Journal of Applied Analysis and Computation Website:http://www.jaac-online.com
Volume 13, Number 4, August 2023, 1720–1738 DOI:10.11948/20210397

FINITE-TIME STABILITY OF NONAUTONOMOUS AND AUTONOMOUS LINEAR SYSTEMS*

Minghui Jiang[1,2,†], Xue Fang[1] and Junhao Hu[3]

Abstract In this research, in view of Lyapunov theory, the finite time stability (FTS) conditions of linear time-delay systems are investigated. Firstly, by using matrix inequality and algebraic inequality methods, the conditions for FTS of nonautonomous and autonomous systems are given respectively. Compared with the existing literature, the judging conditions are easier to verify and have a better conservative type. In addition, by employing the provided FTS theoretical results, several novel criteria for ensuring the stabilization of autonomous delay systems and the stability of impulsive switched nonautonomous time-varying systems are obtained. Eventually, several concrete examples are put forward to validate the theoretical findings.

Keywords Finite time stability, time delay, Lyapunov functional, impulsive switching.

MSC(2010) 34-XX, 93D40.

1. Introduction

As we all know, the stability of system is always the first problem to be considered in control system theory. In 1953, Kamenkov first advanced a new theory of finite time stability (FTS) [22], before this concept was put forwarded, asymptotic stability was the object of widespread research. However, the transient performance of asymptotic stability was very poor, and the transient performance of the system could not be fed back within the specified time interval. It has a bad influence on engineering, and even can not be applied at all in practical engineering, such as communication network system, robot control system, etc. After the concept of FTS is proposed, these have been improved. In the past few decades, this has aroused considerable amount of researchers' interest in the FTS of the system, and obtained relevant conclusions [1–9, 15–18, 25, 28, 32–34, 36–38].

In many practical systems, the delay of system state is unavoidable which has a huge impact on the FTS of the system. Therefore, it is meaningful to take the

† The corresponding author.
Email: jiangminghui@ctgu.edu.cn(M. Jiang), 1712723504@qq.com(X. Fang), junhaomath@163.com(J. Hu)
[1] Institute of Nonlinear Complex Systems, China Three Gorges University, YiChang, Hubei 443000, China
[2] Three Gorges Mathematical Research Center, China Three Gorges University
[3] College of Mathematics and Statistics, South Central University for Nationalities, Wuhan, Hubei,430074 China
* This work was supported by the National Natural Science Foundation of China (61876192).

FTS of time-delay systems into consideration [10–13, 19, 20, 24, 26, 27, 35]. In [19], several new explicit conditions for FTS of nonautonomous linear time-varying delay system have been gleaned from using M-matrix theory. Based on M-matrix method, Hien et al. [20] resulted new FTS conditions for a type of nonautonomous neural networks with non-uniform proportional delay. In [35], a sufficient criterion of FTS for nonautonomous linear time-delay systems has been gleaned from constructing an appropriate function and combining with M-matrix theory. In view of the Lyapunov-Razumikhin method, Li et al. [26] investigated the FTS problem of time-delay systems and obtained a few sufficient criteria. Nevertheless, it is often difficult to verify M-matrix and Lyapunov-Razumikhin condition.

In recent years, these basic work on FTS have been further extended to systems involving time delay [12,24,27], impulsive [7–9,32,36] and switching [15,33]. From a control point of view, the problem of FTS, that is, finite time stabilization, has done a lot of interesting work in the past decades, see [21, 30, 39]. In practice, it is easier for the switching system to jump at each switching moment. Therefore, in this case, we introduce impulsive into the switching system, which is called impulsive switched system [14, 23, 29, 31, 40]. In [14], an indefinite Lyapunov function is employed to verify the FTS of linear time-varying system. After that, the findings are enlarged to the FTS issue of impulsive switched nonautonomous linear time-varying system.

Inspired by the above discussion, on the one hand, in this paper, without using M-matrix theory and Lyapunov-Razumikhin condition, several algebraic criteria on the FTS of the time-delay system, which are easier to examine than those in [26, 27, 35], are obtained by Lyapunov method and inequality techniques. On the other hand, the novel FTS criterion is applied to the stabilization of constant delay system, and the stability of nonautonomous systems with impulsive switching. Among them, the impulsive phenomenon considered in this article not only occurs at the switching time, but also occurs at any time within the subsystem.

This research is structured as follows. The necessary definition as well as model description are presented in Section 2. The main results on FTS of the time-delay system have been put forwarded in Section 3. In Section 4, two applications are put forwarded. In Section 5, numerical simulation examples are put forwarded. Lastly, Section 6 concludes the research.

