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ADMISSIBLE PERTURBATIONS OF THE THREE-DIMENSIONAL HINDMARSH – ROSE NEURON MODEL

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Abstract For the autonomous Hindmarsh – Rose system the set of admissibly perturbed nonautonomous systems have been obtained. Reflecting functions of the Hindmarsh – Rose system and admissibly perturbed system are coinciding. This allows to investigate the admissibly perturbed systems using outcomes of researches of the well-known Hindmarsh – Rose system and the theory of reflecting function. The results are illustrated by numerical examples.

Keywords Reflecting function, admissible perturbation, bifurcation diagram, periodic attractor, strange attractor.

MSC(2010) 34C23, 34C25, 34C60, 34D10.

1. Introduction

Researchers often use systems of ordinary differential equations when modeling various processes and they face certain difficulties, since most of differential systems cannot be integrated in quadratures (in finite terms). In such cases, it is possible to study the properties of solutions of differential systems on the qualitative level. At the same time, complicated systems can be studied based on well-studied simpler systems. In such cases, the method of the reflecting function of professor V.I. Mironenko can help. The theory of reflecting function has been used for studying the qualitative behavior of solutions of differential systems by many authors: M.S. Belokurskii, V.A. Bel'skii, S.V. Maiorovskaya, V.I. Mironenko, V.V. Mironenko, E.V. Varenikova, J. Zhou, Z. Zhou et al. [3, 4, 9, 10, 14, 17, 18, 25, 29, 30].

In Appendix A we present the fundamentals of the theory of the reflecting function.

Since many qualitative properties of solutions of systems with the same reflecting function are common, it is reasonable to look for perturbations that do not change the reflecting function (the so-called *admissible perturbations*) of known (well-studied) systems (see [16, 19, 21]). Thus, we will know which perturbations will not change the qualitative properties of the solutions inherent in the solutions of the original unperturbed system.

This opens up new opportunities for researchers both in modeling processes occurring in the real world and in studying new (not yet studied) systems of ODEs.

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2. Main Results

The object of this study is the Hindmarsh – Rose neuron model [7], which written as an eight-parameter set of three autonomous first-order nonlinear ordinary differential equations given by

$$\begin{aligned} \dot{x} &= y - ax^3 + bx^2 - z + I, \\ \dot{y} &= c - dx^2 - y, \\ \dot{z} &= r\left(s\left(x-\alpha\right)-z\right); \quad x, y, z, a, b, c, d, I, r, s, \alpha \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where a, b, c, d, I, r, s, α are model parameters. The state variable x represents the membrane potential, y is a recovery variable associated with the fast current, and z is a slowly changing adaptation current. This means that x tells us about the dynamics of the membrane potential in the axon of a neuron, while y and z describe the exchanges of ions through the neuron membrane by means of fast and slow ions channels, respectively. Model (2.1) is a well-studied system by many authors; see, for example, [1, 2, 5, 6, 11, 22–24, 26, 27].

The aim of this work is to find admissible perturbations (which do not change the reflecting function) of the system (2.1). On admissible perturbations of other models, see [16, 19, 21].

To search for admissible perturbations, we use Theorem 1 from [14], which we formulate here as the following lemma.

Lemma 2.1. *Let the vector functions $\Delta_i(t,x)$ $(i=\overline{1,m},$ where $m \in \mathbb{N}$ or $m=\infty)$ be solutions of the equation $\frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial x} X - \frac{\partial X}{\partial x}\Delta = 0$ and $\alpha_i(t)$ be any scalar continuous odd functions. Then the reflecting function of every perturbed system of the form $\dot{x} = X(t,x) + \sum_{i=1}^{m} \alpha_i(t)\Delta_i(t,x), \quad t \in \mathbb{R},\ x \in D \subset \mathbb{R}^n$ is equal to the reflecting function of system $\dot{x} = X(t,x), \quad t \in \mathbb{R},\ x \in D \subset \mathbb{R}^n$.*

We search admissible perturbations in the following form

$$\Delta \cdot \alpha(t) = \left(\sum_{i+j+k=0}^{n} q_{ijk} x^i y^j z^k, \sum_{i+j+k=0}^{n} r_{ijk} x^i y^j z^k, \sum_{i+j+k=0}^{n} s_{ijk} x^i y^j z^k \right)^{\mathrm{T}} \cdot \alpha(t),$$

where $q_{ijk}, r_{ijk}, s_{ijk} \in \mathbb{R}$, $i, j, k, n \in \mathbb{N} \cup \{0\}$; $\alpha(t)$ is an arbitrary scalar odd continuous function. We were able to obtain the next result.

