Journal of Applied Analysis and Computation
Volume 13, Number 4, August 2023, 1702–1719
Website:http://www.jaac-online.com
DOI:10.11948/20210295

FILTER REGULARIZATION FOR AN INVERSE SOURCE PROBLEM OF THE TIME-FRACTIONAL DIFFUSION EQUATION*

Wan-Xia Shi[1,†] and Xiang-Tuan Xiong[2]

Abstract In this paper, we focus on a problem of identifying the unknown source of time-fractional diffusion equation. It is known that such problem is ill-posed in the sense that reconstructed solution does not depend continuously on the observation data. Based on this fact, a GFR (general filter regularized method) is proposed. We further give the error convergence estimates under deterministic case and random noise, respectively. Lastly, some special cases and numerical examples are presented to illustrate the efficiency of our method.

Keywords Inverse source problem, time-fractional diffusion equation, ill-posedness, general filter regularization, error estimates.

MSC(2010) 35R30, 65N20, 65M12.

1. Introduction

In this work, we consider a problem of recovering the source function $f(x)$ of the following inhomogeneous time-fractional diffusion equation

$$\begin{cases} D_t^\alpha u(x,t) = (Lu)(x,t) + f(x), & (x,t) \in \Omega \times (0,T),\ 0<\alpha<1 \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = 0, & x \in \Omega \end{cases} \tag{1.1}$$

under indirect observable data $g(x) = u(x,T)$ at the final moment $T = t$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain with sufficient smooth boundary $\partial\Omega$, α is the fractional order of the time derivative and L is a symmetric uniformly elliptic operator defined on $H^2(\Omega) \cap H_0^1(\Omega)$. In (1.1), D_t^α is the Caputo fractional time derivative defined by

$$D_t^\alpha u(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha},$$

where $\Gamma(x)$ denotes the standard Gamma function. Fractional calculus in mathematics is a natural extension of integer-order, and fractional diffusion equation can

[†]The corresponding author.
Email: shi_wan_xia@163.com(W. X. Shi), xiongxt@gmail.com(X. T. Xiong)
[1]School of Science, Lanzhou University of Technology, Langongping, 730050 Lanzhou, China
[2]School of Mathematics and Statistics, Northwest Normal University, Anning East Road, 730070 Lanzhou, China
*The authors were supported by Natural Science Foundation of China (11661072) and Science and Technology Plan Foundation of Gansu Province of China (22JR5RA302).

simulate real field data more accurately than classical diffusion equation [1, 7, 14]. As a special example of (1.1), the following fractional diffusion equation

$$D_t^\alpha u(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}u(x,t)\right), \qquad x \in \Omega,\, t > 0,\, 0 < \alpha < 1, \tag{1.2}$$

where diffusion coefficient $p(x)$ depicts the heterogeneity of the medium, is a macroscopic model from the continuous-time random walk (see, e.g., [3, 25] and the references therein). Taking $\alpha \in (0,1)$, the model (1.2) describes the slow diffusion and the fractional order α is related to the parameter specifying the large-time behavior of the waiting-time distribution function (see, e.g., [3] and the references therein).

Based on physical and practical importance, a backward problem of fractional diffusion equation is of great significance, and there have been a lot of studies on such issue [3–5, 8–10, 13, 25]. As is well known, the problem (1.1) is ill-posed in the sense of Hadamard, that is, minor changes in the measured data may deduce the blow up of the solution. In addition, small noise on the measured data may give rise to large errors. Essentially, such elements make the numerical computation much difficult, hence, a regularization proceed is required to overcome its ill-posedness and recover the stability of the solution.

For the case of exact the initial status from the observation data-provided along the final data: Liu and Yamamoto [6] presented a regularizing scheme by the quasi-reversibility for the homogeneous version of (1.1), i.e. $f \equiv 0$; Ren et al., [12] proposed a regularization by projection method where truncated level plays the role of the regularization parameter; Tuan et al. [16] investigated a general regularization method to recover the stability under deterministic case and with the random noise case for (1.1).

For the case of identifying the unknown source from the observation data provided along the final data for problem (1.1) with $f \neq 0$, we can refer to the following literatures. In [21, 22], the authors employed the quasi-reversibility regularization method and Fourier regularization method to identify the unknown source for the fractional heat diffusion equation, respectively. Wang, Zhou and Wei [18] adopted Tikhonov regularization method and a simplified Tikhonov regularization method to solve (1.1) in one-dimensional case and proposed the convergence estimates. Recently, Xiong and Xue [20] investigated an inverse heat conduction problem by a fractional Tikhonov method. Some other relative research, one can refer to [12, 15, 24]. Note that the aforementioned studies are concerned with backward problems under the deterministic case. However, at a microscopic level, the diffusion is the result of the random motion of individual particles. Therefore, it is necessary to consider the inhomogeneous problem (1.1) of identifying the unknown source under the stochastic case.

Motivated by the L'Hospital rule and [16], we explore a general filter regularization (GFR) to solve the inverse source problem of the fractional diffusion equation (1.1) with variables in a general bounded domain in both cases: the deterministic case and random noise case. Our aim of this work is to estimate (or reconstruct) the unknown source function $f(x)$ by use of the following observations:

The Deterministic case: The observed data $g(x)$ is approximated by the noisy observation data $g^\epsilon(x)$ such that

$$\|g^\epsilon - g\|_{L^2(\Omega)} \le \epsilon,$$

where $\epsilon > 0$ is the noise level.

The Random noise case: The observed data function $g^{\epsilon}(x)$ is replaced by

$$\tilde{g}_{\epsilon}(x) := g(x) + \epsilon\xi(x),$$

where ϵ also represents the noise level and $\xi(x)$ is stochastic errors which is defined in section 4.

In this paper, we mainly propose a priori and a posteriori regularization parameter choice rules using a general filter regularization. The advantage of this method is that some other existing regularization methods such as fractional Tikhonov-Regularization [11] and Landweber iterative regularization [23] can be deduced from GFR by choosing special suitable cases. Therefore, we can improve the convergence rate by comparing some different filters.

The remainder of this paper is organized as follows. In section 2, we state a general filter regularization method and some useful results. In section 3, the error estimate is obtained for the deterministic case. And the error estimates under the random noise case is shown in section 4. In section 5, we especially present that fractional Tikhonov regularization and Landweber iterative regularization can be deduced from the GFR. The numerical examples complete this paper.

2. Preliminaries

In this section, we are mainly consider about some useful lemmas and give a brief introduction of GFR (general filter regularization).