Notations. Throughout this research, $\mathcal{N}$, $\Re$, and $\Re^+$ represent the set of natural numbers, real numbers, and non-negative numbers, respectively. $\Re^{\mathfrak{n}}$ the n-dimensional Euclidean vector space, and $C([-\vartheta,0],\Re^{\mathfrak{n}})$ is the Banach space of all continuous functions, where ϑ is a positive number. For real symmetric matrices Y and Z, $Y<Z(Y\leq Z)$ means that $Z-Y$ is positive definite (positive semi-definite). Y^T denotes the transpose of Y, I denotes identity matrix, and "$*$" represents symmetric terms in a symmetric matrix. $\mu(t)=\big(\mu_1(t),\ \mu_2(t),\ \cdots,\ \mu_n(t)\big)^T\in\Re^{\mathfrak{n}}$ with the $\|\mu(t)\|_W=\mu^T(t)W\mu(t)$, where $W>0$, and $\dot{\mu}(t)$ means $\frac{\mathrm{d}\mu}{\mathrm{d}t}$. $\varphi\in C([-\vartheta,0],\Re^{\mathfrak{n}})$ denote the initial function and writes $\|\varphi\|_W^{\vartheta}=\sup_{\nu\in[-\vartheta,0]}|\varphi^T(\nu)W\varphi(\nu)|$,($\aleph(t)\vee 0$) denotes $\max\{\aleph(t),0\}$.

2. Preliminaries

Take the linear system with time delay into consideration, as follows:

$$\begin{cases} \dot{\mu}(t) = f(t, \mu_t), & t \in [0, \mathcal{T}], \\ \mu_0(t) = \varphi(t), & t \in [-\vartheta, 0], \end{cases} \tag{2.1}$$

where $\mathcal{T} > 0,\ \mu(t) \in \mathfrak{R}^{\mathfrak{n}},\ \mu_t \in C([-\vartheta, \mathcal{T}], \mathfrak{R}^{\mathfrak{n}})$ is represented by $\mu_t(\epsilon) = \mu(t+\epsilon)$ for $-\vartheta \le \epsilon \le 0$, furthermore, $f(t, \mu_t)$ is continuous with t.

We always assume that the solution of the systems discussed in the paper exists.

There are several definitions used in the subsequent parts of this research.

Definition 2.1. Three scalars $0 < \mathcal{T},\ 0 < \beta_1 < \beta_2$ and a matrix $W > 0$ are given. The system (2.1) is described as FTS subject to $(\beta_1, \beta_2, W, \mathcal{T})$, if $\|\varphi\|_W^{\vartheta} < \beta_1 \Rightarrow \|\mu(t)\|_W < \beta_2,\ \forall\ t \in [0, \mathcal{T}]$.

Definition 2.2. $\omega : \mathfrak{R}^+ \to \mathfrak{R}^+$ is called a κ-function if it is strictly increasing and continuous with $\omega(0) = 0$, and denotes $\omega \in \kappa$.

3. Main results

Theorem 3.1. *Suppose functions $\omega_1, \omega_2 \in \kappa$, and give scalars $0 < \mathcal{T},\ 0 < \beta_1 < \beta_2$. The system (2.1) is FTS subject to $(\beta_1, \beta_2, W, \mathcal{T})$, if there exist an integrable function $\aleph(t) : \mathfrak{R}^+ \to \mathfrak{R}$ and a function $\mathcal{G}(t, \mu(t)) : [-\vartheta, \mathcal{T}] \times \mathfrak{R}^{\mathfrak{n}} \to \mathfrak{R}^+$ is differentiable, satisfying*
(a) $\omega_2(\|\mu(t)\|_W) \le \mathcal{G}(t, \mu(t))$, and $\omega_1(\|\varphi\|_W^{\vartheta}) \ge \mathcal{G}(0, \mu(0))$, for $\forall (t, \varphi) \in [-\vartheta, \mathcal{T}] \times \mathfrak{R}^{\mathfrak{n}}$;
(b) $\dot{\mathcal{G}}(t, \mu(t)) \le \aleph(t)\mathcal{G}(t, \mu(t))$, for $t \in [0, \mathcal{T}]$;
(c) $\ln \frac{\omega_2(\beta_2)}{\omega_1(\beta_1)} \ge \int_0^t \aleph(\nu)d\nu$, for $t \in [0, \mathcal{T}]$.

