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THE EFFECT OF AN ADDITIVE NOISE ON SOME SLOW-FAST EQUATION NEAR A TRANSCRITICAL POINT*

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Abstract We consider the effect of small additive noise with intensity σ on trajectories of a slow-fast system with small parameter ε which admits bifurcation delay at a transcritical point. We estimate the probability that the perturbed stochastic paths stay in some tubular neighborhood of the deterministic path to show that small but not exponentially small noise destroys the bifurcation delay caused by transcritical point and obtain a noise intensity threshold value $N(\varepsilon)$ of order $\varepsilon^{\frac{3}{4}}$. When $e^{-\frac{1}{\varepsilon}} \ll \sigma < N(\varepsilon)$, the paths are likely to leave the neighborhood of the corresponding determinate path before some time of order $\sqrt{\varepsilon|log\sigma|}$. When $\sigma > N(\varepsilon)$, the paths are likely to leave before some time of order $\sigma^{\frac{2}{3}}$.

Keywords Stochastic slow-fast differential equation, additive noise, bifurcation delay, transcritical point, sample path.

MSC(2010) 37H20, 60H10, 34E15.

1. Introduction

Consider the planar slow-fast system

$$\begin{aligned} \frac{dx}{dt} &= x' = F(x, y, \varepsilon), \\ \frac{dy}{dt} &= y' = \varepsilon G(x, y, \varepsilon), \end{aligned} \tag{1.1}$$

where F, G are sufficiently smooth. We assume that the system (1.1) has a nondegenerate transcritical bifurcation point (NT-point) at the origin $(0, 0)$. This means:

$$\begin{aligned} &F(0,0,0) = 0,\ \partial_x F(0,0,0) = 0,\ \partial_y F(0.0.0) = 0, \\ &\partial_{xx} F(0,0,0) \neq 0,\ \begin{vmatrix} \partial_{xx} F(0,0,0) & \partial_{xy} F(0,0,0) \\ \partial_{xy} F(0,0,0) & \partial_{yy} F(0,0,0) \end{vmatrix} < 0,\ G_0 = G(0,0,0) \neq 0. \end{aligned} \tag{1.2}$$

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Let $F_{xx} = \partial_{xx}F(0,0,0)$, $F_{xy} = \partial_{xy}F(0,0,0)$, $F_{yy} = \partial_{yy}F(0,0,0)$, $F_\varepsilon = \partial_\varepsilon F(0,0,0)$, then there exists a coordinate change

$$\hat{x} = F_{xx}x - F_{xy}y,\ \hat{y} = sign(G_0)y\sqrt{F_{xy}^2 - F_{yy}Fxx},\ \hat{\varepsilon} = \varepsilon|G_0|\sqrt{F_{xy}^2 - F_{yy}Fxx},$$

transforming (1.1) into the normalized form [19] [23]

$$\begin{aligned} x' &= x^2 - y^2 + \lambda\varepsilon + \tilde{\mathcal{O}}(x^3, x^2y, xy^2, y^3, \varepsilon x, \varepsilon y, \varepsilon^2), \\ y' &= \varepsilon(1 + \tilde{\mathcal{O}}(x, y, \varepsilon)), \end{aligned} \tag{1.3}$$

with

$$\lambda = \frac{F_\varepsilon F_{xx} + G_0 F_{xy}}{|G_0|\sqrt{F_{xy}^2 - F_{xx}F_{yy}}}.$$

Obviously, (1.3) has a critical set $\mathcal{C}_0 = \{(x,y) \in R^2 : x^2 = y^2\}$, which consists of two attracting branches $\mathcal{C}_{a1} = \{x = y, y < 0\}$, $\mathcal{C}_{a2} = \{x = -y, y > 0\}$, two repelling branches $\mathcal{C}_{r1} = \{x = -y, y < 0\}$, $\mathcal{C}_{r2} = \{x = y, y > 0\}$, and a NT-point $(0,0)$. Let $\gamma_1(\varepsilon)$ be a trajectory of the normalized equation (1.3) corresponding to the system (1.1) starting from some fixed point (x_0, y_0) with $x_0 < -y_0$ and $y_0 < 0$. By the value of λ, we can classify the system (1.1) with a NT-point into three cases [4,19,23,24] (See Figure 1):

- $\lambda < 1$: Exchange-of-stability
  For sufficiently small $\varepsilon > 0$, $\gamma_1(\varepsilon)$ is attracted into an ε-neighbourhood of $\mathcal{C}_{a1}$ exponentially fast and remains near $\mathcal{C}_{a1}$ until crossing another attracting critical branch $\mathcal{C}_{a2}$ at some point $(-y_1, -y_1)$ where y_1 is of order $\sqrt{\varepsilon}$. Then it moves towards $\mathcal{C}_{a2}$, staying $O(-\frac{\varepsilon}{y})$ closer;
- $\lambda > 1$: Critical transition
  For sufficiently small $\varepsilon > 0$, $\gamma_1(\varepsilon)$ is attracted into an order ε neighbourhood of $\mathcal{C}_{a1}$ exponentially fast and remains near $\mathcal{C}_{a1}^1$ until crossing the repelling critical branch $\mathcal{C}_{r1}$ at some point $(y_2, -y_2)$ where y_2 is of order $\sqrt{\varepsilon}$. And then $\gamma_1(\varepsilon)$ fast jumps away from the vicinity of the two repelling critical branches $\mathcal{C}_{r1}$ and $\mathcal{C}_{r2}$ though the point (ρ_1, ρ_2) where ρ_1 is a positive constant and ρ_2 is of order $\sqrt{\varepsilon}$;
- $\lambda = 1$: Canard
  For sufficiently small $\varepsilon > 0$, $\gamma_1(\varepsilon)$ is attracted into an ε neighbourhood of $\mathcal{C}_{a1}$ exponentially fast and remains near $\mathcal{C}_{a1}$ moving and extending to $\mathcal{C}_{r2}$, until it follows the repelling branch $\mathcal{C}_{r2}$ over an distance of $O(1)$ before being repelled.

More research conclusions on the deterministic systems with transcritical bifurcation points were presented in [10,11,19,29,38].

The slow-fast equation (1.1) with a NT-point has very important applications in many fields, such as chemistry [22], ecology [5,32,35], epidemiology [18,37], biochemistry [9]. There are many studies about stochastic slow-fast equations (SSFEs) with NT-points. For instance, Hurth and Kuehn proved the existence of invariant probability measures for a particular case with a NT-point satisfying $\lambda = 1$ in [17]; Kuehn calculated the scaling laws of the variance of stochastic sample paths and obtained stationary density near critical transitions for a particular case with a NT-point satisfying $\lambda = 1$ in [20,21], which provides an important basis on finding the early warning signal; Berglund studied the dynamics of (1.3) with $\lambda = 0$ by sample

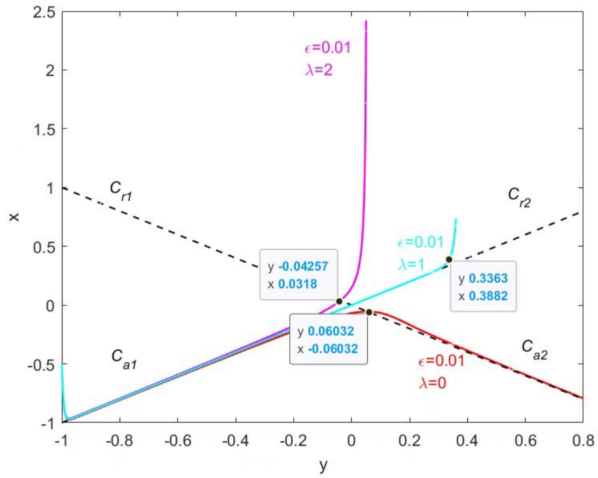

Figure 1

paths approach [4]. As far as we know, majority of the research on the SSFEs with a NT-point is to find early warning signals by using numerical simulation [32,34,37].

It is worth noting that, a sample path study of the dynamical behavior of the SSFEs with NT-points satisfying $\lambda = 1$ has not been studied. We treat this problem here.

We describe our set up next. For a planar slow fast system with a NT-point at $(0, 0)$ satisfying $\lambda = 1$:

$$\begin{aligned} \frac{dx}{dt} &= F(x,y) = f(x,y)x, \\ \frac{dy}{dt} &= \varepsilon G(x,y), \end{aligned} \tag{1.4}$$

restrict $x \in \mathbf{I}_1 \triangleq [-d, d]$, $y \in \mathbf{I}_2 \triangleq [-T, T]$, where d and T are positive constants independent of the parameter ε, and assume that $G(x, y) \in \mathcal{C}^2(\mathbf{I}_1 \times \mathbf{I}_2, \mathbf{R})$, $f(x, y) \in \mathcal{C}^2(\mathbf{I}_1 \times \mathbf{I}_2, \mathbf{R})$, and $0 < \varepsilon \ll 1$. We assume that there exist positive constants d_1, d_2 such that $G(x, y) > d_1$ for all $x \in \mathbf{I}_1$ and $y \in \mathbf{I}_2$, $\min\{\partial_y f(0, 0), \partial_x f(0, 0)\} \geq d_2$ and $f(0, 0) = 0$. We also put the following

Assumption. Let $(x, y) \in \mathbf{I}_1 \times \mathbf{I}_2$. For any fixed $y < 0$, the potential function $V(x, y)$ about x defined as $\partial_x V(x, y) = F(x, y)$, has only one local minimum point $x = 0$ and one local maximum point $x = \varphi(y)$. For any fixed $y > 0$, $V(x, y)$ has only one local minimum point $x = \varphi(y)$ and one local maximum point $x = 0$, where $x = \varphi(y)$ is the only solution function of $f(x, y) = 0$ and satisfies $\varphi(y) > 0$ for $y < 0$ and $\varphi(y) < 0$ for $y > 0$.