Theorem 2.1. *Suppose that $\alpha_i(t)$ $(i=\overline{1,3})$ are arbitrary scalar odd continuous functions. Then:*

1. *the reflecting function of the system* (2.1) *is equal to reflecting function of the system*

$$\begin{aligned} \dot{x} &= \big((b-ax)\,x^2 + y - z + I\big)\left(1+\alpha_1(t)\right), \\ \dot{y} &= \big(c - dx^2 - y\big)\left(1+\alpha_1(t)\right), \\ \dot{z} &= r\left(s\left(x-\alpha\right)-z\right)\left(1+\alpha_1(t)\right); \end{aligned} \tag{2.2}$$

2. *for $d=0$ and $r=1$, the reflecting function of the system* (2.1) *is equal to reflecting function of the system*

$$\dot{x} = \big((b-ax)\,x^2 + y - z + I\big)\left(1+\alpha_1\left(t\right)\right),$$

$$\begin{aligned} \dot{y} &= (c-y)\,(1+\alpha_1(t)-\alpha_2(t))\,, \\ \dot{z} &= (s\,(x-\alpha)-z)\,(1+\alpha_1(t))+(y-c)\,\alpha_2(t)\,; \end{aligned} \tag{2.3}$$

3. *for* $s=0$ *and* $r=1$*, the reflecting function of the system* (2.1) *is equal to reflecting function of the system*

$$\begin{aligned} \dot{x} &= \big((b-ax)\,x^2+y-z+I\big)\,(1+\alpha_1(t))\,, \\ \dot{y} &= \big(c-dx^2-y\big)\,(1+\alpha_1(t))+z\alpha_2(t)\,, \\ \dot{z} &= z\,(\alpha_2(t)-\alpha_1(t)-1)\,; \end{aligned}$$

4. *for* $d=s=0$ *and* $r=1$*, the reflecting function of the system* (2.1) *is equal to reflecting function of the system* [20]

$$\begin{aligned} \dot{x} &= \big((b-ax)\,x^2+y-z+I\big)\,(1+\alpha_1(t))\,, \\ \dot{y} &= (c-y)\,(1+\alpha_1(t)-\alpha_2(t))+z\alpha_3(t)\,, \\ \dot{z} &= z\,(\alpha_3(t)-\alpha_1(t)-1)+(y-c)\,\alpha_2(t)\,. \end{aligned}$$

Proof. The proof follows from Lemma 2.1 by consequentially checking the identity $\frac{\partial\Delta}{\partial t}+\frac{\partial\Delta}{\partial x}X(t,x)-\frac{\partial X(t,x)}{\partial x}\Delta=0$ for each vector-factor Δ multiplying $\alpha_i(t)$ on the right-hand side of the perturbed system; where $X(t,x)$ is the right-hand side of the system (2.1).

Let us show it for the case 2 (for the system (2.3)). Let $d=0$ and $r=1$ then the right-hand side of the system (2.1) is $X=\big(y-ax^3+bx^2-z+I,c-y,s(x-\alpha)-z\big)^{\mathrm{T}}$; for the right-hand side of the system (2.3) the vector-factor multiplying $\alpha_1(t)$ is $\Delta_1=\big((b-ax)x^2+y-z+I,\ c-y,\ s(x-\alpha)-z\big)^{\mathrm{T}}$, the vector-factor multiplying $\alpha_2(t)$ is $\Delta_2=(0,\ y-c,\ y-c)^{\mathrm{T}}$. So, for Δ_1 we have

