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# HERMITE–HADAMARD TYPE INEQUALITIES FOR KATUGAMPOLA FRACTIONAL INTEGRALS

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**Abstract** In the paper, basing on the Katugampola fractional integrals ${}^{\rho}\mathcal{K}^{\alpha}_{a+}f$ and ${}^{\rho}\mathcal{K}^{\alpha}_{b-}f$ with $f \in \mathfrak{X}^p_c(a,b)$, the authors establish the Hermite–Hadamard type inequalities for convex functions, give their left estimates, and apply these newly-established inequalities to special means of real numbers. When $\rho \to 1$, these results become the corresponding ones for the Riemann–Liouville fractional integrals.

**Keywords** Hermite–Hadamard type inequality, convex function, special mean, Katugampola fractional integral.

**MSC(2010)** 26A33, 26D07, 26D10, 26D15.

## 1. Introduction

Fractional Calculus is a field of applied mathematics and deals with derivatives and integrals of arbitrary orders, including complex orders. Although the definitions for fractional derivatives are inconsistent and work in some cases but not in others, there are almost practical applications and profound impact in science, engineering, mathematics, economics, and other fields.

Suppose that $(a, b)$ is a finite or infinite interval of the real line $\mathbb{R}$, where $a < b$ and $a, b \in [-\infty, \infty]$, and that $\alpha$ is a complex number with $\Re(\alpha) > 0$. Let $\Gamma(z)$ is the classical Euler gamma function defined by

$$\Gamma(z) = \int_0^{\infty} \tau^{z-1} e^{-\tau} d\tau.$$

In [13], Podlubny introduced the left-side and right-side Riemann–Liouville fractional integrals of order $\alpha$ for a function $f$ as

$$\mathcal{R}^{\alpha}_{a+} f(\chi) = \frac{1}{\Gamma(\alpha)} \int_a^{\chi} (\chi - \tau)^{\alpha-1} f(\tau) d\tau$$

and

$$\mathcal{R}^{\alpha}_{b-} f(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\chi}^{b} (\tau - \chi)^{\alpha-1} f(\tau) d\tau$$

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respectively.

In [16], Samko introduced the left-side and right-side Hadamard fractional integrals of order $\alpha$ for a function $f$ as

$$\mathcal{H}^{\alpha}_{a+}f(\chi)=\frac{1}{\Gamma(\alpha)}\int_a^{\chi}(\ln\chi-\ln\tau)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}$$

and

$$\mathcal{H}^{\alpha}_{b-}f(\chi)=\frac{1}{\Gamma(\alpha)}\int_{\chi}^{b}(\ln\tau-\ln\chi)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}$$

respectively.

Suppose that $\mathfrak{X}^p_c(a,b)$ is the space of the complex-valued Lebesgue measurable functions $f$ on $[a,b]$ with $\|f\|_{\mathfrak{X}^p_c}<\infty$, that is,

$$\mathfrak{X}^p_c(a,b)=\{f:[a,b]\to\mathbb{C},\|f\|_{\mathfrak{X}^p_c}<\infty\},$$

where the norm $\|f\|_{\mathfrak{X}^p_c}$ is defined by

$$\|f\|_{\mathfrak{X}^p_c}=\left(\int_a^b|\tau^c f(\tau)|^p\frac{d\tau}{\tau}\right)^{1/p},\quad 1\le p<\infty,\quad c\in\mathbb{R}$$

and

$$\|f\|_{\mathfrak{X}^\infty_c}=esssup_{a\le\tau\le b}\big[\tau^c|f(\tau)|\big],\quad p=\infty,\quad c\in\mathbb{R}.$$

In the sense of the above function space, Katugampola introduced in [6, 7] the left-side and right-side fractional integrals of order $\alpha$ for a function $f\in\mathfrak{X}^p_c(a,b)$ by

$${}^{\rho}\mathcal{K}^{\alpha}_{a+}f(\chi)=\frac{1}{\Gamma(\alpha)}\int_a^{\chi}\left(\frac{\chi^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau^{1-\rho}},\quad\rho>0$$

and

$${}^{\rho}\mathcal{K}^{\alpha}_{b-}f(\chi)=\frac{1}{\Gamma(\alpha)}\int_{\chi}^{b}\left(\frac{\tau^{\rho}-\chi^{\rho}}{\rho}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau^{1-\rho}},\quad\rho>0$$

respectively. The above-defined fractional operators are known as the Katugampola fractional integrals in [6, 7], or Erdélyi-Kober fractional integrals in [8, 15], or $\rho$-Riemann–Liouville fractional integral in [4], which generalize fractional integrals of the Riemann–Liouville and Hadamard respectively by

$$\lim_{\rho\to1}[{}^{\rho}\mathcal{K}^{\alpha}_{a+}f(\chi)]=\mathcal{R}^{\alpha}_{a+}f(\chi)$$

and

$$\lim_{\rho\to0}[{}^{\rho}\mathcal{K}^{\alpha}_{a+}f(\chi)]=\mathcal{H}^{\alpha}_{a+}f(\chi).$$

The similar results for right-sided fractional integrals also hold.

For any convex function $f:[a,b]\to\mathbb{R}$, the double inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(\tau)d\tau\le\frac{f(a)+f(b)}{2}$$

is well known as the Hermite–Hadamard inequality, see [2, 3, 5, 10, 11, 17].

In the paper [1], Chen and Katugampola gave the following Hermite–Hadamard type inequalities basing on the Katugampola fractional integrals.

**Theorem 1.1** ( [1]). *For $\rho > 0$ and $0 \le a < b$, suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is a positive function and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $f$ is a convex function on $[a, b]$, then for any $\alpha > 0$*

$$f\left(\frac{a^\rho + b^\rho}{2}\right) \le \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}[{}^\rho\mathcal{K}_{a+}^\alpha f(b^\rho) + {}^\rho\mathcal{K}_{b-}^\alpha f(a^\rho)] \le \frac{f(a^\rho) + f(b^\rho)}{2},$$

*where the fractional integrals are considered for the function $f(\chi^\rho)$ and evaluated at $a$ and $b$, respectively.*

In [1], Chen and Katugampola also gave some right estimates of the Hermite–Hadamard type inequalities for the Katugampola fractional integrals.

**Theorem 1.2** ( [1]). *For $\rho > 0$ and $0 \le a < b$, suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is a differentiable function and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $f'$ is differentiable on $(a^\rho, b^\rho)$, then for any $\alpha > 0$*

$$\begin{aligned}&\left|\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}[{}^\rho\mathcal{K}_{a+}^\alpha f(b^\rho) + {}^\rho\mathcal{K}_{b-}^\alpha f(a^\rho)]\right| \\ \le&\frac{(b^\rho - a^\rho)^2}{2(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right) \sup_{\varsigma \in [a^\rho, b^\rho]} |f''(\varsigma)|,\end{aligned}$$

*where the fractional integrals are considered for the function $f(\chi^\rho)$ and evaluated at $a$ and $b$, respectively.*

**Theorem 1.3.** *Suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is a differentiable function on $(a^\rho, b^\rho)$ with $\rho > 0$, $0 \le a < b$, and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $|f'|$ is convex on $[a^\rho, b^\rho]$, then for any $\alpha > 0$*

$$\begin{aligned}&\left|\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}[{}^\rho\mathcal{K}_{a+}^\alpha f(b^\rho) + {}^\rho\mathcal{K}_{b-}^\alpha f(a^\rho)]\right| \\ \le&\frac{b^\rho - a^\rho}{2(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)[|f'(a^\rho)| + |f'(b^\rho)|],\end{aligned}$$

*where the fractional integrals are considered for the function $f(\chi^\rho)$ and evaluated at $a$ and $b$, respectively.*

In this paper, we will improve some conclusions of the literature [1]. Based on the Katugampola fractional integrals ${}^\rho\mathcal{K}_{a+}^\alpha f$ and ${}^\rho\mathcal{K}_{b-}^\alpha f$ with $f \in \mathfrak{X}_c^p(a, b)$, we will establish the Hermite–Hadamard type inequalities for convex functions and give their left estimates, and we will apply the newly-established inequalities to special means of real numbers. These newly-obtained inequalities generalize the corresponding results for the Riemann–Liouville fractional integrals, which can be demonstrated by taking limit $\rho \to 1$.

