagonal values, and e_i and e_j are the ith and jth columns of I_n, respectively.*

Proof. An easy calculation gives

$$\bar{H} = \begin{bmatrix} g_1 & (m_{12} + \bar{y}_1 + \bar{y}_{n+2})_+ & \cdots & (m_{1n-1} + \bar{y}_1 + \bar{y}_{2n-1})_+ & (m_{1n} + \bar{y}_1)_+ \\ (m_{21} + \bar{y}_2 + \bar{y}_{n+1})_+ & g_2 & \cdots & (m_{2n-1} + \bar{y}_2 + \bar{y}_{2n-1})_+ & (m_{2n} + \bar{y}_2)_+ \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (m_{n1} + \bar{y}_n + \bar{y}_{n+1})_+ & (m_{n2} + \bar{y}_n + \bar{y}_{n+2})_+ & \cdots & (m_{nn-1} + \bar{y}_n + \bar{y}_{2n-1})_+ & g_n \end{bmatrix}, \tag{2.27}$$

where $\bar{y} = [\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{3n-1}]^T \in \mathbb{R}^{3n-1}$ satisfies $G(\bar{y}) = 0$ and $g = \{g_1, g_2, \ldots, g_n\} >$

0 is the given diagonal entries. For any $\tilde{D} \in \partial G(\bar{y})$,

$$\tilde{D} = \left(\begin{array}{ccc|ccc|ccc}
\sum_{i=1}^n d_{1,i} & \cdots & 0 & d_{1,1} & \cdots & d_{1,n-1} & d_{1,1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \sum_{i=1}^n d_{n,i} & d_{n,1} & \cdots & d_{n,n-1} & 0 & \cdots & d_{n,n} \\
\hline
d_{1,1} & \cdots & d_{n,1} & \sum_{i=1}^n d_{i,1} & \cdots & 0 & d_{1,1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d_{1,n-1} & \cdots & d_{n,n-1} & 0 & \cdots & \sum_{i=1}^n d_{i,n-1} & 0 & \cdots & 0 \\
\hline
d_{1,1} & \cdots & 0 & d_{1,1} & \cdots & 0 & d_{1,1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & 0 & \cdots & d_{n-1,n-1} & & & \\
0 & \cdots & d_{n,n} & 0 & \cdots & 0 & 0 & \cdots & d_{n,n}
\end{array}\right), \quad (2.28)$$

where for $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ and $i \neq j$

$$\begin{cases} d_{i,j} = 1, & \text{if} \quad m_{i,j} + \bar{y}_i + \bar{y}_{n+j} > 0, \\ d_{i,j} \in [0,1], & \text{if} \quad m_{i,j} + \bar{y}_i + \bar{y}_{n+j} = 0, \\ d_{i,j} = 0. & \text{if} \quad m_{i,j} + \bar{y}_i + \bar{y}_{n+j} < 0, \end{cases} \quad (2.29)$$

for $i = 1, \ldots, n-1$, $j = n$

$$\begin{cases} d_{i,n} = 1, & \text{if} \quad m_{i,n} + \bar{y}_i > 0, \\ d_{i,n} \in [0,1], & \text{if} \quad m_{i,n} + \bar{y}_i = 0, \\ d_{i,n} = 0, & \text{if} \quad m_{i,n} + \bar{y}_i < 0, \end{cases} \quad (2.30)$$

Besides, since $g = (g_1, g_2, \ldots, g_n) > 0$, we can obtain

$$d_{i,i} = 1, \ i = 1, \ldots, n. \quad (2.31)$$

Assume that $e_i^T \bar{H} > 0$ for some $i = l$. According to (2.29), (2.30) and (2.31), we get that $e_l^T \bar{H} > 0$ if and only if $d_{l1} = d_{l2} = \cdots = d_{ln} = 1$.