Transform (1.4) into an equivalent one-dimensional non-autonomous ordinary differential equation (ODE):

$$\frac{dx}{dy} = \frac{1}{\varepsilon} \frac{F(x,y)}{G(x,y)} = \frac{1}{\varepsilon} H(x,y), \tag{1.5}$$

and add additive noise with intensity $\sigma = \sigma(\varepsilon)$ to (1.5):

$$\frac{dx}{dy} = \frac{1}{\varepsilon} H(x,y) + \frac{\sigma}{\sqrt{\varepsilon}} dW_y, \tag{1.6}$$

where W_y is a double-sided Brownian Motion on some fixed probability space $(\Omega, \mathcal{F}, P)$.

In the study of SDEs, there are several approaches including but not limited to: sample paths [4], random attractors [6], transition probability [25] and random dynamical system [15,26,27]. In this paper, we extend the sample paths approach [4] to study dynamical behavior of the SDE (1.6) near the NT-point with $\lambda = 1$. We describe the dynamics of paths mainly by analyzing the probability of the first passage time of paths. By comparing our result to the result of the case $\lambda = 0$ [4], we find that the effect of additive noises on the dynamical behavior near the two cases NT-points is quite different.

We study the dynamics of the DSFE (1.4) and the effect of additive noise varying with slow variable on the fast variable near the NT-point, and give a critical value y^* where the 'critical transitions' [21] occur before $y < y^*$ with very high probability.

We show that:

- The effect of a small additive noise becomes negligible on the segments of trajectories that undergo large excursions in the fast variable.
- For $\lambda = 1$, the paths under additive noise near a NT point are likely to remain concentrated near the deterministic solution only for $e^{\frac{1}{\varepsilon}} \ll \sigma < \varepsilon^{\frac{3}{4}}$ and y less than $O(\sqrt{\varepsilon})$, and are likely to leave the vicinity of the deterministic solution before y reaching $O(\sqrt{\varepsilon log|\sigma|})$. Therefore small but not exponentially small noise destroys the bifurcation delay caused by the NT-point.
- If the noise intensity $\sigma > \varepsilon^{\frac{3}{4}}$, then critical transition is likely to occur before a time of order $\sigma^{\frac{2}{3}}$.

In Section 2, we state results in detail. In Section 3, we prove our theorems.

Notation. $f(u) = \mathcal{O}(u)$ means that there are two positive constants ξ_1 and ξ_2 such that $\xi_1 u \leq f(u) \leq \xi_2 u$; $f(u) = \tilde{\mathcal{O}}(u)$ means that there are two nonnegative constants ξ_1 and ξ_2 such that $\xi_1 u \leq |f(u)| \leq \xi_2 u$; $x \wedge y$ $(x \vee y)$ means the minimum (maximum) between x and y; $\lceil x \rceil$ means the smallest integer which is greater than or equal to x for $x \geq 0$; $f(u) = o(u)$ means $\lim_{u \to 0} \frac{f(u)}{u} = 0$; $P^{y_0,x_0}\{(x_y, y) \in \cdot\}$ is a probability measure induced by the process $\{x_y\}_{y \geq y_0}$ starting from x_0 at time y_0 on some probability space $(\Omega, \mathcal{F}, P)$; E^{y_0,x_0} denote expectations with respect to P^{y_0,x_0}.

2. The statement of results

First, we recall some useful properties about the equation (1.4). Under the **Assumption**, by Taylor's expansion, we know:

$$H(x,y) = [ay + \tilde{\mathcal{O}}(y^2) + bx + \tilde{\mathcal{O}}(yx) + \tilde{\mathcal{O}}(x^2)]x, \tag{2.1}$$

with $a = \frac{\partial_y f(0,0)}{G(0,0)} > 0$, $b = \frac{\partial_x f(0,0)}{G(0,0)} > 0$. Obviously, (1.4) is locally topologically equivalent to the typical example with a transcritical bifurcation:

$$\frac{dx}{dt} = x(x+y), \quad \frac{dy}{dt} = \varepsilon. \tag{2.2}$$

And there is a critical set

$$\mathcal{C}_0 = \{(x,t) : f(x,t) = 0 \ or \ x = 0\} = \mathcal{C}_{a1} \cup \mathcal{C}_{a2} \cup \mathcal{C}_{r1} \cup \mathcal{C}_{r2} \cup \{(0,0)\},$$

where $\mathcal{C}_{a1} = \{x = 0, y < 0\}$, $\mathcal{C}_{a1} = \{x = \varphi(y), y > 0\}$ are attracting, $\mathcal{C}_{r1} = \{x = \varphi(y), y < 0\}$, $\mathcal{C}_{r2} = \{x = 0, y > 0\}$ are repelling.

For normally hyperbolic critical manifold, one can use Fenichel's theory [12, 13] to obtain the nearby dynamics. (1.4) has a non-hyperbolic point $(0,0)$, where Fenichel's theory fails. We find that the NT-point $p = (0,0)$ of (1.4) is also a loss-of-stability turning point that the normal stability changes from stable to unstable [16, 28, 30, 33, 36]. One has

Lemma 2.1. *Let γ_ε be a trajectory of* (1.4) *starting from a fixed point (x_0, y_0), with $-T < y_0 < 0$ and $x_0 \in (-d, \varphi(y_0))$. Then, for sufficiently small $\varepsilon > 0$, there is a singular orbit γ_0:*

$$\begin{aligned}\gamma_0 =& \gamma_{01} \cup \gamma_{02} \cup \gamma_{03}\\ =& \{(x,y) : x \in [x_0 \wedge 0, x_0 \vee 0], y = y_0\} \cup \{(x,y) : x = 0, y \in [y_0, y_1]\}\\ & \cup \{(x,y) : x \in [x_0 \wedge 0, x_0 \vee 0], y = y_1\},\end{aligned}$$

where y_1 satisfies

$$\int_{y_0}^{y_1} \frac{F(0,y)}{G(0,y)} dy = 0, \tag{2.3}$$

such that $\gamma_\varepsilon \cap U$ converges to γ_0 in the Hausdorff distance as $\varepsilon \to 0$, where $U = \{(x,y) : |x| \le |x_0|, |y| < T\}$. In addition to, $\min\limits_{(x,y)\in\gamma_\varepsilon\cap U} |x| = e^{\frac{-c+o(1)}{\varepsilon}}$, with $c = \int_{y_0}^{0} |\frac{F(0,y)}{G(0,y)}| dy$.

The function (2.3) of y_1 is called an entry-exit [1] or way in-way out [8] function. The lemma means that, there is a phenomenon of 'delay of instability' [28], which is also called 'bifurcation delay' [2], or 'Pointryagin delay' [31] caused by the NT-point $(0,0)$: After the trajectory γ_ε is attracted into the vicinity of C_{a1} and crosses the critical branch $\{x = \varphi(y)\}$, instead of leaving the vicinity of C_{r2} immediately, γ_ε stays near C_{r2} until it reaches near $(y_1, 0)$.

Next, we analyze the paths near the NT-point $(0,0)$ of the equation (1.6) in order to explore the effect of additive noise varying with slow variable y on the fast variable x near the NT-point.

Let $x_y(x_0, y_0)$ be a solution of (1.6) and $x_y^{det}(x_0, y_0)$ be a solution of (1.5) staring from the fixed point (x_0, y_0). Our main theorem is as follows.