## 2. Hermite–Hadamard type inequalities for Katugampola fractional integrals

In this section, basing on the Katugampola fractional integrals ${}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)$ and ${}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)$ for any $\chi \in [a, b]$ with $f \in \mathfrak{X}_c^p(a, b)$, we establish the Hermite–Hadamard type inequalities for convex functions.

**Theorem 2.1.** *Suppose that* $f : [a^\rho, b^\rho] \to \mathbb{R}$ *is a positive function with* $\rho > 0$, $0 \le a < b$, *and* $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. *If* $f$ *is a convex function on* $[a, b]$, *then for any* $\alpha > 0$ *and any* $\chi \in [a, b]$, *we have*

$$\begin{aligned} f\left(\frac{\alpha}{\alpha+1}\frac{a^\rho+b^\rho}{2}+\frac{1}{\alpha+1}\chi^\rho\right) &\le \frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right] \\ &\le \frac{\alpha}{\alpha+1}\left[\frac{f(a^\rho)+f(b^\rho)}{2\alpha}+f(\chi^\rho)\right], \end{aligned} \tag{2.1}$$

*where the fractional integrals are considered for the function* $f(\chi^\rho)$ *and evaluated at* $a$ *and* $b$, *respectively.*

**Proof.** It easy to follow that

$$\begin{aligned} &\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right] \\ =&\frac{\rho\alpha}{2}\int_0^1 \tau^{\rho\alpha-1}\big[f(\tau^\rho a^\rho+(1-\tau^\rho)\chi^\rho)+f((1-\tau^\rho)\chi^\rho+\tau^\rho b^\rho)\big]d\tau. \end{aligned}$$

Then by the convexity of $f$ and Jensen's inequality, we obtain

$$\begin{aligned} &\frac{\rho\alpha}{2}\int_0^1 \tau^{\rho\alpha-1}\big[f(\tau^\rho a^\rho+(1-\tau^\rho)\chi^\rho)+f((1-\tau^\rho)\chi^\rho+\tau^\rho b^\rho)\big]d\tau \\ \ge&\, \rho\alpha\int_0^1 \tau^{\rho\alpha-1} f\left((1-\tau^\rho)\chi^\rho+\tau^\rho\frac{a^\rho+b^\rho}{2}\right)d\tau \\ \ge&\, \rho\alpha f\left(\frac{\int_0^1 \tau^{\rho\alpha-1}\left[(1-\tau^\rho)\chi^\rho+\tau^\rho\frac{a^\rho+b^\rho}{2}\right]d\tau}{\int_0^1\tau^{\rho\alpha-1}d\tau}\right)\int_0^1\tau^{\rho\alpha-1}d\tau \\ =&\, f\left(\frac{\alpha}{\alpha+1}\frac{a^\rho+b^\rho}{2}+\frac{1}{\alpha+1}\chi^\rho\right), \end{aligned}$$

which completes the proof of the first inequality of Theorem 2.1.

On the other hand, by the convexity of $f$ again, we have

$$\begin{aligned} &\frac{\rho\alpha}{2}\int_0^1 \tau^{\rho\alpha-1}\big[f(\tau^\rho a^\rho+(1-\tau^\rho)\chi^\rho)+f((1-\tau^\rho)\chi^\rho+\tau^\rho b^\rho)\big]d\tau \\ \le&\, \rho\alpha\int_0^1 \tau^{\rho\alpha-1}\left[(1-\tau^\rho)f(\chi^\rho)+\tau^\rho\frac{f(a^\rho)+f(b^\rho)}{2}\right]d\tau \\ =&\, \frac{\alpha}{\alpha+1}\left[\frac{f(a^\rho)+f(b^\rho)}{2\alpha}+f(\chi^\rho)\right], \end{aligned}$$

which completes the proof of the second inequality of Theorem 2.1. □

**Corollary 2.1.** *With the assumptions of Theorem 2.1 and taking* $\chi^\rho = \frac{a^\rho+b^\rho}{2}$, *we have*

$$f\left(\frac{a^\rho+b^\rho}{2}\right) \le \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}_{a+}^\alpha f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho\mathcal{K}_{b-}^\alpha f\left(\frac{a^\rho+b^\rho}{2}\right)\right]$$

$$
\begin{aligned}
&\leq \frac{\alpha}{\alpha+1}\left[\frac{f(a^\rho)+f(b^\rho)}{2\alpha}+f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\\
&\leq \frac{f(a^\rho)+f(b^\rho)}{2}.
\end{aligned}
$$

When taking the limits $\chi \to a$ and $\chi \to b$ respectively in the inequality (2.1), and when using the L'Hôpital rule, we deduce the following result.

**Corollary 2.2.** *With the assumptions of Theorem 2.1, we have*

$$
\begin{aligned}
f\left(\frac{a^\rho+b^\rho}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{(\alpha+2)a^\rho+\alpha b^\rho}{2(\alpha+1)}\right)+f\left(\frac{\alpha a^\rho+(\alpha+2)b^\rho}{2(\alpha+1)}\right)\right]\\
&\leq \frac{f(a^\rho)+f(b^\rho)}{4}+\frac{\rho^\alpha\Gamma(\alpha+1)}{4(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{a+}f(b^\rho)+{}^\rho\mathcal{K}^\alpha_{b-}f(a^\rho)\right]\\
&\leq \frac{f(a^\rho)+f(b^\rho)}{2}.
\end{aligned}
$$

# 3. Left estimates of Hermite–Hadamard type inequalities for Katugampola fractional integrals

Now we first establish an interesting equality.

