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HIGH ACCURACY PIECEWISE-ANALYTIC SOLUTIONS AND HIGHER-ORDER NUMERIC SOLUTIONS OF PROJECTILE MOTION WITH A QUADRATIC DRAG FORCE BY THE MULTISTAGE MODIFIED DECOMPOSITION METHOD

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Abstract We apply the multistage modified decomposition method (MDM) of Rach, Adomian, and Meyers to simulate the atmospheric projectile trajectory subject to a quadratic drag force. We readily obtain both the approximate analytic and numeric solutions through means of one-step recurrence algorithms based on the concept of analytic continuation, where the step size h and the order m are used to control errors. Simply put, the numeric solutions become the nodal values of our piecewise-analytic approximations. The realistic mathematical model includes sinusoidal, quadratic, reciprocal, and product nonlinearities which are conveniently treated by the Adomian polynomials without resort to any linearization or perturbation whatsoever. Fast algorithms of the Adomian polynomials guarantee the efficiency of our approach, and both the approximate analytic and numeric higher-order solutions can be readily generated at will unlike the usual Runge-Kutta methods that rely on a crude linearization. Multistage analytic and numeric decomposition algorithms demonstrate the rapid convergence of our new approach, where the MDM is based on the nonlinear transformation of series by the Adomian-Rach theorem. As an example, we also determine several important aerodynamic measures for the trajectory of a baseball such as the time of ascent, the velocity at the trajectory apex, the maximum height of ascent, then the flight range, the impact velocity and the impact angle with respect to the horizontal, the optimal launch angle, and the maximum flight range. We consider the error analyses for the multistage analytic approximations including the remainder error functions and also introduce the accumulative remainder error functions and the accumulative remainder error bounds for the numeric solutions. Our approximate solutions compare most favorably to the exact solution by Bernoulli.

Keywords Projectile motion, quadratic drag, Adomian decomposition method, modified decomposition method, Adomian polynomials.

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1. Introduction

The problem of the motion of a projectile, treated as a point mass and thrown at an angle to the horizon, has a long history. If the aerodynamic drag force is neglected, it is well known that the trajectory of the point mass is a parabola. In the case that the resistance is proportional to the velocity, the problem can be analytically solved. But for the case of a quadratic drag force, the problem is modeled by a system of nonlinear ordinary differential equations which does not have exact analytic solutions for the time and space variables unless the projectile is thrown vertically [17, 29, 36].

In practice, such as throwing a ball, taking into account the effect of the medium, whether the linear or the quadratic resistance law is considered, depends of the Reynolds number Re. The linear law holds for Re < 1, while quadratic law is applied when $10^3 <$ Re $< 2 \times 10^5$ [36]. The quadratic drag force is more commonly seen by students during physical ball games [19,38]. For the two-dimensional projectile problem with a quadratic drag force, Chudinov [17] gave a simple approximate analytic expression for the projectile trajectory. Approximate analytical formulas for the main parameters of the projectile trajectory were obtained in [14]. In [39], the Lambert W function was used to model the projectile thrown with a low angle relative to the horizonal. In order to analytically compute the involved integrations with enough accuracy and using elementary functions, Turkyilmazoglu [37] introduced an interpolating function approximation. Such approximations were further improved in [15, 16] for a wide range of motion. The horizontal distance travelled by the projectile was considered through numerical integration and approximation of the impact angle [38]. In most references such as in [15, 16, 37], the usual fourth-order Runge-Kutta method was used to compare with the analytical approximate results.

In this article, we present high accuracy piecewise-analytic solutions of degree m and higher-order numeric solutions of order m for the projectile motion with a quadratic drag force by the multistage modified decomposition method (MDM) of Rach, Adomian, and Meyers [33]. This method benefits from the convenient algorithms for the Adomian polynomials, which were originally used in the Adomian decomposition method (ADM) [3, 4]. The ADM and its modifications are practical techniques for solving functional equations, especially for nonlinear cases [5, 25–27, 34, 35, 41].

The method supposes a decomposition series solution and decomposes the nonlinear term $Nu = f(u)$ into a series

$$u = \sum_{n=0}^{\infty} u_n, \ Nu = f(u) = \sum_{n=0}^{\infty} A_n, \tag{1.1}$$

where the A_n, depending on the solution components $u_0, u_1, \cdots, u_n$, are called the Adomian polynomials, and are defined for the analytic nonlinearity $Nu = f(u)$ by the definitional formula [6]

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[f\left(\sum_{k=0}^{\infty} u_k \lambda^k\right)\right]_{\lambda=0}, \ n = 0, 1, 2, \cdots, \tag{1.2}$$

where λ is a convenient grouping parameter. The first five Adomian polynomials are

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= f'(u_0)u_1, \\ A_2 &= f'(u_0)u_2 + f''(u_0)\tfrac{u_1^2}{2!}, \\ A_3 &= f'(u_0)u_3 + f''(u_0)u_1u_2 + f^{(3)}(u_0)\tfrac{u_1^3}{3!}, \\ A_4 &= f'(u_0)u_4 + f''(u_0)(u_1u_3 + \tfrac{u_2^2}{2!}) + f^{(3)}(u_0)\tfrac{u_1^2u_2}{2!} + f^{(4)}(u_0)\tfrac{u_1^4}{4!}. \end{aligned} \tag{1.3}$$

The Adomian decomposition method consists in identifying the solution components u_n's by means of a suitable recursion scheme.

For the Adomian polynomials, Rach [30] gave the first formula discarding the concept of analytic parametrization (Rach's Rule in [4,5]),

$$A_n = \sum_{k=1}^{n} f^{(k)}(u_0)C(k,n),\ n \geq 1, \tag{1.4}$$

where the $C(k,n)$ are the sums of all possible products of k components from u_1, u_2, $\cdots$, u_n, whose subscripts sum to n, divided by the factorial of the number of repeated subscripts, that is,

$$C(k,n) = \sum_{\sum_{j=1}^{n} jp_j=n,\ \sum_{j=1}^{n} p_j=k} \frac{u_1^{p_1}u_2^{p_2}\cdots u_n^{p_n}}{p_1!p_2!\cdots p_n!}. \tag{1.5}$$

Additional contributions have been advanced for algorithms of the Adomian polynomials such as in [2,13,31,40,42]. Recently new, more efficient algorithms to generate the Adomian polynomials quickly and to high orders, including the single variable and multivariable cases, have been proposed in [20–22]. Here we recall Duan's efficient Corollary 3 algorithm in [22] as follows:

$$A_0 = f(u_0),\ A_n = \sum_{k=1}^{n} f^{(k)}(u_0)C_n^k,\ n \geq 1, \tag{1.6}$$

where

$$C_n^1 = u_n,\ n \geq 1 \tag{1.7}$$

and

$$C_n^k = \frac{1}{n}\sum_{j=0}^{n-k}(j+1)u_{j+1}C_{n-1-j}^{k-1},\ 2 \leq k \leq n. \tag{1.8}$$

We point out that the recursion relation (1.8) in this algorithm requires only the operations of addition and multiplication, which is eminently convenient for computer algebra systems.

The MDM proposed by Rach, Adomian and Meyers [33] is a simplified form of the ADM. The MDM is based on the nonlinear transformation of series by the Adomian–Rach theorem [7,10]:

$$\text{If} \quad u(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n, \quad \text{then} \quad f(u(x)) = \sum_{n=0}^{\infty} A_n(x-x_0)^n, \tag{1.9}$$

where the $A_n = A_n(a_0, a_1, \ldots, a_n)$ are the Adomian polynomials in terms of the solution coefficients. The MDM and its multistage forms have been efficiently applied to solve various nonlinear models [5, 9, 11, 12, 23–26, 28]. For the multivariable Adomian polynomials and new efficient algorithms and applications, see [1, 5, 8, 20, 22]. For the MATHEMATICA code generating the single-variable and multivariable Adomian polynomials, see [20, 22, 25]. For systematic references, see [32].

In this work, by using the multistage form of MDM [33], we develop multistage analytic approximate solutions and higher-order numeric solutions for projectile motion subject to quadratic drag. The error of the multistage analytic approximations can be directly examined by using the proposed remainder error functions, accumulative remainder error functions and accumulative remainder error bounds.

2. The projectile equation and description of the method

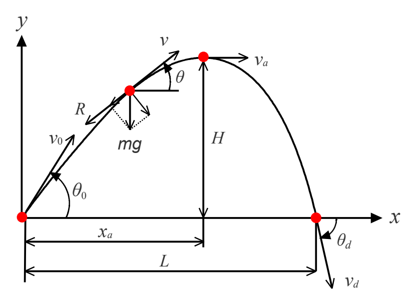

Figure 1. Schematic diagram of a moving projectile.

Let us consider the motion of a projectile as a point mass with the initial velocity v_0 and launch angle θ_0 subject to a quadratic resistance force $R = mgkv^2$; see Figure 1 for a schematic diagram. Here v is the velocity, m is the mass of the projectile, g is the gravity acceleration constant and $k > 0$ is the resistance factor with the dimension $\mathrm{s}^2\mathrm{m}^{-2}$. Considering the tangential direction and the normal direction of the point mass on the trajectory, also known as the natural axes, Newton's second law leads to the following system of nonlinear differential equations

$$\frac{dv}{dt} = -g\sin(\theta) - gkv^2, \tag{2.1}$$

$$\frac{d\theta}{dt} = -gv^{-1}\cos(\theta), \tag{2.2}$$

with the initial conditions $v(t = 0) = v_0$ and $\theta(t = 0) = \theta_0$, where θ is the angle of slope of the tangent and the initial values satisfy $v_0 > 0$ and $-\frac{\pi}{2} < \theta_0 < \frac{\pi}{2}$. In practice, a positive value θ_0 is commonly used.