$$\begin{aligned} &\frac{\partial\Delta_1}{\partial t}+\frac{\partial\Delta_1}{\partial x}X-\frac{\partial X}{\partial x}\Delta_1 \\ &\equiv \begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}-3ax^2+2bx & 1 & -1\\ 0 & -1 & 0\\ s & 0 & -1\end{pmatrix}\begin{pmatrix}y-ax^3+bx^2-z+I\\ c-y\\ s\,(x-\alpha)-z\end{pmatrix} \\ &\quad-\begin{pmatrix}2bx-3ax^2 & 1 & -1\\ 0 & -1 & 0\\ s & 0 & -1\end{pmatrix}\begin{pmatrix}(b-ax)\,x^2+y-z+I\\ c-y\\ s\,(x-\alpha)-z\end{pmatrix}\equiv\begin{pmatrix}0\\0\\0\end{pmatrix}; \end{aligned}$$

for Δ_2 we have

$$\begin{aligned} &\frac{\partial\Delta_2}{\partial t}+\frac{\partial\Delta_2}{\partial x}X-\frac{\partial X}{\partial x}\Delta_2 \\ &\equiv \begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}0&0&0\\0&1&0\\0&1&0\end{pmatrix}\begin{pmatrix}y-ax^3+bx^2-z+I\\ c-y\\ s\,(x-\alpha)-z\end{pmatrix} \end{aligned}$$

$$-\begin{pmatrix} 2bx-3ax^2 & 1 & -1 \\ 0 & -1 & 0 \\ s & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ y-c \\ y-c \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then it follows from Lemma 2.1 that the reflecting function of the system $\dot{x} = X$ is equal to reflecting function of the system $\dot{x} = X + \Delta_1\alpha_1(t) + \Delta_2\alpha_2(t)$, where $\alpha_i(t)$ $(i = \overline{1,2})$ are arbitrary scalar odd continuous functions. This proves the truth of the second statement of the theorem.

The rest of the cases are proved analogously. □

Since many qualitative properties of solutions of systems with the same reflecting function are common, these results enable us to know which perturbations of the Hindmarsh − Rose system will not affect the qualitative behavior of the solutions of the perturbed system.

Note that the non-autonomous perturbation $\alpha(t)X(x)$ of the autonomous system $\dot{x} = X(x)$ (which takes place in system (2.2)) can preserve the qualitative properties of the solutions even when the function $\alpha(t)$ is not odd (i.e., perturbation is not admissible) [15]. Let us reformulate Theorem 2.1 and Theorem 2.2 from [15] for systems (2.1) and (2.2).

Theorem 2.2. *Let a solution $\eta(t)$ of system (2.1) and a scalar continuous function $\alpha_1(t)$ (which is not necessarily odd) be ω-periodic. If additionally $\int_0^\omega \alpha_1(s)ds = 0$, then solution $\eta(t + \int_0^t \alpha_1(s)ds)$ of system (2.2) is ω-periodic (the period ω is not necessarily minimal).*

Theorem 2.3. *Let $\alpha_1(t)$ be an arbitrary scalar continuous function (not necessarily odd) such that $\int_0^t \alpha_1(s)ds \geq -t \quad \forall t \geq 0$.*

1. *If the equilibrium point (x^*, y^*, z^*) of system (2.1) is Lyapunov stable, then the equilibrium point (x^*, y^*, z^*) of system (2.2) is uniformly Lyapunov stable.*
2. *If $\int_0^{+\infty} (\alpha_1(s) + 1)\, ds = +\infty$ and the equilibrium point (x^*, y^*, z^*) of system (2.1) is Lyapunov unstable, then the equilibrium point (x^*, y^*, z^*) of system (2.2) is Lyapunov unstable.*
3. *If $\int_0^{+\infty} (\alpha_1(s) + 1)\, ds = +\infty$ and the equilibrium point (x^*, y^*, z^*) of system (2.1) is asymptotically Lyapunov stable, then the equilibrium point (x^*, y^*, z^*) of system (2.2) is uniformly asymptotically Lyapunov stable.*

3. Numerical Examples

With the help of the theory of the reflecting function, Theorem 2.1 can be used to study the qualitative behavior of solutions of admissible perturbed systems. Many qualitative properties of solutions of admissibly perturbed systems coincide with those for the original system, which is confirmed by the following examples.