**Lemma 3.1.** *Suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is differentiable with $\rho > 0$, $0 \leq a < b$, and $f \in \mathfrak{X}^p_c(a^\rho, b^\rho)$. If the Katugampola fractional integrals exist, then for any $\alpha > 0$ and any $\chi \in [a, b]$, the equality*

$$
\begin{aligned}
&\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{a+}f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{b-}f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\\
=&\frac{\rho}{2}\int_0^1(\tau^{\rho\alpha}-1)\tau^{\rho-1}\big[(\chi^\rho-a^\rho)f'(\tau^\rho a^\rho+(1-\tau^\rho)\chi^\rho)\\
&-(b^\rho-\chi^\rho)f'((1-\tau^\rho)\chi^\rho+\tau^\rho b^\rho)\big]d\tau \qquad (3.1)
\end{aligned}
$$

*holds, where the fractional integrals are considered for the function $f(\chi^\rho)$ and evaluated at $a$ and $b$, respectively.*

**Proof.** It is easy to see that

$$
\begin{aligned}
&\frac{\rho}{2}\int_0^1(\tau^{\rho\alpha}-1)\tau^{\rho-1}\big[(\chi^\rho-a^\rho)f'(\tau^\rho a^\rho+(1-\tau^\rho)\chi^\rho)-(b^\rho-\chi^\rho)f'((1-\tau^\rho)\chi^\rho+\tau^\rho b^\rho)\big]d\tau\\
=&\frac{\rho(\chi^\rho-a^\rho)}{2}\int_0^1(\tau^{\rho\alpha}-1)\tau^{\rho-1}f'(\tau^\rho a^\rho+(1-\tau^\rho)\chi^\rho)d\tau\\
&-\frac{\rho(b^\rho-\chi^\rho)}{2}\int_0^1(\tau^{\rho\alpha}-1)\tau^{\rho-1}f'((1-\tau^\rho)\chi^\rho+\tau^\rho b^\rho)d\tau.
\end{aligned}
$$

Integrating by parts, we obtain

$$
\begin{aligned}
&\frac{\rho(\chi^\rho-a^\rho)}{2}\int_0^1(\tau^{\rho\alpha}-1)\tau^{\rho-1}f'(\tau^\rho a^\rho+(1-\tau^\rho)\chi^\rho)d\tau\\
=&-\frac{f(\chi^\rho)}{2}+\frac{\rho\alpha}{2}\int_0^1\tau^{\rho\alpha-1}f(\tau^\rho a^\rho+(1-\tau^\rho)\chi^\rho)d\tau \qquad (3.2)
\end{aligned}
$$

$$= -\frac{f(\chi^\rho)}{2} + \frac{\rho^\alpha\Gamma(\alpha+1)}{2(\chi^\rho - a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{a+}f(\chi^\rho)\right]$$

and

$$\begin{aligned}
&\frac{\rho(b^\rho - \chi^\rho)}{2}\int_0^1 (\tau^{\rho\alpha}-1)\tau^{\rho-1}f'((1-\tau^\rho)\chi^\rho + \tau^\rho b^\rho)d\tau \\
&= \frac{f(\chi^\rho)}{2} - \frac{\rho\alpha}{2}\int_0^1 \tau^{\rho\alpha-1}f(\tau^\rho b^\rho + (1-\tau^\rho)\chi^\rho)d\tau \\
&= \frac{f(\chi^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho - \chi^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{b-}f(\chi^\rho)\right].
\end{aligned} \tag{3.3}$$

Subtracting the result in (3.3) from the result in (3.2) completes the proof of the Lemma 3.1. □

**Remark 3.1.** If taking the limits $\chi \to a$ and $\chi \to b$ respectively in the identity (3.1), by using the L'Hôpital rule and Corollary 3 in [9], we obtain

$$\begin{aligned}
&\frac{\rho^\alpha\Gamma(\alpha+1)\left[{}^\rho\mathcal{K}^\alpha_{b-}f(a^\rho)\right]}{2(b^\rho - a^\rho)^\alpha} - \frac{f(a^\rho)}{2} \\
&= -\frac{\rho(b^\rho - a^\rho)}{2}\int_0^1 (\tau^{\rho\alpha}-1)\tau^{\rho-1}f'((1-\tau^\rho)a^\rho + \tau^\rho b^\rho)d\tau
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&\frac{\rho^\alpha\Gamma(\alpha+1)\left[{}^\rho\mathcal{K}^\alpha_{a+}f(b^\rho)\right]}{2(b^\rho - a^\rho)^\alpha} - \frac{f(b^\rho)}{2} \\
&= \frac{\rho(b^\rho - a^\rho)}{2}\int_0^1 (\tau^{\rho\alpha}-1)\tau^{\rho-1}f'(\tau^\rho a^\rho + (1-\tau^\rho)b^\rho)d\tau.
\end{aligned} \tag{3.5}$$

Adding the identities (3.4) and (3.5), we arrive at

$$\begin{aligned}
&\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{a+}f(b^\rho) + {}^\rho\mathcal{K}^\alpha_{b-}f(a^\rho)\right] - \frac{f(a^\rho)+f(b^\rho)}{2} \\
&= \frac{\rho(b^\rho - a^\rho)}{2}\int_0^1 (\tau^{\rho\alpha}-1)\tau^{\rho-1}\left[f'(\tau^\rho a^\rho + (1-\tau^\rho)b^\rho) - f'((1-\tau^\rho)a^\rho + \tau^\rho b^\rho)\right]d\tau.
\end{aligned}$$

By the substitution of integral variables, it is easy to follow that

$$\begin{aligned}
&\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{a+}f(b^\rho) + {}^\rho\mathcal{K}^\alpha_{b-}f(a^\rho)\right] - \frac{f(a^\rho)+f(b^\rho)}{2} \\
&= \frac{\rho(b^\rho - a^\rho)}{2}\int_0^1 [\tau^{\rho\alpha} - (1-\tau^\rho)^\alpha]\tau^{\rho-1}f'(\tau^\rho a^\rho + (1-\tau^\rho)b^\rho)d\tau,
\end{aligned}$$

which recovers the equality in [1, Lemma 2.4].

**Remark 3.2.** If taking $\chi^\rho = \frac{a^\rho + b^\rho}{2}$ in the identity (3.1), then we obtain a midpoint equality

$$\begin{aligned}
&\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{a+}f\left(\frac{a^\rho+b^\rho}{2}\right) + {}^\rho\mathcal{K}^\alpha_{b-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right] - f\left(\frac{a^\rho+b^\rho}{2}\right) \\
&= \frac{\rho(b^\rho - a^\rho)}{4}\int_0^1 (\tau^{\rho\alpha}-1)\tau^{\rho-1}\left[f'\left(\frac{1+\tau^\rho}{2}a^\rho + \frac{(1-\tau^\rho)}{2}b^\rho\right)\right.
\end{aligned} \tag{3.6}$$

$$- f'\left(\frac{(1-\tau^\rho)}{2}a^\rho + \frac{1+\tau^\rho}{2}b^\rho\right)\bigg]d\tau.$$

Next, basing on the Katugampola fractional integrals, by virtue of the differentiability, the convexity, and Lemma 3.1, we will establish the left estimates of the Hermite–Hadamard type inequalities.

**Theorem 3.1.** *Suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is differentiable with $\rho > 0$, $0 \le a < b$, and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $f'$ is differentiable, then the inequality*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}_{a+}^\alpha f\left(\frac{a^\rho+b^\rho}{2}\right) + {}^\rho\mathcal{K}_{b-}^\alpha f\left(\frac{a^\rho+b^\rho}{2}\right)\right] - f\left(\frac{a^\rho+b^\rho}{2}\right)\right|\\ \le&\frac{\alpha(b^\rho-a^\rho)^2}{8(\alpha+2)}\sup_{\zeta\in\left(\frac{a^\rho}{2},\frac{b^\rho}{2}\right)}\left|f''(\zeta)\right|\end{aligned}$$

*holds for any $\alpha > 0$.*

**Proof.** Using the identity (3.6) and considering mean value theorem for the function $f'$, it is easy to see that

$$\begin{aligned}&\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}_{a+}^\alpha f\left(\frac{a^\rho+b^\rho}{2}\right) + {}^\rho\mathcal{K}_{b-}^\alpha f\left(\frac{a^\rho+b^\rho}{2}\right)\right] - f\left(\frac{a^\rho+b^\rho}{2}\right)\\ =&\frac{\rho(b^\rho-a^\rho)^2}{4}\int_0^1(1-\tau^{\rho\alpha})\tau^{2\rho-1}f''(\zeta(\tau))d\tau,\end{aligned}$$

where $\zeta(\tau) \in \left(\frac{a^\rho}{2}, \frac{b^\rho}{2}\right)$. Consequently, we find out