For any $h = (h_1, h_2, \ldots, h_{3n-1})^T \in \mathbb{R}^{3n-1}$, if $h^T \tilde{D} h = 0$, that is,

$$\begin{aligned} h^T \tilde{D} h = & \sum_{i=1}^{n-1} d_{ii}(h_i + h_{n+i} + h_{2n-1+i})^2 + \sum_{j=2}^{n-1} d_{1j}(h_1 + h_{n+j})^2 \\ & + \sum_{i=2}^{n} \sum_{j=1, j\neq i}^{n-1} d_{ij}(h_i + h_{n+j})^2 + \sum_{i=1}^{n-1} d_{in} h_i^2 + d_{nn}(h_n + h_{3n-1})^2 = 0. \end{aligned} \quad (2.32)$$

Then,

$$\sum_{j=2}^{n-1} d_{1j}(h_1 + h_{n+j})^2 + d_{1n} h_1^2 = 0,$$

$$\sum_{j=1,j\neq i}^{n-1} d_{ij}(h_i+h_{n+j})^2+d_{in}h_i^2=0, \quad i=2,\dots,n-1, \tag{2.33}$$
$$\sum_{j=1}^{n-1} d_{nj}(h_n+h_{n+j})^2+d_{nn}(h_n+h_{3n-1})^2=0.$$

In what follows, we divide our proof in two cases.
Case 1: $d_{l1}=d_{l2}=\cdots=d_{ln}=1$, for some $l\neq n$. It follows that

$$h_{n+1}=h_{n+2}=\cdots=h_{n+l-1}=h_{n+l+1}=\cdots=h_{2n-1}=-h_l=0. \tag{2.34}$$

Hence, (2.33) can be rewritten as

$$\begin{aligned}&\sum_{j=2,j\neq l}^{n} d_{1j}h_1^2=0, \qquad \sum_{j=1,j\neq i,j\neq l}^{n-1} d_{ij}h_i^2=0, \quad i=2,\dots,n-1,\\ &\sum_{j=1,j\neq l}^{n-1} d_{nj}h_n^2=0.\end{aligned} \tag{2.35}$$

Since $\bar{H}e=e$ and $0<g+\bar{H}e_l<1$, we have

$$\begin{aligned}&\sum_{j=2,j\neq l}^{n-1}(m_{ij}+\bar{y}_i+\bar{y}_{n+j})_+ +(m_{in}+\bar{y}_i)_+\neq 0,\\ &\sum_{j=1,j\neq i,j\neq l}^{n-1}(m_{ij}+\bar{y}_i+\bar{y}_{n+j})_+ +(m_{in}+\bar{y}_i)_+\neq 0, \quad i=2,\dots,n-1,\\ &\sum_{j=1,j\neq l}^{n-1}(m_{nj}+\bar{y}_n+\bar{y}_{n+j})_+\neq 0,\end{aligned} \tag{2.36}$$

and then,

$$\begin{aligned}&\sum_{j=2,j\neq l}^{n} d_{1j}\neq 0, \qquad \sum_{j=1,j\neq i,j\neq l}^{n-1} d_{ij}\neq 0, \quad i=2,\dots,n-1,\\ &\sum_{j=1,j\neq l}^{n-1} d_{nj}\neq 0.\end{aligned} \tag{2.37}$$

Hence,

$$h_i=0, \qquad i=1,2,\dots,n. \tag{2.38}$$

Again $e^T\bar{H}=e^T$ and $0<g_l<1$, we further obtain

$$\sum_{i=1,i\neq l}^{n}(m_{il}+\bar{y}_i+\bar{y}_{n+l})_+\neq 0. \tag{2.39}$$

Then

$$\sum_{i=1,i\neq l}^{n} d_{il}\neq 0. \tag{2.40}$$

Recall that

$$\sum_{i=1, i\neq l}^{n} d_{il}(h_i + h_{n+l})^2 = 0, \quad \text{and} \quad h_i = 0, i = 1, 2, \ldots, n, \tag{2.41}$$

we have,

$$h_{n+l} = 0. \tag{2.42}$$

According to (2.31) and (2.32), we can see that

$$\begin{aligned} &d_{ii}(h_i + h_{n+i} + h_{2n-1+i})^2 = 0, \quad \text{and} \quad d_{ii} = 1 \qquad \forall i = 1, 2, \ldots, n-1, \\ &d_{nn}(h_n + h_{3n+1})^2 = 0, \quad \text{and} \quad d_{nn} = 1. \end{aligned} \tag{2.43}$$