Theorem 2.1. *There is a positive constant $\tilde{T} \in (0, T)$ and a sufficiently small ε_0. For any $\varepsilon \in (0, \varepsilon_0]$, we have:*

(1) Given $y_0 \in (-T, 0)$, $x_0 \in (-d, \varphi(y_0))$. There is a $h_0 = \mathcal{O}(\varepsilon^{\frac{3}{4}})$, and a $y_0^* = y_0 + \mathcal{O}(\varepsilon|log\varepsilon|)$. For any $h \in (0, h_0]$, positive constant r and $y \in (y_0^*, \sqrt{r\varepsilon}]$, we have

$$\begin{aligned}&P^{x_0,y_0}\{\sup_{y_0^* \le u \le y} |x_y(x_0,y_0)| > \frac{h}{\sqrt{|u| \vee \sqrt{r\varepsilon}}}\}\\ \le& Q_1(y, y_0^*, \varepsilon) e^{-L_1 \frac{h^2}{\sigma^2}[1-\tilde{\mathcal{O}}(\sqrt{\varepsilon})]} + Q_2(y_0^*, y_0, \varepsilon) e^{-L_1 \frac{h^2}{\sqrt{r\varepsilon}\sigma^2}[1-\sqrt{\varepsilon}]} + \frac{3}{2} e^{-\frac{L_2}{\sigma^2}},\end{aligned} \tag{2.4}$$

with $Q_1(y,y_0^*)=\mathcal{O}(\frac{(y_0^*)^2-(y\vee 0)^2}{\varepsilon^2}+\frac{1}{\varepsilon}+\frac{1}{\sqrt{\varepsilon}}+1)$, $Q_2(y_0^*,y_0,\varepsilon)=\mathcal{O}(\frac{|log\varepsilon|}{\varepsilon}+1)$, $L_1=\frac{r}{8c_0^+(r+4)}\wedge\frac{b_1}{16c_0^+}$, $L_2=k_1[c_0^2\vee(d-|x_0|)^2\vee(2(\varphi(y_0)-|x_0|)^2)]$, *where* $c_0\le d\wedge x_0$ *satisfies*

$$\max_{\{y_0\le u<y_0^*,|x|\le c_0\}}\partial_x H(x,u)=-\mathcal{O}(1),$$

and

$$k_1=\min_{\{c_0\le|x|\le x_0\wedge[\varphi(u)-\frac{[\varphi(y_0)-x_0]\wedge\varphi(\frac{1}{2}y_0)}{2}],y_0\le u\le\frac{y_0}{2}\}}|\frac{H(x,u)}{x}|=\mathcal{O}(1).$$

(2) There is a positive constant $\bar r$ *and a function* $\bar x(y)=\bar a y$ *with* $0<\bar a<\frac{a}{b}$. *If* $\sqrt{\bar r\varepsilon}\le y_0\le y\le\tilde T$, $|x_0|<\bar x(y_0)$, *and* $\sigma|log\sigma|^{\frac{3}{2}}\le\mathcal{O}(\varepsilon^{\frac{3}{4}})$, *then*

$$P^{y_0,x_0}\{\sup_{u\in[y_0,y]}\frac{|x_u(x_0,y_0)|}{|\bar x(y)|}<1\}\le Q_3(y,y_0,\varepsilon)\frac{e^{-k_2\alpha_0(y,y_0)/\varepsilon}}{\sqrt{1-e^{-2k_2\alpha_0(y,y_0)/\varepsilon}}},\qquad(2.5)$$

where $Q_3(y,y_0,\varepsilon)=\frac{\bar x(y)\sqrt{k_2a_0(y)}}{\sqrt{\pi}\sigma}(1+\frac{1}{k_2}+\frac{2k_2\alpha_0(y,y_0)}{\varepsilon})$ *and* k_2 *is a positive constant satisfying* $(a-b\bar a)u\ge k_2\partial_xH(0,u)$ *for* $u\in(0,\tilde T]$.

(3) If $\sqrt{\bar r\varepsilon}\le y_0\le y\le\tilde T$ *and* $x_0=-\bar x(y_0)$, *then there is a positive constant* L_3 *such that*

$$\begin{aligned}&P^{y_0,x_0}\{\sup_{u\in[y_0,y]}|x_u(x_0,y_0)-x_u^{det}(x_0,y_0)|>\frac{h}{\sqrt{u}}\}\\&\le Q_4(y,y_0,\varepsilon)e^{-\frac{L_3h^2}{\sigma^2}[1-\mathcal{O}(\sqrt{\varepsilon})]}.\end{aligned}\qquad(2.6)$$

where $Q_4(y,y_0,\varepsilon)=\frac{\int_{y_0}^{y}\partial_xH(x_v^{det}(x_0,y_0))dv}{\varepsilon^2}+2$. *If* $\sqrt{\bar r\varepsilon}\le y_0\le y\le\tilde T$ *and* $x_0=\bar x(y_0)$, *then*

$$\begin{aligned}&P^{y_0,x_0}\{\sup_{u\in[y_0,y]}x_u(x_0,y_0)\le d\}\\&\le e^{-a_0(y_0)\bar x(y_0)^2/\sigma^2}+\frac{d\sqrt{a_0(y)}e^{-\alpha_0(y,y_0)/\varepsilon}}{\sqrt{\pi}\sigma\sqrt{1-e^{-2\alpha_0(y,y_0)/\varepsilon}}},\end{aligned}\qquad(2.7)$$

where $a_0(u)=\partial_xH(0,u)$ *and* $\alpha_0(y,y_0)=\int_{y_0}^{y}a_0(u)du$.

(4) Assume that $\sigma\ge\varepsilon^{\frac{3}{4}}$. *There are positive constants* ρ, k_3 *and* L_4, *and a* $h_0=\mathcal{O}(\varepsilon^{\frac{3}{4}})$ *such that*

$$\begin{aligned}&P^{x_0,t_0}\{\sup_{y_0\le u\le y}x_u\le d\}\\&\le L_4[\frac{|\alpha_0(y_0,y)|}{\varepsilon^2}+1]e^{-\frac{h^2}{\sigma^2}[1-\tilde{\mathcal{O}}(\sqrt{\varepsilon})]}+\frac{3}{2}e^{-\rho\frac{|\alpha_0(y\wedge(n\sigma^{\frac{2}{3}}),-n\sigma^{\frac{2}{3}})|}{\varepsilon(|log\sigma|\vee|log\frac{h}{\sigma}|)}},\end{aligned}\qquad(2.8)$$

for fixed $x_0\in(-d,\varphi(y_0))$, $y_0\in(-T,-k_3\sigma^{\frac{2}{3}}]$ *and* $y\in[-k_3\sigma^{\frac{2}{3}},k_3\sigma^{\frac{2}{3}}]$, $0<h<h_0$.

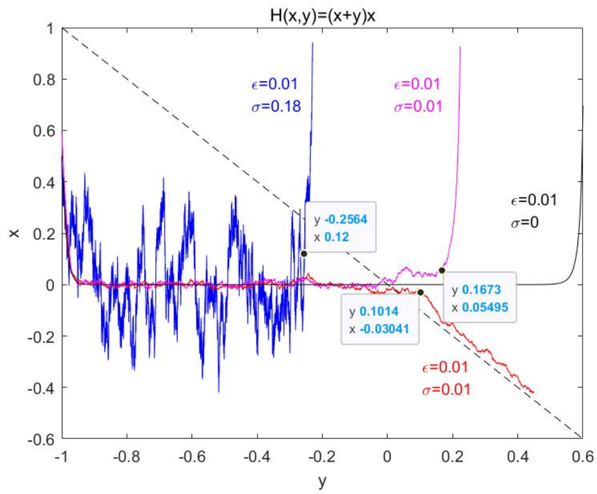

Figure 2

From the above theorem, we know that, there is a threshold value $N(\varepsilon) = \mathcal{O}(\varepsilon^{\frac{3}{4}})$ of noise intensity which makes the estimations of the probability useful for $\mathcal{O}(e^{-\frac{1}{\varepsilon}}) < \sigma < N(\varepsilon)$. When $\sigma < N(\varepsilon)$, similar to the dynamic behavior of the deterministic solution $x_y^{det}(x_0, y_0)$, the sample paths are likely to be rapidly attracted into a $\mathcal{O}(\varepsilon^{\frac{3}{4}})$ neighbourhood of $x = 0$ with exponential speed, and remain in the small neighbourhood until $y = \mathcal{O}(\sqrt{\varepsilon})$. And then, different from the deterministic solution remaining in the $\mathcal{O}(\varepsilon)$ neighbourhood of $x = 0$ until near $y = y_1 = \mathcal{O}(1)$ where critical transition happens, the sample paths are likely to transit in advance near $y = \mathcal{O}(\sqrt{\varepsilon|log\sigma|})$. And if $\sigma \geq N(\varepsilon)$ the critical transition is likely to occur earlier at some $y \in [-\sigma^{\frac{2}{3}}, \sigma^{\frac{2}{3}}]$. And when $\sigma \ll N(\varepsilon)$ or $\sigma \gg N(\varepsilon)$, the probability that the corresponding events as described above occur is almost one.

Remark 2.1. Note that, the threshold value $N(\varepsilon)$ of noise intensity is different to the threshold value $\mathcal{O}(\sqrt{\varepsilon})$ in the study of the pitchfork bifurcations in [3]. And the difference is caused by the difference in the lowest order of the higher order term of their drift term.

3. The proof of theorem

We compare the equation

$$dx = \frac{1}{\varepsilon} H(x, y)dy, \tag{3.1}$$

and

$$dx = \frac{1}{\varepsilon} H(x, y)dy + \frac{\sigma}{\sqrt{\varepsilon}} dW_y. \tag{3.2}$$

Let $D = \{(x, y) : |x| < d, |y| < T\}$, our discussion is confined to the region D.

Let $x_y(x_0, y_0)$ be a solution of (3.2) and $x_y^{det}(x_0, y_0)$ be a solution of (3.1) staring from the fixed point (x_0, y_0). The dynamical behavior of the solution $x_y^{det}(x_0, y_0)$ in D follows from Lemma 2.1.