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}_{a+}^\alpha f\left(\frac{a^\rho+b^\rho}{2}\right) + {}^\rho\mathcal{K}_{b-}^\alpha f\left(\frac{a^\rho+b^\rho}{2}\right)\right] - f\left(\frac{a^\rho+b^\rho}{2}\right)\right|\\ \le&\frac{\rho(b^\rho-a^\rho)^2}{4}\int_0^1(1-\tau^{\rho\alpha})\tau^{2\rho-1}\left|f''(\zeta(\tau))\right|d\tau\\ \le&\frac{\alpha(b^\rho-a^\rho)^2}{8(\alpha+2)}\sup_{\zeta\in\left(\frac{a^\rho}{2},\frac{b^\rho}{2}\right)}\left|f''(\zeta)\right|.\end{aligned}$$

The proof of Theorem 3.1 is complete. □

**Theorem 3.2.** *Suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is differentiable with $\rho > 0$, $0 \le a < b$, and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $|f'|$ is convex, then the inequality*

$$\begin{aligned}&\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right] - f(\chi^\rho)\right|\\ \le&\frac{\alpha}{4(\alpha+1)(\alpha+2)}\big[(\alpha+1)(\chi^\rho-a^\rho)|f'(a^\rho)|\\ &+(\alpha+3)(b^\rho-a^\rho)|f'(\chi^\rho)| + (\alpha+1)(b^\rho-\chi^\rho)|f'(b^\rho)|\big]\end{aligned}\tag{3.7}$$

*holds for any $\alpha > 0$ and any $\chi \in [a, b]$.*

**Proof.** Using Lemma 3.1 and the convexity of the function $|f'|$, we have

$$\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right] - f(\chi^\rho)\right|$$

$$
\begin{aligned}
\leq & \frac{\rho(\chi^\rho - a^\rho)}{2} \int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1} |f'(\tau^\rho a^\rho + (1-\tau^\rho)\chi^\rho)| d\tau \\
& + \frac{\rho(b^\rho - \chi^\rho)}{2} \int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1} |f'((1-\tau^\rho)\chi^\rho + \tau^\rho b^\rho)| d\tau \\
\leq & \frac{\rho(\chi^\rho - a^\rho)}{2} \int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1} \left[\tau^\rho |f'(a^\rho)| + (1-\tau^\rho)|f'(\chi^\rho)|\right] d\tau \\
& + \frac{\rho(b^\rho - \chi^\rho)}{2} \int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1} \left[(1-\tau^\rho)|f'(\chi^\rho)| + \tau^\rho |f'(b^\rho)|\right] d\tau \\
= & \frac{\alpha}{4(\alpha+1)(\alpha+2)} \left[(\alpha+1)(\chi^\rho - a^\rho)|f'(a^\rho)| + (\alpha+3)(b^\rho - a^\rho)|f'(\chi^\rho)| \right. \\
& \left. + (\alpha+1)(b^\rho - \chi^\rho)|f'(b^\rho)|\right].
\end{aligned}
$$

Theorem 3.2 is thus proved. □

**Remark 3.3.** If taking the limits $\chi \to a$ and $\chi \to b$ respectively in the inequality (3.7), then

$$
\begin{aligned}
& \left| \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left[{}^\rho\mathcal{K}_{b-}^\alpha f(a^\rho)\right] - \frac{f(a^\rho)}{2} \right| \\
\leq & \frac{\alpha(b^\rho - a^\rho)}{4(\alpha+1)(\alpha+2)} \left[(\alpha+3)|f'(a^\rho)| + (\alpha+1)|f'(b^\rho)|\right]
\end{aligned}
$$

and

$$
\begin{aligned}
& \left| \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left[{}^\rho\mathcal{K}_{a+}^\alpha f(b^\rho)\right] - \frac{f(b^\rho)}{2} \right| \\
\leq & \frac{\alpha(b^\rho - a^\rho)}{4(\alpha+1)(\alpha+2)} \left[(\alpha+1)|f'(a^\rho)| + (\alpha+3)|f'(b^\rho)|\right].
\end{aligned}
$$

If taking $\chi^\rho = \frac{a^\rho + b^\rho}{2}$ in the inequality (3.7), then

$$
\begin{aligned}
& \left| \frac{2^{\alpha-1}\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left[{}^\rho\mathcal{K}_{a+}^\alpha f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho\mathcal{K}_{b-}^\alpha f\left(\frac{a^\rho + b^\rho}{2}\right)\right] - f\left(\frac{a^\rho + b^\rho}{2}\right) \right| \\
\leq & \frac{\alpha(b^\rho - a^\rho)}{8(\alpha+1)(\alpha+2)} \left[(\alpha+1)|f'(a^\rho)| + 2(\alpha+3)\left|f'\left(\frac{a^\rho + b^\rho}{2}\right)\right| + (\alpha+1)|f'(b^\rho)|\right].
\end{aligned} \tag{3.8}
$$

**Theorem 3.3.** *Suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is differentiable with $\rho > 0$, $0 \leq a < b$, and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $|f'|$ is convex, then the inequality*

$$
\begin{aligned}
& \left| \frac{2^{\alpha-1}\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} \left[{}^\rho\mathcal{K}_{a+}^\alpha f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho\mathcal{K}_{b-}^\alpha f\left(\frac{a^\rho + b^\rho}{2}\right)\right] - f\left(\frac{a^\rho + b^\rho}{2}\right) \right| \\
\leq & \frac{\alpha(b^\rho - a^\rho)}{4(\alpha+1)} [|f'(a^\rho)| + |f'(b^\rho)|]
\end{aligned} \tag{3.9}
$$

*holds for any $\alpha > 0$.*

**Proof.** Using the identity (3.6) and the convexity of the function $|f'|$, we acquire

$$
\left| \frac{2^{\alpha-1}\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} \left[{}^\rho\mathcal{K}_{a+}^\alpha f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho\mathcal{K}_{b-}^\alpha f\left(\frac{a^\rho + b^\rho}{2}\right)\right] - f\left(\frac{a^\rho + b^\rho}{2}\right) \right|
$$

$$
\begin{aligned}
\leq & \frac{\rho(b^\rho-a^\rho)}{4}\int_0^1(1-\tau^{\rho\alpha})\tau^{\rho-1}\left[\left|f'\left(\frac{1+\tau^\rho}{2}a^\rho+\frac{1-\tau^\rho}{2}b^\rho\right)\right|\right. \\
& \left.+\left|f'\left(\frac{1-\tau^\rho}{2}a^\rho+\frac{1+\tau^\rho}{2}b^\rho\right)\right|\right]d\tau \\
\leq & \frac{\rho(b^\rho-a^\rho)}{4}\int_0^1(1-\tau^{\rho\alpha})\tau^{\rho-1}[|f'(a^\rho)|+|f'(b^\rho)|]d\tau \\
= & \frac{\alpha(b^\rho-a^\rho)}{4(\alpha+1)}[|f'(a^\rho)|+|f'(b^\rho)|],
\end{aligned}
$$

which completes the proof of Theorem 3.3. □

**Theorem 3.4.** *Suppose that $f:[a^\rho,b^\rho]\to\mathbb{R}$ is differentiable with $\rho>0$, $0\leq a<b$, and $f\in\mathfrak{X}_c^p(a^\rho,b^\rho)$. If $|f'|^q$ for $q>1$ is convex, then the inequality*

$$
\begin{aligned}
& \left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right| \\
\leq & \frac{1}{2\alpha}\left[B\left(\frac{2q-r-1}{q-1},\frac{\rho q-r-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}\left\{(\chi^\rho-a^\rho)\left[B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)|f'(a^\rho)|^q\right.\right. \\
& \left.+\left(B\left(r+1,\frac{r+1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\right]^{1/q} \\
& +(b^\rho-\chi^\rho)\left[\left(B\left(r+1,\frac{r+1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\right. \\
& \left.\left.+B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)|f'(b^\rho)|^q\right]^{1/q}\right\} \qquad (3.10)
\end{aligned}
$$