Then

$$h_{2n} = h_{2n+1} = \cdots = h_{3n-2} = h_{3n-1} = 0. \tag{2.44}$$

In view of (2.34), (2.38), (2.42) and (2.44), we finally have

$$h_1 = h_2 = \cdots = h_{3n-1} = 0. \tag{2.45}$$

Case 2: $d_{n1} = d_{n2} = \cdots = d_{nn} = 1$. It follows that

$$h_{n+1} = h_{n+2} = \cdots = h_{2n-1} = h_{3n-1} = -h_n. \tag{2.46}$$

Since $\bar{H}e = e$ and $0 < g + \bar{H}e_n < 1$, we have

$$\begin{aligned} &\sum_{j=2}^{n-1} (m_{1j} + \bar{y}_1 + \bar{y}_{n+j})_+ \neq 0, \\ &\sum_{j=1, j\neq i,}^{n-1} (m_{ij} + \bar{y}_i + \bar{y}_{n+j})_+ + (m_{in} + \bar{y}_i)_+ \neq 0, \quad i = 2, \ldots, n-1, \end{aligned} \tag{2.47}$$

and then

$$\sum_{j=2}^{n-1} d_{1j} \neq 0, \qquad \sum_{j=1, j\neq l}^{n-1} d_{ij} \neq 0, \quad i = 2, \ldots, n-1. \tag{2.48}$$

By (2.33) and (2.46), we have

$$h_1 = h_2 = \cdots = h_{n-1} = -h_n. \tag{2.49}$$

Due to (2.32),

$$\sum_{i=1}^{n-1} d_{in} h_i^2 = 0, \tag{2.50}$$

we can obtain that,

$$h_1 = h_2 = \cdots = h_{2n-1} = h_{3n-1} = 0. \tag{2.51}$$

Moreover,

$$d_{ii}(h_i + h_{n+i} + h_{2n-1+i})^2 = 0, \quad \text{and} \quad d_{ii} = 1 \qquad \forall i = 1, 2, \ldots, n-1. \tag{2.52}$$

Finally, we have

$$h_1 = h_2 = \cdots = h_{3n-1} = 0. \tag{2.53}$$

Now, we have shown that $h^T \tilde{D} h = 0$ only if $h = 0$ under the assumption that $e_i^T \bar{H} > 0$ and $0 < g + \bar{H} e_i < 1$ for some $1 \leq i \leq n$. Therefore, $\tilde{D}$ is positive definite.

In addition, it can be verified by (2.19) and (2.23) that

$$G(y + \delta y) - G(y) - D\delta y = 0, \quad \forall D \in \partial_B G(y + \delta y), \quad \delta y \to 0. \tag{2.54}$$

Since any $D \in \partial G(y + \delta y)$ is just a convex combination of elements in $\partial_B G(y + \delta y)$, so,

$$G(y + \delta y) - G(y) - D\delta y = 0, \quad \forall D \in \partial G(y + \delta y), \quad \delta y \to 0. \tag{2.55}$$

Hence, G is strongly semismooth at any $y \in \mathbb{R}^{3n-1}$.

By Theorem 3.2 in [24], the sequence y^l generated by nonsmooth Newton method (2.22) converges to $\bar{y}$ quadratically provided that the starting point y^0 is sufficiently close to $\bar{y}$. We have completed the proof. $\square$

Briefly speaking, the second projection of M_{k+1} onto $\mathcal{K}$ is given by $H_{k+1} = P_{\mathcal{N}}(\mathcal{F}^*(\bar{y}) + M_{k+1})$, where $\bar{y}$ is the solution of $G(y) = 0$ (2.17) which is solved by nonsmooth Newton method (2.22).

3. Convergency analysis

In this section, we will study the convergence of alternating projection method for the DISVP-PE.

From [14], we have the following important property of projection operator on convex set.

Proposition 3.1 (Escalante and Raydan [14])**.** *If $\mathcal{C}$ is a closed and convex set in a Hilbert space $\mathcal{U}$, then for all $z \in \mathcal{C}$ and $y \in \mathcal{U}$*

$$\langle z - y, P_{\mathcal{C}}(y) - y \rangle \geq \|P_{\mathcal{C}}(y) - y\|^2.$$

Since the set $\mathcal{K} = \{A \in \mathbb{R}^{n \times n} | Ae = e, e^T A = e^T, A \geq 0, A_{ii} = g_i, i = 1, 2, \cdots, n\}$ is convex, the following distance reduction property always holds for projections on $\mathcal{K}$.