Given $y_0 \in (-T, 0)$ and $x_0 \in (-d, \varphi(y_0))$. And let $z_y = x_y(x_0, y_0) - x_y^{det}(x_0, y_0)$. We know z_y obeys the SDE

$$dz_y = \frac{1}{\varepsilon}\{H(x_y(x_0, y_0), y) - H(x_y^{det}(x_0, y_0), y)\}dy + \frac{\sigma}{\sqrt{\varepsilon}}dW_y,\ z_{y_0} = 0. \tag{3.3}$$

The linearization of (3.3) around $z = 0$ is

$$dz_y^0 = \frac{1}{\varepsilon}a(y)z_y^0 dy + \frac{\sigma}{\sqrt{\varepsilon}}dW_y,\ z_{y_0}^0 = 0, \tag{3.4}$$

with $a(y) = \partial_x H(x_y^{det}(x_0, y_0), y)$. Obviously,

$$z_y^0 = z_{y_0}e^{\alpha(y,y_0)/\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}}\int_{y_0}^{y} e^{\alpha(y,v)/\varepsilon}dW_v$$

is a Gaussian process obeying the Gaussian distribution $\mathcal{N}(E_y(x_0, y_0), V_y(x_0, y_0))$, with

$$E_y(x_0, y_0) = z_{y_0}e^{\alpha(y,y_0)/\varepsilon},\ V_y(x_0, y_0) = \frac{\sigma^2}{\varepsilon}\int_{y_0}^{y} e^{2\alpha(y,v)/\varepsilon}dv,$$

with $\alpha(y, v) = \int_u^y a(u)du$. We define a function $\zeta(y)$ that approximates the variance function and has an advantage over the variance function of being bounded away from zero,

$$\zeta(y) = \frac{1}{|a(y_0)|}e^{2\alpha(y,y_0)/\varepsilon} + \frac{1}{\varepsilon}\int_{y_0}^{y} e^{2\alpha(y,v)/\varepsilon}dv. \tag{3.5}$$

Under our **Assumption**, it is easy to know the following lemma from the argument about the stable case in the subsection 3 in [3].

Lemma 3.1. *There exist positive constants ε_0, h_0, b_1, b_2 and c_0 depending only on $H(x, y)$ and y_0, such that for $0 < \varepsilon \le \varepsilon_0$, $0 < h \le h_0$, $|x_0| \le c_0$ and $y \in [y_0, \frac{1}{2}y_0]$,*

$$P^{0,y_0}\{\sup_{y_0 \le u \le y} \frac{|z_y|}{\sqrt{\zeta(y)}} > h\} \le C_1(y, y_0, \varepsilon)exp\{-\frac{h^2}{2\sigma^2}[1 - \tilde{\mathcal{O}}(\varepsilon) - \tilde{\mathcal{O}}(h)]\}, \tag{3.6}$$

with $C_1(y, y_0, \varepsilon) = \frac{|\alpha(y,y_0)|}{\varepsilon^2} + 2$. And $\frac{1}{2b_1} \le \zeta(y) \le \frac{1}{2b_2}$.

Note that, the limitation $|x_0| < c_0$ is to make sure $a(y) = -\mathcal{O}(1)$. If $c_0 < d$, then we use the sample paths approach to obtain the following lemma, which relaxes the region of the initial point.

Proposition 3.1. *Given $y_0 \in (-T, 0)$, $x_0 \in (-d, \varphi(y_0))/\{0\}$ and $c_1 \in (0, |x_0|]$. There is a sufficiently small ε_0 and a positive constant $\bar{c}_1$, such that for any $\varepsilon \in (0, \varepsilon_0)$ and $y \in (y_0 + \bar{c}_1\varepsilon, \frac{1}{2}y_0)$, we have*

$$\begin{aligned}
&P^{x_0,y_0}\{\sup_{u\in[y_0,y]} x_y(x_0, y_0) < -c_1\} \le \frac{3}{2}e^{-\frac{k_1[c_1^2 \wedge (d-|x_0|)^2))]}{\sigma^2}},\ for\ x_0 \in (-d, 0),\\
&P^{x_0,y_0}\{\inf_{u\in[y_0,y]} x_y(x_0, y_0) > c_1\} \le \frac{3}{2}e^{-\frac{k_1[c_1^2 \wedge (2(|\varphi(y_0)|-|x_0|)^2)]}{\sigma^2}},\ for\ x_0 \in (0, \varphi(y_0)).
\end{aligned} \tag{3.7}$$

Proof. First, we assume $x_0 \in (-d, 0)$ be fixed, given a constant $c_1 \in (0, |x_0|]$. Under the **Assumption**, it easy to know that there is a positive constant k_1 such that $\frac{H(x,y)}{x} \le -k_1$ for all $x \in [x_0, -c_1]$. Let $x_y^{det,0}(x_0, y_0)$ be a solution of the linear differential equation

$$dx = -\frac{1}{\varepsilon} k_1 x dy,\ x_{y_0} = x_0. \tag{3.8}$$

Obviously,

$$x_y^{det,0}(x_0, y_0) = x_0 e^{-k_1(y-y_0)/\varepsilon}. \tag{3.9}$$

Let $x_y^0(x_0, y_0)$ be a solution of the linear equation

$$dx = -\frac{1}{\varepsilon} k_1 x dy + \frac{\sigma}{\sqrt{\varepsilon}} dW_y, \qquad x_{y_0} = x_0, \tag{3.10}$$

then

$$x_u^0(x_0, y_0) = x_0 e^{-k_1(y-y_0)/\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}} \int_{y_0}^{y} e^{-k_1(y-u)/\varepsilon} dW_u. \tag{3.11}$$

This means $x_u^0(x_0, y_0)$ is a Gaussian process and follows the distribution $\mathcal{N}(E_u, V_u)$ with $E_u = x_0 e^{-k_1(y-y_0)/\varepsilon}$ and $V_u = \frac{\sigma^2}{\varepsilon} \int_{y_0}^{y} e^{-2k_1(y-u)/\varepsilon} du = \frac{\sigma^2}{2k_1}(1 - e^{-2k_1(y-y_0)/\varepsilon})$.

We define some stopping times

$$\tau_{-d}^0 = \inf\{u \in [y_0, y] : x_u^0(x_0, y_0) \le -d\}, \tau_{-c_1}^0 = \inf\{u \in [y_0, y] : x_u^0(x_0, y_0) \ge -c_1\},$$
$$\tau_{-d} = \inf\{u \in [y_0, y] : x_u(x_0, y_0) \le -d\}, \tau_{-c_1} = \inf\{u \in [y_0, y] : x_u(x_0, y_0) \ge -c_1\}.$$

If $x_u^0(x_0, y_0) > -d$ for all $u \in [y_0, y]$, then $\tau_{-d}^0 = \infty$. And $\tau_{-c_1}^0$, τ_{-d}, τ_{-c_1} are similarly defined.

Let $z_u = x_u(x_0, y_0) - x_u^0(x_0, y_0)$ with $z_{y_0} = 0$. Obviously, z_u satisfies

$$dz_u = \frac{1}{\varepsilon}\{-k_1 z_u + H(x_u(x_0, y_0), u) + k_1 x_u(x_0, y_0)\} du. \tag{3.12}$$

If $u \in [y_0, y \wedge \tau_{-d} \wedge \tau_{-c_1}]$, then

$$\begin{aligned} z_u &= z_{y_0} + \frac{1}{\varepsilon} \int_{y_0}^{u} -k_1 z_u + [F(x_u(x_0, y_0), u) + k_1 x_u(x_0, y_0)] du \\ &\ge z_{y_0} + \frac{1}{\varepsilon} \int_{y_0}^{u} -k_1 z_u du. \end{aligned} \tag{3.13}$$

By Gronwall's inequality, we know that

$$z_u \ge z_{y_0} e^{-k_1(u-y_0)/\varepsilon} \ge 0 \text{ as } z_{y_0} \ge 0 \Longrightarrow x_u(x_0, y_0) \ge x_u^0(x_0, y_0). \tag{3.14}$$

Thus $\tau_{-d} \ge \tau_{-d}^0$ and $\tau_{-c_1}^0 \ge \tau_{-c_1}$. Then

$$\begin{aligned} &P^{x_0,y_0}\{\sup_{y_0 \le u \le y} x_y(x_0, y_0) < -c_1\} \\ \le &P^{x_0,y_0}\{\sup_{y_0 \le u \le y} x_y(x_0, y_0) < -c_1, \tau_{-d} \le y\} \\ &+ P^{x_0,y_0}\{\sup_{y_0 \le u \le y} x_y(x_0, y_0) < -c_1, \tau_{-d} = \infty\} \\ \le &P^{x_0,y_0}\{\tau_{-d}^0 \le y\} + P^{x_0,y_0}\{\tau_{-c_1}^0 > y\}. \end{aligned} \tag{3.15}$$