Eqs. (2.1) and (2.2) have an asymptotically stable equilibrium solution

$$v(t) \equiv v_{\lim} = \frac{1}{\sqrt{k}},\ \theta(t) \equiv \theta_{\lim} = -\frac{\pi}{2}, \tag{2.3}$$

which correspond to the limit velocity and slope angle as $t \to +\infty$. From Eqs. (2.1) and (2.2), the ordinary differential equation about v and θ is derived as

$$\frac{dv}{d\theta} = \tan(\theta)\, v + \frac{k}{\cos(\theta)} v^3. \tag{2.4}$$

This is the Bernoulli equation and the solution is deduced with the initial condition $v(\theta_0) = v_0$ as

$$v(\theta) = \frac{v_0 \cos(\theta_0)}{\cos(\theta)\sqrt{1 + k v_0^2 \cos^2(\theta_0)(f(\theta_0) - f(\theta))}}, \tag{2.5}$$

where

$$f(\theta) = \frac{\sin(\theta)}{\cos^2(\theta)} + \ln\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right). \tag{2.6}$$

In practical measures, the time t and the Cartesian coordinates x and y of the point mass have to be calculated. From Eq. (2.2), the time has the expression

$$t = \frac{1}{g}\int_{\theta}^{\theta_0} \frac{v}{\cos(\theta)} d\theta. \tag{2.7}$$

The Cartesian coordinates satisfy the differential equations expressed by v and θ as

$$\frac{dx}{dt} = v\cos(\theta), \tag{2.8}$$

$$\frac{dy}{dt} = v\sin(\theta). \tag{2.9}$$

Combining Eq. (2.2) respectively with Eqs. (2.8) and (2.9), we have

$$x = x_0 + \frac{1}{g}\int_{\theta}^{\theta_0} v^2 d\theta,\ y = y_0 + \frac{1}{g}\int_{\theta}^{\theta_0} v^2 \tan(\theta) d\theta, \tag{2.10}$$

where $x(t = 0) = x_0$ and $y(t = 0) = y_0$ are the initial Cartesian coordinates. The Cartesian coordinates satisfy the limit

$$x \to x_{\lim} < \infty,\ y \to -\infty \quad \text{as} \quad t \to +\infty. \tag{2.11}$$

In fact, $t \to +\infty$ means $\theta \to -\frac{\pi}{2}$, and on the interval $(-\frac{\pi}{2}, \theta_0]$, v^2 is continuous and $v^2 \to \frac{1}{k}$ as $\theta \to \left(-\frac{\pi}{2}\right)^+$, so $x_{\lim}$ is finite. But

$$\tan(\theta) \sim \frac{-1}{\theta + \frac{\pi}{2}} \quad \text{as} \quad \theta \to \left(-\frac{\pi}{2}\right)^+,$$

thus $y \to -\infty$ as $t \to +\infty$.

Nevertheless, since the integrals in Eqs. (2.7) and (2.10) can not be exactly calculated in terms of elementary or special functions, one has to approximately calculate these integrals or approximately solve the differential equations to obtain quantitative and explicit formulations for the time t and the Cartesian coordinates x and y.

In this work, we approximately solve the differential equations (2.1), (2.2), (2.8) and (2.9) simultaneously by using the multistage MDM. The exact expression of $v(\theta)$ will be used for comparison. First, we take the initial values v_0, θ_0, x_0 and y_0 as parameters, and generate the Taylor polynomial solutions of degree m depending on the initial values. Then they are applied to consecutive subintervals to produce the multistage analytic approximate solutions or numeric solutions which characterize the fundamental process of analytic continuation.

We decompose the solutions of the differential equations (2.1), (2.2), (2.8) and (2.9) into their respective power series as

$$v(t) = \sum_{n=0}^{\infty} v_n t^n,\ \theta(t) = \sum_{n=0}^{\infty} \theta_n t^n, \tag{2.12}$$

$$x(t) = \sum_{n=0}^{\infty} x_n t^n,\ y(t) = \sum_{n=0}^{\infty} y_n t^n. \tag{2.13}$$

Their derivatives are thus

$$\frac{dv}{dt} = \sum_{n=0}^{\infty} (n+1) v_{n+1} t^n,\ \frac{d\theta}{dt} = \sum_{n=0}^{\infty} (n+1) \theta_{n+1} t^n, \tag{2.14}$$

$$\frac{dx}{dt} = \sum_{n=0}^{\infty} (n+1) x_{n+1} t^n,\ \frac{dy}{dt} = \sum_{n=0}^{\infty} (n+1) y_{n+1} t^n. \tag{2.15}$$

By using the Adomian polynomials, we decompose the nonlinear functions of the solutions into power series as

$$\sin(\theta) = \sum_{n=0}^{\infty} A_n t^n,\ v^2 = \sum_{n=0}^{\infty} B_n t^n,\ v^{-1} = \sum_{n=0}^{\infty} C_n t^n,\ \cos(\theta) = \sum_{n=0}^{\infty} D_n t^n, \tag{2.16}$$

where the coefficients are the Adomian polynomials,

$$A_n = A_n(\theta_0, \ldots, \theta_n),\ B_n = B_n(v_0, \ldots, v_n), \tag{2.17}$$

$$C_n = C_n(v_0, \ldots, v_n),\ D_n = D_n(\theta_0, \ldots, \theta_n), \tag{2.18}$$

and they can be conveniently calculated from Eqs. (1.6)–(1.8). Here we list the formula for B_n,

$$B_n = \sum_{m=0}^{n} v_{n-m} v_m,\ n = 0, 1, \ldots.$$

The first six Adomian polynomials for $\sin(\theta)$, $\cos(\theta)$ and v^{-1} are listed in Appendix A. The decompositions of the nonlinearities $v^{-1}\cos(\theta)$, $v\cos(\theta)$ and $v\sin(\theta)$ are calculated by their respective Cauchy products as

$$v^{-1}\cos(\theta) = \sum_{n=0}^{\infty} E_n t^n,\ v\cos(\theta) = \sum_{n=0}^{\infty} F_n t^n,\ v\sin(\theta) = \sum_{n=0}^{\infty} G_n t^n, \tag{2.19}$$

where

$$E_n = E_n(v_0, \ldots, v_n; \theta_0, \ldots, \theta_n) = \sum_{m=0}^{n} C_{n-m} D_m, \tag{2.20}$$

$$F_n = F_n(v_0, \ldots, v_n; \theta_0, \ldots, \theta_n) = \sum_{m=0}^{n} v_{n-m} D_m, \tag{2.21}$$

$$G_n = G_n(v_0, \ldots, v_n; \theta_0, \ldots, \theta_n) = \sum_{m=0}^{n} v_{n-m} A_m. \tag{2.22}$$

Substituting these decompositions into Eqs. (2.1), (2.2), (2.8) and (2.9), we have

$$\sum_{n=0}^{\infty}(n+1)\,v_{n+1}t^n = -g\sum_{n=0}^{\infty}A_nt^n - gk\sum_{n=0}^{\infty}B_nt^n,$$
$$\sum_{n=0}^{\infty}(n+1)\,\theta_{n+1}t^n = -g\sum_{n=0}^{\infty}E_nt^n,$$
$$\sum_{n=0}^{\infty}(n+1)\,x_{n+1}t^n = \sum_{n=0}^{\infty}F_nt^n,$$
$$\sum_{n=0}^{\infty}(n+1)\,y_{n+1}t^n = \sum_{n=0}^{\infty}G_nt^n.$$

Comparing the coefficients of like powers on both sides leads to the system of four coupled nonlinear recurrence relations as

$$v_{n+1} = \frac{-gA_n - gkB_n}{n+1},\ n=0,1,\ldots, \tag{2.23}$$
$$\theta_{n+1} = \frac{-gE_n}{n+1},\ n=0,1,\ldots, \tag{2.24}$$
$$x_{n+1} = \frac{F_n}{n+1},\ n=0,1,\ldots, \tag{2.25}$$
$$y_{n+1} = \frac{G_n}{n+1},\ n=0,1,\ldots. \tag{2.26}$$

Thus for $n \geq 0$, the solution coefficients $v_{n+1}, \theta_{n+1}, x_{n+1}$ and y_{n+1} are all determined by v_0 and θ_0. The $(m+1)$-term truncated approximations of the solutions (2.12) and (2.13) become their respective Taylor polynomials of degree m, which we denote as

$$\phi_{v,m+1}(t;v_0,\theta_0) = \sum_{n=0}^{m}v_nt^n,\ \ \phi_{\theta,m+1}(t;v_0,\theta_0) = \sum_{n=0}^{m}\theta_nt^n, \tag{2.27}$$
$$\phi_{x,m+1}(t;v_0,\theta_0,x_0) = \sum_{n=0}^{m}x_nt^n,\ \ \phi_{y,m+1}(t;v_0,\theta_0,y_0) = \sum_{n=0}^{m}y_nt^n. \tag{2.28}$$

Here $\phi_{v,m+1}(t;v_0,\theta_0)$, $\phi_{\theta,m+1}(t;v_0,\theta_0)$, $\phi_{x,m+1}(t;v_0,\theta_0,x_0)$ and $\phi_{y,m+1}(t;v_0,\theta_0,y_0)$ are functions of t and the related initial values used to generate the multistage approximate analytical solutions of degree m and numerical solutions of order m.