Lemma 3.1. *Let $H_k = P_{\mathcal{K}}(M_k)$, $M_{k+1} = P_{\mathcal{M}}(H_k)$ and $H_{k+1} = P_{\mathcal{K}}(M_{k+1})$. If $H_{k+1} \neq H_k$, then*

$$\|H_{k+1} - M_{k+1}\| < \|H_k - M_{k+1}\|. \tag{3.1}$$

Proof. By Proposition 3.1, we obtain

$$\begin{aligned}
&\|H_{k+1} - H_k\|^2 \\
=&\|H_{k+1} - M_{k+1} + M_{k+1} - H_k\|^2 \\
=&\|H_{k+1} - M_{k+1}\|^2 + \|M_{k+1} - H_k\|^2 + 2\langle H_{k+1} - M_{k+1}, M_{k+1} - H_k\rangle \\
=&\|H_{k+1} - M_{k+1}\|^2 + \|M_{k+1} - H_k\|^2 \\
&+ 2\langle P_{\mathcal{K}}(M_{k+1}) - M_{k+1}, M_{k+1} - H_k\rangle \\
\leq&\|H_{k+1} - M_{k+1}\|^2 + \|M_{k+1} - H_k\|^2 - 2\|P_{\mathcal{K}}(M_{k+1}) - M_{k+1}\|^2 \\
\leq&\|M_{k+1} - H_k\|^2 - \|H_{k+1} - M_{k+1}\|^2.
\end{aligned} \tag{3.2}$$

This implies that

$$\|H_{k+1} - M_{k+1}\|^2 \leq \|M_{k+1} - H_k\|^2 - \|H_{k+1} - H_k\|^2. \tag{3.3}$$

If $H_{k+1} \neq H_k$, then

$$\|H_{k+1} - M_{k+1}\|^2 < \|H_k - M_{k+1}\|^2.$$

We have completed the proof. □

Here, we would like to present an important fact. If $H_{k+1} = H_k$ and H_k is not in the solution set $\mathcal{M} \cap \mathcal{K}$, the algorithm will not make any progress. But it does not necessarily mean that the alternating method can not obtain a solution in the set $\mathcal{M} \cap \mathcal{K}$. Since the projection of H_k onto $\mathcal{M}$ are not unique, it may be possible to escape from H_{k+1} by a different projection point $M_{k+1} = P_{\mathcal{M}}(H_k)$.

By Theorem 2.1 and Lemma 3.1, the following distance reduction property always holds for alternating projection method for the DISVP-PE.

Corollary 3.1. *Let $M_k = P_{\mathcal{M}}(H_{k-1})$, $H_k = P_{\mathcal{K}}(M_k)$, $M_{k+1} = P_{\mathcal{M}}(H_k)$ and $H_{k+1} = P_{\mathcal{K}}(M_{k+1})$. If $H_{k+1} \neq H_k$, then*

$$\|M_{k+1} - H_{k+1}\| < \|M_k - H_k\|. \tag{3.4}$$

Proof. By theorem 2.1, we have

$$\|H_{k+1} - M_{k+1}\| \leq \|M_{k+1} - H_k\| \leq \|M_k - H_k\|. \tag{3.5}$$

If $H_{k+1} \neq H_k$, it follows from Lemma 3.1 that

$$\|M_{k+1} - H_{k+1}\| < \|M_{k+1} - H_k\|. \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\|M_{k+1} - H_{k+1}\| < \|M_k - H_k\|.$$

□

As presented in Section 2.2, the projection $P_{\mathcal{K}}(M_{k+1})$ is solved by nonsmooth Newton method. $H_{k+1} = P_{\mathcal{K}}(M_{k+1})$ is an approximate solution. Since the sequence y^l generated by nonsmooth Newton method converges to $\bar{y}$, i.e., $\|y^l - \bar{y}\| \to 0$ as $l \to \infty$, then for any $\varepsilon_k > 0$, there exists an l, such that $\|H_{k+1} - P_{\mathcal{K}}(M_{k+1})\| \leq \varepsilon_k$. To ensure that the distance $\|M_k - H_k\|$ is decreasing with k, the approximate solution H_{k+1} should satisfy $\|H_{k+1} - P_{\mathcal{K}}(M_{k+1})\| \leq \varepsilon_k$ and $\|M_{k+1} - H_{k+1}\| < \|M_k - H_k\|$.