Since $-L$ is a symmetric uniformly elliptic operator, the eigenvalues of $-L$ satisfy

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_m \le \cdots$$

with $\lambda_m \to \infty$ as $m \to \infty$. Then, we have

$$\begin{cases} L\phi_m(x) = -\lambda_m \phi_m(x), & x \in \Omega, \\ \phi_m(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\phi_m \in H_0^1(\Omega) \cap H^2(\Omega)$ denotes the corresponding eigenfunctions and $\{\phi_m\}_{m=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$. Let

$$H^k(\Omega) = \left\{ u \in L^2(\Omega) : \sum_{m=1}^{\infty} \lambda_m^{2k} \left|\langle u, \phi_m\rangle\right|^2 < \infty \right\}.$$

Obviously, $H^k(\Omega)$ is a Hilbert space equipped with norm

$$\|u\|_{H^k(\Omega)} = \left(\sum_{m=1}^{\infty} \lambda_m^{2k} \left|\langle u, \phi_m\rangle\right|^2 \right)^{1/2}.$$

From [20], we can obtain the formal solution of (1.1)

$$u(x,t) = \sum_{m=1}^{\infty} \langle f, \phi_m\rangle t^{\alpha} E_{\alpha,1+\alpha}\left(-\lambda_m T^{\alpha}\right) \phi_m(x).$$

Denote $f_m = \langle f, \phi_m \rangle$ and $g_m = \langle g, \phi_m \rangle$. Then we get

$$f = \sum_{m=1}^{\infty} \frac{g_m}{T^\alpha E_{\alpha,1+\alpha}\left(-\lambda_m T^\alpha\right)} \phi_m. \tag{2.1}$$

In other words, there holds

$$\left(Kf\right)(x) := \int_\Omega k(x,\xi) f(\xi) d\xi = g(x), \qquad x \in \Omega, \tag{2.2}$$

where the kernel is $k(x,\xi) = \sum_{m=1}^{\infty} T^\alpha E_{\alpha,1+\alpha}\left(-\lambda_m T^\alpha\right) \phi_m(x)\phi_m(\xi)$. Obviously, K is a self-adjoint operator and (1.1) is ill-posed. In fact, $\frac{1}{E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)}$ is the magnifying factor of the problem, so the leading idea is to replace $\frac{1}{E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)}$ by $\frac{R(r,m)}{E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)}$ in order to approximate $f(x)$ by $f_r^\epsilon(x)$ as follows:

$$f_r^\epsilon(x) = \sum_{m=1}^{\infty} \frac{R(r,m)}{T^\alpha E_{\alpha,1+\alpha}\left(-\lambda_m T^\alpha\right)} g_m^\epsilon \phi_m(x), \tag{2.3}$$

$$f_r(x) = \sum_{m=1}^{\infty} \frac{R(r,m)}{T^\alpha E_{\alpha,1+\alpha}\left(-\lambda_m T^\alpha\right)} g_m \phi_m(x). \tag{2.4}$$

Hence, the function $R(r,m)$ is called a regularizing filter and r plays the role of the regularization parameter, which should be chosen respect to ϵ. For the stability, we make the following assumptions about $R(r,m)$.

Assumption 2.1. *For any $r \in (0,\infty)$ and $m \in \mathbb{N}$, the function $R_m(r) = R(m,r)$ satisfies*

1. *$R_m(r)$ is continuous;*
2. $\lim_{r\to\infty} R_m(r) = 0, \quad \lim_{r\to 0} R_m(r) = 1$;
3. *$R_m(r)$ is a strictly decreasing function over $(0,+\infty)$.*

Assumption 2.2. *Let $R_m(r)$ be a function such that for any $r > 0$, there exist a decreasing function $B_1(r)$ and an increasing function $B_2(r)$ satisfying*

$$\left|\frac{R_m(r)}{T^\alpha E_{\alpha,1+\alpha}\left(-\lambda_m T^\alpha\right)}\right| \le B_1(r), \qquad \left|R_m(r) - 1\right| \left|\lambda_m\right|^{-k} \le B_2(r)$$

for $k > 0, n > 0$.

Lemma 2.1 ([19]). *For any λ_m satisfying $\lambda_m \ge \lambda_1 > 0$, there exist positive constants c_1 and c_2 such that*

$$\frac{c_1}{\lambda_m T^\alpha} \le E_{\alpha,1+\alpha}\left(-\lambda_m T^\alpha\right) \le \frac{c_2}{1+\lambda_m T^\alpha} \le \frac{1}{\lambda_m T^\alpha},$$

i.e.

$$\frac{c_1}{\lambda_m} \le T^\alpha E_{\alpha,1+\alpha}\left(-\lambda_m T^\alpha\right) \le \frac{1}{\lambda_m}.$$

3. Regularization with the deterministic case

In this whole section, the observed data function $g(x)$ is approximated by $g^{\epsilon}(x)$ such that

$$\|g^{\epsilon} - g\|_{L^2(\Omega)} \leq \epsilon \tag{3.1}$$

for some known error level $\epsilon > 0$.

3.1. A-priori regularization method and convergence analysis

To derive the convergence estimates under a-priori regularization parameter choice rules, the following lemma is needed.

Lemma 3.1. *Suppose that there exist positive constants $k(>0)$ and $M(>)0$ such that $\|f\|_{H^k(\Omega)} \leq M$, and let Assumption 2.2 is valid. Then there hold*

$$\|f_r^{\epsilon} - f_r\|^2_{L^2(\Omega)} \leq B_1^2(r)\epsilon^2, \tag{3.2}$$

$$\|f_r - f\|^2_{L^2(\Omega)} \leq B_2^2(r)M^2, \tag{3.3}$$

where f_r and f_r^{ϵ} defined by (2.4) *and* (2.3)*respectively, $B_1(r)$ and $B_2(r)$ defined in Assumption 2.2.*

Proof. Firstly, it is easy to see from (3.1) that

$$\begin{aligned}
&\|f_r^{\epsilon} - f_r\|^2_{L^2(\Omega)} \\
=&\left\| \sum_{m=1}^{\infty} \frac{R_m(r)}{T^{\alpha} E_{\alpha,1+\alpha}(-\lambda_m T^{\alpha})} g_m^{\epsilon} \phi_m - \sum_{m=1}^{\infty} \frac{R_m(r)}{T^{\alpha} E_{\alpha,1+\alpha}(-\lambda_m T^{\alpha})} g_m \phi_m \right\|^2_{L^2(\Omega)} \\
=&\left\| \sum_{m=1}^{\infty} \frac{R_m(r)}{T^{\alpha} E_{\alpha,1+\alpha}(-\lambda_m T^{\alpha})} (g_m^{\epsilon} - g_m) \phi_m \right\|^2_{L^2(\Omega)} \\
=&\sum_{m=1}^{\infty} \left(\frac{R_m(r)}{T^{\alpha} E_{\alpha,1+\alpha}(-\lambda_m T^{\alpha})} \right)^2 (g_m^{\epsilon} - g_m)^2 \\
\leq& B_1^2(r)\epsilon^2.
\end{aligned}$$

By a simple calculation, we obtain

$$\begin{aligned}
\|f_r - f\|^2_{L^2(\Omega)} &= \left\| \sum_{m=1}^{\infty} \frac{R_m(r) - 1}{T^{\alpha} E_{\alpha,\alpha+1}(-\lambda_m T^{\alpha})} g_m(x) \phi_m(x) \right\|^2_{L^2(\Omega)} \\
&= \sum_{m=1}^{\infty} \left(\frac{R_m(r) - 1}{T^{\alpha} E_{\alpha,\alpha+1}(-\lambda_m T^{\alpha})} \right)^2 g_m^2(x) \\
&= \sum_{m=1}^{\infty} (R_m(r) - 1)^2 \langle f(x), \phi_m(x) \rangle^2 \\
&= \sum_{m=1}^{\infty} (R_m(r) - 1)^2 \cdot \lambda_m^{2k} \cdot \lambda_m^{-2k} \langle f(x), \phi_m(x) \rangle^2
\end{aligned}$$

$$\leq B_2^2(r)M^2.$$

The proof is complete. □

Combining (3.2) and (3.3) and by the triangle inequality, we obtain the main result of this section.