If we take $m=7$, the 8-term truncated approximations are

$$\begin{aligned}
&\phi_{v,8}(t;v_0,\theta_0)\\
=&v_0 - gt\left(\sin(\theta_0) + k{v_0}^2\right) + \frac{g^2t^2}{2v_0}\left(\cos^2(\theta_0) + 2k^2{v_0}^4 + 2k{v_0}^2\sin(\theta_0)\right)\\
&-\frac{g^3t^3}{6{v_0}^2}\left(-3\sin(\theta_0)\cos^2(\theta_0) + 6k^3{v_0}^6 + 8k^2{v_0}^4\sin(\theta_0) + 2k{v_0}^2\sin^2(\theta_0)\right.\\
&\left.+k{v_0}^2\cos^2(\theta_0)\right) + \frac{g^4t^4}{24{v_0}^3}\left(-3\cos^4(\theta_0) + 12\sin^2(\theta_0)\cos^2(\theta_0) + 24k^4{v_0}^8\right.\\
&\left.+40k^3{v_0}^6\sin(\theta_0) + 16k^2{v_0}^4\sin^2(\theta_0) + 8k^2{v_0}^4\cos^2(\theta_0) + 8k{v_0}^2\sin(\theta_0)\cos^2(\theta_0)\right)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{g^5t^5}{120{v_0}^4}\left(-30\sin^3(\theta_0)+45\sin(\theta_0)\cos^4(\theta_0)-30\sin^3(\theta_0)\cos(2\theta_0)+120k^5{v_0}^{10}\right.\\
&+240k^4{v_0}^8\sin(\theta_0)+136k^3{v_0}^6\sin^2(\theta_0)+48k^3{v_0}^6\cos^2(\theta_0)+16k^2{v_0}^4\sin^3(\theta_0)\\
&\left.+16k^2{v_0}^4\sin(\theta_0)\cos^2(\theta_0)+17k{v_0}^2\cos^4(\theta_0)-60k{v_0}^2\sin^2(\theta_0)\cos^2(\theta_0)\right)\\
&+\frac{g^6t^6}{720{v_0}^5}\left(45\cos^6(\theta_0)-540\sin^2(\theta_0)\cos^4(\theta_0)+360\sin^4(\theta_0)\cos^2(\theta_0)+720k^6{v_0}^{12}\right.\\
&+1680k^5{v_0}^{10}\sin(\theta_0)+1232k^4{v_0}^8\sin^2(\theta_0)+336k^4{v_0}^8\cos^2(\theta_0)+272k^3{v_0}^6\sin^3(\theta_0)\\
&+272k^3{v_0}^6\sin(\theta_0)\cos^2(\theta_0)-18k^2{v_0}^4\cos^4(\theta_0)+136k^2{v_0}^4\sin^2(\theta_0)\cos^2(\theta_0)\\
&\left.-402k{v_0}^2\sin(\theta_0)\cos^4(\theta_0)+480k{v_0}^2\sin^3(\theta_0)\cos^2(\theta_0)\right)\\
&-\frac{g^7t^7}{5040{v_0}^6}\left(-1575\sin(\theta_0)\cos^6(\theta_0)+6300\sin^3(\theta_0)\cos^4(\theta_0)-2520\sin^5(\theta_0)\cos^2(\theta_0)\right.\\
&+5040k^7{v_0}^{14}+13440k^6{v_0}^{12}\sin(\theta_0)+12096k^5{v_0}^{10}\sin^2(\theta_0)+2688k^5{v_0}^{10}\cos^2(\theta_0)\\
&+3968k^4{v_0}^8\sin^3(\theta_0)+3072k^4{v_0}^8\sin(\theta_0)\cos^2(\theta_0)+272k^3{v_0}^6\sin^4(\theta_0)\\
&+290k^3{v_0}^6\cos^4(\theta_0)+408k^3{v_0}^6\sin^2(\theta_0)\cos^2(\theta_0)+1568k^2{v_0}^4\sin(\theta_0)\cos^4(\theta_0)\\
&-1848k^2{v_0}^4\sin^3(\theta_0)\cos^2(\theta_0)-627k{v_0}^2\cos^6(\theta_0)+6954k{v_0}^2\sin^2(\theta_0)\cos^4(\theta_0)\\
&\left.-4200k{v_0}^2\sin^4(\theta_0)\cos^2(\theta_0)\right),\\
&\phi_{\theta,8}(t;v_0,\theta_0)\\
=&\theta_0-\frac{gt\cos(\theta_0)}{v_0}-\frac{g^2t^2}{2{v_0}^2}\left(\sin(2\theta_0)+k{v_0}^2\cos(\theta_0)\right)\\
&-\frac{g^3t^3}{12{v_0}^3}\left(5k{v_0}^2\sin(2\theta_0)-4\cos(3\theta_0)\right)-\frac{g^4t^4}{96{v_0}^4}\left(-24\sin(4\theta_0)+10k^2{v_0}^4\sin(2\theta_0)\right.\\
&\left.-5k{v_0}^2\cos(\theta_0)-39k{v_0}^2\cos(3\theta_0)\right)+\frac{g^5t^5}{960{v_0}^5}\left(-192\cos(5\theta_0)+40k^2{v_0}^4\cos(\theta_0)\right.\\
&\left.+176k^2{v_0}^4\cos(3\theta_0)+387k{v_0}^2\sin(4\theta_0)+108k{v_0}^2\sin(\theta_0)\cos(\theta_0)\right)\\
&+\frac{g^6t^6}{11520{v_0}^6}\left(-1920\sin(6\theta_0)+80k^3{v_0}^6\cos(\theta_0)+352k^3{v_0}^6\cos(3\theta_0)+756k^2{v_0}^4\sin(2\theta_0)\right.\\
&\left.+3026k^2{v_0}^4\sin(4\theta_0)+54k{v_0}^2\cos(\theta_0)-657k{v_0}^2\cos(3\theta_0)-4629k{v_0}^2\cos(5\theta_0)\right)\\
&+\frac{g^7t^7}{161280{v_0}^7}\left(23040\cos(7\theta_0)+4160k^3{v_0}^6\sin(2\theta_0)+13160k^3{v_0}^6\sin(4\theta_0)\right.\\
&+432k^2{v_0}^4\cos(\theta_0)-14332k^2{v_0}^4\cos(3\theta_0)-55188k^2{v_0}^4\cos(5\theta_0)\\
&\left.+927k{v_0}^2\sin(2\theta_0)-9228k{v_0}^2\sin(4\theta_0)-64701k{v_0}^2\sin(6\theta_0)\right),\\
&\phi_{x,8}(t;v_0,\theta_0,x_0)\\
=&x_0+tv_0\cos(\theta_0)-\frac{1}{2}gkt^2{v_0}^2\cos(\theta_0)+\frac{1}{12}g^2kt^3v_0(\sin(2\theta_0)\\
&+4k{v_0}^2\cos(\theta_0))-\frac{1}{48}g^3kt^4\left(2\cos^3(\theta_0)+12k^2{v_0}^4\cos(\theta_0)+5k{v_0}^2\sin(2\theta_0)\right)\\
&+\frac{g^4kt^5}{120v_0}\left(-3\sin(\theta_0)\cos^3(\theta_0)+24k^3{v_0}^6\cos(\theta_0)+14k^2{v_0}^4\sin(2\theta_0)+5k{v_0}^2\cos^3(\theta_0)\right.\\
&\left.+5k{v_0}^2\sin^2(\theta_0)\cos(\theta_0)\right)-\frac{g^5kt^6}{720{v_0}^2}\left(-3\cos^5(\theta_0)+6\cos^3(\theta_0)-6\cos^3(\theta_0)\cos(2\theta_0)\right.\\
&+120k^4{v_0}^8\cos(\theta_0)+90k^3{v_0}^6\sin(2\theta_0)+33k^2{v_0}^4\cos^3(\theta_0)+61k^2{v_0}^4\sin^2(\theta_0)\cos(\theta_0)
\end{aligned}
$$

$$
\begin{aligned}
&+3kv_0{}^2\sin(\theta_0)\cos^3(\theta_0)\big) + \frac{g^6kt^7}{5040v_0{}^3}\big(45\sin(\theta_0)\cos^5(\theta_0) - 60\sin^3(\theta_0)\cos^3(\theta_0)\\
&+ 720k^5v_0{}^{10}\cos(\theta_0) + 660k^4v_0{}^8\sin(2\theta_0) + 246k^3v_0{}^6\cos^3(\theta_0)\\
&+ 662k^3v_0{}^6\sin^2(\theta_0)\cos(\theta_0) + 89k^2v_0{}^4\sin(\theta_0)\cos^3(\theta_0) + 61k^2v_0{}^4\sin^3(\theta_0)\cos(\theta_0)\\
&+9kv_0{}^2\cos^5(\theta_0) - 33kv_0{}^2\sin^2(\theta_0)\cos^3(\theta_0)\big)\,,\\
&\phi_{y,8}(t; v_0, \theta_0, y_0)\\
=&y_0 + tv_0\sin(\theta_0) - \frac{1}{2}gt^2\left(\sin^2(\theta_0) + \cos^2(\theta_0) + kv_0{}^2\sin(\theta_0)\right)\\
&+ \frac{1}{6}g^2kt^3v_0\left(2\sin^2(\theta_0) + \cos^2(\theta_0) + 2kv_0{}^2\sin(\theta_0)\right)\\
&- \frac{1}{24}g^3kt^4\left(\sin(\theta_0) + kv_0{}^2\right)\left(2\sin^2(\theta_0) + 3\cos^2(\theta_0) + 6kv_0{}^2\sin(\theta_0)\right)\\
&+ \frac{g^4kt^5}{120v_0}\big(3\cos^4(\theta_0) + 24k^3v_0{}^6\sin(\theta_0) + 40k^2v_0{}^4\sin^2(\theta_0) + 12k^2v_0{}^4\cos^2(\theta_0)\\
&+16kv_0{}^2\sin^3(\theta_0) + 16kv_0{}^2\sin(\theta_0)\cos^2(\theta_0)\big) - \frac{g^5kt^6}{720v_0{}^2}\big(-15\sin(\theta_0)\cos^4(\theta_0)\\
&+120k^4v_0{}^8\sin(\theta_0) + 240k^3v_0{}^6\sin^2(\theta_0) + 60k^3v_0{}^6\cos^2(\theta_0) + 136k^2v_0{}^4\sin^3(\theta_0)\\
&+108k^2v_0{}^4\sin(\theta_0)\cos^2(\theta_0) + 16kv_0{}^2\sin^4(\theta_0) + 8kv_0{}^2\sin^2(2\theta_0) + 13kv_0{}^2\cos^4(\theta_0)\big)\\
&+ \frac{g^6kt^7}{5040v_0{}^3}\big(-15\cos^6(\theta_0) + 90\sin^2(\theta_0)\cos^4(\theta_0) + 720k^5v_0{}^{10}\sin(\theta_0)\\
&+1680k^4v_0{}^8\sin^2(\theta_0) + 360k^4v_0{}^8\cos^2(\theta_0) + 1232k^3v_0{}^6\sin^3(\theta_0)\\
&+816k^3v_0{}^6\sin(\theta_0)\cos^2(\theta_0) + 272k^2v_0{}^4\sin^4(\theta_0)\ +102k^2v_0{}^4\sin^2(2\theta_0)\\
&+108k^2v_0{}^4\cos^4(\theta_0) + 42kv_0{}^2\sin(\theta_0)\cos^4(\theta_0)\big)\,.
\end{aligned}
$$

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. Denote the approximations of the solutions $v(t)$, $\theta(t)$, $x(t)$ and $y(t)$ at $t = t_i$ as $v^{\langle i\rangle}$, $\theta^{\langle i\rangle}$, $x^{\langle i\rangle}$ and $y^{\langle i\rangle}$ for $i = 0, 1, \ldots, N$.