There is a positive constant $\bar{c}_1$, if $y - y_0 \geq \bar{c}_1\varepsilon$, then

$$\begin{aligned}
P^{x_0,y_0}\{\tau^0_{-c_1} > y\} &= P^{x_0,y_0}\{\sup_{y_0\leq u\leq y} x^0_u(x_0,y_0) < -c_1\} \\
&\leq P^{x_0,y_0}\{x^0_y(x_0,y_0) \leq -c_1\} \\
&= \frac{1}{\sqrt{2\pi V_u}}\int_{-\infty}^{-c_1} e^{\frac{(x-E_u)^2}{2V_u}}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-c_1-E_u}{\sqrt{V_u}}} e^{\frac{v^2}{2}}dv \\
&\leq \frac{1}{2}e^{-\frac{(-c_1-E_u)^2}{2V_u}} \leq \frac{1}{2}e^{-\frac{2k_1(-c_1-x_0e^{-k_1(y-y_0)/\varepsilon})^2}{\sigma^2(1-e^{-2k_1(y-y_0)/\varepsilon})}} \leq \frac{1}{2}e^{-\frac{k_1c_1^2}{\sigma^2}}.
\end{aligned} \tag{3.16}$$

Let $X_u \triangleq \frac{\sigma}{\sqrt{\varepsilon}}\int_{y_0}^u e^{-k_1(u-v)/\varepsilon}dW_v$. X_u is a Gaussian process starting from $(0,y_0)$ and obeys the distribution $\mathcal{N}(0,V_u)$. The symmetry and the strong Markov property of $\{X_u\}_{u\geq y_0}$ implies that

$$\begin{aligned}
&P^{x_0,y_0}\{\tau^0_{-d} \leq y\} = P^{x_0,y_0}\{\exists u\in[y_0,y]\ s.t.\ x^0_u(x_0,y_0) \leq -d\} \\
=&P^{x_0,y_0}\{\exists u\in[y_0,y]\ s.t.\ x^0_u(x_0,y_0) - x^{det,0}_u(x_0,y_0) \leq -d - x^{det,0}_u(x_0,y_0)\} \\
=&P^{0,y_0}\{\exists u\in[y_0,y]\ s.t.\ X_u \leq -d - x^{det,0}_u(x_0,y_0)\} \\
\leq&P^{0,y_0}\{\exists u\in[y_0,y]\ s.t.\ X_u \leq -d + |x_0|\} \\
=&2P^{0,y_0}\{X_y < -d + |x_0|\} \\
=&\frac{2}{\sqrt{2\pi V_y}}\int_{-\infty}^{-d+|x_0|} e^{\frac{-x^2}{2V_y}}dx = \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-d+|x_0|}{\sqrt{V_y}}} e^{\frac{-v^2}{2}}dv \\
\leq&e^{-\frac{(-d-|x_0|)^2}{2V_y}} \leq e^{-\frac{2k_1(-d+|x_0|)^2}{\sigma^2(1-e^{-2k_1(y-y_0)/\varepsilon})}} \leq e^{-\frac{2k_1(-d+|x_0|)^2}{\sigma^2}}.
\end{aligned} \tag{3.17}$$

Substitute (3.16) and (3.17) into (3.15), we obtain the first inequality of (3.7). And if $x_0 \in (0,\varphi(y_0))$ is fixed, the second inequality of (3.7) is obtained by an analysis similar to the above. The only difference is the defined stopping time. □

We know that the deterministic solution $x^{det}_y(x_0,y_0)$ approaches and crosses the line $x = c_0$ at $y^*_0 = y_0 + \tilde{\mathcal{O}}(\varepsilon)$ exponentially fast. The above lemma means that the paths of the process $x_y(x_0,y_0)$ cross the line $x = c_0$ near y^*_0 with probability close to one for σ sufficiently small. Note that, the similar result holds for autonomous equations and has been proved in chapter 2 of [14] and in chapter 5 of [7], respectively, by the method of expansion in powers of a small parameter and the large deviation theory.

Let $a_0(y) = \partial_x H(0,y) = ay + \tilde{\mathcal{O}}(y^2)$. By the **Assumption**, we know there are positive constants $a^+_0 > a^-_0 > 0$, such that $a^+_0 y \leq a_0(y) \leq a^-_0 y$ for $-T \leq y \leq 0$, and $a^-_0 y \leq a_0(y) \leq a^+_0 y$ for $0 < y < T$. We know the linearization of (3.2) around $x = 0$ satisfies

$$dx^0_y = \frac{1}{\varepsilon}a_0(y)x^0_y dy + \frac{\sigma}{\sqrt{\varepsilon}}dW_y,\ x^0_{y_0} = x_0. \tag{3.18}$$

We define $\zeta_0(y) = \frac{1}{|a_0(y_0)|}e^{2\alpha_0(y,y_0)/\varepsilon} + \frac{1}{\varepsilon}\int_{y_0}^y e^{2\alpha_0(y,v)/\varepsilon}dv$ with $\alpha_0(y,v) = \int_v^y a_0(u)du$. There exist positive constants $c^-_0 \leq c^+_0$ such that $\zeta_0(y)$ satisfies

$$\begin{aligned}
&\frac{c^-_0}{|y|\vee\sqrt{r\varepsilon}} \leq \zeta_0(y) \leq \frac{c^+_0}{|y|\vee\sqrt{r\varepsilon}},\ \text{for } -T \leq y \leq \sqrt{r\varepsilon}, \\
&\frac{c^-_0}{\sqrt{\varepsilon}}e^{2\alpha_0(y,0)/\varepsilon} \leq \zeta_0(y) \leq \frac{c^+_0}{\sqrt{\varepsilon}}e^{2\alpha_0(y,0)/\varepsilon},\ \text{for } \sqrt{r\varepsilon} \leq y \leq T.
\end{aligned} \tag{3.19}$$

Let $z_1(y) = x_y(x_0, y_0) - x_y^0$ which satisfies

$$dz_1(y) = \frac{1}{\varepsilon}\{a_0(y)z_1(y) + H_0(x_y(x_0, y_0), y)\}dy, \text{ for } z_1(y_0) = 0, \tag{3.20}$$

with $H_0(x_y(x_0, y_0), y) = H(x_y(x_0, y_0), y) - a_0(y)x_y(x_0, y_0)$. Note that, there is a positive constant M_0 such that $|H_0(x_y(x_0, y_0), y)| \le M_0 x_y(x_0, y_0)^2$ as long as $(x_y(x_0, y_0), y) \in D$. By analyzing the linear process x_y^0 and the difference process $z_1(y)$, we obtain the following

Lemma 3.2. *There exists a constant* $L = \frac{\sqrt{2}M_0 c_0^+(\sqrt{c_0^+} + e^{\frac{ra_0^+}{2}})^2}{c_0^-} \vee M_0\sqrt{c_0^+}(2\sqrt{c_0^+} + e^{\frac{ra_0^+}{2}})^2$. *If* $0 < h \le \frac{d(r\varepsilon)^{\frac{1}{4}}}{2\sqrt{c_0^+} + e^{\frac{ra_0^+}{2}}} \wedge \frac{\varepsilon^{\frac{3}{4}}}{L}$, $|x_0| \le \frac{h}{(r\varepsilon)^{\frac{1}{4}}}$, *then for* $y \in [y_0, \sqrt{r\varepsilon}]$, *there is*

$$P^{y_0,x_0}\{\sup_{u\in[y_0,y]} \frac{|x_u(x_0, y_0) - x_0 e^{\alpha_0(u,y_0)/\varepsilon}|}{\zeta_0(u)} > h\} \le C_2(y, y_0, \varepsilon)e^{\frac{r}{2(r+4)}\frac{h^2}{\sigma^2}}[1 - \delta(\varepsilon)],$$

with $C_2(y, y_0, \varepsilon) = \frac{|\alpha_0(y,y_0)|}{\varepsilon^2} + \frac{r}{2a_0^-\varepsilon} + \frac{4\sqrt{r}}{\sqrt{\varepsilon}} + 4$, $\delta(\varepsilon) = \tilde{\mathcal{O}}(\varepsilon)$ *for* $y_0 \le y \le -\sqrt{r\varepsilon}$ *and* $\delta(\varepsilon) = \tilde{\mathcal{O}}(\sqrt{\varepsilon})$ *for* $|y| \le \sqrt{r\varepsilon}$.

We omit the long but straightforward proof of the lemma, which follows the same analytical process as that for Lemma 4.2 and Theorem 2.10 in [3]. The difference lies in the inclusion relation between these two measurable sets of $(\Omega, \mathcal{F}, P)$:

$$\mathcal{A}_y^0(h) \subseteq \mathcal{A}_y[(1 + \frac{L}{(r\varepsilon)^{\frac{3}{4}}})h], \ \mathcal{A}_y(h) \subseteq \mathcal{A}_y^0[(1 + \frac{L}{(r\varepsilon)^{\frac{3}{4}}})h], \tag{3.21}$$

with

$$\begin{aligned} \mathcal{A}_y^0(h) &= \{\omega : |x_u^0(\omega) - x_0 e^{\alpha_0(u,y_0)/\varepsilon}| \le h\sqrt{\zeta_0(y)}, \ \forall\, u \in [y_0, y]\}, \\ \mathcal{A}_y(h) &= \{\omega : |x_u(x_0, y_0)(\omega) - x_0 e^{\alpha_0(u,y_0)/\varepsilon}| \le h\sqrt{\zeta_0(y)}, \ \forall\, u \in [y_0, y]\}. \end{aligned} \tag{3.22}$$

The difference is due to the difference between the lowest order of the higher order term of the drift term near the transcritical bifurcation and pitchfork bifurcation. The remainder of the proof is similar, except that a lot of more careful calculation is involved.