Theorem 3.1. *Assume that the conditions in Lemma 3.1 and* (3.1) *are satisfied. Then*

$$\|f_r^\epsilon(x) - f(x)\|_{L^2(\Omega)} \leq \sqrt{2}\left(B_1(r)\epsilon + B_2(r)M\right).$$

3.2. A-posteriori regularization method and convergence analysis

In this subsection, we apply a modified discrepancy principle in the following form:

$$\|Kf_r^\epsilon - g^\epsilon\|_{L^2(\Omega)} = \tau\epsilon, \tag{3.4}$$

where $\tau > 1$ is a constant.

Lemma 3.2 ([16]). *Let $R_m(r)$ satisfy Assumption 2.1 and*

$$T(r) = \|Kf_r^\epsilon - g^\epsilon\|_{L^2(\Omega)},$$

where f_r^ϵ is defined by (2.3). *Then the following results hold:*

1. *$T(r)$ is a continuous function.*
2. $\lim_{r\to 0} T(r) = 0, \quad \lim_{r\to\infty} T(r) = \|g^\epsilon\|_{L^2(\Omega)}$.
3. *$T(r)$ is a strictly increasing function over $(0, +\infty)$.*

According to the above lemma, there exists a unique solution for (3.4) provided that $\|g^\epsilon\|_{L^2(\Omega)} \geq \tau\epsilon > 0$.

Theorem 3.2. *Assume that $\|f\|_{H^k(\Omega)} \leq M (k > 0)$ and the equality* (3.4) *holds. Let $\tau > 1$ be such that*

$$\|g^\epsilon\|_{L^2(\Omega)} \geq \tau\epsilon > 0.$$

The regularization parameter $r(\epsilon)$ is chosen appropriately. Then we have

1. *If $0 < k < 1$, then*

$$\|f_r^\epsilon - f\|_{L^2(\Omega)} \leq \sqrt{2}B_1(r)\epsilon + C_1\epsilon^{\frac{k}{k+1}}, \tag{3.5}$$

where $C_1 = C_1(k, c_1, \tau, M) = \sqrt{2}M^{\frac{1}{k+1}}\left(\frac{\tau+1}{c_1}\right)^{\frac{k}{k+1}}$.

2. *If $k > 1$, then*

$$\|f_r^\epsilon - f\|_{L^2(\Omega)} \leq \sqrt{2}B_1(r)\epsilon + C_2\epsilon^{\frac{1}{2}}, \tag{3.6}$$

where $C_2 = C_2(c_1, \tau, M) = \sqrt{2}\sqrt{\tau+1}M^{\frac{1}{2}}\left(\frac{a}{c_1}\right)^{\frac{1}{2}}$.

Proof. Since

$$\|f_r^\epsilon - f\|_{L^2(\Omega)}^2$$

$$
\begin{aligned}
=&\left\|\sum_{m=1}^{\infty}\frac{R_m(r)}{T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)}g_m^\epsilon\phi_m-\sum_{m=1}^{\infty}\frac{1}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)}g_m\phi_m\right\|_{L^2(\Omega)}^2\\
\le&2\sum_{m=1}^{\infty}\left(\frac{R_m(r)}{T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)}\left(g_m^\epsilon-g_m\right)\right)^2+2\sum_{m=1}^{\infty}\left(\frac{(R_m(r)-1)}{T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)}g_m\right)^2\\
\le&2B_1^2(r)\epsilon^2+2\sum_{m=1}^{\infty}\left(\frac{(R_m(r)-1)}{T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)}g_m\right)^2,
\end{aligned}\tag{3.7}
$$

we just pay more attention to the estimate for the term $\sum_{m=1}^{\infty}\left(\frac{(R_m(r)-1)}{T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)}g_m\right)^2$.

In the case where $0<k<1$, we have

$$
\left\|\sum_{m=1}^{\infty}(R_m(r)-1)\cdot\langle f,\phi_m\rangle\cdot\phi_m\right\|_{L^2(\Omega)}\le I_1^{\frac{k}{k+1}}I_2^{\frac{1}{k+1}},\tag{3.8}
$$

where

$$
\begin{aligned}
I_1=&\left\|\sum_{m=1}^{\infty}\left((R_m(r)-1)\,T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)\right)^{k+1}\cdot(R_m(r)-1)^{1-k}\right.\\
&\left.\times\frac{\langle f,\phi_m\rangle}{(T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha))^k}\cdot\phi_m\right\|_{L^2(\Omega)},\\
I_2=&\left\|\sum_{m=1}^{\infty}(R_m(r)-1)^{1-k}\cdot\frac{\langle f,\phi_m\rangle}{(T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha))^k}\cdot\phi_m\right\|_{L^2(\Omega)}.
\end{aligned}
$$

The first term I_1 is estimated as follows:

$$
\begin{aligned}
I_1=&\left\|\sum_{m=1}^{\infty}(R_m(r)-1)^2\cdot(T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha))\cdot\langle f,\phi_m\rangle\cdot\phi_m\right\|_{L^2(\Omega)}\\
=&\left\|\sum_{m=1}^{\infty}(R_m(r)-1)^2\cdot\langle g,\phi_m\rangle\cdot\phi_m\right\|_{L^2(\Omega)}\\
\le&\left\|\sum_{m=1}^{\infty}(R_m(r)-1)^2\cdot\langle g-g^\epsilon,\phi_m\rangle\cdot\phi_m\right\|_{L^2(\Omega)}\\
&+\left\|\sum_{m=1}^{\infty}(R_m(r)-1)^2\cdot\langle g^\epsilon,\phi_m\rangle\cdot\phi_m\right\|_{L^2(\Omega)}\\
\le&(\tau+1)\epsilon,
\end{aligned}\tag{3.9}
$$

where we utilize the fact that $\|g-g^\epsilon\|_{L^2(\Omega)}\le\epsilon$ and the equality (3.4). On the other hand, we estimate I_2 as follows:

$$
\begin{aligned}
I_2=&\left\|\sum_{m=1}^{\infty}(R_m(r)-1)^{1-k}\cdot\frac{\langle f,\phi_m\rangle}{(T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha))^k}\cdot\phi_m\right\|_{L^2(\Omega)}\\
\le&\left\|\sum_{m=1}^{\infty}\frac{\langle f,\phi_m\rangle}{(T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha))^k}\cdot\phi_m\right\|_{L^2(\Omega)}
\end{aligned}
$$

$$\le \frac{1}{c_1^k}\left\|\sum_{m=1}^{\infty}\lambda_m^k\langle f,\phi_m\rangle\phi_m\right\|_{L^2(\Omega)}$$
$$\le \frac{1}{c_1^k}M. \tag{3.10}$$

Combining (3.7), (3.8), (3.9) and (3.10), we have

$$\|f_r^{\epsilon}-f\|_{L^2(\Omega)} \le \sqrt{2}B_1(r)\epsilon+\sqrt{2}\epsilon^{\frac{k}{k+1}}\cdot(\tau+1)^{\frac{k}{k+1}}\cdot\left(\frac{M}{c_1^k}\right)^{\frac{1}{k+1}},$$

which implies that (3.5) holds true.