Example 3.1. Following [23] we draw the bifurcation diagram of the system (2.2), where $a = c = 1$, $d = 5$, $r = 0.01$, $s = 4$, $\alpha = -1.6$, $\alpha_1(t) = \sin t$, $I = -3.4b + 13.24$ and $2.6 \leq b \leq 3.3$. Bifurcation diagram depicting local maxima x was drawn using the Wolfram Mathematica software [28] for $49000 \leq t \leq 50000$ with time step $\Delta t = 100$, parameter step $\Delta b = 0.0005$ and initial conditions $x(0) = y(0) = z(0) = 0.1$.

Comparing Fig. 2 in [23] and Fig. 1, we can see that the bifurcation diagrams of systems (2.1) and (2.2) have the same structure. Regions 1, 2, 3, 4, 5, 6, etc. (in Fig. 1) form a bifurcation sequence that increases the period by the unity (which is confirmed by Fig. 2), and in each region a period doubling bifurcation is observed, which, starting from region 2, passes into a chaotic subregion and then again into a periodic one. Fig. 2 shows periodic attractors (from regions 1–6 in Fig. 1) of the system (2.2), where $a = c = 1$, $d = 5$, $r = 0.01$, $s = 4$, $\alpha = -1.6$, $\alpha_1(t) = \sin t$, $I = -3.4b + 13.24$. The figures were drawn using the Wolfram Mathematica software for $4500 \leq t \leq 5000$ with initial conditions $x(0) = y(0) = z(0) = 0.1$.

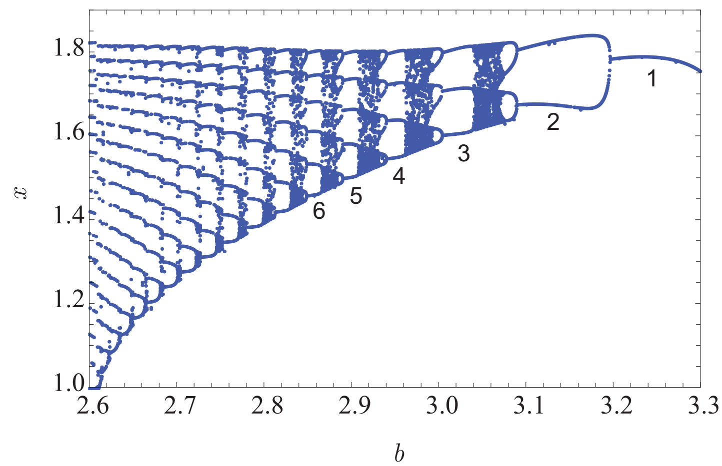

Figure 1. Bifurcation diagram of the system (2.2) depicting local maxima x versus b for $I = -3.4b + 13.24$.

Example 3.2. Following [6] we draw the bifurcation diagram (but of a different type) of the system (2.2), where $a = c = 1$, $b = 3$, $d = 5$, $s = 4$, $\alpha = -1.6$, $I = 3.25$, $\alpha_1(t) = \sin t$. Bifurcation diagram depicting local maxima x in Fig. 3 was drawn using the Wolfram Mathematica software [28] for $49000 \leq t \leq 50000$ with time step $\Delta t = 100$, parameter step $\Delta r = 10^{-4}$ and initial conditions $x(0) = y(0) = z(0) = 0$. Moreover we draw the magnification of the region $0 \leq r \leq 0.0011$ of the bifurcation diagram with parameter step $\Delta r = 10^{-6}$. Fig. 2b in [6] and Fig. 3 illustrate the same structure of bifurcation diagrams of the systems (2.1) and (2.2).

Example 3.3. Following [22] we draw the bifurcation diagram of the system (2.2), where $a = c = 1$, $b = 3$, $d = 5$, $r = 0.007$, $s = 4$, $\alpha = -1.6$, $\alpha_1(t) = \sin t$. Bifurcation diagram depicting local maxima z in Fig. 4 was drawn using the Wolfram Mathematica software [28] for $49000 \leq t \leq 50000$ with time step $\Delta t = 100$, parameter step $\Delta I = 10^{-3}$ and initial conditions $x(0) = y(0) = 0.1$, $z(0) = -0.1$. We also draw a strange attractor of the system (2.2), where $a = c = 1$, $b = 3$, $d = 5$, $I = 3.27$, $r = 0.007$, $s = 4$, $\alpha = -1.6$, $\alpha_1(t) = \sin t$. Strange attractor in Fig. 4 was drawn using the Wolfram Mathematica software for $500 \leq t \leq 5000$ with initial conditions $x(0) = y(0) = 0.1$, $z(0) = -0.1$. Fig. 5 in [22] and Fig. 4 illustrate the same structure of bifurcation diagrams and similar strange attractors of the systems (2.1) and (2.2).