We are now ready to proceed with

Proof of (1) **of Theorem 2.1.** Without loss of generality, we assume $-T < y_0 < 0$ and $0 < x_0 < \varphi(y_0)$ fixed. Let

$$\begin{aligned} y_{01}^* &= \inf\{u \ge y_0 : x_0 e^{\alpha_0(u,y_0)/\varepsilon} \le \frac{h}{2\sqrt{|u| \vee \sqrt{r\varepsilon}}}\}, \\ y_{02}^* &= \inf\{u \ge \frac{1}{2}y_{01}^* : x_u^{det}(c_0, \frac{1}{2}y_{01}^*) \le \frac{h}{2\sqrt{|u| \vee \sqrt{r\varepsilon}}}\}. \end{aligned}$$

We know that $x_0 e^{\alpha_0(u,y_0)/\varepsilon}$ and $x_u^{det}(c_0, \frac{1}{2}y_{01}^*)$ are attracted to a $\tilde{\mathcal{O}}(\varepsilon)$ neighborhood of $x = 0$ with velocity $\mathcal{O}(e^{\frac{1}{\varepsilon}})$. So $y_{01}^* = \mathcal{O}(\varepsilon|log\varepsilon|)$ and $y_{02}^* = \mathcal{O}(\varepsilon|log\varepsilon|)$. Let $y_0^* = \max(y_{01}^*, y_{02}^*)$, then $y_0^* \ll \frac{1}{2}y_0$ for sufficiently small ε. Define $\tau_{c_0} = \inf\{u >$

$y_0 : x_u(x_0, y_0) \le c_0\}$, by the above lemmas and the strong Markov property of $x_u(x_0, y_0)$, we have

$$
\begin{aligned}
& P^{y_0,x_0}\{\exists u \in [y_0^*, y] : |x_u(x_0, y_0)| > \frac{h}{\sqrt{|u| \vee \sqrt{r\varepsilon}}}\} \\
\le & P^{y_0,x_0}\{\exists u \in [y_0^*, y] : |x_u(x_0, y_0) - x_0 e^{\alpha_0(u,y_0)/\varepsilon}| > \frac{h}{2\sqrt{|u| \vee \sqrt{r\varepsilon}}}\} \\
\le & P^{y_0,x_0}\{\exists u \in [y_0^*, y] : |x_u(x_0, y_0) - x_0 e^{\alpha_0(u,y_0)/\varepsilon}| > h_1\sqrt{\zeta_0(u)}\} \\
\le & P^{y_0,x_0}\{\exists u \in [y_0^*, y] : |x_u(x_0, y_0) - x_0 e^{\alpha_0(u,y_0)/\varepsilon}| > h_1\sqrt{\zeta_0(u)}, \\
& |x_{y_0^*}(x_0, y_0)| \le \frac{h_1}{(r\varepsilon)^{\frac{1}{4}}}\} + P^{y_0,x_0}\{|x_{y_0^*}(x_0, y_0)| > \frac{h_1}{(r\varepsilon)^{\frac{1}{4}}}, \tau_{c_0} < \frac{1}{2}y_{01}^*\} \\
& + P^{y_0,x_0}\{\tau_{c_0} \ge \frac{1}{2}y_{01}^*\} \\
= & E^{y_0,x_0}\{I_{\{|x_{y_0^*}(x_0,y_0) < \frac{h_1}{(r\varepsilon)^{\frac{1}{4}}}\}} P^{y_0^*, x_{y_0^*}(x_0,y_0)}\{\exists u \in [y_0^*, y] : |x_u(x_0, y_0) - x_0 e^{\alpha_0(u,y_0)/\varepsilon}| \\
& > h_1\sqrt{\zeta_0(u)}\}\} + E^{y_0,x_0}\{I_{\{\tau_{c_0} < \frac{1}{2}y_{01}^*\}} P^{\tau_{c_0}, c_0}\{|x_{y_0^*}(x_0, y_0)| > \frac{h_1}{(r\varepsilon)^{\frac{1}{4}}}\}\} \\
& + P^{y_0,x_0}\{\tau_{c_0} \ge \frac{1}{2}y_{01}^*\} \\
\le & E^{y_0,x_0}\{I_{\{|x_{y_0^*}(x_0,y_0) < \frac{h_1}{(r\varepsilon)^{\frac{1}{4}}}\}} P^{y_0^*, x_{y_0^*}(x_0,y_0)}\{\exists u \in [y_0^*, y] : |x_u(x_0, y_0) - x_0 e^{\alpha_0(u,y_0)/\varepsilon}| \\
& > h_1\sqrt{\zeta_0(u)}\}\} + E^{y_0,x_0}\{I_{\{\tau_{c_0} < \frac{1}{2}y_{01}^*\}} P^{\tau_{c_0}, c_0}\{\exists u \in [\tau_{c_0}, y_0^*] : |x_u(x_0, y_0) \\
& - x_u^{det}(x_0, y_0)| > h_2\sqrt{\zeta(u)}\}\} + P^{y_0,x_0}\{\tau_{c_0} \ge \frac{1}{2}y_{01}^*\} \\
\le & Q_1(y, y_0^*, \varepsilon) e^{-L_1 \frac{h^2}{\sigma^2}[1 - \tilde{\mathcal{O}}(\sqrt{\varepsilon})]} + Q_2(y_0^*, y_0, \varepsilon) e^{-L_1 \frac{h^2}{\sqrt{r\varepsilon}\sigma^2}[1 - \sqrt{\varepsilon}]} + \frac{3}{2} e^{-\frac{L_2}{\sigma^2}},
\end{aligned}
\tag{3.23}
$$

where $h_1 = \frac{1}{2\sqrt{c_0^+}}h$, $h_2 = \frac{\sqrt{b_2}}{\sqrt{2}(r\varepsilon)^{\frac{1}{4}}}h_1$. □

Next, we look for an appropriate region $\bar{D} \triangleq \{(x, y) : |x| \le \bar{x}(y), \tilde{T} \ge y \ge \sqrt{\bar{r}(\varepsilon)}\}$ which approximates the region $\tilde{D} \triangleq \{(x, y) : |x| \le |\varphi(y)|, \tilde{T} \ge y \ge \sqrt{\bar{r}(\varepsilon)}\}$ and is contained in $\tilde{D}$, in order to show that the sample paths starting from the region are likely to leave the region rapidly. Here $\bar{r}$ is a positive constant to be determined later. This will be used in proving that the sample paths starting from some $\mathcal{O}(\sqrt{\varepsilon})$ neighbourhood of $x = 0$ jump away from the vicinity of $x = 0$ with a high probability after $y > \sqrt{\bar{r}\varepsilon}$. Note that, from now on, without loss of generality, we can assume d is a suitably small constant so that the following conditions holds. Otherwise, we can supplement our proof with the conclusion of Proposition 3.1.

We summarize conditions according to the subsequent proof requirements:

(A_1) $\bar{D}$ approximates the region $\tilde{D} \triangleq \{(x, y) : |x| \le |\varphi(y)|, \sqrt{\bar{r}(\varepsilon)} \le y \le \tilde{T}\}$ where $\tilde{T} \le T$ is a positive constant such that

$$
\begin{aligned}
& \min_{y \in [0, \tilde{T}]} [-\varphi'(y)] = \mathcal{O}(1), \min_{\{|x| \le d, y \in (0, \tilde{T}]\}} \partial_x \frac{H(x, y)}{x} > 0, \\
& \min_{\{|x| \le d, y \in [-\tilde{T}, \tilde{T}]\}} \partial_{xx} H(x, y) > 0;
\end{aligned}
$$

(A_2) The solution $\varphi(y) \le x_y^{det}(-\bar{x}(y_0), y_0) \le -\bar{x}(y)$ for $\sqrt{\bar{r}(\varepsilon)} \le y_0 \le y \le \tilde{T}$;

(A_3) $\partial_x H(-\bar{x}(y), y) = \mathcal{O}(\partial_x H(\varphi(y), y))$ in order to maximize the range of values of σ in which our conclusion holds.