In the case where $k \ge 1$, we have

$$\left\|\sum_{m=1}^{\infty}(R_m(r)-1)\cdot\langle f,\phi_m\rangle\cdot\phi_m\right\|_{L^2(\Omega)} \le L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}, \tag{3.11}$$

where

$$L_1=\left\|\sum_{m=1}^{\infty}\left((R_m(r)-1)\left(T^{\alpha}E_{\alpha,1+\alpha}(-\lambda_mT^{\alpha})\right)\right)^2\cdot\frac{\langle f,\phi_m\rangle}{T^{\alpha}E_{\alpha,1+\alpha}(-\lambda_mT^{\alpha})}\cdot\phi_m\right\|_{L^2(\Omega)},$$
$$L_2=\left\|\sum_{m=1}^{\infty}\frac{\langle f,\phi_m\rangle}{T^{\alpha}E_{\alpha,1+\alpha}(-\lambda_mT^{\alpha})}\cdot\phi_m\right\|_{L^2(\Omega)}.$$

In the following, we show the estimates for L_1 and L_2, respectively, that is,

$$L_1=\left\|\sum_{m=1}^{\infty}(R_m(r)-1)^2\cdot\langle g,\phi_m\rangle\cdot\phi_m\right\|_{L^2(\Omega)} \le (\tau+1)\epsilon, \tag{3.12}$$

while

$$L_2=\left\|\sum_{m=1}^{\infty}\frac{\langle f,\phi_m\rangle}{T^{\alpha}E_{\alpha,\alpha+1}(-\lambda_mT^{\alpha})}\cdot\phi_m\right\|_{L^2(\Omega)}$$
$$\le \frac{1}{c_1}\left\|\sum_{m=1}^{\infty}\lambda_m\langle f,\phi_m\rangle\phi_m\right\|_{L^2(\Omega)}$$
$$\le \frac{a}{c_1}M, \tag{3.13}$$

where we use the fact that for $k \ge 1$, there exists a positive number a such that $\|f\|_{H^1(\Omega)} \le a\|f\|_{H^k(\Omega)} \le aM$ [16]. Combining (3.7), (3.11), (3.12) and (3.13), we obtain

$$\|f_r^{\epsilon}-f\|_{L^2(\Omega)} \le \sqrt{2}B_1(r)\epsilon+\sqrt{2}\epsilon^{\frac{1}{2}}(\tau+1)^{\frac{1}{2}}\cdot\left(\frac{aM}{c_1}\right)^{\frac{1}{2}}.$$

This means (3.6) holds true. We complete the proof. □

Now, we consider the information about the regularization parameter r. From (3.4), it is not difficult to see that

$$m\epsilon=\left\|\sum_{m=1}^{\infty}(R_m(r)-1)\cdot\langle g^{\epsilon},\phi_m\rangle\cdot\phi_m\right\|_{L^2(\Omega)}$$

$$\leq \left\| \sum_{m=1}^{\infty} (R_m(r)-1) \cdot \langle g^\epsilon - g, \phi_m \rangle \cdot \phi_m \right\|_{L^2(\Omega)} + \left\| \sum_{m=1}^{\infty} (R_m(r)-1) \cdot \langle g, \phi_m \rangle \cdot \phi_m \right\|_{L^2(\Omega)}$$

$$\leq \epsilon + \left\| \sum_{m=1}^{\infty} (R_m(r)-1) \cdot (T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)) \cdot \langle f, \phi_m \rangle \cdot \phi_m \right\|_{L^2(\Omega)}$$

$$\leq \epsilon + \left\| \sum_{m=1}^{\infty} (R_m(r)-1) \cdot (T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)) \cdot \lambda_m^{-k} \cdot \lambda_m^k \cdot \langle f, \phi_m \rangle \cdot \phi_m \right\|_{L^2(\Omega)}$$

$$\leq \epsilon + \sup_{m \in \mathbb{N}} \left[|(R_m(r)-1) T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)| \, |\lambda_m|^{-k} \right] \cdot \left\| \sum_{m=1}^{\infty} \lambda_m^k \langle f, \phi_m \rangle \phi_m \right\|_{L^2(\Omega)}$$

$$\leq \epsilon + c_2 M B_2(r) T^\alpha.$$

Then there holds $B_2(r) \geq \left(\frac{\epsilon(\tau-1)}{c_2 M T^\alpha} \right)$. Since $B_2(r)$ is an increasing function, we get

$$r = B_2^{-1}(B_2(r)) \geq B_2^{-1} \left(\frac{\epsilon(\tau-1)}{c_2 M T^\alpha} \right).$$

4. Regularization with the Random noise

Let $\tilde{g}^\epsilon(x) = g(x) + \epsilon\xi$. Here $\xi_i := \langle \xi, \phi_i \rangle \sim N(0,1)$ and mutually independent. Note that

$$\tilde{f}_r^\epsilon = \sum_{m=1}^{\infty} \frac{R_m(r)}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)} \langle \tilde{g}^\epsilon, \phi_m \rangle \phi_m \tag{4.1}$$

and $\tilde{g}_m^\epsilon = \langle \tilde{g}^\epsilon, \phi_m \rangle$. According to [2], we show the following definition.

Definition 4.1. The MISE (Mean Integrated Squared Error) of $\tilde{f}_r^\epsilon$ is define by

$$E \left\| \tilde{f}_r^\epsilon - f \right\|_{L^2(\Omega)}^2 = E \left(\sum_{m=1}^{\infty} \left(\tilde{f}_{rm}^\epsilon - f_m \right)^2 \right).$$

where $\tilde{f}_{rm}^\epsilon = \langle \tilde{f}_r^\epsilon, \phi_m \rangle \text{and} f_m = \langle f, \phi_m \rangle$.

Obviously, the following result holds.

Lemma 4.1. *Assume that*

$$J_{1,m} = \frac{R_m(r)-1}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)} \langle g, \phi_m \rangle \quad \textit{and} \quad J_{2,m} = \frac{R_m(r)}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)} \langle \xi, \phi_m \rangle.$$

Then there hold

$$\sum_{m=1}^{\infty} J_{1,m}^2 = \sum_{m=1}^{\infty} \left(\frac{(R_m(r)-1)}{T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)} g_m \right)^2 \leq B_2^2(r) M^2,$$

$$E \left(\sum_{m=1}^{\infty} J_{2,m}^2 \right) = \sum_{m=1}^{\infty} \left(\frac{R_m(r)}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)} \langle \xi, \phi_m \rangle \right)^2 \leq B_1^2(r) E \|\xi\|_{L^2(\Omega)}^2$$

and $E\left(\sum_{m=1}^{\infty} J_{1,m} \cdot J_{2,m}\right) = 0$.