Proof. Based on the condition (b), one has

$$\dot{\mathcal{G}}(x, \mu(x)) - \aleph(x)\mathcal{G}(x, \mu(x)) \le 0, \text{ for } x \in [0, \mathcal{T}]. \tag{3.1}$$

Multiplying (3.1) by $e^{-\int_0^x \aleph(\nu)d\nu}$, it yields

$$e^{-\int_0^x \aleph(\nu)d\nu}\dot{\mathcal{G}}(x, \mu(x)) - e^{-\int_0^x \aleph(\nu)d\nu}\aleph(x)\mathcal{G}(x, \mu(x)) \le 0. \tag{3.2}$$

Integrating from 0 to t on both sides of (3.2), one has

$$\mathcal{G}(t, \mu(t)) \le e^{\int_0^t \aleph(\nu)d\nu}\mathcal{G}(0, \varphi(0)). \tag{3.3}$$

Applying the conditions (a) and (c) to (3.3) yields

$$\begin{aligned} \omega_2(\|\mu(t)\|_W) &\le \mathcal{G}(t, \mu(t)) \le \mathcal{G}(0, \varphi(0))e^{\int_0^t \aleph(\nu)d\nu} \\ &\le \omega_1(\|\varphi\|_W^{\vartheta})e^{\ln \frac{\omega_2(\beta_2)}{\omega_1(\beta_1)}}, \text{ for } t \in [0, \mathcal{T}]. \end{aligned} \tag{3.4}$$

If $\|\varphi\|_W^{\vartheta} \le \beta_1$, for $\omega_1 \in \kappa$, then

$$\omega_2(\|\mu(t)\|_W) \le \omega_1(\beta_1)\frac{\omega_2(\beta_2)}{\omega_1(\beta_1)} = \omega_2(\beta_2), \text{ for } t \in [0, \mathcal{T}]. \tag{3.5}$$

Due to $\omega_2 \in \kappa$, we get

$$\|\mu(t)\|_W \leq \beta_2, \text{ for } t \in [0, \mathcal{T}]. \tag{3.6}$$

By the definition 2.1, we know that system (2.1) is FTS subject to $(\beta_1, \beta_2, W, \mathcal{T})$. This concludes the proof. $\square$

Remark 3.1. The function $\aleph(t) : \mathfrak{R}^+ \to \mathfrak{R}$ in condition (b) of Theorem 3.1 may be a sign-changing function. That is to say, to ensure FTS, there is no requirement for $\dot{\mathcal{G}}(t, \mu(t))$ to be negative definite or semi-negative definite. The FTS result of system (2.1) with time delay in Theorem 3.1 is simpler than Theorem 1 in [26], and it does not require the restriction of Razumikhin type condition which is difficult to verify. Therefore, the result in the article improve these shortcomings and offer new methods to handle time-delay systems.

Take the linear system with time-varying delay into consideration, as follows:

$$\begin{cases} \dot{\mu}(t) = A(t)\mu(t) + Z(t)\mu(t - \rho(t)), & 0 \leq t, \\ \mu_0(t) = \varphi(t), & t \in [-\vartheta, 0], \end{cases} \tag{3.7}$$

where $\mu(t) \in \mathfrak{R}^{\mathrm{n}}$ is the state vector, $\varphi \in C([-\vartheta, 0], \mathfrak{R}^{\mathrm{n}})$ is the initial condition and $A(t), Z(t) \in \mathfrak{R}^{\mathrm{n}\times\mathrm{n}}$ are matrix-valued functions.

Applying Theorem 3.1 to the system (3.7) with constant delay $\rho(t) = \vartheta$, we have

Corollary 3.1. *Given three scalars $0 < \mathcal{T},\ 0 < \beta_1 < \beta_2$ and a matrix $W > 0$. If there exist scalar function $\aleph(t)$ and matrices $H > 0,\ Q > 0$, such that*

(a) $\begin{pmatrix} A^T(t)H + HA(t) + \frac{1}{\vartheta}Q - \aleph(t)H & Z^T(t)H \\ * & -\frac{1}{\vartheta}Q \end{pmatrix} \leq 0,\ t \in [0, \mathcal{T}],$

(b) $\int_0^t (\aleph(\nu) \vee 0) d\nu \leq \ln \frac{\lambda_2\beta_2}{\lambda_1\beta_1},\ t \in [0, \mathcal{T}]$

where $\lambda_1 = \lambda_{\max}(\widetilde{H}) + \lambda_{\max}(\widetilde{Q}),\ \lambda_2 = \lambda_{\min}(\widetilde{H}),\ \widetilde{H} = W^{-\frac{1}{2}}HW^{-\frac{1}{2}},\ \widetilde{Q} = W^{-\frac{1}{2}}QW^{-\frac{1}{2}}$. *Then system (3.7) with constant $\vartheta > 0$ is FTS subject to $(\beta_1, \beta_2, W, \mathcal{T})$.*