The multistage analytic approximate solutions of degree m

On the subinterval $t_0 \le t < t_1$, we assign the initial values $v^{\langle 0\rangle} = v_0$, $\theta^{\langle 0\rangle} = \theta_0$, $x^{\langle 0\rangle} = x_0$, $y^{\langle 0\rangle} = y_0$, so the analytical approximations are

$$
\begin{aligned}
&\phi_{v,m+1}\left(t - t_0; v^{\langle 0\rangle}, \theta^{\langle 0\rangle}\right),\ \phi_{\theta,m+1}\left(t - t_0; v^{\langle 0\rangle}, \theta^{\langle 0\rangle}\right),\\
&\phi_{x,m+1}\left(t - t_0; v^{\langle 0\rangle}, \theta^{\langle 0\rangle}, x^{\langle 0\rangle}\right),\ \phi_{y,m+1}\left(t - t_0; v^{\langle 0\rangle}, \theta^{\langle 0\rangle}, y^{\langle 0\rangle}\right).
\end{aligned}
\tag{2.29}
$$

On the subinterval $t_i \le t < t_{i+1}$, for $i = 1, 2, \ldots, N-1$, the initial values are determined as

$$
\begin{aligned}
v^{\langle i\rangle} &= \phi_{v,m+1}\left(t_i - t_{i-1}; v^{\langle i-1\rangle}, \theta^{\langle i-1\rangle}\right),\\
\theta^{\langle i\rangle} &= \phi_{\theta,m+1}\left(t_i - t_{i-1}; v^{\langle i-1\rangle}, \theta^{\langle i-1\rangle}\right),\\
x^{\langle i\rangle} &= \phi_{x,m+1}\left(t_i - t_{i-1}; v^{\langle i-1\rangle}, \theta^{\langle i-1\rangle}, x^{\langle i-1\rangle}\right),\\
y^{\langle i\rangle} &= \phi_{y,m+1}\left(t_i - t_{i-1}; v^{\langle i-1\rangle}, \theta^{\langle i-1\rangle}, y^{\langle i-1\rangle}\right),
\end{aligned}
\tag{2.30}
$$

and thus the analytical approximations are

$$\begin{aligned}&\phi_{v,m+1}\left(t-t_i;v^{\langle i\rangle},\theta^{\langle i\rangle}\right),\ \phi_{\theta,m+1}\left(t-t_i;v^{\langle i\rangle},\theta^{\langle i\rangle}\right),\\&\phi_{x,m+1}\left(t-t_i;v^{\langle i\rangle},\theta^{\langle i\rangle},x^{\langle i\rangle}\right),\ \phi_{y,m+1}\left(t-t_i;v^{\langle i\rangle},\theta^{\langle i\rangle},y^{\langle i\rangle}\right).\end{aligned}\tag{2.31}$$

Finally, on the whole interval $t_0 \le t \le t_N$, we can express the analytic approximations of the solutions as

$$\Phi_{v,m+1}(t)=\Phi_{v,m+1}\left(t;v_0,\theta_0\right)=\sum_{i=0}^{N-1}\phi_{v,m+1}\left(t-t_i;v^{\langle i\rangle},\theta^{\langle i\rangle}\right)\Pi(t;t_i,t_{i+1}),\tag{2.32}$$

$$\Phi_{\theta,m+1}(t)=\Phi_{\theta,m+1}\left(t;v_0,\theta_0\right)=\sum_{i=0}^{N-1}\phi_{\theta,m+1}\left(t-t_i;v^{\langle i\rangle},\theta^{\langle i\rangle}\right)\Pi(t;t_i,t_{i+1}),\tag{2.33}$$

$$\begin{aligned}\Phi_{x,m+1}(t)=&\Phi_{x,m+1}\left(t;v_0,\theta_0,x_0\right)\\=&\sum_{i=0}^{N-1}\phi_{x,m+1}\left(t-t_i;v^{\langle i\rangle},\theta^{\langle i\rangle},x^{\langle i\rangle}\right)\Pi(t;t_i,t_{i+1}),\end{aligned}\tag{2.34}$$

$$\begin{aligned}\Phi_{y,m+1}(t)=&\Phi_{y,m+1}\left(t;v_0,\theta_0,y_0\right)\\=&\sum_{i=0}^{N-1}\phi_{y,m+1}\left(t-t_i;v^{\langle i\rangle},\theta^{\langle i\rangle},y^{\langle i\rangle}\right)\Pi(t;t_i,t_{i+1}),\end{aligned}\tag{2.35}$$

which use the boxcar and modified boxcar functions,

$$\Pi(t;t_i,t_{i+1})=\begin{cases}1,\ t_i\le t<t_{i+1},\\0,\ \text{otherwise},\end{cases}\tag{2.36}$$

for $i = 0, 1, \ldots, N-2$, and

$$\Pi(t;t_{N-1},t_N)=\begin{cases}1,\ t_{N-1}\le t\le t_N,\\0,\ \text{otherwise}.\end{cases}\tag{2.37}$$

The numeric solutions of order m

We assign the initial values at $t = t_0$,

$$v^{\langle 0\rangle}=v_0,\ \theta^{\langle 0\rangle}=\theta_0,\ x^{\langle 0\rangle}=x_0,\ y^{\langle 0\rangle}=y_0,\tag{2.38}$$

and at $t = t_i$,

$$\begin{aligned}v^{\langle i\rangle}&=\phi_{v,m+1}\left(t_i-t_{i-1};v^{\langle i-1\rangle},\theta^{\langle i-1\rangle}\right),\\\theta^{\langle i\rangle}&=\phi_{\theta,m+1}\left(t_i-t_{i-1};v^{\langle i-1\rangle},\theta^{\langle i-1\rangle}\right),\\x^{\langle i\rangle}&=\phi_{x,m+1}\left(t_i-t_{i-1};v^{\langle i-1\rangle},\theta^{\langle i-1\rangle},x^{\langle i-1\rangle}\right),\\y^{\langle i\rangle}&=\phi_{y,m+1}\left(t_i-t_{i-1};v^{\langle i-1\rangle},\theta^{\langle i-1\rangle},y^{\langle i-1\rangle}\right),\end{aligned}\tag{2.39}$$

for $i = 1, 2, \ldots, N$. For the case of equal step size, $h = t_i - t_{i-1}$, $i = 1, 2, \ldots, N$, Eqs. (2.39) becomes

$$\begin{aligned} v^{\langle i \rangle} &= \phi_{v,m+1}\left(h; v^{\langle i-1 \rangle}, \theta^{\langle i-1 \rangle}\right), \\ \theta^{\langle i \rangle} &= \phi_{\theta,m+1}\left(h; v^{\langle i-1 \rangle}, \theta^{\langle i-1 \rangle}\right), \\ x^{\langle i \rangle} &= \phi_{x,m+1}\left(h; v^{\langle i-1 \rangle}, \theta^{\langle i-1 \rangle}, x^{\langle i-1 \rangle}\right), \\ y^{\langle i \rangle} &= \phi_{y,m+1}\left(h; v^{\langle i-1 \rangle}, \theta^{\langle i-1 \rangle}, y^{\langle i-1 \rangle}\right), \end{aligned} \tag{2.40}$$

for $i = 1, 2, \ldots, N$. We observe that the numeric solutions are just the values of the multistage analytic approximations evaluated on the nodes.

The procedure of the algorithm for generating multistage analytic approximations of degree m or numeric solutions of order m
(i) For $n = 0$ to $n = m - 1$, give the Adomian polynomials A_n, B_n, C_n, D_n, E_n, F_n and G_n. Taking v_0, θ_0, x_0 and y_0 as parameters, for $n = 0$ to $n = m - 1$, from Eqs (2.23)–(2.26) give v_{n+1}, θ_{n+1}, x_{n+1} and y_{n+1} in terms of v_0 and θ_0; then write down the $(m + 1)$-term approximations for the four solutions in Eqs. (2.27) and (2.28) as functions of t and initial values.
(ii) Partition the concerned range, $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, or for the case of equal step size, give the step size h and the number of subintervals N, and specify the initial velocity v_0, the launch angle θ_0 and the initial location $(x_0, y_0) = (0, 0)$; then give the multistage analytic approximations of degree m by Eqs. (2.32)–(2.35) or the numeric solutions of order m by Eqs. (2.38)–(2.40).

MATHEMATICA 11 code for generating the multistage analytic approximations of degree m or numeric solutions of order m are presented in Appendix B.

3. Error analyses for multistage approximate solutions

For the multistage analytic approximations, we introduce the following remainder error functions

$$REF_{v,m+1}(t) = \left| \frac{d}{dt}\Phi_{v,m+1}(t) + g\sin\left(\Phi_{\theta,m+1}(t)\right) + gk\Phi^2_{v,m+1}(t) \right|, \tag{3.1}$$

$$REF_{\theta,m+1}(t) = \left| \frac{d}{dt}\Phi_{\theta,m+1}(t) + g\Phi^{-1}_{v,m+1}(t)\cos\left(\Phi_{\theta,m+1}(t)\right) \right|, \tag{3.2}$$

$$REF_{x,m+1}(t) = \left| \frac{d}{dt}\Phi_{x,m+1}(t) - \Phi_{v,m+1}(t)\cos\left(\Phi_{\theta,m+1}(t)\right) \right|, \tag{3.3}$$

$$REF_{y,m+1}(t) = \left| \frac{d}{dt}\Phi_{y,m+1}(t) - \Phi_{v,m+1}(t)\sin\left(\Phi_{\theta,m+1}(t)\right) \right|. \tag{3.4}$$

We note that the multistage analytic approximations are continuous functions of t, but their first-order derivatives can exhibit jumps at the nodes t_i, $i = 1, 2, \ldots, N-1$, so they are piecewise continuous. Thus the remainder error functions in Eqs. (3.1)–(3.4) are also piecewise continuous.