*holds for any $\alpha>0$, $0\leq r\leq\min\{q,q(\rho-1)\}$ and $\chi\in[a,b]$, where $B$ denotes the beta function defined by*

$$
B(\omega,\lambda)=\int_0^1 u^{\omega-1}(1-u)^{\lambda-1}du,\quad \Re(\omega),\Re(\lambda)>0. \qquad (3.11)
$$

**Proof.** Employing Lemma 3.1, Hölder's inequality, and the convexity of the function $|f'|^q$ for $q>1$, we figure out

$$
\begin{aligned}
& \left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right| \\
\leq & \frac{\rho(\chi^\rho-a^\rho)}{2}\int_0^1(1-\tau^{\rho\alpha})\tau^{\rho-1}|f'(\tau^\rho a^\rho+(1-\tau^\rho)\chi^\rho)|d\tau \\
& +\frac{\rho(b^\rho-\chi^\rho)}{2}\int_0^1(1-\tau^{\rho\alpha})\tau^{\rho-1}|f'((1-\tau^\rho)\chi^\rho+\tau^\rho b^\rho)|d\tau \\
\leq & \frac{\rho(\chi^\rho-a^\rho)}{2}\left[\int_0^1(1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}\tau^{\frac{q(\rho-1)-r}{q-1}}d\tau\right]^{1-1/q} \\
& \times\left[\int_0^1(1-\tau^{\rho\alpha})^r\tau^r\big(\tau^\rho|f'(a^\rho)|^q+(1-\tau^\rho)|f'(\chi^\rho)|^q\big)d\tau\right]^{1/q} \\
& +\frac{\rho(b^\rho-\chi^\rho)}{2}\left[\int_0^1(1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}\tau^{\frac{q(\rho-1)-r}{q-1}}d\tau\right]^{1-1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[\int_0^1(1-\tau^{\rho\alpha})^r\tau^r\big((1-\tau^\rho)|f'(\chi^\rho)|^q+\tau^\rho|f'(b^\rho)|^q\big)d\tau\right]^{1/q}\\
=&\frac{1}{2\alpha}\left[B\left(\frac{2q-r-1}{q-1},\frac{\rho q-r-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}\Bigg\{(\chi^\rho-a^\rho)\Bigg[B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)|f'(a^\rho)|^q\\
&+\left(B\left(r+1,\frac{r+1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\Bigg]^{1/q}\\
&+(b^\rho-\chi^\rho)\Bigg[\left(B\left(r+1,\frac{r+1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\\
&+B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)|f'(b^\rho)|^q\Bigg]^{1/q}\Bigg\}.
\end{aligned}
$$

Theorem 3.4 is thus proved. □

**Remark 3.4.** Taking $r=0$ and $r=1$ in the inequality (3.10) respectively, we obtain

$$
\begin{aligned}
&\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{a+}f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{b-}f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right|\\
\le&\frac{\rho^{1/q}}{2\alpha^{1-1/q}(\rho+1)^{1/q}}\left[B\left(\frac{2q-1}{q-1},\frac{\rho q-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}\Big[(\chi^\rho-a^\rho)\big(|f'(a^\rho)|^q\\
&+\rho|f'(\chi^\rho)|^q\big)^{1/q}+(b^\rho-\chi^\rho)\big(\rho|f'(\chi^\rho)|^q+|f'(b^\rho)|^q\big)^{1/q}\Big] \qquad (3.12)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{a+}f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{b-}f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right|\\
\le&\frac{\rho^2\alpha(q+1)^{2-2/q}}{2^{1+1/q}[(\rho+2)(\rho+2+\rho\alpha)(2+\rho\alpha)]^{1/q}}\left[\frac{1}{(\rho q-2)[\rho q-2+\rho\alpha(q-1)]}\right]^{1-1/q}\\
&\times\Big\{(\chi^\rho-a^\rho)\big[2(2+\rho\alpha)|f'(a^\rho)|^q+\rho(\rho\alpha+\rho+2)|f'(\chi^\rho)|^q\big]^{1/q}\\
&+(b^\rho-\chi^\rho)\big[\rho(\rho\alpha+\rho+2)|f'(\chi^\rho)|^q+2(2+\rho\alpha)|f'(b^\rho)|^q\big]^{1/q}\Big\}.
\end{aligned}
$$

**Remark 3.5.** If taking $\chi^\rho=\frac{a^\rho+b^\rho}{2}$ in the inequality (3.10), then

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{a+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho\mathcal{K}^\alpha_{b-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]-f\left(\frac{a^\rho+b^\rho}{2}\right)\right|\\
\le&\frac{(b^\rho-a^\rho)}{4\alpha}\left[B\left(\frac{2q-r-1}{q-1},\frac{\rho q-r-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}\Bigg\{\Bigg[B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)|f'(a^\rho)|^q\\
&+\left(B\left(r+1,\frac{r+1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)\right)\left|f'\left(\frac{a^\rho+b^\rho}{2}\right)\right|^q\Bigg]^{1/q}\\
&+\Bigg[\left(B\left(r+1,\frac{r+1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)\right)\left|f'\left(\frac{a^\rho+b^\rho}{2}\right)\right|^q\\
&+B\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)|f'(b^\rho)|^q\Bigg]^{1/q}\Bigg\}.
\end{aligned}
$$

**Theorem 3.5.** *Suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is differentiable with $\rho > 0$, $0 \le a < b$, and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $|f'|^q$ for $q > 1$ is convex, then the inequality*

$$\begin{aligned}
&\left|\frac{\rho^\alpha \Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho - a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho - \chi^\rho)^\alpha}\right] - f(\chi^\rho)\right| \\
\le& \frac{1}{2\alpha}\left[B\left(\frac{2q-r-1}{q-1}, \frac{\rho(q-r)+r-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q} \\
&\times \Bigg\{(\chi^\rho - a^\rho)\bigg[B\left(r+1, \frac{\rho(r+1)-r+1}{\rho\alpha}\right)|f'(a^\rho)|^q \\
&+ \left(B\left(r+1, \frac{\rho r - r + 1}{\rho\alpha}\right) - B\left(r+1, \frac{\rho(r+1)-r+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\bigg]^{1/q} \\
&+ (b^\rho - \chi^\rho)\bigg[\left(B\left(r+1, \frac{\rho r - r + 1}{\rho\alpha}\right) - B\left(r+1, \frac{\rho(r+1)-r+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q \\
&+ B\left(r+1, \frac{\rho(r+1)-r+1}{\rho\alpha}\right)|f'(b^\rho)|^q\bigg]^{1/q}\Bigg\} \qquad (3.13)
\end{aligned}$$

*holds for $\alpha > 0$, $0 \le r \le q$, and $\chi \in [a, b]$, where $B$ denotes the beta function defined in* (3.11).