Example 3.4. Fig. 5–7 (which drawn using the Wolfram Mathematica software

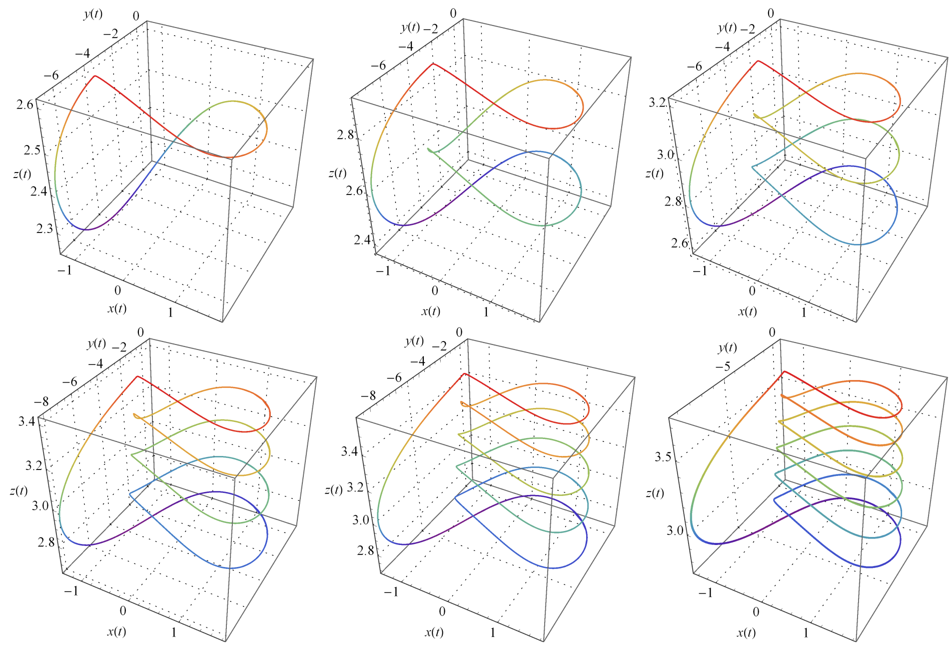

Figure 2. Periodic attractors of the system (2.2) for $b = 3.25$, $I = 2.19$; $b = 3.15$, $I = 2.53$; $b = 3.02$, $I = 2.972$; $b = 2.95$, $I = 3.21$; $b = 2.9$, $I = 3.38$; $b = 2.85$, $I = 3.55$ (left to right, top to bottom).

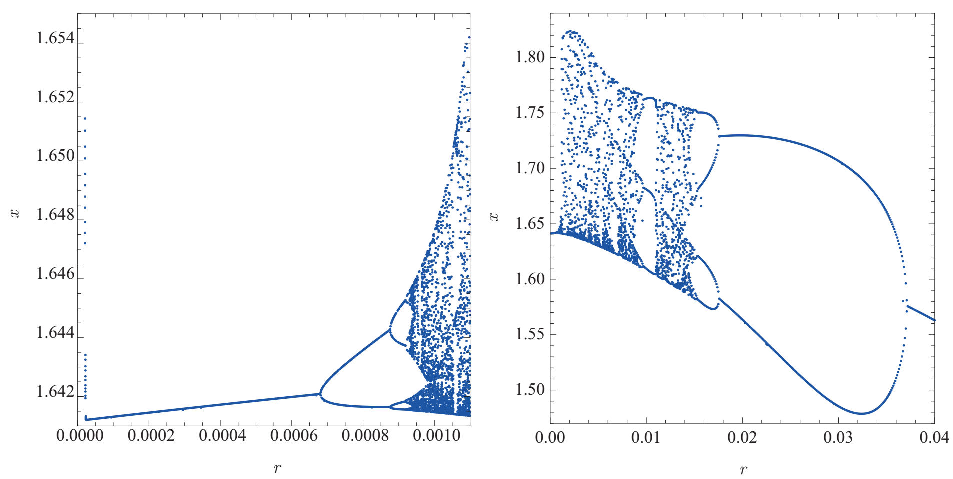

Figure 3. Bifurcation diagram of the system (2.2) depicting local maxima x versus r and its magnification of the region $0 \leq r \leq 0.0011$ (right and left respectively).