We also introduce the following accumulative remainder error functions for $\Phi_{v,m+1}(t; v_0, \theta_0)$,

$$AREF_{v,m+1}(t) = \sum_{i=0}^{N-1} AREF_{v,m+1}^{\langle i \rangle}(t)\Pi(t; t_i, t_{i+1}), \tag{3.5}$$

where for $i = 0, 1, \ldots, N-1$,

$$\begin{aligned} AREF_{v,m+1}^{\langle i \rangle}(t) =& AREF_{v,m+1}^{\langle i-1 \rangle}(t_i) + \left| \frac{d}{dt}\phi_{v,m+1}\left(t - t_i; v^{\langle i \rangle}, \theta^{\langle i \rangle}\right) \right. \\ & \left. + g\sin\left(\phi_{\theta,m+1}\left(t - t_i; v^{\langle i \rangle}, \theta^{\langle i \rangle}\right)\right) + gk\phi_{v,m+1}^2\left(t - t_i; v^{\langle i \rangle}, \theta^{\langle i \rangle}\right) \right|, \end{aligned} \tag{3.6}$$

and $AREF_{v,m+1}^{\langle -1 \rangle}(t_0) = 0$; for $\Phi_{\theta,m+1}(t; v_0, \theta_0)$,

$$AREF_{\theta,m+1}(t) = \sum_{i=0}^{N-1} AREF_{\theta,m+1}^{\langle i \rangle}(t)\Pi(t; t_i, t_{i+1}), \tag{3.7}$$

where

$$\begin{aligned} AREF_{\theta,m+1}^{\langle i \rangle}(t) = AREF_{\theta,m+1}^{\langle i-1 \rangle}(t_i) + & \left| \frac{d}{dt}\phi_{\theta,m+1}\left(t - t_i; v^{\langle i \rangle}, \theta^{\langle i \rangle}\right) \right. \\ & \left. + \frac{g\cos\left(\phi_{\theta,m+1}\left(t - t_i; v^{\langle i \rangle}, \theta^{\langle i \rangle}\right)\right)}{\phi_{v,m+1}\left(t - t_i; v^{\langle i \rangle}, \theta^{\langle i \rangle}\right)} \right|, \ i = 0, 1, \ldots, N-1, \end{aligned} \tag{3.8}$$

and $AREF_{\theta,m+1}^{\langle -1 \rangle}(t_0) = 0$. Accumulative remainder error functions for $\Phi_{x,m+1}(t; v_0, \theta_0, x_0)$ and $\Phi_{y,m+1}(t; v_0, \theta_0, y_0)$ are similar and denoted as $AREF_{x,m+1}(t)$ and $AREF_{y,m+1}(t)$.

Here, the accumulative remainder error function on the ith subinterval includes the final value of the function on the previous subinterval. So the accumulative remainder error function is continuous on the interval $t_0 \le t \le t_N$.

Further, we can consider the accumulative remainder error bounds as

$$\begin{aligned} AREB_{v,m+1} &= \max_{t_0 \le t \le t_N} AREF_{v,m+1}(t), \\ AREB_{\theta,m+1} &= \max_{t_0 \le t \le t_N} AREF_{\theta,m+1}(t), \end{aligned} \tag{3.9}$$

and definitions of $AREB_{x,m+1}$ and $AREB_{y,m+1}$ are similar.

We take a group of typical data for the motion of a baseball as [14, 18]

$$v_0 = 40 \text{ m/s}, \ \theta_0 = 45^\circ, \ k = 0.000625 \text{ s}^2/\text{m}^2, \ g = 9.81 \text{ m/s}^2, \tag{3.10}$$

and take the degree $m = 6$, the equal step size $h = 0.2$ and the number of subintervals $N = 60$. The multistage analytic approximations in (3.1)–(3.4) and their remainder error and accumulative remainder error functions are plotted in Figures 2–13. For the four multistage analytic approximations, the accumulative remainder error bounds are respectively

$$4.03224 \times 10^{-5}, \ 4.21957 \times 10^{-6}, \ 4.43075 \times 10^{-6}, \ 8.65837 \times 10^{-6}.$$

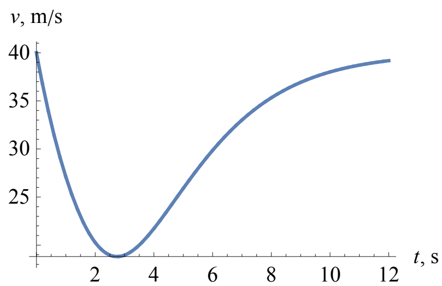

Figure 2. Solution $\Phi_{v,m+1}(t)$ on the interval $0 \le t \le 12$.

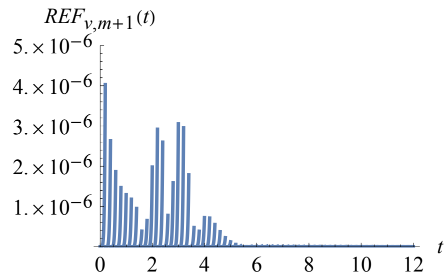

Figure 3. Remainder error function $REF_{v,m+1}(t)$.

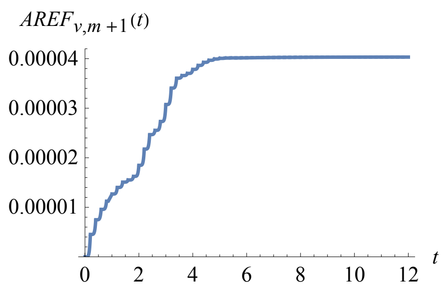

Figure 4. Accumulative remainder error function $AREF_{v,m+1}(t)$.

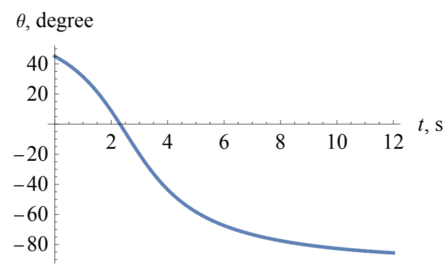

Figure 5. Solution $\Phi_{\theta,m+1}(t)$ on the interval $0 \le t \le 12$.

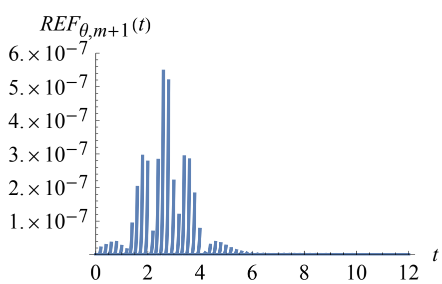

Figure 6. Remainder error function $REF_{\theta,m+1}(t)$.

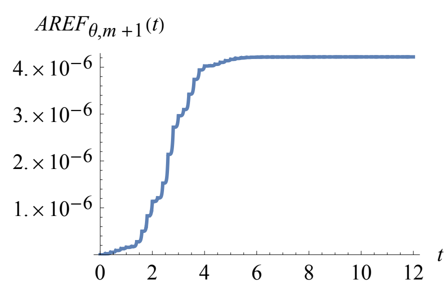

Figure 7. Accumulative remainder error function $AREF_{\theta,m+1}(t)$.

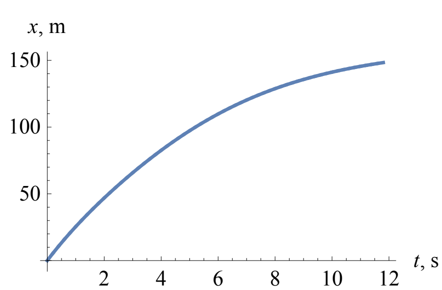

Figure 8. Solution $\Phi_{x,m+1}(t)$ on the interval $0 \le t \le 12$.

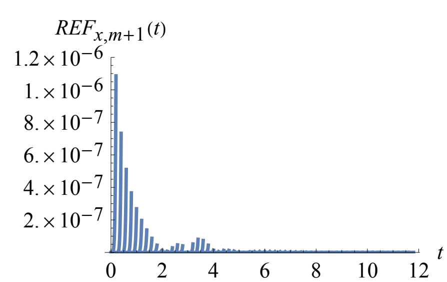

Figure 9. Remainder error function $REF_{x,m+1}(t)$.

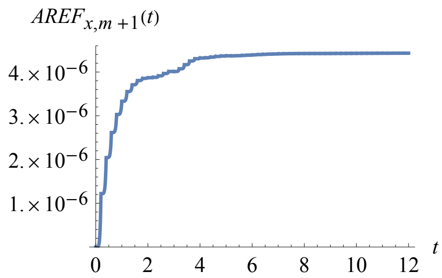

Figure 10. Accumulative remainder error function $AREF_{x,m+1}(t)$.

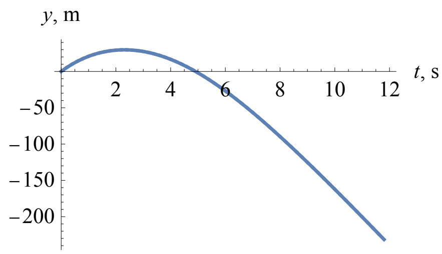

Figure 11. Solution $\Phi_{y,m+1}(t)$ on the interval $0 \leq t \leq 12$.

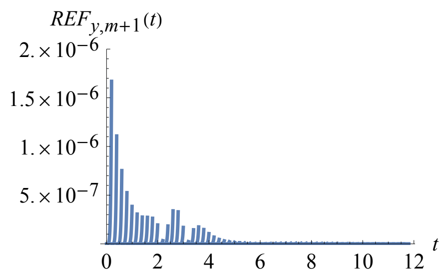

Figure 12. Remainder error function $REF_{y,m+1}(t)$.

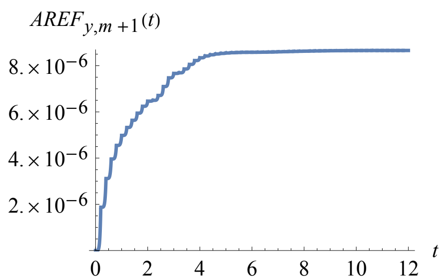

Figure 13. Accumulative remainder error function $AREF_{y,m+1}(t)$.