**Proof.** Employing Lemma 3.1, Hölder's inequality, and the convexity of the function $|f'|^q$ for $q > 1$, we have

$$\begin{aligned}
&\left|\frac{\rho^\alpha \Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho - a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho - \chi^\rho)^\alpha}\right] - f(\chi^\rho)\right| \\
\le& \frac{\rho(\chi^\rho - a^\rho)}{2}\int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1}|f'(\tau^\rho a^\rho + (1-\tau^\rho)\chi^\rho)|d\tau \\
&+ \frac{\rho(b^\rho - \chi^\rho)}{2}\int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1}|f'((1-\tau^\rho)\chi^\rho + \tau^\rho b^\rho)|d\tau \\
\le& \frac{\rho(\chi^\rho - a^\rho)}{2}\left[\int_0^1 (1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}\tau^{\frac{(\rho-1)(q-r)}{q-1}}d\tau\right]^{1-1/q} \\
&\times \left[\int_0^1 (1-\tau^{\rho\alpha})^r \tau^{(\rho-1)r}\big(\tau^\rho|f'(a^\rho)|^q + (1-\tau^\rho)|f'(\chi^\rho)|^q\big)d\tau\right]^{1/q} \\
&+ \frac{\rho(b^\rho - \chi^\rho)}{2}\left[\int_0^1 (1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}\tau^{\frac{(\rho-1)(q-r)}{q-1}}d\tau\right]^{1-1/q} \\
&\times \left[\int_0^1 (1-\tau^{\rho\alpha})^r \tau^{(\rho-1)r}\big((1-\tau^\rho)|f'(\chi^\rho)|^q + \tau^\rho|f'(b^\rho)|^q\big)d\tau\right]^{1/q} \\
=& \frac{1}{2\alpha}\left[B\left(\frac{2q-r-1}{q-1}, \frac{\rho(q-r)+r-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q} \\
&\times \Bigg\{(\chi^\rho - a^\rho)\bigg[B\left(r+1, \frac{\rho(r+1)-r+1}{\rho\alpha}\right)|f'(a^\rho)|^q \\
&+ \left(B\left(r+1, \frac{\rho r - r + 1}{\rho\alpha}\right) - B\left(r+1, \frac{\rho(r+1)-r+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\bigg]^{1/q} \\
&+ (b^\rho - \chi^\rho)\bigg[\left(B\left(r+1, \frac{\rho r - r + 1}{\rho\alpha}\right) - B\left(r+1, \frac{\rho(r+1)-r+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q
\end{aligned}$$

$$+B\left(r+1,\frac{\rho(r+1)-r+1}{\rho\alpha}\right)|f'(b^\rho)|^q\Bigg]^{1/q}\Bigg\}.$$

The proof of Theorem 3.5 is complete. □

**Remark 3.6.** Letting $r=1$ and $r=q$ in the inequality (3.13) respectively gives

$$\begin{aligned}&\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{a+}f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{b-}f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right|\\ \le&\frac{\alpha}{2^{1+1/q}(\alpha+1)(\alpha+2)^{1/q}}\big\{(\chi^\rho-a^\rho)\big[(\alpha+1)|f'(a^\rho)|^q+(\alpha+3)|f'(\chi^\rho)|^q\big]^{1/q}\\ &+(b^\rho-\chi^\rho)\big[(\alpha+3)|f'(\chi^\rho)|^q+(\alpha+1)|f'(b^\rho)|^q\big]^{1/q}\big\}\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{a+}f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{b-}f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right|\\ \le&\frac{\rho^{1-1/q}}{2\alpha^{1/q}}\Bigg\{(\chi^\rho-a^\rho)\Bigg[B\left(q+1,\frac{\rho(q+1)-q+1}{\rho\alpha}\right)|f'(a^\rho)|^q\\ &+\left(B\left(q+1,\frac{\rho q-q+1}{\rho\alpha}\right)-B\left(q+1,\frac{\rho(q+1)-q+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\Bigg]^{1/q}\\ &+(b^\rho-\chi^\rho)\Bigg[\left(B\left(q+1,\frac{\rho q-q+1}{\rho\alpha}\right)-B\left(q+1,\frac{\rho(q+1)-q+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\\ &+B\left(q+1,\frac{\rho(q+1)-q+1}{\rho\alpha}\right)|f'(b^\rho)|^q\Bigg]^{1/q}\Bigg\}.\end{aligned}$$

It is clear that, when setting $r=0$ in the inequality (3.13), we can also obtain the inequality (3.12).

**Remark 3.7.** Taking $\chi^\rho=\frac{a^\rho+b^\rho}{2}$ in the inequality (3.12), we obtain

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{a+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho\mathcal{K}^\alpha_{b-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]-f\left(\frac{a^\rho+b^\rho}{2}\right)\right|\\ \le&\frac{(b^\rho-a^\rho)}{4\alpha}\left[B\left(\frac{2q-r-1}{q-1},\frac{\rho(q-r)+r-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}\Bigg\{\Bigg[B\left(r+1,\frac{\rho(r+1)-r+1}{\rho\alpha}\right)|f'(a^\rho)|^q\\ &+\left(B\left(r+1,\frac{\rho r-r+1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho(r+1)-r+1}{\rho\alpha}\right)\right)\left|f'\left(\frac{a^\rho+b^\rho}{2}\right)\right|^q\Bigg]^{1/q}\\ &+\Bigg[\left(B\left(r+1,\frac{\rho r-r+1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho(r+1)-r+1}{\rho\alpha}\right)\right)\left|f'\left(\frac{a^\rho+b^\rho}{2}\right)\right|^q\\ &+B\left(r+1,\frac{\rho(r+1)-r+1}{\rho\alpha}\right)|f'(b^\rho)|^q\Bigg]^{1/q}\Bigg\}.\end{aligned}$$

**Theorem 3.6.** *Suppose that $f:[a^\rho,b^\rho]\to\mathbb{R}$ is differentiable with $\rho>0$, $0\le a<b$, and $f\in\mathfrak{X}^p_c(a^\rho,b^\rho)$. If $|f'|^q$ for $q>1$ is convex, then the inequality*

$$\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{a+}f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{b-}f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right|$$

$$\leq \frac{\rho^{1/q}}{2\alpha^{1-1/q}[(\rho r - r + 1)(\rho(r+1) - r + 1)]^{1/q}}$$
$$\times \left[B\left(\frac{2q-1}{q-1}, \frac{(\rho-1)(q-r)+q-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}$$
$$\times \big\{(\chi^\rho - a^\rho)\big[(\rho r - r + 1)|f'(a^\rho)|^q + \rho|f'(\chi^\rho)|^q\big]^{1/q}$$
$$+ (b^\rho - \chi^\rho)\big[\rho|f'(\chi^\rho)|^q + (\rho r - r + 1)|f'(b^\rho)|^q\big]^{1/q}\big\} \tag{3.14}$$

*holds for $\alpha > 0$, $0 \leq r \leq q$, and $\chi \in [a,b]$, where $B$ denotes the beta function defined in* (3.11)*.*