On the basis of the above conclusion, we have the following statement.

Theorem 4.1. *Assume that ξ satisfies*

$$E\|\xi\|^2_{L^2(\Omega)} = c_3 < \infty. \tag{4.2}$$

Then the following inequality holds:

$$E\left\|\tilde{f}^\epsilon_r - f\right\|^2_{L^2(\Omega)} \le B_2^2(r)M^2 + c_3\epsilon^2 B_1^2(r).$$

Proof. Note that

$$\begin{aligned}
\left\|\tilde{f}^\epsilon_r - f\right\|^2_{L^2(\Omega)} &= \left\|\sum_{m=1}^{\infty}\left(\tilde{f}^\epsilon_{r,m} - f_m\right)\phi_m\right\|^2_{L^2(\Omega)} \\
&= \left\|\sum_{m=1}^{\infty}\left(\frac{R_m(r)\langle g+\epsilon\xi, \phi_m\rangle}{T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)} - \frac{\langle g, \phi_m\rangle}{T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)}\right)\phi_m\right\|^2_{L^2(\Omega)} \\
&= \left\|\sum_{m=1}^{\infty}\left(\frac{(R_m(r)-1)\langle g, \phi_m\rangle}{T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)} + \frac{\epsilon R_m(r)\langle \xi, \phi_m\rangle}{T^\alpha E_{\alpha,1+\alpha}(-\lambda_m T^\alpha)}\right)\phi_m\right\|^2_{L^2(\Omega)} \\
&= \sum_{m=1}^{\infty} J_{1,m}^2 + \epsilon^2\sum_{m=1}^{\infty} J_{2,m}^2 + 2\epsilon\sum_{m=1}^{\infty} J_{1,m}J_{2,m}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
E\left\|\tilde{f}^\epsilon_r - f\right\|^2_{L^2(\Omega)} &= \sum_{m=1}^{\infty} J_{1,m}^2 + \epsilon^2\sum_{m=1}^{\infty} J_{2,m}^2 + 2\epsilon E\left(\sum_{m=1}^{\infty} J_{1,m}J_{2,m}\right) \\
&\le B_2^2(r)M^2 + c_3\epsilon^2 B_1^2(r).
\end{aligned}$$

The proof is completed. □

In what follows, we are concerned with the convergence estimate under an a-posteriori regularization parameter choice role by using the same modified discrepancy principle in the form

$$E\|K\tilde{f}^\epsilon_r - \tilde{g}^\epsilon\|_{L^2(\Omega)} = \tilde{\tau}\epsilon, \tag{4.3}$$

where $\tilde{\tau} > 1$ is a positive constant.

Lemma 4.2. *Assume that $R_m(r)$ satisfies Assumption 2.1 and let*

$$\tilde{T}(r) = E\|K\tilde{f}^\epsilon_r - \tilde{g}^\epsilon\|_{L^2(\Omega)},$$

where $\tilde{f}^\epsilon_r$ is defined by (4.1)*. Then the following statements hold.*

(1) $\tilde{T}(r)$ *is a continuous function.*

(2) $\lim_{r\to 0}\tilde{T}(r) = 0, \quad \lim_{r\to\infty}\tilde{T}(r) = E\|\tilde{g}^\epsilon\|^2_{L^2(\Omega)}$.

(3) $\tilde{T}(r)$ *is a strictly increasing function over* $(0, +\infty)$.

Proof. By virtue of

$$\|K\tilde{f}_r^\epsilon - \tilde{g}^\epsilon\|^2_{L^2(\Omega)} = \sum_{m=1}^{\infty} (R_m(r)-1)^2 \langle \tilde{g}^\epsilon, \phi_m\rangle^2$$

and the properties of $R_m(r)$, the above conclusion can be proven easily. □

According to the above lemma, there exists a unique solution for (4.3) provided that $E\|\tilde{g}^\epsilon\|^2_{L^2(\Omega)} > \tilde{\tau}\epsilon > 0$.

Theorem 4.2. *Assume that there exist positive constants M and k such that $\|f\|_{H^k(\Omega)} \le M$. Let $\tilde{f}_r^\epsilon$ satisfies*

$$E\|K\tilde{f}_r^\epsilon - \tilde{g}^\epsilon\|^2_{L^2(\Omega)} = \tilde{\tau}\epsilon$$

and

$$E\|\tilde{g}^\epsilon\|^2_{L^2(\Omega)} > \tilde{\tau}\epsilon > 0.$$

Then, there exists a regularization parameter $r(\epsilon)$ such that

$$E\left\|\tilde{f}_r^\epsilon - f\right\|^2_{L^2(\Omega)} \le C_3\epsilon^{\frac{2k}{k+1}} + c_3\epsilon^2 B_1^2(r) \qquad \text{for } k\in(0,1), \tag{4.4}$$

while

$$E\left\|\tilde{f}_r^\epsilon - f\right\|^2_{L^2(\Omega)} \le C_4\epsilon + c_3\epsilon^2 B_1^2(r) \qquad \text{for } k\ge 1, \tag{4.5}$$

where $C_3 = (\frac{c_3+\tilde{\tau}}{c_1})^{\frac{2k}{k+1}} \cdot M^{\frac{2}{k+1}}$ and $C_4 = M(\frac{a(c_3+\tilde{\tau})}{c_1})$.

Proof. Note that

$$E\left\|\tilde{f}_r^\epsilon - f\right\|^2_{L^2(\Omega)} = \sum_{m=1}^{\infty} J_{1,m}^2 + \epsilon^2 E\left\|\sum_{m=1}^{\infty} J_{2,m}^2\right\|_{L^2(\Omega)} \le \sum_{m=1}^{\infty} J_{1,m}^2 + c_3\epsilon^2 B_1^2(r).$$

When $0<k<1$, we have

$$\sum_{m=1}^{\infty} J_{1,m}^2 \le I_3^{\frac{2k}{k+1}} I_4^{\frac{2}{k+1}}, \tag{4.6}$$

where

$$I_3 = \left\|\sum_{m=1}^{\infty} (R_m(r)-1)^2 \left(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)\right) \langle f,\phi_m\rangle \phi_m\right\|_{L^2(\Omega)},$$

$$I_4 = \left\|\sum_{m=1}^{\infty} \frac{(R_m(r)-1)^{1-k}\langle f,\phi_m\rangle\phi_m}{\left(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)\right)^k}\right\|_{L^2(\Omega)}.$$