Proof. The Lyapunov functional candidates are constructed as

$$\mathcal{G}(t, \mu(t)) = \mu^T(t)H\mu(t) + \frac{1}{\vartheta}\int_{t-\vartheta}^{t} \mu^T(\nu)Q\mu(\nu)d\nu. \tag{3.8}$$

Then, we have

$$\begin{aligned} \mathcal{G}(0, \mu(0)) &= \mu^T(0)H\mu(0) + \frac{1}{\vartheta}\int_{-\vartheta}^{0} \mu^T(\nu)Q\mu(\nu)d\nu \\ &= \mu^T(0)W^{\frac{1}{2}}(W^{-\frac{1}{2}}HW^{-\frac{1}{2}})W^{\frac{1}{2}}\mu(0) + \\ &\quad \frac{1}{\vartheta}\int_{-\vartheta}^{0} \mu^T(\nu)W^{\frac{1}{2}}(W^{-\frac{1}{2}}QW^{-\frac{1}{2}})W^{\frac{1}{2}}\mu(\nu)d\nu \\ &\leq \lambda_{\max}(\widetilde{H})\mu^T(0)W\mu(0) + \frac{\lambda_{\max}(\widetilde{Q})}{\vartheta}\int_{-\vartheta}^{0} \mu^T(\nu)W\mu(\nu)d\nu \\ &\leq (\lambda_{\max}(\widetilde{H}) + \lambda_{\max}(\widetilde{Q}))\|\varphi\|_W^{\vartheta}. \end{aligned} \tag{3.9}$$

Moreover, one can get

$$\mathcal{G}(t,\mu(t)) \geq \mu^T(t)W^{\frac{1}{2}}(W^{-\frac{1}{2}}HW^{-\frac{1}{2}})W^{\frac{1}{2}}\mu(t) \geq \lambda_{\min}(\widetilde{H})\|\mu(t)\|_W. \tag{3.10}$$

Let $\omega_1(\nu)=\lambda_1\nu, \omega_2(\nu)=\lambda_2\nu$, the condition (a) of Theorem 3.1 holds. Taking the derivative of $\mathcal{G}(t,\mu(t))$ in equation (3.8) along system (3.7) and calculate as

$$\begin{aligned}\dot{\mathcal{G}}(t,\mu(t)) &= \mu^T(t)H\dot{\mu}(t) + \dot{\mu}^T(t)H\mu(t) + \frac{1}{\vartheta}\mu^T(t)Q\mu(t) - \frac{1}{\vartheta}\mu^T(t-\vartheta)Q\mu(t-\vartheta)\\ &= \mu^T(t)(A^T(t)H + HA(t) + \frac{1}{\vartheta}Q)\mu(t) + \mu^T(t)(Z^T(t)H + HZ(t))\mu(t-\vartheta)\\ &\quad -\frac{1}{\vartheta}\mu^T(t-\vartheta)Q\mu(t-\vartheta).\end{aligned} \tag{3.11}$$

Then, according to condition (a), it can be concluded that

$$\begin{aligned}&\dot{\mathcal{G}}(t,\mu(t)) - \aleph(t)\mathcal{G}(t,\mu(t))\\ =\ &\mu^T(t)(A^T(t)H + HA(t) + \frac{1}{\vartheta}Q - \aleph(t)H)\mu(t) + \mu^T(t)(Z^T(t)H + HZ(t))\mu(t-\vartheta)\\ &-\frac{1}{\vartheta}\mu^T(t-\vartheta)Q\mu(t-\vartheta) + \frac{-\aleph(t)}{\vartheta}\int_{t-\vartheta}^{t}\mu^T(\nu)Q\mu(\nu)d\nu\\ \leq\ &\xi^T(t)\begin{pmatrix} A^T(t)H + HA(t) + \frac{1}{\vartheta}Q - \aleph(t)H & Z^T(t)H\\ * & -\frac{1}{\vartheta}Q\end{pmatrix}\xi(t)\\ &+\Big((-\aleph(t))\vee 0\Big)\mathcal{G}(t,\mu(t))\\ \leq\ &\Big((-\aleph(t))\vee 0\Big)\mathcal{G}(t,\mu(t)),\end{aligned} \tag{3.12}$$

where $\xi(t) = \Big(\mu(t) \quad \mu(t-\vartheta)\Big)^T$. Therefore,

$$\dot{\mathcal{G}}(t,\mu(t)) \leq \Big(\aleph(t)\vee 0\Big)\mathcal{G}(t,\mu(t)). \tag{3.13}$$

So, the condition (b) of Theorem 3.1 is also true. Moreover, the condition (b) of Corollary 3.1 is in common with the condition (c) of Theorem 3.1. For this reason, system (3.7) with constant delay is FTS. The proof is accomplished. □

Remark 3.2. The algebraic criteria of FTS for time-delay systems are proposed based on M-matrix theory in [19,20], but they are difficult to verify. In this paper, that one are given by LMI (linear matrix inequality) matrix theory, which are easy to be implemented by algorithm.