[28]) illustrate the same behavior of solutions of the systems (2.1) and (2.3) for $a = 1/2$, $b = 2$, $c = I = r = \alpha = 1$, $s = 5$, $d = 0$, $\alpha_i(t) = \sin(i\,t)$, $i = \overline{1, 2}$ and with initial conditions $x(0) = y(0) = z(0) = 0$.

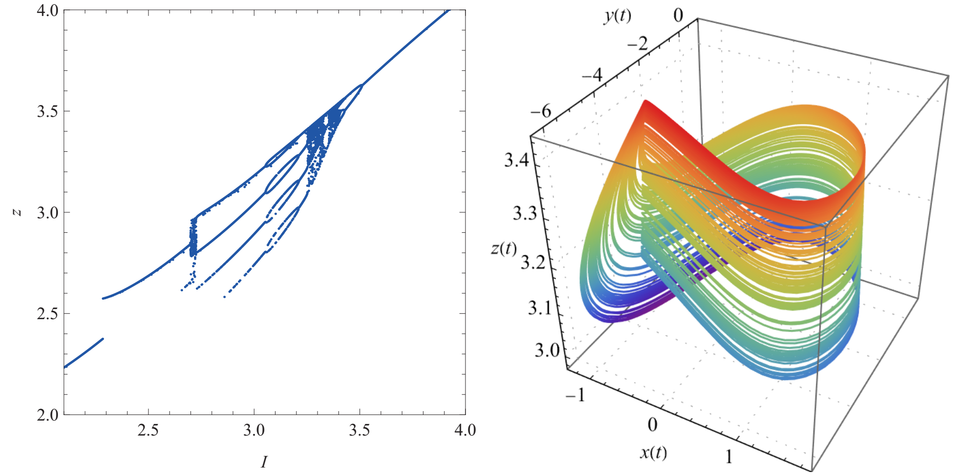

Figure 4. Bifurcation diagram of the system (2.2) depicting local maxima z versus I and strange attractor of the system (2.2) (left and right respectively).

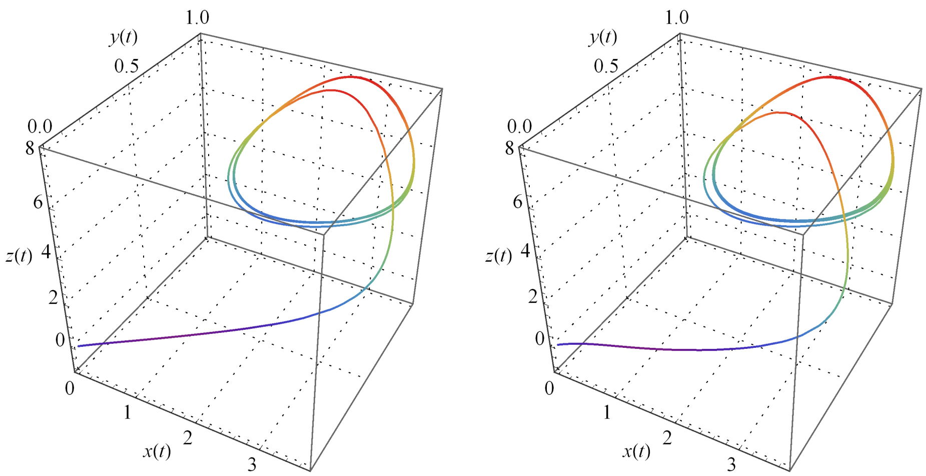

Figure 5. Solution of the systems (2.1) and (2.3) (left and right respectively) in the phase space.