Let us first show the estimate for I_3,

$$\begin{aligned} I_3 &= \left\|\sum_{m=1}^{\infty} (R_m(r)-1)^2 \left(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)\right) \langle f,\phi_m\rangle \phi_m\right\|_{L^2(\Omega)} \\ &\le \left\|\sum_{m=1}^{\infty} (R_m(r)-1)^2 \langle g-\tilde{g}^\epsilon,\phi_m\rangle\phi_m\right\|_{L^2(\Omega)} + \left\|\sum_{m=1}^{\infty} (R_m(r)-1)^2 \langle \tilde{g}^\epsilon,\phi_m\rangle\phi_m\right\|_{L^2(\Omega)} \end{aligned}$$

$$\leq \|g-\tilde{g}^\epsilon\|_{L^2(\Omega)} + \left\| \sum_{m=1}^{\infty} (R_m(r)-1)^2 \langle \tilde{g}^\epsilon, \phi_m \rangle \phi_m \right\|_{L^2(\Omega)}$$
$$\leq (c_3+\tilde{\tau})\epsilon. \tag{4.7}$$

In addition, we have

$$\begin{aligned} I_4 &= \left\| \sum_{m=1}^{\infty} \frac{(R_m(r)-1)^{1-k} \langle f, \phi_m \rangle \phi_m}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha))^k} \right\|_{L^2(\Omega)} \\ &\leq \left\| \sum_{m=1}^{\infty} \frac{\langle f, \phi_m \rangle \phi_m}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha))^k} \right\|_{L^2(\Omega)} \\ &\leq \frac{M}{c_1^k}. \end{aligned} \tag{4.8}$$

Combining (4.6), (4.7) and (4.8), we have

$$\sum_{m=1}^{\infty} J_{1,m}^2 \leq [(c_3+\tilde{\tau})\epsilon]^{\frac{2k}{k+1}} \left[\frac{M}{c_1^k}\right]^{\frac{2}{k+1}} = \left(\frac{c_3+\tilde{\tau}}{c_1}\right)^{\frac{2k}{k+1}} \cdot \epsilon^{\frac{2k}{k+1}} \cdot M^{\frac{2}{k+1}}.$$

Thus, (4.4) holds true.

When $k \geq 1$, it follows from Hölder inequality that

$$\left\| \sum_{m=1}^{\infty} (R_m(r)-1) \langle f, \phi_m \rangle \phi_m \right\|_{L^2(\Omega)} \leq L_3^{\frac{1}{2}} L_4^{\frac{1}{2}}, \tag{4.9}$$

where

$$\begin{aligned} L_3 &= \left\| \sum_{m=1}^{\infty} (R_m(r)-1)^2 (T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha))^2 \frac{\langle f, \phi_m \rangle \phi_m}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)} \right\|_{L^2(\Omega)}, \\ L_4 &= \left\| \sum_{m=1}^{\infty} \frac{\langle f, \phi_m \rangle \phi_m}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)} \right\|_{L^2(\Omega)}. \end{aligned}$$

In the following, we show the estimates for L_3 and L_4, respectively, that is,

$$\begin{aligned} L_3 &= \left\| \sum_{m=1}^{\infty} (R_m(r)-1)^2 (T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)) \langle f, \phi_m \rangle \phi_m \right\|_{L^2(\Omega)} \\ &= \left\| \sum_{m=1}^{\infty} (R_m(r)-1)^2 \langle g, \phi_m \rangle \phi_m \right\|_{L^2(\Omega)} \\ &\leq (c_3+\tilde{\tau})\epsilon \end{aligned} \tag{4.10}$$

and

$$L_4 = \left\| \sum_{m=1}^{\infty} \frac{\langle f, \phi_m \rangle \phi_m}{T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)} \right\|_{L^2(\Omega)} \leq \frac{1}{c_1} \left\| \sum_{m=1}^{\infty} \lambda_m \langle f, \phi_m \rangle \phi_m \right\|_{L^2(\Omega)} \leq \frac{aM}{c_1}, \tag{4.11}$$

where a is a constant such that

$$\|f\|_{H^1(\Omega)} \le a\|f\|_{H^k(\Omega)} \le aM.$$

It then follows from (4.9), (4.10) and (4.11) that

$$\sum_{m=1}^{\infty} J_{1,m}^2 = \left\| \sum_{m=1}^{\infty} (R_m(r)-1)\langle f, \phi_m\rangle \phi_m \right\|_{L^2(\Omega)}^2 \le \frac{a\epsilon M}{c_1} \cdot (c_3 + \tilde{\tau}).$$

This implies that (4.5) holds true. The proof is complete. □

5. Two specific filters with their regularization solutions

Lemma 5.1. *Let* $F(z) := \frac{rz^{2\beta-k}}{rz^{2\beta}+c}(z>0)$. *Then*

$$F(z) \le \begin{cases} G_1(k)r := \frac{r}{cz^{k-2\beta}}, & k > 2\beta, \\ G_2(k)r^{\frac{k}{2\beta}} := \left(\frac{k}{c(2\beta-k)}\right)^{\frac{k}{2\beta}} \cdot r^{\frac{k}{2\beta}}, & k < 2\beta. \end{cases}$$

Proof. For $k > 2\beta$, we have

$$\frac{rz^{2\beta-k}}{rz^{2\beta}+c} = \frac{r}{rz^k + cz^{k-2\beta}} \le \frac{r}{cz^{k-2\beta}}.$$

Note that

$$\lim_{z\to 0} F(z) = \lim_{z\to\infty} F(z) = 0$$

for $k \in (0, 2\beta)$. Then there exists a point z_0 such that $F'(z_0) = 0$. Simple calculation yields $z_0 = \left(\frac{c(2\beta-k)}{kr}\right)^{\frac{1}{2\beta}}$. Hence, for all $z > 0$, there holds

$$\begin{aligned} F(z) \le F(z_0) = F\left(\left(\frac{c(2\beta-k)}{kr}\right)^{\frac{1}{2\beta}}\right) &= \frac{kr\left(\frac{c(2\beta-k)}{kr}\right)^{\frac{2\beta-k}{2\beta}}}{c(2\beta-k)+ck} \\ &\le \frac{(kr)^{\frac{k}{2\beta}}(2\beta-k)^{1-\frac{k}{2\beta}}c^{1-\frac{k}{2\beta}}}{c(2\beta-k)} \\ &= \left(\frac{k}{c(2\beta-k)}\right)^{\frac{k}{2\beta}} \cdot r^{\frac{k}{2\beta}}. \end{aligned}$$

The proof is complete. □

Corollary 5.1. *Let*

$$R_m(r) = \frac{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha))^{2\beta}}{(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha))^{2\beta} + r} \qquad \frac{1}{2} \le \beta < 1, r > 0.$$

Then $R_m(r)$ satisfies Assumption 2.2 with admissible regularization $r = r(\epsilon)$. Here $B_1(r) = \frac{1}{r}$ and

$$B_2(r) = \begin{cases} \max G_1(k), G_2(k)r, & k > 2\beta, \\ \max G_1(k), G_2(k)r^{\frac{k}{2\beta}}, & k < 2\beta, \end{cases}$$

where $G_1(k)$ and $G_2(k)$ defined in Lemma 5.1.