**Proof.** Employing Lemma 3.1, Hölder's inequality, and the convexity of the function $|f'|^q$ for $q > 1$, we arrive at

$$\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho - a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho - \chi^\rho)^\alpha}\right] - f(\chi^\rho)\right|$$
$$\leq \frac{\rho(\chi^\rho - a^\rho)}{2}\int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1}|f'(\tau^\rho a^\rho + (1-\tau^\rho)\chi^\rho)|d\tau$$
$$+ \frac{\rho(b^\rho - \chi^\rho)}{2}\int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1}|f'((1-\tau^\rho)\chi^\rho + \tau^\rho b^\rho)|d\tau$$
$$\leq \frac{\rho(\chi^\rho - a^\rho)}{2}\left[\int_0^1 (1-\tau^{\rho\alpha})^{\frac{q}{q-1}}\tau^{\frac{(\rho-1)(q-r)}{q-1}}d\tau\right]^{1-1/q}$$
$$\times \left[\int_0^1 \tau^{(\rho-1)r}\big(\tau^\rho|f'(a^\rho)|^q + (1-\tau^\rho)|f'(\chi^\rho)|^q\big)d\tau\right]^{1/q}$$
$$+ \frac{\rho(b^\rho - \chi^\rho)}{2}\left[\int_0^1 (1-\tau^{\rho\alpha})^{\frac{q}{q-1}}\tau^{\frac{(\rho-1)(q-r)}{q-1}}d\tau\right]^{1-1/q}$$
$$\times \left[\int_0^1 \tau^{(\rho-1)r}\big((1-\tau^\rho)|f'(\chi^\rho)|^q + \tau^\rho|f'(b^\rho)|^q\big)d\tau\right]^{1/q}$$
$$= \frac{\rho^{1/q}}{2\alpha^{1-1/q}[(\rho r - r + 1)(\rho(r+1) - r + 1)]^{1/q}}$$
$$\times \left[B\left(\frac{2q-1}{q-1}, \frac{(\rho-1)(q-r)+q-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}$$
$$\times \big\{(\chi^\rho - a^\rho)\big[(\rho r - r + 1)|f'(a^\rho)|^q + \rho|f'(\chi^\rho)|^q\big]^{1/q}$$
$$+ (b^\rho - \chi^\rho)\big[\rho|f'(\chi^\rho)|^q + (\rho r - r + 1)|f'(b^\rho)|^q\big]^{1/q}\big\}.$$

The proof of Theorem 3.6 is thus complete. □

**Remark 3.8.** Assuming $r = 1$ and $r = q$ in the inequality (3.14) respectively results in

$$\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho - a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho - \chi^\rho)^\alpha}\right] - f(\chi^\rho)\right|$$
$$\leq \frac{1}{2^{1+1/q}\alpha^{1-1/q}}\left[B\left(\frac{2q-1}{q-1}, \frac{1}{\alpha}\right)\right]^{1-1/q}$$
$$\times \big\{(\chi^\rho - a^\rho)\big[|f'(a^\rho)|^q + |f'(\chi^\rho)|^q\big]^{1/q} + (b^\rho - \chi^\rho)\big[|f'(\chi^\rho)|^q + |f'(b^\rho)|^q\big]^{1/q}\big\}$$

and

$$\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{a+}f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{b-}f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right|$$
$$\leq\frac{\rho^{1/q}}{2\alpha^{1-1/q}[(\rho q-q+1)(\rho(q+1)-q+1)]^{1/q}}\left[B\left(\frac{2q-1}{q-1},\frac{1}{\rho\alpha}\right)\right]^{1-1/q}$$
$$\times\big\{(\chi^\rho-a^\rho)\big[(\rho q-q+1)|f'(a^\rho)|^q+\rho|f'(\chi^\rho)|^q\big]^{1/q}$$
$$+(b^\rho-\chi^\rho)\big[\rho|f'(\chi^\rho)|^q+(\rho q-q+1)|f'(b^\rho)|^q\big]^{1/q}\big\}.$$

It is obvious that taking $r=0$ in the inequality (3.14) leads to the inequality (3.12).

**Remark 3.9.** Letting $\chi^\rho=\frac{a^\rho+b^\rho}{2}$ in the inequality (3.14) yields

$$\left|\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{a+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho\mathcal{K}^\alpha_{b-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]-f\left(\frac{a^\rho+b^\rho}{2}\right)\right|$$
$$\leq\frac{\rho^{1/q}(b^\rho-a^\rho)}{4\alpha^{1-1/q}[(\rho r-r+1)(\rho(r+1)-r+1)]^{1/q}}\left[B\left(\frac{2q-1}{q-1},\frac{(\rho-1)(q-r)+q-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}$$
$$\times\left\{\left[(\rho r-r+1)|f'(a^\rho)|^q+\rho\left|f'\left(\frac{a^\rho+b^\rho}{2}\right)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\rho\left|f'\left(\frac{a^\rho+b^\rho}{2}\right)\right|^q+(\rho r-r+1)|f'(b^\rho)|^q\right]^{1/q}\right\}.$$

**Theorem 3.7.** *Suppose that $f:[a^\rho,b^\rho]\to\mathbb{R}$ is differentiable with $\rho>0$, $0\leq a<b$, and $f\in\mathfrak{X}^p_c(a^\rho,b^\rho)$. If $|f'|^q$ for $q>1$ is convex, then the inequality*

$$\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{a+}f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{b-}f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right|$$
$$\leq\frac{1}{2\alpha}\left[B\left(\frac{2q-r-1}{q-1},\frac{\rho q-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}\left\{(\chi^\rho-a^\rho)\left[B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)|f'(a^\rho)|^q\right.\right.$$
$$\left.+\left(B\left(r+1,\frac{1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\right]^{1/q}+(b^\rho-\chi^\rho)\left[\left(B\left(r+1,\frac{1}{\rho\alpha}\right)\right.\right.$$
$$\left.\left.\left.-B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q+B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)|f'(b^\rho)|^q\right]^{1/q}\right\} \tag{3.15}$$

*holds for $\alpha>0$, $0\leq r\leq q$, and $\chi\in[a,b]$, where $B$ denotes the beta function defined in* (3.11).

**Proof.** Employing Lemma 3.1, Hölder's inequality, and the convexity of the function $|f'|^q$ for $q>1$, we have

$$\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{a+}f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{b-}f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right|$$
$$\leq\frac{\rho(\chi^\rho-a^\rho)}{2}\int_0^1(1-\tau^{\rho\alpha})\tau^{\rho-1}|f'(\tau^\rho a^\rho+(1-\tau^\rho)\chi^\rho)|d\tau$$
$$+\frac{\rho(b^\rho-\chi^\rho)}{2}\int_0^1(1-\tau^{\rho\alpha})\tau^{\rho-1}|f'((1-\tau^\rho)\chi^\rho+\tau^\rho b^\rho)|d\tau$$