Similarly, for system (3.7) with time-varying delay satisfying $\vartheta \geq \rho(t) > 0, 1 > \delta \geq \dot{\rho}(t)$, we have

Corollary 3.2. *Three scalars $0 < \mathcal{T},\ 0 < \beta_1 < \beta_2$ and a matrix $W > 0$ are given. The system (3.7) with varying-time delay $\rho(t)$ is FTS subject to $(\beta_1, \beta_2, W, \mathcal{T})$, if there exist scalar function $\aleph(t)$ and matrices $0 < H,\ 0 < Q$ satisfying*

(a) $$\begin{pmatrix} A^T(t)H + HA(t) + Q - \aleph(t)H & Z^T(t)H\\ * & -(1-\delta)Q\end{pmatrix} \leq 0, \quad t \in [0, \mathcal{T}],$$

(b) $\int_0^t (\aleph(\nu)\vee 0)d\nu \le \ln\frac{\lambda_2\beta_2}{\lambda_1\beta_1}, \quad t\in[0,\mathcal{T}]$,
where $\lambda_1 = \lambda_{\max}(\widetilde{H}) + \vartheta\lambda_{\max}(\widetilde{Q})$, $\lambda_2 = \lambda_{\min}(\widetilde{H})$, $\widetilde{H} = W^{-\frac{1}{2}}HW^{-\frac{1}{2}}$, $\widetilde{Q} = W^{-\frac{1}{2}}QW^{-\frac{1}{2}}$.

Proof. Select Lyapunov functional

$$\mathcal{G}(t,\mu(t)) = \mu^T(t)H\mu(t) + \int_{t-\rho(t)}^{t}\mu^T(\nu)Q\mu(\nu)d\nu. \tag{3.14}$$

So, we can get

$$\begin{aligned}\mathcal{G}(0,\mu(0)) &= \mu^T(0)H\mu(0) + \int_{-\rho(t)}^{0}\mu^T(\nu)Q\mu(\nu)d\nu\\ &\le \mu^T(0)H\mu(0) + \int_{-\vartheta}^{0}\mu^T(\nu)Q\mu(\nu)d\nu\\ &\le (\lambda_{\max}(\widetilde{H}) + \vartheta\lambda_{\max}(\widetilde{Q}))\|\varphi\|_W^{\vartheta}. \end{aligned}\tag{3.15}$$

Moreover, one has

$$\mathcal{G}(t,\mu(t)) \ge \lambda_{\min}(\widetilde{H})\|\mu(t)\|_W. \tag{3.16}$$

Taking the derivative of $\mathcal{G}(t,\mu(t))$ in equation (3.14) along system (3.7) and calculate as

$$\begin{aligned}\dot{\mathcal{G}}(t,\mu(t)) \le{}& \mu^T(t)(A^T(t)H + HA(t) + Q)\mu(t) + \mu^T(t)(Z^T(t)H + HZ(t))\mu(t-\rho(t))\\ &-(1-\delta)\mu^T(t-\rho(t))Q\mu(t-\rho(t)).\end{aligned}\tag{3.17}$$

Therefore, similar to Corollary 3.1, one has

$$\begin{aligned}&\dot{\mathcal{G}}(t,\mu(t)) - \aleph(t)\mathcal{G}(t,\mu(t))\\ \le{}& \xi^T(t)\begin{pmatrix} A^T(t)H + HA(t) + Q - \aleph(t)H & Z^T(t)H\\ * & -(1-\delta)Q\end{pmatrix}\xi(t)\\ &+\Big((-\aleph(t))\vee 0\Big)\mathcal{G}(t,\mu(t))\\ \le{}& \Big((-\aleph(t))\vee 0\Big)\mathcal{G}(t,\mu(t)),\end{aligned}\tag{3.18}$$

where $\xi(t) = \Big(\mu(t) \quad \mu(t-\rho(t))\Big)^T$. The latter proof is analogue to Corollary 3.1, and is omitted here. □

Remark 3.3. Corollary 3.1 and 3.2 can deal with non-autonomous systems, but the conclusions in [27] cannot. If $0<\aleph(t),\ t\in[0,\mathcal{T}]$, the condition (b) in Corollary 3.1 and 3.2 can be simply written as $\int_0^{\mathcal{T}}\aleph(\nu)d\nu \le \ln\frac{\lambda_2\beta_2}{\lambda_1\beta_1}$. Furthermore, we can similarly achieve the following result of the Theorem 3.1.