Proof. Note that

$$R_m(r)=\frac{\left(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)\right)^{2\beta}}{\left(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)\right)^{2\beta}+r}\leq \frac{1}{r}\left(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)\right).$$

Then $B_1(r)=\frac{1}{r}$. On the other hand, it easy to see that

$$|R_m(r)-1|\,|\lambda_m|=\frac{r\lambda_m^{-k}}{\left(T^\alpha E_{\alpha,\alpha+1}(-\lambda_m T^\alpha)\right)^{2\beta}+r}\leq\frac{r\lambda_m^{2\beta-k}}{r\lambda_m^{2\beta}+c_1^{2\beta}}.$$

Then the desired result can be obtained by Lemma 5.1. □

Based on the above results, it follows that $R_m(r)$ satisfies Assumptions 2.1 and 2.2. Hence, the corresponding regularized solution implies fractional Tikhonov regularization method, which is given in [20].

The remainder of this section will show that GFR can also be adopted to recover the ill-posedness of the time-fractional diffusion equation with unknown source type $F(x,t)=f(x)q(t)$.

Lemma 5.2 ([17]). *Denote Mittag-Leffler function by*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\qquad z\in\mathbb{C}.$$

Then

$$0<\int_0^T (T-S)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_m(T-S)^\alpha\right)ds\leq\frac{1}{\lambda_m}.$$

Lemma 5.3. *Let $f(x)=1-(1-x^2)^m$. Then for all $0<x<1$, there holds*

$$f(x)\leq\max\left\{\frac{x}{m},\left(2+\frac{1}{2m^2}\right)x\right\}.$$

Proof. We divided the proof into two parts.

If $2m^2x(1-x^2)^{m-1}<1$, then $(1-x^2)^m>1-\frac{x}{m}$, i.e., $f(x)<\frac{x}{m}$.

If $2m^2x(1-x^2)^{m-1}\geq 1$, then

$$\begin{aligned}f(x)&=1-(1-x^2)^m=1-(1-x^2)^{m-1}+x^2(1-x^2)^{m-1}\\&\leq 2-x^2(1-x^2)^{m-1}\leq\left(2+\frac{1}{2m^2}\right)x.\end{aligned}$$

The proof is complete. □

By the above results, it is easy to obtain the following corollary.

Corollary 5.2. *Let $R_m(r)=1-\left(1-aH_m^2(T)\right)^r$, where*

$$H_m(T):=\int_0^T q(s)(T-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_m(T-s)^\alpha\right)ds$$

and $a=\frac{1}{\|K\|^2}$. Then $R_m(r)$ satisfies Assumption 2.2 with admissible regularization parameter $r=r(\epsilon)$. Here $B_1(r)=\max\{\frac{\sqrt{a}}{r},\sqrt{a}(2+\frac{1}{2r^2})\}$ and $B_2(r)=\frac{1}{a\lambda_1^{2+k}}(1+ra^2\|q\|)$.

Proof. Let $x=\sqrt{a}H_m(T)$. It then follows from Lemma 5.3 that

$$\left|\frac{R_m(r)}{H_m(T)}\right| \le \max\left\{\frac{\sqrt{a}}{r}, \sqrt{a}\left(2+\frac{1}{2r^2}\right)\right\},$$

that is, $B_1(r)=\max\{\frac{\sqrt{a}}{r}, \sqrt{a}(2+\frac{1}{2r^2})\}$. Next, we consider the estimates for $B_2(r)$. Note that

$$|1-R_m(r)||\lambda_m|^{-k} = \left(1-aH_m^2(T)\right)^r \lambda_m^{-k} \le \left(1+raH_m^2(T)\right)^r \lambda_m^{-k}.$$

In view of Lemma 5.2, we have

$$|1-R_m(r)||\lambda_m|^{-k} \le \frac{\lambda_m^2+ra|q|}{\lambda_m^{2+k}} \le \frac{1+ra^2|q|}{a\lambda_1^{2+k}}.$$

The proof is complete. □

As a consequence, $R_m(r)$ satisfies Assumptions 2.1 and 2.2, and the corresponding regularized solution implies the Landweber iterative regularization solution of the following problem [23]:

$$\begin{cases} \partial_t^\alpha u(x,t) = Lu(x,t)+f(x)q(t), & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=0, & x\in\Omega,\\ u(x,T)=g(x), & x\in\Omega. \end{cases}$$

6. Numerical example

In this section, we provide some examples to show the effectiveness of our method with random noise. For the deterministic case, we can refer to [23] and so on. Consider the following problem:

$$\begin{cases} D_t^\alpha u(x,t) = u_{xx}(x,t)+f(x), & (x,t)\in\Omega\times(0,T),\ 0<\alpha<1,\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=0, & x\in\Omega. \end{cases} \tag{6.1}$$

It is easy to see that $u(x,t)=\frac{\pi}{2}t^\alpha E_{\alpha,1+\alpha}(-T^\alpha)\sin x$ is the exact solution of (6.1) with $f(x)=-\sin x$ and $u(x,T)=g(x)=\frac{\pi}{2}T^\alpha E_{\alpha,1+\alpha}(-T^\alpha)\sin 3x$. Denote the observed data defined in section 4 by $\tilde{g}^\epsilon(x)$ and let $\tilde{f}_r^\epsilon(x)$ be the regularized solution of $f(x)$ defined by (4.1). By using the same procedure of Theorem 4.1, we have

$$\begin{aligned} &E\left\|\tilde{f}_r^\epsilon - f\right\|_{L^2(\Omega)}^2\\ &=\sum_{m=1}^\infty\left(\frac{R_m(r)-1}{T^\alpha E_{\alpha,\alpha+1}\left(-\lambda_m T^\alpha\right)}\langle g,\phi_m\rangle\right)^2+\epsilon^2\sum_{m=1}^\infty\left(\frac{R_m(r)}{T^\alpha E_{\alpha,\alpha+1}\left(-\lambda_m T^\alpha\right)}\right)^2\\ &=\sum_{m=1}^\infty\left(\frac{R_m(r)-1}{T^\alpha E_{\alpha,\alpha+1}\left(-m^2 T^\alpha\right)}\right)^2 g_m^2+\epsilon^2\sum_{m=1}^\infty\left(\frac{R_m(r)}{T^\alpha E_{\alpha,\alpha+1}\left(-m^2 T^\alpha\right)}\right)^2. \end{aligned}$$

If we choose $R_m(r) = \frac{T^\alpha E_{\alpha,\alpha+1}(-m^2T^\alpha)}{r+T^\alpha E_{\alpha,\alpha+1}(-m^2T^\alpha)}$, then

$$E\left\|\tilde{f}_r^\epsilon - f\right\|^2_{L^2(\Omega)} = \sum_{m=1}^{\infty}\left(\frac{r}{T^\alpha E_{\alpha,\alpha+1}\left(-m^2T^\alpha\right)\left(r+T^\alpha E_{\alpha,\alpha+1}\left(-m^2T^\alpha\right)\right)}\right)^2 g_m^2 + \epsilon^2\sum_{m=1}^{\infty}\left(\frac{1}{r+T^\alpha E_{\alpha,\alpha+1}\left(-m^2T^\alpha\right)}\right)^2.$$