$$
\begin{aligned}
\leq&\frac{\rho(\chi^\rho-a^\rho)}{2}\left[\int_0^1\tau^{\frac{q(\rho-1)}{q-1}}(1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}d\tau\right]^{1-1/q}\\
&\times\left[\int_0^1(1-\tau^{\rho\alpha})^r\big(\tau^\rho|f'(a^\rho)|^q+(1-\tau^\rho)|f'(\chi^\rho)|^q\big)d\tau\right]^{1/q}\\
&+\frac{\rho(b^\rho-\chi^\rho)}{2}\left[\int_0^1\tau^{\frac{q(\rho-1)}{q-1}}(1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}d\tau\right]^{1-1/q}\\
&\times\left[\int_0^1(1-\tau^{\rho\alpha})^r\big((1-\tau^\rho)|f'(\chi^\rho)|^q+\tau^\rho|f'(b^\rho)|^q\big)d\tau\right]^{1/q}\\
=&\frac{1}{2\alpha}\left[B\left(\frac{2q-r-1}{q-1},\frac{\rho q-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}\left\{(\chi^\rho-a^\rho)\left[B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)|f'(a^\rho)|^q\right.\right.\\
&\left.+\left(B\left(r+1,\frac{1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\right]^{1/q}\\
&+(b^\rho-\chi^\rho)\left[\left(B\left(r+1,\frac{1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)\right),|f'(\chi^\rho)|^q\right.\\
&\left.\left.+B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)|f'(b^\rho)|^q\right]^{1/q}\right\},
\end{aligned}
$$

which proves Theorem 3.7. □

**Remark 3.10.** Assuming $r=1$ and $r=q$ in the inequality (3.15) respectively shows

$$
\begin{aligned}
&\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right|\\
\leq&\frac{\rho^2\alpha(q-1)^{2-2/q}}{2[(\rho+1)(\rho\alpha+1)(\rho\alpha+\rho+1)]^{1/q}}\left[\frac{1}{(\rho q-1)[\rho q-1+\rho\alpha(q-1)]}\right]^{1-1/q}\\
&\times\big\{(\chi^\rho-a^\rho)\big[(1+\rho\alpha)|f'(a^\rho)|^q+\rho(\rho\alpha+\rho+2)|f'(\chi^\rho)|^q\big]^{1/q}\\
&+(b^\rho-\chi^\rho)\big[\rho(\rho\alpha+\rho+2)|f'(\chi^\rho)|^q+(1+\rho\alpha)|f'(b^\rho)|^q\big]^{1/q}\big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}_{a+}^\alpha f(\chi^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}_{b-}^\alpha f(\chi^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]-f(\chi^\rho)\right|\\
\leq&\frac{\rho^{1-1/q}}{2\alpha^{1/q}}\left(\frac{q-1}{\rho q-1}\right)^{1-1/q}\left\{(\chi^\rho-a^\rho)\left[B\left(q+1,\frac{\rho+1}{\rho\alpha}\right)|f'(a^\rho)|^q\right.\right.\\
&\left.+\left(B\left(q+1,\frac{1}{\rho\alpha}\right)-B\left(q+1,\frac{\rho+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\right]^{1/q}\\
&+(b^\rho-\chi^\rho)\left[\left(B\left(q+1,\frac{1}{\rho\alpha}\right)-B\left(q+1,\frac{\rho+1}{\rho\alpha}\right)\right)|f'(\chi^\rho)|^q\right.\\
&\left.\left.+B\left(q+1,\frac{\rho+1}{\rho\alpha}\right)|f'(b^\rho)|^q\right]^{1/q}\right\}.
\end{aligned}
$$

It is clear that taking $r=0$ in the inequality (3.15) immediately leads to the inequality (3.12).

**Remark 3.11.** Taking $\chi^\rho = \frac{a^\rho+b^\rho}{2}$ in the inequality (3.15), we obtain

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{a+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho\mathcal{K}^\alpha_{b-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]-f\left(\frac{a^\rho+b^\rho}{2}\right)\right|\\
\le&\frac{(b^\rho-a^\rho)}{4\alpha}\left[B\left(\frac{2q-r-1}{q-1},\frac{\rho q-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}\left\{\left[B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)|f'(a^\rho)|^q\right.\right.\\
&+\left.\left(B\left(r+1,\frac{1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)\right)\left|f'\left(\frac{a^\rho+b^\rho}{2}\right)\right|^q\right]^{1/q}\\
&+\left[\left(B\left(r+1,\frac{1}{\rho\alpha}\right)-B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)\right)\left|f'\left(\frac{a^\rho+b^\rho}{2}\right)\right|^q\right.\\
&\left.\left.+B\left(r+1,\frac{\rho+1}{\rho\alpha}\right)|f'(b^\rho)|^q\right]^{1/q}\right\}.
\end{aligned}$$

# 4. Applications to special means

In this section, we will consider the following special means for arbitrary real numbers $\alpha, \beta$:

$$\begin{aligned}
H(\alpha,\beta)&=\frac{2\alpha\beta}{\alpha+\beta},\quad \alpha,\beta\in\mathbb{R}\setminus\{0\};\\
A(\alpha,\beta)&=\frac{\alpha+\beta}{2},\quad \alpha,\beta\in\mathbb{R};\\
L(\alpha,\beta)&=\frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|},\quad \alpha,\beta\in\mathbb{R},\quad |\alpha|\ne|\beta|,\quad \alpha\beta\ne 0;\\
L_n(\alpha,\beta)&=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{1/n},\quad n\in\mathbb{Z}\setminus\{-1,0\},\quad \alpha,\beta\in\mathbb{R},\quad \alpha\ne\beta.
\end{aligned}$$

For more information on special means, please refer to [12,14] and references cited therein.

**Theorem 4.1.** *Suppose that $b > a > 0$, $\rho > 0$, and $n \in \mathbb{Z}\setminus\{-1,-2\}$. Then*

$$\left|\frac{\rho(b^{n+\rho+1}-a^{n+\rho+1})}{(n+\rho+1)(b^\rho-a^\rho)}-2A^{n+1}(a^\rho,b^\rho)\right|\le\frac{(n+1)(b^\rho-a^\rho)}{6}\left[A(a^{n\rho},b^{n\rho})+2A^n(a^\rho,b^\rho)\right]$$

*and*

$$\left|\frac{\rho(b^{n+\rho+1}-a^{n+\rho+1})}{(n+\rho+1)(b^\rho-a^\rho)}-A^{n+1}(a^\rho,b^\rho)\right|\le\frac{(n+1)(b^\rho-a^\rho)A(a^{n\rho},b^{n\rho})}{4}.$$

*In particular, if $\rho = 1$, then*

$$\left|L^{n+1}_{n+1}(a,b)-2A^{n+1}(a,b)\right|\le\frac{(n+1)(b-a)}{6}\left[A(a^n,b^n)+2A^n(a,b)\right]$$

*and*

$$\left|L^{n+1}_{n+1}(a,b)-A^{n+1}(a,b)\right|\le\frac{(n+1)(b-a)A(a^n,b^n)}{4}.$$

**Proof.** This follows from applying $f(x)=\frac{x^{n+1}}{n+1}$ and $\alpha = 1$ to the inequalities (3.8) and (3.9). □

**Theorem 4.2.** *For $b > a > 0$, we have*

$$\left|2A^{-1}(a,b) - L^{-1}(a,b\right| \leq \frac{b-a}{6}\left[H^{-1}(a^2,b^2) + 2A^{-2}(a,b)\right]$$

*and*

$$\left|A^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{(b-a)H^{-1}(a^2,b^2)}{4}.$$

**Proof.** This follows from applying $f(x) = -\frac{1}{x}$, $\alpha = 1$, and $\rho = 1$ to the inequalities (3.8) and (3.9). □

# 5. Conclusions

In this paper, we generalized inequalities for convex functions as in [1]. Making use of these generalizations, we derived some new inequalities by choosing specific parameters. These newly-established results are refinements and extensions of earlier results. It is an interesting and new problem that the upcoming researchers can offer similar inequalities for generalized convex functions via different fractional integrals in their future research.

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