Since $-\varphi(y) = \mathcal{O}(y)$, we naturally assume that $\bar{x}(y) = \bar{a}y$. By the above conditions, it is clear that $\frac{a}{2b} < \bar{a} < \min_{y\in[0,\tilde{T}]}[-\varphi'(y)] < \frac{a}{b}$, and $\frac{1}{a-b\bar{a}} < \bar{r}$. Let $\sqrt{\bar{r}\varepsilon} \le y_0 \le u \le \tilde{T}$ and $z_u = x_u(-\bar{x}(y_0), y_0) - x_u^{det}(-\bar{x}(y_0), y_0)$. z_u satisfies

$$dz_u = \frac{1}{\varepsilon}\{a_2(u)z_u + H_2(z_u, u)\}du + \frac{\sigma}{\sqrt{\varepsilon}}dW_u, \quad z_{y_0} = 0, \tag{3.24}$$

with $a_2(u) = \partial_x H(x_u^{det}(-\bar{x}(y_0), y_0))$ and

$$H_2(z_u, u) = H(x_u(-\bar{x}(y_0), y_0), u) - H(x_u^{det}(-\bar{x}(y_0), y_0), u) - a_2(u)z_u.$$

It is easy to prove that there is a positive constant M_2 such that $|H_2(z_u, u)| \le M_2 z_u^2$ for $(z_u, u) \in D$. Let z_u^0 be a solution of the linear equation

$$dz_u^0 = \frac{1}{\varepsilon}a_2(u)z_u^0 du + \frac{\sigma}{\sqrt{\varepsilon}}dW_u,\ z_{y_0}^0 = 0, \tag{3.25}$$

and $\zeta_2(y) = \frac{1}{|a_2(y_0)|}e^{2\alpha_2(y,y_0)/\varepsilon} + \frac{1}{\varepsilon}\int_{y_0}^{y} e^{2\alpha_2(y,v)/\varepsilon}dv$ with $\alpha_2(y, v) = \int_v^y a_2(u)du$. It satisfies

$$\frac{1}{2|a_-(u)|} \le \zeta_2(u) \le \frac{1}{2|\bar{a}(u)|},\ \zeta'(u) \le \frac{1}{\varepsilon} \tag{3.26}$$

with $a_-(u) = \partial_x H(\varphi(u), u) = -\mathcal{O}(u)$ and $\bar{a}(u) = \partial_x H(-\bar{x}(u), u) = -\mathcal{O}(u)$. Then, we can obtain the following

Lemma 3.3. *For $\sqrt{\bar{r}\varepsilon} \le y_0 \le y < \tilde{T}$, there is a $h_0 = \mathcal{O}(y_0^{\frac{3}{2}} \wedge y_0)$. For $0 < h < h_0$ and sufficiently small ε, we have*

$$P^{y_0,-\bar{x}(y_0)}\{\sup_{y_0\le u\le y} \frac{|z_u|}{\sqrt{\zeta_2(u)}} \ge h\} \le Q_4(y, y_0, \varepsilon)e^{-\frac{1}{2}\frac{h^2}{\sigma^2}[1-\tilde{\mathcal{O}}(\sqrt{\varepsilon})]}, \tag{3.27}$$

with $Q_4(y, y_0, \varepsilon) = \frac{\alpha_2(y,y_0)}{\varepsilon^2} + 2$.

We omit the proof of the lemma, which follows similar steps of that of Theorem 2.12 in [3].

Proof of (2) and (3) of Theorem 2.1. Since there is a positive constant $\bar{a}_0 = a - b\bar{a}$ such that $\frac{H(x,y)}{x} \ge \bar{a}_0 y$, one can prove (2) basically following the steps of Theorem 2.11 in [3]. The only difference is the restriction on σ that $\sigma|log\sigma|^{\frac{3}{2}} \le \mathcal{O}(\varepsilon^{\frac{3}{4}})$. (2.6) can be obtained directly by the above Lemma 3.3 and the fact that $\zeta_2(u) = \mathcal{O}(\frac{1}{u})$ for $u \in [\sqrt{\bar{r}\varepsilon}, \tilde{T}]$.

Next, we prove (2.7). Let x_u^0 be a solution of (3.18) with $x_{y_0}^0 = \bar{x}(y_0)$. $z_u = x_u(\bar{x}(y_0), y_0) - x_u^0(\bar{x}(y_0), y_0)$ satisfying (3.20). Since there is a $\theta \in [0, 1]$, such that $H_0(x_u(\bar{x}(y_0), y_0), y_0) = \frac{1}{2}\partial_{xx}H(\theta x_u(\bar{x}(y_0), y_0), u)[x_u(\bar{x}(y_0), y_0)]^2 > 0$ for $0 < x_u(\bar{x}(y_0), y_0), u) < b, u \in [0, \tilde{T}]$. Repeating the analysis of z_u in Proposition 3.1, we obtain that $z_u(\omega) > 0$ for $\omega \in \{x_u \in [0, d], \forall u \in [y_0, y]\}$. Define $\tau_0^0 = \inf\{u \ge y_0 :$

$x^0_u = 0\}$, then

$$\begin{aligned}
&P^{y_0,\bar{x}(y_0)}\{x_u(\bar{x}(y_0),y_0),u) < d,\ \forall u \in [y_0,y]\}\\
=&P^{y_0,\bar{x}(y_0)}\{x_u(\bar{x}(y_0),y_0),u) < d,\ \forall u \in [y_0,y], \tau_0^0 \le y\}\\
&+P^{y_0,\bar{x}(y_0)}\{x_u(\bar{x}(y_0),y_0),u) < d,\ \forall u \in [y_0,y], \tau_0^0 > y\}\\
\le&P^{y_0,\bar{x}(y_0)}\{\tau_0^0 \le y\} + P^{y_0,\bar{x}(y_0)}\{x_u(\bar{x}(y_0),y_0),u) < d,\ \forall u \in [y_0,y], \tau_0^0 > y\}\\
\le&P^{y_0,\bar{x}(y_0)}\{\tau_0^0 \le y\} + P^{y_0,\bar{x}(y_0)}\{0 < x^0_u < d,\ \forall u \in [y_0,y]\}\\
\le&P^{y_0,\bar{x}(y_0)}\{\tau_0^0 \le y\} + P^{y_0,\bar{x}(y_0)}\{0 < x^0_y < d\}\\
\le&e^{-a_0(y_0)\bar{x}(y_0)^2/\sigma^2} + \frac{d\sqrt{a_0(y)}e^{-\alpha_0(y,y_0)/\varepsilon}}{\sqrt{\pi}\sigma\sqrt{1-e^{-2\alpha_0(y,y_0)/\varepsilon}}}.
\end{aligned} \tag{3.28}$$

The last inequality follows from Lemma 4.9 in [3] and the fact that x^0_y obeys the distribution $\mathcal{N}(\bar{x}(y_0)e^{\alpha_0(y,y_0)/\varepsilon}, \frac{\sigma^2}{\varepsilon}\int_{y_0}^{y} e^{2\alpha_0(y,v)/\varepsilon}dv)$. □

We show next that when $\sigma \ge \varepsilon^{\frac{3}{4}}$, the bifurcation delay caused by a NT-point is destroyed earlier than $y = \sqrt{\varepsilon|log\sigma|}$ in the case $e^{-\frac{1}{\varepsilon}} \ll \sigma < \mathcal{O}(\varepsilon^{\frac{3}{4}})$. And we get the critical value $y* = \sigma^{\frac{2}{3}}$, which means that the solution x_y starting from the attracting domain of $x = 0$ is very likely to enter the respelling sub-domain $[\mathcal{O}(1), +\infty)$ of $x = \varphi(y)$ before $y = y^*$.

Proof of (4) **of Theorem 2.1.** This proof is similar to the proof of Theorem 3.3.4 in [4] studied near a fold point. We outline the main steps. First

$$\begin{aligned}
&P^{x_0,y_0}\{\sup_{y_0\le u\le y} x_u(x_0,y_0) \le d\}\\
=&P^{x_0,y_0}\{\sup_{y_0\le u\le y} x_u(x_0,y_0) \le d, \inf_{y_0\le u\le y}[x_u(x_0,y_0) - x_u^{det}(x_0,y_0) + h\sqrt{\zeta_0(u)}] > 0\}\\
&+P^{x_0,y_0}\{\sup_{y_0\le u\le y} x_u(x_0,y_0) \le d, \inf_{y_0\le u\le y}[x_u(x_0,y_0) - x_u^{det}(x_0,y_0) + h\sqrt{\zeta_0(u)}] \le 0\},
\end{aligned} \tag{3.29}$$

where $h = \mathcal{O}(\varepsilon^{\frac{1}{4}})$ so that $\max_{y_0\le u\le y}[x_u^{det}(x_0,y_0) - h\sqrt{\zeta_0(u)}] > -d$. Next, we estimate the two terms on the right hand side of the equality (3.29).

Step 1. We prove that there is a positive constant ρ such that

$$\begin{aligned}
&P^{y_0,x_0}\{\sup_{y_0\le u\le y} x_u(x_0,y_0) \le d, \inf_{y_0\le u\le y}[x_u(x_0,y_0) - x_u^{det}(x_0,y_0) - h\sqrt{\zeta_0(u)}] > 0\}\\
\le&\frac{3}{2}e^{-\rho\frac{|\alpha_0(y\wedge(n\sigma^{\frac{2}{3}}),-n\sigma^{\frac{2}{3}})|}{\varepsilon(|log\sigma|\vee|log\frac{h}{\sigma}|)}}.
\end{aligned} \tag{3.30}$$

Since

$$\begin{aligned}
&P^{y_0,x_0}\{\sup_{y_0\le u\le y} x_u(x_0,y_0) \le d, \inf_{y_0\le u\le y}[x_u(x_0,y_0) - h\sqrt{\zeta_0(u)}] > 0\}\\
=&E^{y_0,x_0}\{I_{\{-h\sqrt{\zeta_0(u)}<x_u(x_0,y_0)<d,\ \forall u\in[y_0,-n\sigma^{\frac{2}{3}}]\}}(\omega)\\
&\times P^{-n\sigma^{\frac{2}{3}},x_1}\{-h\sqrt{\zeta_0(u)} < x_u(x_0,y_0) < d,\ \forall u \in [-n\sigma^{\frac{2}{3}}, y]\}\}
\end{aligned}$$

$$\leq \sup_{-h\sqrt{\zeta_0(-n\sigma^{\frac{2}{3}})}<x_1<d} P^{-n\sigma^{\frac{2}{3}},x_1}\{-h\sqrt{\zeta_0(u)} < x_u(x_0,y_0) < d,\ \forall u \in [-n\sigma^{\frac{2}{3}}, y]\}, \tag{3.31}$$

where n is a proper constant determined by (3.36).