Theorem 3.2. *Suppose functions* $\psi_1,\psi_2\in\kappa$, *and give three scalars* $0<\mathcal{T},\ 0<\beta_1<\beta_2$ *and a matrix* $W>0$. *If there exist nonegative integrable function* $\aleph(t):\mathfrak{R}^+\to\mathfrak{R}^+$ *and function* $\mathcal{G}:[-\vartheta,\mathcal{T}]\times\mathfrak{R}^{\mathfrak{n}}\to\mathfrak{R}^+$ *is differentiable, such that*
(a) $\psi_2(\|\mu(t)\|_W)\le\mathcal{G}(t,\mu(t))$, *and* $\psi_1(\|\varphi\|_W^{\vartheta})\ge\mathcal{G}(0,\mu(0))$, *for* $\forall(t,\varphi)\in[-\vartheta,\mathcal{T}]\times$

$\Re^n$;
(b) $\dot{\mathcal{G}}(t,\mu(t)) \le \aleph(t)\mathcal{G}(t,\mu(t)), \quad t \in [0,\mathcal{T}]$;
(c) $\int_0^{\mathcal{T}} \aleph(\nu)d\nu \le \ln\frac{\psi_2(\beta_2)}{\psi_1(\beta_1)}$,
then system (2.1) is FTS subject to $(\beta_1,\beta_2,W,\mathcal{T})$.

If $A(t)=A, Z(t)=Z$ in system (3.7), by Theorem 3.2, we have

Corollary 3.3. *Three scalars* $0<\mathcal{T},\ 0<\beta_1<\beta_2$ *and a matrix* $W>0$ *are given. If there exist a scalar* $\aleph>0$ *and matrices* $0<H,\ 0<Q$, *such that*

$$(a)\ \begin{pmatrix} A^TH+HA+\frac{1}{\vartheta}Q-\aleph H & Z^TH \\ * & -\frac{1}{\vartheta}Q \end{pmatrix} \le 0,$$

(b) $\aleph \le \frac{1}{\mathcal{T}}\ln\frac{\lambda_2\beta_2}{\lambda_1\beta_1}$,
where $\lambda_1=\lambda_{\max}(\widetilde{H})+\lambda_{\max}(\widetilde{Q}),\ \lambda_2=\lambda_{\min}(\widetilde{H}),\ \widetilde{H}=W^{-\frac{1}{2}}HW^{-\frac{1}{2}},\ \widetilde{Q}=W^{-\frac{1}{2}}QW^{-\frac{1}{2}}$.

Then system (3.7) with constant delay ϑ *is FTS subject to* $(\beta_1,\beta_2,W,\mathcal{T})$.

Corollary 3.4. *Three scalars* $0<\mathcal{T},\ 0<\beta_1<\beta_2$ *and a matrix* $W>0$ *are given. If there exist a scalar* $\aleph>0$ *and matrices* $0<H,\ 0<Q$, *such that*

$$(a)\ \begin{pmatrix} A^TH+HA+Q-\aleph H & Z^TH \\ * & -(1-\delta)Q \end{pmatrix} \le 0,$$

(b) $\aleph \le \frac{1}{\mathcal{T}}\ln\frac{\lambda_2\beta_2}{\lambda_1\beta_1}$,
where $\lambda_1=\lambda_{\max}(\widetilde{H})+\vartheta\lambda_{\max}(\widetilde{Q}),\ \lambda_2=\lambda_{\min}(\widetilde{H}),\ \widetilde{H}=W^{-\frac{1}{2}}HW^{-\frac{1}{2}},\ \widetilde{Q}=W^{-\frac{1}{2}}QW^{-\frac{1}{2}}, \vartheta\ge\rho(t)>0,\ 1>\delta\ge\dot{\rho}(t)$.

Then system (3.7) with time-varying delay $\rho(t)$ *is FTS subject to* $(\beta_1,\beta_2,W,\mathcal{T})$.