Corollary 3.1 ensures that the distance $\|M_k - H_k\|$ is decreasing with k, which implies that $\|M_k - H_k\|$ converges. In such a case, there is no guarantee that the iteration in algorithm 1 will terminate. Moreover, $\mathcal{M}$ is compact [16] and $\mathcal{K}$ is a closed convex set. The next theorem gives a convergence result for Algorithm 1.

Theorem 3.1. *If $\mathcal{M} = \{U\Sigma V | U, V \in \mathcal{O}(n)\}$ and $\mathcal{K} = \{H \in \mathbb{R}^{n\times n} | He = e, e^T H = e^T, H \geq 0, H_{ii} = g_i, i = 1, 2, \ldots, n\}$, then the sequence generated by Algorithm 1 converges in norm to some point in $\mathcal{M} \cap \mathcal{K}$ under the assumption that $\lim\limits_{k\to\infty} \|M_k - H_k\| = 0$.*

Proof. In Algorithm 1, the first projection is $M_{k+1} = P_{\mathcal{M}}(H_k)$, where $\mathcal{M} = \{U\Sigma V | U, V \in \mathcal{O}(n)\}$. Since $\mathcal{M}$ is compact and $M_k \in \mathcal{M}$, $\{M_k\}$ has a convergent subsequence, say $\{M_{k_i}\}$. Define

$$\lim_{i\to\infty} M_{k_i} = M, \tag{3.7}$$

for some $M \in \mathcal{M}$.

As for the second projection $H_k = P_{\mathcal{K}}(M_k)$, the projection operator $P_{\mathcal{K}}$ is non-expansive, i.e., for

$$\|P_{\mathcal{K}}(M_{k_i}) - P_{\mathcal{K}}(M)\| \leq \|M_{k_i} - M\|. \tag{3.8}$$

This implies that $\lim\limits_{i\to\infty} H_{k_i} = \lim\limits_{i\to\infty} P_{\mathcal{K}}(M_{k_i}) = P_{\mathcal{K}}(M)$ and $P_{\mathcal{K}}(M) \in \mathcal{K}$. Moreover,

$$\|M - P_{\mathcal{K}}(M)\| \leq \|M - M_{k_i}\| + \|H_{k_i} - P_{\mathcal{K}}(M)\| + \|M_{k_i} - H_{k_i}\|. \tag{3.9}$$

If $\lim\limits_{k\to\infty} \|M_k - H_k\| = 0$, then $\lim\limits_{i\to\infty} \|M_{k_i} - H_{k_i}\| = 0$. So we have

$$M = P_{\mathcal{K}}(M). \tag{3.10}$$

It follows that

$$M \in \mathcal{M} \cap \mathcal{K}. \tag{3.11}$$

□

4. Numerical tests

This section contains some numerical results for solving the doubly stochastic inverse singular value problem with prescribed entries by Algorithm 1. All computational results are obtained using MATLAB 2015b running on a Intel(R) Core(TM) PC of i7-8550U CPU. In our numerical experiments, the initial starting H_0 is always randomly generated by the built-in functions rand. In order to reflect the results of numerical experiments intuitively, we use the following stopping criterion:

$$(\|\sigma(H_k)-\sigma\|^2+\|H_k e-e\|^2+\|e^T H_k-e^T\|^2+\|\min(H_k,0)\|^2+\|\mathrm{diag}(H_k)-g\|^2)^{\frac{1}{2}} \leq \epsilon, \tag{4.1}$$

where $\epsilon > 0$ is a given tolerance, and $\sigma(H_k)$ denotes the vector consisting of the singular values, arranged in descending order, of the matrix H_k.

First, we report a numerical test of small-scale DISVP-PE. We set $\epsilon = 1.0 \times 10^{-12}$.

Example 4.1. To ensure the feasibility of test data, we start with doubly stochastic matrix generated by the Sinkhorn-Knopp algorithm and use its singular values as the objective singular value. Considering the DISVP-PE with $n = 8$, we first generate a positive doubly stochastic matrix $\overline{H}$ by the Sinkhorn-Knopp algorithm [27] as follows:

$$\overline{H} = \begin{pmatrix} \mathbf{0.1542} & 0.0509 & 0.1688 & 0.1149 & 0.1620 & 0.2025 & 0.1321 & 0.0146 \\ 0.1670 & \mathbf{0.1952} & 0.1060 & 0.0727 & 0.2125 & 0.0459 & 0.1033 & 0.0973 \\ 0.2189 & 0.0732 & \mathbf{0.0556} & 0.0569 & 0.0043 & 0.2300 & 0.0891 & 0.2721 \\ 0.0255 & 0.1450 & 0.1573 & \mathbf{0.1565} & 0.2629 & 0.1875 & 0.0154 & 0.0498 \\ 0.0927 & 0.0097 & 0.2699 & 0.1426 & \mathbf{0.1687} & 0.0079 & 0.1547 & 0.1538 \\ 0.1797 & 0.1113 & 0.2199 & 0.0428 & 0.1659 & \mathbf{0.1095} & 0.1170 & 0.0539 \\ 0.1103 & 0.0966 & 0.0000 & 0.2134 & 0.0004 & 0.0822 & \mathbf{0.3390} & 0.1582 \\ 0.0517 & 0.3180 & 0.0225 & 0.2002 & 0.0235 & 0.1344 & 0.0495 & \mathbf{0.2002} \end{pmatrix} \tag{4.2}$$

The singular value of $\overline{H}$ is $\{1.0000, 0.4138, 0.3252, 0.2656, 0.1928, 0.1731, 0.0907, 0.0243\}$. We use the singular value of $\overline{H}$ as the given singular value and the diagonal elements of $\overline{H}$ as the given diagonal elements. By using the alternating direction method, we obtain the following solution:

$$H = \begin{pmatrix} \mathbf{0.1542} & 0.0109 & 0.3220 & 0.0944 & 0.0463 & 0.1407 & 0.1713 & 0.0602 \\ 0.0900 & \mathbf{0.1952} & 0.2099 & 0.2138 & 0.0414 & 0.0590 & 0.1085 & 0.0821 \\ 0.1543 & 0.0824 & \mathbf{0.0556} & 0.0593 & 0.2895 & 0.3217 & 0.0256 & 0.0115 \\ 0.0651 & 0.1014 & 0.1560 & \mathbf{0.1565} & 0.1503 & 0.0568 & 0.1306 & 0.1834 \\ 0.1075 & 0.0990 & 0.0115 & 0.1048 & \mathbf{0.1687} & 0.1597 & 0.0501 & 0.2987 \\ 0.2636 & 0.1054 & 0.0450 & 0.0333 & 0.1964 & \mathbf{0.1095} & 0.1342 & 0.1126 \\ 0.0389 & 0.1543 & 0.0610 & 0.1742 & 0.0760 & 0.1055 & \mathbf{0.3390} & 0.0512 \\ 0.1262 & 0.2514 & 0.1389 & 0.1639 & 0.0315 & 0.0472 & 0.0407 & \mathbf{0.2002} \end{pmatrix} \tag{4.3}$$

H is a nonnegative matrix with $\|He - e\| = 1.6338 \times 10^{-27}$, $\|e^T H - e^T\| = 1.0839 \times 10^{-26}$, $\|\mathrm{diag}(H) - g\| = 6.2800 \times 10^{-14}$ and $(\sum_{i=1}^{n}(\sigma_i(H_k) - \sigma_i)^2)^{\frac{1}{2}} = 8.0689 \times 10^{-14}$.

Example 4.2. Results for various problem sizes n are given in this example to illustrate the feasibility of our approach for relatively large problems. Let $\widetilde{H}$ be an $n \times n$ positive matrix with random entries uniformly distributed on the interval $[0, 1]$. To ensure that the problem is solvable, we first generate a doubly stochastic matrix $\overline{H}$ by

$$\overline{H} := (\mathrm{diag}(\widetilde{H}\widetilde{H}^T))^{-\frac{1}{2}}\widetilde{H}.$$

We use the singular value of $\overline{H}$ as the given singular value and the diagonal elements of $\overline{H}$ as the given diagonal elements. Alternating projection method is applied to solve the DISVP-PE. We find that all the numerical solution H is nonnegative. Detailed numerical experimental results are presented in Table 1, where "CT." stands for the total computing time. And $\sigma(H) = (\sigma_1(H), \sigma_2(H), \dots, \sigma_n(H))$, where $\sigma_i(H)$ denotes the ith largest singular value of H.