Next, we need to estimate

$$P^{-n\sigma^{\frac{2}{3}},x_1}\{-h\sqrt{\zeta_0(u)} < x_u(x_0,y_0) < d,\ \forall u \in [-n\sigma^{\frac{2}{3}}, y]\}.$$

We assume $-d \leq -h\sqrt{\zeta_0(y_0)} < x_0 < 0$, and $y > \sqrt{r\varepsilon}$. The analysis for $y < \sqrt{r\varepsilon}$ is included in the following analysis. If $x_1 > 0$ then we know

$$\begin{aligned}&P^{-n\sigma^{\frac{2}{3}},x_1}\{-h\sqrt{\zeta_0(u)} < x_u(x_1,y_0) < d,\ \forall u \in [-n\sigma^{\frac{2}{3}}, y]\}\\ <&P^{-n\sigma^{\frac{2}{3}},x_0}\{-h\sqrt{\zeta_0(u)} < x_u(x_0,y_0) < d,\ \forall u \in [-n\sigma^{\frac{2}{3}}, y]\},\end{aligned} \tag{3.32}$$

by the uniqueness of the solution. We divide the interval $[-n\sigma^{\frac{2}{3}}, y]$ into K subintervals: $-n\sigma^{\frac{2}{3}} < y_1 < y_2 < \cdots < y_{K_1-1} < y_{K_1} = -\sqrt{r\varepsilon} < y_{K_1+1} = \sqrt{r\varepsilon} < \cdots < y_{K-1} < y_K = y$, satisfying $K_1 = \frac{\lceil|\alpha_0(-\sqrt{r\varepsilon}, -n\sigma^{\frac{2}{3}})|\rceil}{m\varepsilon}$, $K = \frac{|\alpha_0(y,\sqrt{r\varepsilon})|}{m\varepsilon}$, and $\alpha_0(y_{k+1}, y_k) = m\varepsilon$ for $0 \leq k \leq K_1 - 2$ and $K_1 + 1 \leq k \leq K - 2$ with $m = \mathcal{O}(|log\frac{h}{\sigma}| + |log\sigma|)$. Let

$$A_k = \{\omega \in \Omega : -h\sqrt{\zeta_0(u)} < x_u(x_0,y_0)(\omega) \leq d, \forall u \in [y_k, y_{k+1}]\}. \tag{3.33}$$

Then

$$\begin{aligned}&P^{-n\sigma^{\frac{2}{3}},x_0}\{-h\sqrt{\zeta_0(u)} < x_u(x_0,y_0) < d,\ \forall u \in [-n\sigma^{\frac{2}{3}}, y]\}\\ \leq&P^{-n\sigma^{\frac{2}{3}},x_0}\{\cap_{k=0}^{K-1}A_k\} = E^{-n\sigma^{\frac{2}{3}},x_0}\{I_{\cap_{k=0}^{K-1}A_k}\}\\ =&E^{-n\sigma^{\frac{2}{3}},x_0}\{I_{\cap_{k=0}^{K-1}A_k}\}\\ =&E^{-n\sigma^{\frac{2}{3}},x_0}\{E^{-n\sigma^{\frac{2}{3}},x_0}[I_{\cap_{k=0}^{K-1}A_k}|(x_u(x_0,y_0))_{y_0\leq u\leq y_{K-1}}]\}\\ =&E^{-n\sigma^{\frac{2}{3}},x_0}[I_{\cap_{k=0}^{K-2}A_k}E^{x_{y_{K-1}},y_{K-1}}(A_{K-1})] \leq q_{K-1}E^{-n\sigma^{\frac{2}{3}},x_0}\{I_{\cap_{k=0}^{K-2}A_k}\}\\ \leq&\prod_{k=0}^{K-1}q_k \leq \prod_{k=0}^{K-2}q_k,\end{aligned} \tag{3.34}$$

where $q_k = \sup_{-h\sqrt{\zeta_0(y_k)}\leq x_{y_k}\leq d} P^{x_{y_k},y_k}(A_k)$.

Let $y_{k-1} < y_k^{(1)} < y_k^{(2)} < y_k$ satisfying $|\alpha_0(y_k^{(1)}, y_{k-1})| = |\alpha_0(y_k^{(2)}, y_k^{(1)})| = |\alpha_0(y_k, y_k^{(2)})| = \frac{1}{3}m\varepsilon$, and define

$$\begin{aligned}\tau_k^+ &= \inf\{u \in [y_{k-1}, y_k] : x_u(x_0, -n\sigma^{\frac{2}{3}}) \leq -h\sqrt{\zeta_0(u)}\},\\ \tau_{k1} &= \inf\{u \in [y_{k-1}, y_k] : x_u(x_0, -n\sigma^{\frac{2}{3}}) > 0\},\\ \tau_{k2} &= \inf\{u \in [\tau_{k1}, y_k] : x_u(x_0, -n\sigma^{\frac{2}{3}}) > \varphi(u)\},\\ \tau_{k3} &= \inf\{u \in [\tau_{k2}, y_k] : x_u(x_0, -n\sigma^{\frac{2}{3}}) > d\}.\end{aligned} \tag{3.35}$$

Then, by following the proof of Theorem 3.3.5 in [3], we know that there are positive constants $\tilde{L}_1$, $\tilde{L}_2 \geq 1$ and $\tilde{L}_3$ such that

$$\begin{aligned} q_k \leq & \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{m}{3} h\sqrt{\zeta_0(y_{k-1})}\sqrt{2|a_0(y_{k-1})|}}}{\sigma\sqrt{1-e^{-\frac{2m}{3}}}} + \sqrt{\frac{2}{\pi}} \frac{(|\varphi(y_k)| \vee |\varphi(y_{k-1})|)\sqrt{2a_-(y_{k-1})}}{\sigma\sqrt{1-e^{-\frac{2m}{3\tilde{L}_2}}}} \\ & + \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \frac{de^{-\frac{m}{3\tilde{L}_2}}\sqrt{a_-(y_{k-1})}}{\sigma\sqrt{1-e^{-\frac{2m}{3\tilde{L}_2}}}} \\ \leq & \frac{1}{2} + \tilde{L}_3[(\frac{h}{\sigma} + \sqrt{n}\frac{d}{\sigma^{\frac{2}{3}}})e^{-\frac{m}{3\tilde{L}_2}} + n^{\frac{3}{2}}] \leq \frac{2}{3}. \end{aligned} \tag{3.36}$$

Taking (3.35) into (3.34), we obtain the estimation (3.30).

Step 2. We prove

$$\begin{aligned} & P^{y_0,x_0}\{\sup_{y_0\leq u\leq y} x_u(x_0,y_0) \leq d, \inf_{y_0\leq u\leq y}[x_u(x_0,y_0) + h\sqrt{\zeta_0(u)}] \leq 0\} \\ & \leq L_4[\frac{|\alpha_0(-n\sigma^{\frac{2}{3}},y)|}{\varepsilon^2} + 1]e^{-\frac{h^2}{\sigma^2}[1-\tilde{\mathcal{O}}(\sqrt{\varepsilon})]}. \end{aligned} \tag{3.37}$$

Let x_u^0 be a solution of (3.18) with $x^0_{-n\sigma^{\frac{2}{3}}} = x_0$. Since $\partial_{xx}H(x,y) > 0$ for $|x| < d$ and $|y| \leq \tilde{T}$. If $y_0 \leq \tilde{T}$, we can supplement our proof with the conclusion of Lemma 3.1. Thus, we neglect the case. Let h satisfy $\max_{u\in[y_0,y]} h\sqrt{\zeta_0(u)} < d$, that is $h < \mathcal{O}(\varepsilon^{\frac{1}{4}})$. Then by the same arguments as in the proof of Proposition 3.1, we know that there is a positive constant L_4 such that

$$\begin{aligned} & P^{y_0,x_0}\{\sup_{-n\sigma^{\frac{2}{3}}\leq u\leq y} x_u(x_0,y_0) \leq d, \inf_{y_0\leq u\leq y}[x_u(x_0,y_0) + h\sqrt{\zeta_0(u)}] \leq 0\} \\ \leq & P^{y_0,x_0}\{\inf_{y_0\leq u\leq y} \frac{x_u(x_0,y_0)}{\sqrt{\zeta_0(u)}} \leq -h\} \\ < & L_4[\frac{|\alpha_0(y_0,y)|}{\varepsilon^2} + 1]e^{-\frac{h^2}{\sigma^2}[1-\tilde{\mathcal{O}}(\sqrt{\varepsilon})]}, \end{aligned} \tag{3.38}$$

where the last inequality is obtained by the same arguments as in the proof of Proposition 4.3 in [3]. This finishes the proof of (2.8). □

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