4. Conclusions

Thus, for the autonomous three-dimensional Hindmarsh – Rose neuron model (2.1), we obtained a set of nonautonomous systems with the same reflecting function. The coincidence of the reflecting functions means the coincidence of some of the qualitative properties (such as the presence of periodic solutions, the stability of solutions in the sense of Lyapunov, and others [16, 19, 21]) of such systems. On the one hand, it allows us to know the types of perturbations of the Hindmarsh – Rose system that will not affect the qualitative behavior of solutions, and on the other hand, to use the results of studying the qualitative behavior of solutions of the well-known Hindmarsh – Rose system to study more complicated nonautonomous perturbed systems that can be used when modeling. Numerical examples show that

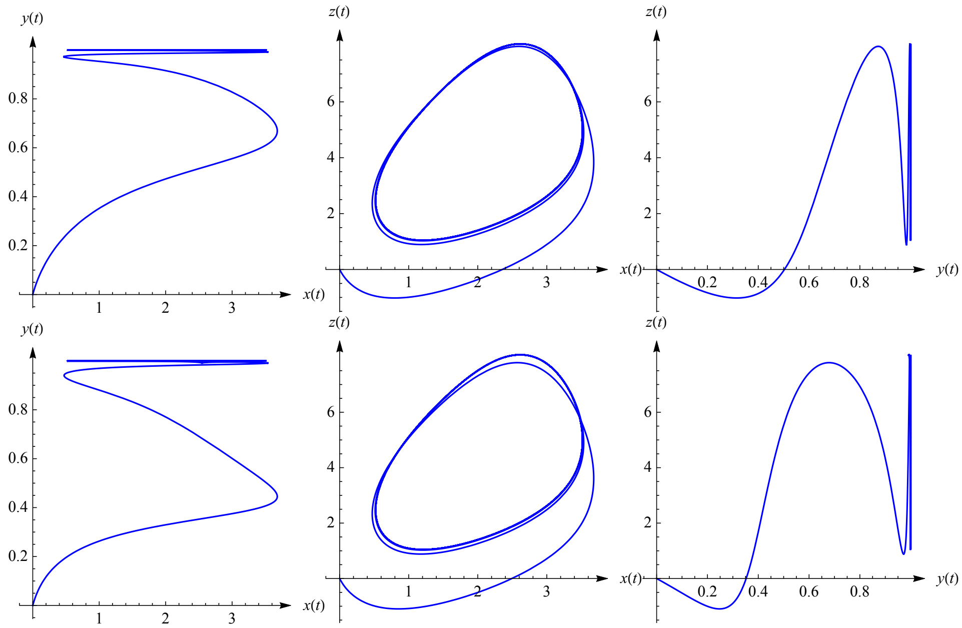

Figure 6. Phase planes projections of the solution of the systems (2.1) and (2.3) (top and bottom rows respectively).

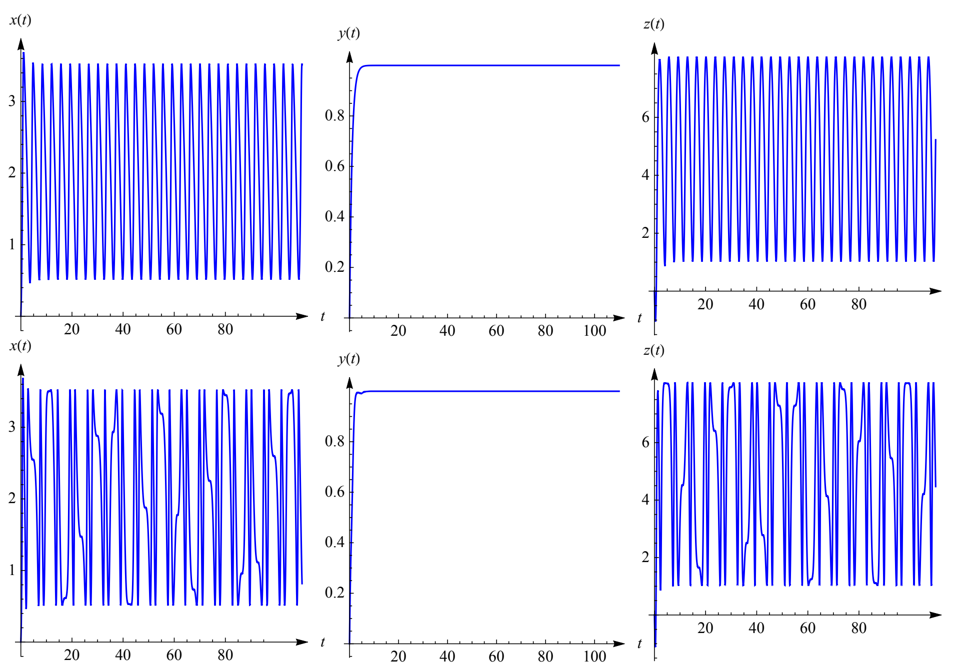

Figure 7. Solution components of the system (2.1) and (2.3) (top and bottom rows respectively).