In Figure 14, the exact value and our computed results of v versus θ are plotted together for comparison, where the solid line is drawn using the exact relation (2.5), while the dots are plotted using our numeric solutions $(v^{\langle i\rangle}, \theta^{\langle i\rangle})$, $i = 0, 1, \ldots, 60$. In Figure 15, the absolute errors of the numeric solutions $v^{\langle i\rangle}$ versus θ in Figure 14 are plotted. In Figure 16, the absolute errors of the numeric solutions $v^{\langle i\rangle}$ versus t are plotted. From Figures 3 and 16, we find that the remainder error function and absolute errors display similar trends. In Figure 17, the curve of y versus x using the multistage analytic approximations was depicted.

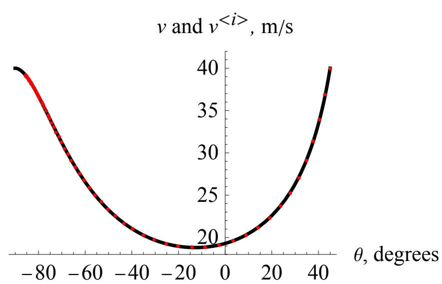

Figure 14. Comparison of the exact value (solid line) and numeric results (dots) of v versus θ.

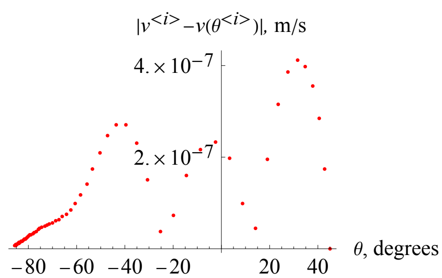

Figure 15. Absolute errors of the numeric solutions $v^{\langle i\rangle}$ versus θ in Figure 14.

In our method, there are two parameters used to control errors, the step size h and the order m. In Table 1, we display the accumulative remainder error bounds of the solution $\Phi_{v,m+1}(t; v_0, \theta_0)$ on the interval $0 \leq t \leq 32$ to show the influence of the parameters h and m on the errors.

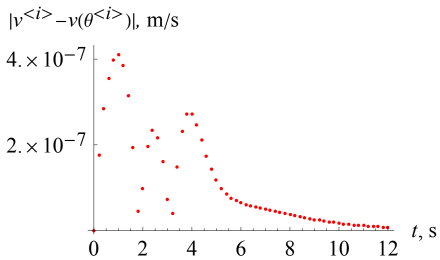

Figure 16. Absolute errors of the numeric solutions $v^{\langle i \rangle}$ versus t.

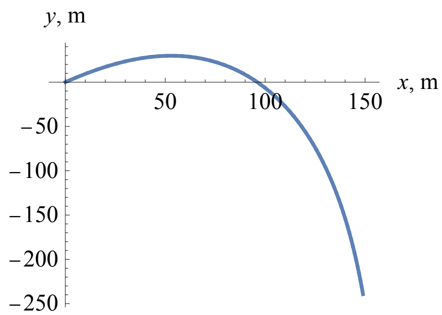

Figure 17. Curve of the Cartesian coordinates y versus x.

Table 1. The accumulative remainder error bounds of $\Phi_{v,m+1}(t; v_0, \theta_0)$ for different h and m on the interval $0 \leq t \leq 32$ for the motion of a baseball in (3.10).

h, N	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$
0.8, 40	0.474336	0.143635	0.0375755	0.017908	2.17055×10^{-3}	2.14259×10^{-3}
0.4, 80	0.059315	7.93862×10^{-3}	1.33385×10^{-3}	1.9725×10^{-4}	3.30217×10^{-5}	5.86641×10^{-6}
0.2, 160	7.15468×10^{-3}	4.56436×10^{-4}	4.03224×10^{-5}	3.07044×10^{-6}	2.62651×10^{-7}	2.24818×10^{-8}
0.1, 320	8.7059×10^{-4}	2.7722×10^{-5}	1.21805×10^{-6}	4.63575×10^{-8}	1.99314×10^{-9}	8.85405×10^{-11}
0.05, 640	1.07569×10^{-4}	1.70344×10^{-6}	3.76301×10^{-8}	7.17569×10^{-10}	1.56906×10^{-11}	6.01609×10^{-13}

4. Computation of the trajectory parameters

Using the multistage analytic approximations, we can determine the important measures for the trajectory. Let $\Phi_{\theta,m+1}(t) = 0$, we obtain the time of ascent t_a. Then the velocity at the trajectory apex, the abscissa of the trajectory apex and the maximum height of ascent are

$$v_a = \Phi_{v,m+1}(t_a),\ x_a = \Phi_{x,m+1}(t_a),\ H = \Phi_{y,m+1}(t_a). \tag{4.1}$$

Let $\Phi_{y,m+1}(t) = 0$, we obtain the motion time T, then the flight range, the impact velocity and the impact angle with respect to the horizontal are

$$L = \Phi_{x,m+1}(T),\ v_d = \Phi_{v,m+1}(T),\ \theta_d = \Phi_{\theta,m+1}(T). \tag{4.2}$$

From Figure 14, we see that the minimal velocity appears after the apex. Let $\frac{\Phi_{v,m+1}(t)}{dt} = 0$, we can obtain the moment t_m and the corresponding minimal velocity $v_{\min} = \Phi_{v,m+1}(t_m)$.

We take $v_0 = 40$ m/s, $\theta_0 = 45°$, $g = 9.81$ m/s^2, $k = 0.000625$ s^2/m^2 (for a baseball), 0.002 s^2/m^2 (for a tennisball) and 0.022 s^2/m^2 (for a shuttlecock) [16,18], respectively, to calculate the piecewise-analytic approximations by setting $m = 7$, $h = 0.1$ and $N = 60$. The results are plotted in Figure 18. Using the aforementioned method, the trajectory parameters of the three projectiles are computed and listed in Table 2.

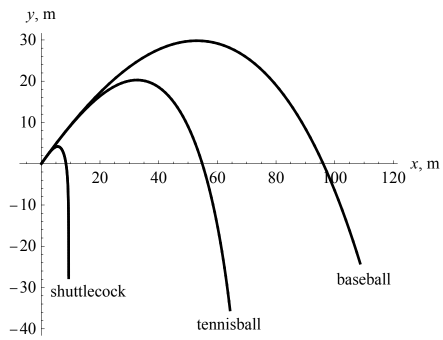

Figure 18. Trajectories of a baseball, tennisball and shuttlecock for $v_0 = 40$ m/s and $\theta_0 = 45°$.

Table 2. The trajectory parameters of a baseball, tennisball and shuttlecock for $v_0 = 40$ m/s and $\theta_0 = 45°$.

Parameters	Baseball	Tennisball	Shuttlecock	Units
t_a	2.31119	1.78169	0.660514	s
v_a	19.2996	13.0843	4.18424	m/s
x_a	53.0237	32.629	5.56502	m
H	29.8085	20.2759	4.16997	m
T	4.91223	4.0233	1.7598	s
L	96.0672	54.8577	8.41689	m
v_d	25.5291	17.4852	6.17661	m/s
θ_d	−57.273	−66.4005	−77.8645	degree (°)
t_m	2.74621	2.19267	0.804753	s
$v_{\min}$	18.8133	12.4232	3.92583	m/s

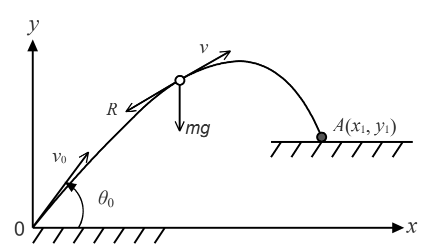

Figure 19. Schematic diagram for the flight range with the impact point on the horizontal line $y = y_1$.

5. The optimal launch angle θ_0^{opt} for the maximum flight range

Let the impact point A be on a horizontal straight line defined by the equation $y = y_1 =$ constant; see Figure 19. For given k and v_0, the flight horizontal range x is a function of the launch angle θ_0. The optimal launch angle θ_0^{opt} means the flight horizontal range has the maximum $x_{\max}$.

For specified θ_0, let $\Phi_{y,m+1}(t) = y_1$, we obtain the flight time t_1, then the flight horizontal range is $x = x(\theta_0) = \Phi_{x,m+1}(t_1)$. Next we consider the motion of baseball with the following parameters

$$v_0 = 40 \text{ m/s},\ k = 0.000625 \text{ s}^2/\text{m}^2,\ g = 9.81 \text{ m/s}^2,\ y_1 = 20, 0, -20 \text{ m},$$

and the piecewise-analytic approximations by setting $m = 7$ and $h = 0.2$.

For $y_1 = 20$, we take $N = 40$ and consider the launch angle θ_0 between 35° and 65° with 20 equal divisions by the gap $\Delta\theta = 1.5°$. For each θ_0 and corresponding flight horizontal range $x(\theta_0)$, we depict them in Figure 20. The three points with maximum x are $(47, 79.501)$, $(48.5, 79.619)$ and $(50, 79.439)$. Then we repeat the above equal division procedure for launch angle θ_0 between 47° and 50° with the new gap $\Delta\theta = 3°/20 = 0.15°$, and the results are plotted in Figure 21. The dot with maximum x is (48.35, 79.621). So $\theta_0^{\text{opt}} = 48.35°$ and $x_{\max} = 79.621$m. Here the calculation error of θ_0^{opt} is no more than 0.15°, and it can be further decreased by repeating the above method. In a similar manner, we calculate the optimal launch angle θ_0^{opt} for the maximum flight range for $y_1 = 0$ and -20, and the results are listed in Table 3.

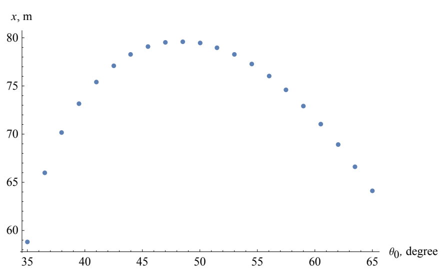

Figure 20. Plots of the horizontal flight range versus the launch angle θ_0 between 35° and 65° .