Obviously, $g_m = \frac{\pi}{2}E_{\alpha,\alpha+1}(-1)$ if $T=1$. In view of $r=\epsilon^{\frac{1}{3}}$, we have

$$E\left\|\tilde{f}_r^\epsilon - f\right\|^2_{L^2(\Omega)} = \left(\frac{\epsilon^{\frac{1}{3}}}{E_{\alpha,\alpha+1}(-1)\left(\epsilon^{\frac{1}{3}}+E_{\alpha,\alpha+1}(-1)\right)}\right)^2\left(\frac{\pi}{2}(E_{\alpha,\alpha+1}(-1))\right)^2 + \left(\frac{\epsilon}{\epsilon^{\frac{1}{3}}+E_{\alpha,\alpha+1}(-1)}\right)^2.$$

In the following, we represent the MISE for Example (6.1) of different parameters where the observation data is obtained at $T=1$, see Table 1.

Table 1. Parameters values in simulation

$E\|f_r^\epsilon - f\|^2_{L^2(\Omega)}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$	$\epsilon = 10^{-9}$
$\alpha = 0.2$	0.2069	0.0161	8.4962e-004	8.7873e-006
$\alpha = 0.4$	0.1916	0.0146	7.6496e-004	7.8980e-006
$\alpha = 0.8$	0.1670	0.0122	6.3561e-004	6.5438e-006
$\alpha = 0.9$	0.1627	0.0118	6.1398e-004	6.3179e-006

The above computations yield the following important facts: The regularization methods given in this paper work well for even acceptable error levels. The regularized solution converges to the exact solution with different values of α. The interest significance is that the numerical accuracy becomes better as the order of the fractional derivative increases.

7. Conclusion

An inverse source problem of the time-fractional diffusion equation is considered in this paper. Based on the conditional stability, we propose a general filter regularization method to deal with it and prove the error estimate under the a-priori and a-posterior regularization parameters choice rules. The numerical examples also illustrate the effectiveness of this method.

Acknowledgements

The authors would like to thank the reviewers for their comments which improves the quality of this manuscript.

This research is partially supported by the Technology Plan Foundation of Gansu Province of China (No. 22JR5RA302) and Natural Science Foundation of China (No. 11661072).

References

[1] B. Berkowitz, H. Scher and S. E. Silliman, *Anomalous transport in laboratory-scale heterogeneous porous media,* Water Resour. Res., 2000, 36(1), 149–158.

[2] L. Cavalier, *Inverse problems in statistics,* Heidelberg, April, 2007.

[3] J. Cheng, J. Nakagawa, M. Yamamoto and T. Yamazaki, *Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation,* Inverse Probl., 2009, 25(11), 115002.

[4] G. Chi, G. Li and X. Jia, *Numerical inversions of a source term in the FADE with a Dirichlet boundary condition using final observations,* Comput. Math. Appl., 2011, 62(4), 1619–1626.

[5] B. Jin and W. Rundell, *An inverse problem for a one-dimensional time-fractional diffusion problem,* Inverse Prob., 2012, 28, 075010.

[6] J. Liu and M. Yamamoto, *A backward problem for the time-fractional diffusion equation,* Appl. Anal., 2010, 89(11), 1769–1788.

[7] R. Metzler and J. Klafter, *Subdiffusive transport close to thermal equilibrium: from the Langevin equation to fractional diffusion,* Phys. Rev. E., 2000, 61, 6308–6311.

[8] D. Murio, *Stable numerical solution of a fractional-diffusion inverse heat conduction problem,* Comput. Math. Appl., 2007, 53, 1492–1501.

[9] D. Murio and C. E. Mejia, *Source terms identification for time fractional diffusion equation,* Rev. Colomb. Mat., 2008, 42(1), 25–46.

[10] D. Murio, *Stable numerical solution of a fractional-diffusion inverse heat conduction problem,* Comput. Math. Appl., 2007, 53, 1492–1501.

[11] H. Qin and T. Wei, *Some filter regularization methods for a backward heat conduction problem,* Appl. Math. Comput., 2011, 217(24) , 10317–10327.

[12] C. Ren, X. Xu and S. Lu, *Regularization by projection for a backward problem of the time fractional diffusion equation,* J. Inverse Ill-Posed Probl., 2014, 22, 121–139.

[13] K. Sakamoto and M. Yamamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems,* J. Math. Anal. Appl., 2011, 382(1), 426–447.

[14] I. Sokolov and J. Klafter, *From diffusion to anomalous diffusion: a century after Einsteins Brownian motion,* Chaos., 2005, 15(2), 1–7.

[15] S. Tatar, R. Tinaztepe and S. Ulusoy, *Determination of an unknown source term in a space-time fractional diffusion equation,* J. Frac. Calc. Appl., 2015, 6(2), 94–101.

[16] N. Tuan, M. Kirane, B. Bin-Mohsin and P. Tam, *Filter regularization for final value fractional diffusion problem with deterministic and random noise,* Comput. Math. Appl., 2017, 74, 1340–1361.

[17] N. Tuan, L. Long, N. Thinh and T. Tran, *On a final value problem for the time-fractional diffusion equation with inhomogeneous source,* Inverse Probl. Sci. Eng., 2017, 25(9), 1367–1395.

[18] J. Wang, Y. Zhou and T. Wei, *Two regularization methods to identify a space-dependent source for the time-fractional diffusion equation,* Appl. Numer. Math., 2013, 68, 39–57.

[19] J. Wang and T. Wei, *Quasi-reversibility method to identify a space-dependent source for the time-fractional diffusion equation,* Appl. Math. Model., 2015, 39, 6139–6149.

[20] X. Xiong and X. Xue, *A fractional Tikhonov regularization method for identifying a space-dependent source in the time-fractional diffusion equation,* Appl. Math. Comput., 2019, 349, 292–303.

[21] F. Yang and C. Fu, *The quasi-reversibility regularization method for identifying the unknown source for time fractional diffusion equation,* Appl. Math. Model., 2015, 39, 1500–1512.

[22] F. Yang, C. Fu and X. Li, *The inverse source problem for time fractional diffusion equation: stability analysis and regularization,* Inverse Probl. Sci. Eng., 2015, 23(6), 969–996.

[23] F. Yang, X. Liu, X. Li and C. Ma, *Landweber iterative regularization method for identifying the unknown source of the time-fractional diffusion equation,* Adv. Differ. Equ., 2017, 388.

[24] S. Yeganeh, R. Mokhtari and J. Hesthaven, *Space-dependent source determination in a time-fractional diffusion equation using a local discontinuous Galerkin method,* BIT Numer. Math., 2017, 57, 685–707.

[25] Y. Zhang and X. Xu, *Inverse source problem for a fractional diffusion equation,* Inverse Probl., 2011, 27, 035010.