4. Application to finite time stabilization and FTS

4.1. Finite time stabilization for linear time delay systems

In the subsection, we consider the controlled system

$$\begin{cases} \dot{\mu}(t)=A\mu(t)+Z\mu(t-\vartheta)+Su(t), & 0\le t, \\ u(t)=U\mu(t), & \\ \mu_0(t)=\varphi(t), & t\in[-\vartheta,0], \end{cases} \tag{4.1}$$

where matrices A,Z,S are known, and U is unknown gain matrix which will be determined in designing controller $u(t)$. The system (4.1) can be abbreviated as

$$\begin{cases} \dot{\mu}(t)=(A+SU)\mu(t)+Z\mu(t-\vartheta), & t\ge 0, \\ \mu_0(t)=\varphi(t), & t\in[-\vartheta,0]. \end{cases} \tag{4.2}$$

Applying Corollary 3.3 to the system (4.2), we obtain

Corollary 4.1. *Three scalars $0 < \mathcal{T},\ 0 < \beta_1 < \beta_2$ and a matrix $W > 0$ are given. If there exist matrix $R > 0$, symmetric matrices E, Y_1, Y_2 and positive scalar $\aleph$ such that*

$$(a) \begin{pmatrix} RA^T + M^T S^T + AR + SM + \frac{1}{\vartheta}E - \aleph R & RZ^T \\ * & -\frac{1}{\vartheta}E \end{pmatrix} \leq 0, \tag{4.3}$$

$$(b)\ \mathcal{T}\aleph - \ln \frac{Y_2\beta_2}{Y_1\beta_1} \leq 0,$$

where

$$R = H^{-1}, \quad M = UH^{-1}, \quad E = H^{-1}QH^{-1}, \quad Q = HEH,$$
$$Y_1 = H^{-1}\lambda_1 H^{-1}, \quad Y_2 = H^{-1}\lambda_2 H^{-1}, \quad \lambda_1 = \lambda_{\max}(\widetilde{H}) + \lambda_{\max}(\widetilde{Q}),$$
$$\lambda_2 = \lambda_{\min}(\widetilde{H}), \quad \widetilde{H} = W^{-\frac{1}{2}}HW^{-\frac{1}{2}}, \quad \widetilde{Q} = W^{-\frac{1}{2}}QW^{-\frac{1}{2}}.$$

Then system (4.1) with controller is FTS subject to $(\beta_1, \beta_2, W, \mathcal{T})$.

Remark 4.1. If the system (3.7) with constant delay ϑ is not FTS, Corollary 4.1 may offer us the method to design the controller which can take the constant delay system FTS subject to $(\beta_1, \beta_2, W, \mathcal{T})$. An example will be given to test the validity of Corollary 4.1 in Section 5.

Remark 4.2. If the term $A+SU$ is directly used instead of the term A in Corollary 3.3, it should be noted that the term U^TS^TH is nonlinear. To eliminate these nonlinearity, we multiply the inequality in condition (a) of Corollary 3.3 by the following diagonal matrix from both left and right sides

$$diag\{H^{-1} \quad H^{-1}\},$$

similarly, the inequality in condition (b) of Corollary 3.3 multiplies by H^{-1} from both left and right sides. Thus Corollary 4.1 is obtained, then the gain matrix $U = MR^{-1}$.

4.2. FTS for impulsive switched linear time-varying systems

Let $\vartheta = 0$, consider the linear switched impulsive systems

$$\begin{cases} \dot{\mu}(t) = A_{\varrho(t)}(t)\mu(t), & t \neq t_{\alpha_k} \\ \mu(t^+_{\alpha_k}) = B(t_{\alpha_k})\mu(t_{\alpha_k}), & t = t_{\alpha_k} \end{cases} \tag{4.4}$$

where $\mu(t) \in \mathfrak{R}^{\mathtt{n}}$ is the state, initial condition $\mu(t_0^+) = \mu_0$. Switching signal $\varrho(\cdot) : [t_0, t_0 + \mathcal{T}] \to \mathcal{M} = \{1, 2, \cdots, m\}$ is a left-continuous piecewise constant function. Accordingly, $\{\mu_0; (i_0, t_0), (i_1, t_1), \cdots, (i_\alpha, t_\alpha), \cdots, |i_\alpha \in \mathcal{M}, \alpha \in \mathcal{N}\}$ is the switching sequence, in which $\varrho(t^+_\alpha) = i_\alpha$, and we say that when $t \in (t_\alpha, t_{\alpha+1}],\ t_\alpha \in [t_0, t_0+\mathcal{T}]$, the i_α-th subsystem is activated, a total of k_α impulsive have occurred. t_{α_k} is the point when the k-th impulsive occurred in time interval $(t_\alpha, t_{\alpha+1}]$, where $(0 \leq k \leq k_\alpha)$, and the impulsive and switching occur at the same time at t_α, here $\alpha \neq 0$ (t_1 is the first impulsive-switching time).