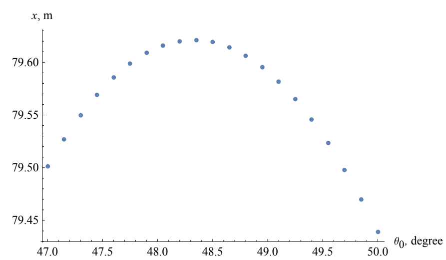

Figure 21. Plots of the horizontal flight range versus the launch angle θ_0 between 47° and 50°.

Table 3. The optimal launch angle θ_0^{opt} for the maximum flight range.

Values of y_1 (m)	Optimal launch angle θ_0^{opt} (degree)	Maximum flight range $x_{\max}$ (m)
20	48.35	79.621
0	41.05	96.816
-20	35.75	110.136

6. Conclusions

We have modeled the atmospheric flight dynamics of a projectile, using a baseball as an exemplar, subject to a quadratic drag force and then solved the resulting system of four coupled nonlinear ordinary differential equations by the multistage MDM. Next we derive the system of four coupled nonlinear recurrence relations as an intermediate step. After simultaneously obtaining the approximate analytic solutions of degree m and the numeric solutions of order m for the velocity, the tangential slope angle, and the two Cartesian coordinates x and y, the numeric solutions become the nodal values of our piecewise-analytic approximations. This approach is incorporated in the new MATHEMATICA code SOL; see Appendix B. Importantly, fast, efficient, cost-effective and accurate solutions can be found without the need to resort to high performance computing. As the nonlinear terms are not ignored or crudely linearized, a much better appreciation of the physics of a particular problem ensues. We then calculate several important aerodynamic measures or flight parameters for the trajectory of a baseball including the time of ascent, the velocity at the trajectory apex, the maximum height of ascent, the flight range, the impact velocity and the impact angle with respect to the horizontal, the optimal launch angle, and the maximum flight range. The fast algorithms by Duan for the Adomian polynomials guarantee the efficiency of our approach and yield the nonlinear effects without resort to any linearization or perturbation whatsoever unlike the usual Runge-Kutta methods which rely on a crude linearization. We find that our residual error terms are indeed quite small as are the new accumulative remainder error functions and the accumulative remainder error bounds for the numeric solutions. For example, our approximate solutions compare most favorably to the well-known exact solution due to Bernoulli. Beginning with the concept of analytic continuation, we submit that the results of all other numeric one-step methods ought to be judged by the multistage MDM and not the other way around, including both the discrete and continuous Runge-Kutta methods, since the multistage MDM has been shown to be more robust let alone more accurate.

A key concept is that the ADM series, and its subset the MDM series, are any computationally advantageous rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function, which permits solution by recursion. Our new modified recursion scheme yields an easily computable, readily verifiable and rapidly convergent sequence of analytic approximate solutions and the associated numeric solutions. Thus the ADM and the MDM subsume the classic power series method while extending the class of amenable nonlinearities to include any analytic nonlinearity and not just polynomial nonlinearities.

We have developed new, higher-order one-step methods for nonlinear differential equations. These new algorithms are derived from the Rach-Adomian-Meyers MDM and present an alternative to such classic schemes as the explicit Runge-Kutta methods for engineering models, which require high accuracy with low computational costs for repetitive simulations in contrast to a one-size-fits-all philosophy. This new approach incorporates the notion of analytic continuation, which extends the region of convergence without resort to other techniques such as the diagonal Padé approximants or the iterated Shanks transforms. Hence global solutions instead of only local solutions are directly realized in both approximate piecewise-analytic and numeric formulations. We observe that one of the difficulties in applying explicit Runge-Kutta one-step methods is that there is no general procedure to generate higher-order methods. It becomes a time-consuming, tedious endeavor to generate such higher-order formulas, because it is constrained by the traditional Picard formalism, whereas the MDM rely instead upon Adomian's operator-theoretic representation and the Adomian polynomials to permit a straightforward universal procedure to generate piecewise-analytic and higher-order numeric algorithms at will such as even a 12th-order or 24th-order one-step method, if need be. All one-step methods are derived from the notion of analytic continuation, thus the capability to extend the effective region of convergence is designed into and is therefore intrinsic to all such algorithms. Essentially, we have extended the concept of the traditional one-step method to encompass both the analytic and numeric algorithms simultaneously to adduce the new concept of the generalized Adomian one-step method, which naturally blends the two into one.

Appendix A. The first several Adomian polynomials

I. For the sine nonlinearity $Nu = \sin(u) = \sum\limits_{n=0}^{\infty} A_n$

$$
\begin{aligned}
A_0 =& \sin(u_0), \\
A_1 =& u_1 \cos(u_0), \\
A_2 =& u_2 \cos(u_0) - \tfrac{1}{2} u_1^2 \sin(u_0), \\
A_3 =& u_3 \cos(u_0) - u_1 u_2 \sin(u_0) - \tfrac{1}{6} u_1^3 \cos(u_0), \\
A_4 =& u_4 \cos(u_0) - \left(\tfrac{1}{2} u_2^2 + u_1 u_3\right) \sin(u_0) - \tfrac{1}{2} u_1^2 u_2 \cos(u_0) + \tfrac{1}{24} u_1^4 \sin(u_0), \\
A_5 =& u_5 \cos(u_0) - (u_2 u_3 + u_1 u_4) \sin(u_0) - \tfrac{1}{2}\left(u_1 u_2^2 + u_1^2 u_3\right) \cos(u_0) \\
&+ \tfrac{1}{6} u_1^3 u_2 \sin(u_0) + \tfrac{1}{120} u_1^5 \cos(u_0), \ \ldots .
\end{aligned}
$$

II. For the cosine nonlinearity $Nu = \cos(u) = \sum\limits_{n=0}^{\infty} A_n$

$$
\begin{aligned}
A_0 =& \cos(u_0), \\
A_1 =& - u_1 \sin(u_0), \\
A_2 =& - u_2 \sin(u_0) - \tfrac{1}{2} u_1^2 \cos(u_0), \\
A_3 =& - u_3 \sin(u_0) - u_1 u_2 \cos(u_0) + \tfrac{1}{6} u_1^3 \sin(u_0), \\
A_4 =& - u_4 \sin(u_0) - \cos(u_0)\left(\tfrac{1}{2} u_2^2 + u_1 u_3\right) + \tfrac{1}{2} u_1^2 u_2 \sin(u_0) + \tfrac{1}{24} u_1^4 \cos(u_0), \\
A_5 =& - u_5 \sin(u_0) - \cos(u_0)(u_2 u_3 + u_1 u_4) + \tfrac{1}{2} \sin(u_0)\left(u_1 u_2^2 + u_1^2 u_3\right) \\
&+ \tfrac{1}{6} u_1^3 u_2 \cos(u_0) - \tfrac{1}{120} u_1^5 \sin(u_0), \ \ldots .
\end{aligned}
$$

III. For the reciprocal nonlinearity $Nu = u^{-1} = \sum\limits_{n=0}^{\infty} A_n$, $u \neq 0$

$$
\begin{aligned}
A_0 =& u_0^{-1}, \\
A_1 =& - u_0^{-2} u_1, \\
A_2 =& - u_0^{-2} u_2 + u_0^{-3} u_1^2, \\
A_3 =& - u_0^{-2} u_3 + 2u_0^{-3} u_1 u_2 - u_0^{-4} u_1^3, \\
A_4 =& - u_0^{-2} u_4 + u_0^{-3}(u_2^2 + 2u_1 u_3) - 3u_0^{-4} u_1^2 u_2 + u_0^{-5} u_1^4, \\
A_5 =& - u_0^{-2} u_5 + 2u_0^{-3}(u_2 u_3 + u_1 u_4) - 3u_0^{-4}\left(u_1 u_2^2 + u_1^2 u_3\right) + 4u_0^{-5} u_1^3 u_2 - u_0^{-6} u_1^5, \\
& \ldots .
\end{aligned}
$$

Appendix B. MATHEMATICA code SOL for generating the multistage analytic approximate solutions of degree m and numeric solutions of order m

```
SOL[v0_, th0_, k_, h_, N1_, m_] :=
```

```
Module[{g, i, A, B, C1, D1, E1, F, G, v, \[Theta]}, g = 9.81;
Adco3[f_, v_, A_, M_] :=
Module[{n, j, der, c, k1}, Table[c[n, j], {n, 1, M}, {j, 1, n}];
A[0] = f[v[0]]; der = Table[D[f[v[0]], {v[0], j}], {j, 1, M}];
For[n = 1, n <= M, n++, c[n, 1] = v[n];
For[k1 = 2, k1 <= n, k1++,
c[n, k1] = Expand[1/n*
Sum[(j + 1)*v[j + 1]* c[n - 1 - j, k1 - 1], {j, 0, n - k1}]]];
A[n] = Take[der, n].Table[c[n, j], {j, 1, n}]]];
Adco3[Sin, \[Theta], A, m - 1]; Adco3[#^2 &, v, B, m - 1];
Adco3[(#^(-1)) &, v, C1, m - 1]; Adco3[Cos, \[Theta], D1, m - 1];
For[n = 0, n <= m - 1, n++,
E1[n] = Sum[C1[n - m1]*D1[m1], {m1, 0, n}];
F[n] = Sum[v[n - m1]*D1[m1], {m1, 0, n}];
G[n] = Sum[v[n - m1]*A[m1], {m1, 0, n}]; ];
For[n = 0, n <= m - 1, n++,
v[n + 1] = (-g A[n] - g k B[n])/(n + 1);
\[Theta][n + 1] = -g E1[n]/(n + 1);
x[n + 1] = F[n]/(n + 1); y[n + 1] = G[n]/(n + 1) ];
phiv[t_, v00_, \[Theta]0_] :=
Module[{}, v[0] = v00; \[Theta][0] = \[Theta]0;
Sum[v[n]*t^n, {n, 0, m}]];
phith[t_, v00_, \[Theta]0_] :=
Module[{}, v[0] = v00; \[Theta][0] = \[Theta]0;
Sum[\[Theta][n]*t^n, {n, 0, m}]];
phix[t_, v00_, \[Theta]0_, x0_] :=
Module[{}, v[0] = v00; \[Theta][0] = \[Theta]0; x[0] = x0;
Sum[x[n]*t^n, {n, 0, m}]];
phiy[t_, v00_, \[Theta]0_, y0_] :=
Module[{}, v[0] = v00; \[Theta][0] = \[Theta]0; y[0] = y0;
Sum[y[n]*t^n, {n, 0, m}]];
hh[t_, t1_, t2_] := If[t1 <= t < t2, 1, 0];
hh2[t_, t1_, t2_] := If[t1 <= t <= t2, 1, 0];
t[0] = 0; vi[0] = v0; thi[0] = th0; xi[0] = 0; yi[0] = 0;
For[i = 1, i <= N1, i++,  t[i] = i*h;
vi[i] = phiv[h, vi[i - 1], thi[i - 1]];
thi[i] = phith[h, vi[i - 1], thi[i - 1]];
xi[i] = phix[h, vi[i - 1], thi[i - 1], xi[i - 1]];
yi[i] = phiy[h, vi[i - 1], thi[i - 1], yi[i - 1]]  ];
phv = Sum[phiv[t - t[i], vi[i], thi[i]]*hh[t, t[i], t[i + 1]],
{i, 0, N1 - 2}] + phiv[t - t[N1 - 1], vi[N1 - 1], thi[N1 - 1]]*
    hh2[t, t[N1 - 1], t[N1]];
phth = Sum[phith[t - t[i], vi[i], thi[i]]*hh[t, t[i], t[i + 1]],
{i, 0, N1 - 2}] + phith[t - t[N1 - 1], vi[N1 - 1], thi[N1 - 1]]*
    hh2[t, t[N1 - 1], t[N1]];
phx = Sum[phix[t - t[i], vi[i], thi[i], xi[i]]*hh[t, t[i], t[i + 1]],
{i, 0, N1 - 2}] + phix[t - t[N1 - 1], vi[N1 - 1], thi[N1 - 1]]*
    hh2[t, t[N1 - 1], t[N1]];
phy = Sum[phiy[t - t[i], vi[i], thi[i], yi[i]]*hh[t, t[i], t[i + 1]],
```

```
{i, 0, N1 - 2}] + phiy[t - t[N1 - 1], vi[N1 - 1], thi[N1 - 1]]*
    hh2[t, t[N1 - 1], t[N1]]; ]
```