Table 1. Numerical results for Example 4.2

n	CT.	$\|He - e\|$	$\|e^T H - e^T\|$	$\|\mathrm{diag}(H) - g\|$	$\|\sigma - \sigma(H)\|$
100	0.6549s	2.56×10^{-25}	8.68×10^{-24}	2.42×10^{-13}	5.06×10^{-14}
200	2.1534s	3.47×10^{-25}	1.90×10^{-23}	2.92×10^{-13}	4.17×10^{-14}
400	14.2068s	2.64×10^{-25}	1.50×10^{-23}	4.04×10^{-13}	2.17×10^{-14}
600	37.9099s	2.64×10^{-25}	1.92×10^{-23}	2.99×10^{-13}	1.84×10^{-14}
800	1min24s	2.04×10^{-25}	9.45×10^{-23}	3.94×10^{-13}	1.33×10^{-14}
1000	2min13s	2.52×10^{-25}	1.84×10^{-22}	3.67×10^{-13}	1.35×10^{-14}
2000	12min2s	3.05×10^{-25}	1.25×10^{-22}	2.65×10^{-13}	1.22×10^{-14}
3000	37min20s	3.05×10^{-25}	4.34×10^{-22}	4.24×10^{-13}	6.54×10^{-15}
4000	2h45min41s	7.70×10^{-26}	1.90×10^{-22}	3.80×10^{-14}	4.79×10^{-15}

From Table 1, one can see that the alternating projection method can solve the doubly stochastic inverse singular value problem with prescribed diagonal entries efficiently. Moreover, from the numerical tests, we observe the fact that alternating projection method converges to different solutions for different starting points.

Example 4.3. For the convenience of description, we only consider the doubly stochastic inverse singular value problem with prescribed diagonal entries. This is an assumption without loss of generality, since the framework we establish in this paper can be easily applied to the problem with any kind of index subset $\mathcal{L}$. In this case, the DISVP-PE can be reformulated as the problem of finding a point in the intersection of two convex sets $\mathcal{M} = \{U\Sigma V | U, V \in \mathcal{O}(n)\}$ and $\mathcal{K} = \{H \in \mathbb{R}^{n\times n} | He = e, e^T H = e^T, H \geq 0, H_{i_v j_v} = g_v\}$ where $(i_v, j_v) \in \mathcal{L}$. The first projection $M_{k+1} = P_{\mathcal{M}}(H_k)$ does not change. The second projection $H_{k+1} = P_{\mathcal{K}}(M_{k+1})$ can be reformulated as the following problem:

$$\begin{aligned} \min\ & \frac{1}{2}\|H - M\|^2 \\ \text{s.t.}\ & He = e, e^T H = e^T, \\ & H \geq 0, \\ & H_{i_v j_v} = g_v, \quad \forall (i_v, j_v) \in \mathcal{L} \end{aligned} \tag{4.4}$$

where $\mathcal{L} = \{(i_v, j_v)\}_{v=1}^{l}$ is the given an index subset of locations and $g = \{g_1, g_2, \dots, g_l\}$ is the prescribed nonnegative values. Let

$$\mathcal{F}(H) = \begin{bmatrix} He \\ \begin{bmatrix} I_{n-1} & 0 \end{bmatrix} H^T e \\ e_{i_1}^T H e_{i_1} \\ \vdots \\ e_{i_l}^T H e_{i_l} \end{bmatrix}, \quad b = \begin{bmatrix} e \\ \begin{bmatrix} I_{n-1} & 0 \end{bmatrix} e \\ g_1 \\ \vdots \\ g_l \end{bmatrix}. \tag{4.5}$$

Hence, we can extend all results that we derived for the problem (2.12) to the problem (4.5), and apply nonsmooth Newton method to solve the dual problem of (4.5) and then obtain the desired solution of the problem.

In this example, we apply our approach to construct the doubly stochastic inverse singular value problem with any kind of index subset $\mathcal{L}$. We first generate a positive doubly stochastic matrix $\overline{H}$ by the Sinkhorn-Knopp algorithm [27] as follows:

$$\overline{H} = \begin{pmatrix} \mathbf{0.1859} & 0.3110 & 0.1866 & 0.0093 & 0.0501 & 0.0402 & 0.0423 & 0.1745 \\ 0.2784 & 0.1346 & 0.1125 & 0.1096 & \mathbf{0.0741} & 0.0923 & 0.0305 & 0.1680 \\ 0.0548 & 0.0478 & 0.1518 & \mathbf{0.1589} & 0.1118 & 0.1773 & 0.0904 & 0.2063 \\ 0.1048 & 0.0335 & 0.0575 & 0.0172 & 0.2070 & \mathbf{0.2547} & 0.1343 & 0.1911 \\ 0.1796 & 0.0023 & 0.1932 & 0.0824 & 0.1214 & 0.1418 & \mathbf{0.1899} & 0.0896 \\ 0.1256 & \mathbf{0.0570} & 0.2019 & 0.3381 & 0.0489 & 0.0708 & 0.1377 & 0.0200 \\ 0.0575 & 0.3138 & 0.0898 & 0.0464 & 0.1649 & 0.0685 & 0.2118 & \mathbf{0.0474} \\ 0.0133 & 0.1000 & \mathbf{0.0068} & 0.2373 & 0.2218 & 0.1544 & 0.1632 & 0.1032 \end{pmatrix} \tag{4.6}$$

The singular value of $\overline{H}$ is $\{1.0000, 0.3976, 0.3313, 0.3069, 0.1864, 0.1368, 0.0298, 0.0116\}$. We use the singular values of $\overline{H}$ as the given singular values and index subset $\mathcal{L} = \{(1,1),(2,5),(3,4),(4,6),(5,7),(6,2),(7,8),(8,3)\}$. Moreover, the prescribed nonnegative values $g = \{0.1859, 0.0741, 0.1589, 0.2547, 0.1899, 0.0570, 0.0474, 0.0068\}$. By using the alternating direction method, we obtain the following solution:

$$H = \begin{pmatrix} \mathbf{0.1859} & 0.2396 & 0.2043 & 0.0733 & 0.0137 & 0.0200 & 0.0351 & 0.2281 \\ 0.2048 & 0.0613 & 0.1339 & 0.0808 & \mathbf{0.0741} & 0.1024 & 0.1560 & 0.1868 \\ 0.0669 & 0.2151 & 0.1290 & \mathbf{0.1598} & 0.1616 & 0.0734 & 0.1064 & 0.0879 \\ 0.0612 & 0.1026 & 0.2560 & 0.0090 & 0.1961 & \mathbf{0.2547} & 0.0506 & 0.0699 \\ 0.2392 & 0.0642 & 0.0799 & 0.0335 & 0.1183 & 0.1275 & \mathbf{0.1899} & 0.1474 \\ 0.1066 & \mathbf{0.0570} & 0.0923 & 0.3821 & 0.1318 & 0.0462 & 0.0604 & 0.1236 \\ 0.0424 & 0.2175 & 0.0979 & 0.1761 & 0.2409 & 0.0733 & 0.1045 & \mathbf{0.0474} \\ 0.0930 & 0.0428 & \mathbf{0.0068} & 0.0853 & 0.0636 & 0.3025 & 0.2971 & 0.1089 \end{pmatrix} \tag{4.7}$$

H is a nonnegative matrix with $\|He - e\| = 2.1716 \times 10^{-25}$, $\|e^T H - e^T\| = 2.5597 \times 10^{-25}$, and $(\sum_{i=1}^{n}(\sigma_i(H) - \sigma_i)^2)^{\frac{1}{2}} = 1.3485 \times 10^{-13}$.

5. Conclusion

In this paper, we have considered the doubly stochastic inverse singular value problem with prescribed entries(DISVP-PE). The inverse problem is reformulated as the problem of finding a point in the intersection of the compact set $\mathcal{M} = \{U\Sigma V | U, V \in \mathcal{O}(n)\}$ and the closed convex set $\mathcal{K} = \{H \in \mathbb{R}^{n\times n} | He = e, e^T H = e^T, H \geq 0, H_{i_v j_v} = g_v\}$. Alternating projection method is applied to solve the problem.

Numerical experiments show that the proposed algorithm is effective for solving the DISVP-PE. If the sets M and K are both closed convex sets, the alternating projection method can be accelerated by $\tilde{H}_{k+1} = t_k H_{k+1} + (1 - t_k) H_k$ where $t_k = \frac{\langle H_k, H_k - H_{k+1} \rangle}{\|H_k - H_{k+1}\|^2}$. But the set $\mathcal{M}$ is nonconvex in the DISVP-PE. And how to accelerated the alternating projection method for the DISVP-PE is still an interesting problem. We will leave this for future research.

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