admissibly perturbed systems have similar bifurcation diagrams, periodic attractors and strange attractor as the original Hindmarsh – Rose system.

Note that since a dynamic process is usually modeled on a nonnegative time semi–axis, therefore the condition of oddness of functions $\alpha_i(t)$ in Theorem 2.1 is not essential, since arbitrary functions $\alpha_i(t)$ ($\alpha_i(0) = 0$) can be extended continuously to the negative semi–axis in an odd way.

A. Brief Theory of the Reflecting Function

In this appendix we give a brief information on the theory of the reflecting function from [12, 13].

Consider the system of ordinary differential equations

$$\dot{x} = X(t, x), \quad t \in \mathbb{R},\ x \in D \subset \mathbb{R}^n, \tag{A.1}$$

the solutions of which are uniquely determined by the initial conditions. Let $x = \varphi(t; t_0, x_0)$ be the general solution in Cauchy form of the system (A.1). The reflecting function for each system (A.1) is defined in some region near the hyperplane $t = 0$ as the function $F(t, x) := \varphi(-t; t, x)$.

The differentiable function $F(t, x)$ is the reflecting function of the system (A.1) if and only if it is the solution of the Cauchy problem

$$\frac{\partial F(t,x)}{\partial t} + \frac{\partial F(t,x)}{\partial x} X(t,x) + X\left(-t, F(t,x)\right) = 0, \quad F(0,x) \equiv x,$$

which is called *basic relation*.

For every extendible solution $x(t)$ of the system (A.1) which exist on $[-\omega; \omega]$ the identity $F\left(-t, x(-t)\right) \equiv x(t)$, $\forall t \in [-\omega; \omega]$, is hold. Thus, using the reflecting function for the past state $x(-t)$ of the system, one can find out its future state $x(t)$ and vice versa.

If $X(t, x)$ continuously differentiable and $X(t, x + 2\omega) \equiv X(t, x)$, then mapping over the period $[-\omega; \omega]$ (Poincaré map) of the system (A.1) is given by formula $F(-\omega, x) = \varphi(\omega; -\omega, x)$, where $F(t, x)$ is reflecting function of the system. In this case the solution $x = \varphi(t; -\omega, x_0)$ of the system (A.1) will be 2ω-periodic if and only if the solution is extendible on $[-\omega; \omega]$ and $F(-\omega, x_0) = x_0$. This solution is stable if and only if the fixed point of the Poincaré map $x \mapsto F(-\omega, x)$ is stable.

If a continuously differentiable function $F(t, x)$ (or it's restriction) is defined in a domain of $\mathbb{R}^{1+n}$, which contained the hyperplane $t = 0$, and the identity $F\left(-t, F(t, x)\right) \equiv F(0, x) \equiv x$ is hold, then $F(t, x)$ is a reflecting function of the class of systems of the form

$$\dot{x} = -\frac{1}{2} \frac{\partial F}{\partial x} \left(-t, F(t,x)\right) \left(\frac{\partial F(t,x)}{\partial t} - 2S(t,x) \right) - S\left(-t, F(t,x)\right), \tag{A.2}$$

where $S(t, x)$ is an arbitrary vector-function for which the solutions of the system (A.2) are uniquely determined by the initial conditions. Therefore, all systems of the form (A.1) are split into equivalence classes of the form (A.2), so that each class is specified by a certain reflecting function, called *the reflecting function of the class*. For all systems of one class, the shift operator [8, pp. 11–13] on the interval $[-\omega; \omega]$ is the same. Consequently, all equivalent 2ω-periodic systems have a common mapping over the period $[-\omega; \omega]$.

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