Note: The code defines a MATHEMATICA function SOL. For the parameters of the motion of a baseball in (3.10), running SOL[40, Pi/4, 0.000625, 0.1, 320, 7] generates the multistage analytic approximate solutions of degree 7 and higher-order numeric solutions of order 7 with the step size $h = 0.1$ and the number of subintervals $N = 320$ on the time interval $0 \leq t \leq 32$. The test time is 18 senconds on personal computer (Intel(R) Core(TM) i5-4570 CPU @3.20GHz, RAM 4 GB with 64-bit operating system). The multistage analytic solutions of v, θ, x and y are saved in the variables phv, phth, phx and phy, respectively. The higher-order numeric solutions are saved in the variables vi[i], thi[i], xi[i] and yi[i], i=0,1,. . . ,N.

References

[1] K. Abbaoui, Y. Cherruault and V. Seng, *Practical formulae for the calculus of multivariable Adomian polynomials*, Math. Comput. Modelling, 1995, 22, 89–93.

[2] F. Abdelwahid, *A mathematical model of Adomian polynomials*, Appl. Math. Comput., 2003, 141, 447–453.

[3] G. Adomian, *Nonlinear Stochastic Operator Equations*, Academic, Orlando, FL, 1986.

[4] G. Adomian, *Nonlinear Stochastic Systems Theory and Applications to Physics*, Kluwer Academic, Dordrecht, 1989.

[5] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic, Dordrecht, 1994.

[6] G. Adomian and R. Rach, *Inversion of nonlinear stochastic operators*, J. Math. Anal. Appl., 1983, 91, 39–46.

[7] G. Adomian and R. Rach, *Transformation of series*, Appl. Math. Lett., 1991, 4, 69–71.

[8] G. Adomian and R. Rach, *Generalization of Adomian polynomials to functions of several variables*, Comput. Math. Appl., 1992, 24, 11–24.

[9] G. Adomian and R. Rach, *Modified decomposition solution of nonlinear partial differential equations*, Appl. Math. Lett., 1992, 5, 29–30.

[10] G. Adomian and R. Rach, *Nonlinear transformation of series–Part II*, Comput. Math. Appl., 1992, 23, 79–83.

[11] G. Adomian and R. Rach, *Modified decomposition solution of linear and nonlinear boundary-value problems*, Nonlinear Anal., 1994, 23, 615–619.

[12] G. Adomian, R. Rach and N. T. Shawagfeh, *On the analytic solution of the Lane-Emden equation*, Found. Phys. Lett., 1995, 8, 161–181.

[13] M. Azreg-Aïnou, *A developed new algorithm for evaluating Adomian polynomials*, CMES-Comput. Model. Eng. Sci., 2009, 42, 1–18.

[14] P. Chudinov, *Approximate analytical description of the projectile motion with a quadratic drag force*, Athens J. Sci., 2014, 1(2), 97–106.

[15] P. Chudinov, V. Eltyshev and Y. Barykin, *Simple and convenient analytical formulas for studying the projectile motion in midair*, Rev. Bras. de Ens. de Física, 2018, 40(1), e1308.

[16] P. Chudinov, V. Eltyshev and Y. Barykin, *Analytical construction of the projectile motion trajectory in midair*, Revista de Física, 2021, 62, 79–96.

[17] P. S. Chudinov, *The motion of a heavy particle in a medium with quadratic drag force*, Int. J. Nonlinear Sci. Numer. Simul., 2002, 3, 121–129.

[18] C. Cohen, B. Darbois-Texier, G. Dupeux et al., *The aerodynamic wall*, Proc. R. Soc. A, 2014, 470, 20130497.

[19] R. Cross, *Physics of Baseball and Softball*, Springer, New York, 2011.

[20] J. S. Duan, *An efficient algorithm for the multivariable Adomian polynomials*, Appl. Math. Comput., 2010, 217, 2456–2467.

[21] J. S. Duan, *Recurrence triangle for Adomian polynomials*, Appl. Math. Comput., 2010, 216, 1235–1241.

[22] J. S. Duan, *Convenient analytic recurrence algorithms for the Adomian polynomials*, Appl. Math. Comput., 2011, 217, 6337–6348.

[23] J. S. Duan, T. Chaolu and R. Rach, *Solutions of the initial value problem for nonlinear fractional ordinary differential equations by the Rach-Adomian-Meyers modified decomposition method*, Appl. Math. Comput., 2012, 218, 8370–8392.

[24] J. S. Duan and R. Rach, *New higher-order numerical one-step methods based on the Adomian and the modified decomposition methods*, Appl. Math. Comput., 2011, 218, 2810–2828.

[25] J. S. Duan, R. Rach and A. M. Wazwaz, *A new modified Adomian decomposition method for higher-order nonlinear dynamical systems*, CMES-Comput. Model. Eng. Sci., 2013, 94, 77–118.

[26] J. S. Duan, R. Rach and A. M. Wazwaz, *Higher order numeric solutions of the Lane-Emden-type equations derived from the multi-stage modified Adomian decomposition method*, Int. J. Comput. Math., 2017, 94(1), 197–215.

[27] J. S. Duan, R. Rach and A. M. Wazwaz, *Simulation of the eigenvalue problem for tapered rotating beams by the modified decomposition method*, Int. J. Comput. Methods Eng. Sci. Mech., 2022, 23, 20–28.

[28] J. C. Hsu, H. Y. Lai and C. K. Chen, *An innovative eigenvalue problem solver for free vibration of uniform Timoshenko beams by using the Adomian modified decomposition method*, J. Sound Vibration, 2009, 325, 451–470.

[29] R. Kantrowitz and M. M. Neumann, *Parabolic sandwiches for functions on a compact interval and an application to projectile motion*, Int. J. Math. Math. Sci., 2019, 2019, 4868106.

[30] R. Rach, *A convenient computational form for the Adomian polynomials*, J. Math. Anal. Appl., 1984, 102, 415–419.

[31] R. Rach, *A new definition of the Adomian polynomials*, Kybernetes, 2008, 37, 910–955.

[32] R. Rach, *A bibliography of the theory and applications of the Adomian decomposition method, 1961-2011*, Kybernetes, 2012, 41, 1087–1148.

[33] R. Rach, G. Adomian and R. E. Meyers, *A modified decomposition*, Comput. Math. Appl., 1992, 23, 17–23.

[34] R. Rach, J. S. Duan and A. M. Wazwaz, *Simulation of large deflections of a flexible cantilever beam fabricated from functionally graded materials by the Adomian decomposition method*, Int. J. Dyn. Syst. Differ. Equ., 2020, 10(4), 287–298.

[35] S. E. Serrano, *Differential Equations: Applied Mathematical Modeling, Nonlinear Analysis, and Computer Simulation in Engineering and Science*, Hydro-Science Inc., Ambler, PA, 2016.

[36] P. Timmerman and J. P. van der Weele, *On the rise and fall of a ball with linear or quadratic drag*, Am. J. Phys., 1999, 67(6), 538–546.

[37] M. Turkyilmazoglu, *Highly accurate analytic formulae for projectile motion subjected to quadratic drag*, Eur. J. Phys., 2016, 37, 035001.

[38] A. Vial, *Horizontal distance travelled by a mobile experiencing a quadratic drag force: normalized distance and parametrization*, Eur. J. Phys., 2007, 28, 657–663.

[39] R. D. H. Warburton, J. Wang and J. Burgdörfer, *Analytic approximations of projectile motion with quadratic air resistance*, J. Serv. Sci. Manag., 2010, 3, 98–105.

[40] A. M. Wazwaz, *A new algorithm for calculating Adomian polynomials for nonlinear operators*, Appl. Math. Comput., 2000, 111, 53–69.

[41] A. M. Wazwaz, *Linear and Nonlinear Integral Equations: Methods and Applications*, Higher Education Press, Beijing, and Springer-Verlag, Berlin, 2011.

[42] Y. Zhu, Q. Chang and S. Wu, *A new algorithm for calculating Adomian polynomials*, Appl. Math. Comput., 2005, 169, 402–416.