For each $i_\alpha \in \mathcal{M}$, a matrix-valued function $A_{i_\alpha}(\cdot) : [t_0, t_0 + \mathcal{T}] \to \mathfrak{R}^{\mathtt{n}\times\mathtt{n}}$ is continuous. $B(\cdot) : [t_0, t_0 + \mathcal{T}] \to \mathfrak{R}^{\mathtt{n}\times\mathtt{n}}$ is the matrix-valued function. $\triangle\mu(t)$ is defined as $\triangle\mu(t_{\alpha_k}) = \mu(t^+_{\alpha_k}) - \mu(t_{\alpha_k})$, among them, we assume that $\mu(t)$ remains left continuous every time t_{α_k}, that is, $\mu(t_{\alpha_k}) = \mu(t^-_{\alpha_k}) = \lim_{r\to 0^-}\mu(t_{\alpha_k}+r)$, $\mu(t^+_{\alpha_k}) = \lim_{r\to 0^+}\mu(t_{\alpha_k}+r)$.

Theorem 4.1. *Three scalars $0 < \mathcal{T}$, $0 < \beta_1 < \beta_2$ are given. If there exist a function $\aleph(t) : [t_0, t_0+\mathcal{T}] \to \Re$ is piecewise continuous, a function $b(t) : [t_0, t_0+\mathcal{T}] \to \Re^+$, and positive definite matrice P_{i_α} such that*

$$(a) \qquad \aleph_{\varrho(t)}(t)P_{\varrho(t)} \geq A^T_{\varrho(t)}(t)P_{\varrho(t)} + P_{\varrho(t)}A_{\varrho(t)}(t), \quad t \neq t_{\alpha_k}, \tag{4.5}$$

$$(b) \qquad b(t_{\alpha_k})P_{\varrho(t_{\alpha_k})} \geq B^T(t_{\alpha_k})P_{\varrho(t^+_{\alpha_k})}B(t_{\alpha_k}), \quad t = t_{\alpha_k}, \tag{4.6}$$

$$(c) \qquad \prod_{n=0}^{\alpha}\prod_{z=1}^{k_n} b(t_{n_z})b(t_1)\cdots b(t_\alpha)e^{\int_{t_0}^{t}\aleph(\nu)d\nu} \leq \frac{a_2\beta_2}{a_1\beta_1}, \quad t \in [t_0, t_0+\mathcal{T}], \tag{4.7}$$

where $\int_{t_0}^{t}\aleph(\nu)d\nu = \int_{t_0}^{t_1}\aleph_{i_0}(\nu)d\nu + \cdots + \int_{t_\alpha}^{t}\aleph_{i_\alpha}(\nu)d\nu$,$a_1 = \max_{i_\alpha\in M}\{\lambda_{\max}(\widetilde{P}_{i_\alpha})\}$, $a_2 = \min_{i_\alpha\in M}\{\lambda_{\min}(\widetilde{P}_{i_\alpha})\}$, $\widetilde{P}_{i_\alpha} = W^{-\frac{1}{2}}P_{i_\alpha}W^{-\frac{1}{2}}$, *for any* $i_\alpha \in \mathcal{M}$, $\alpha \in \mathcal{N}$.

Then system (4.4) is FTS subject to $(\beta_1, \beta_2, W, t_0, \mathcal{T})$.

Proof. Choose a Lyapunov functional

$$\mathcal{G}(t) = \mathcal{G}_{\varrho(t)}(t) = \mu^T(t)P_{\varrho(t)}\mu(t).$$

For any $t \in (t_\alpha, t_{\alpha+1}]$, $\varrho(t^+_\alpha) = i_\alpha$. Dividing interval $(\alpha, \alpha+1]$ into the following subintervals $(\alpha, \alpha_1], \cdots, (\alpha_{k-1}, \alpha_k], (\alpha_k, \alpha+1]$, $k = \{1, 2, \cdots, l\}$. Taking $t \in (t_{\alpha_{k-1}}, t_{\alpha_k}]$, from the first formula of system (4.4) and condition (4.5), one has

$$\dot{\mathcal{G}}_{i_\alpha}(t) = \mu^T(t)[A^T_{i_\alpha}(t)P_{i_\alpha} + P_{i_\alpha}A_{i_\alpha}(t)]\mu(t) \leq \aleph_{i_\alpha}(t)\mathcal{G}_{i_\alpha}(t), \quad i_\alpha \in \mathcal{M}. \tag{4.8}$$

It means that the condition (b) of Theorem 3.1 holds, except possibly at the impulsive and switching points. Now, let us look at these time points. Which implies that

$$\mathcal{G}_{i_\alpha}(t) \leq e^{\int_{t_{\alpha_k}}^{t}\aleph_{i_\alpha}(\nu)d\nu}\mathcal{G}_{i_\alpha}(t^+_{\alpha_k}), \tag{4.9}$$

from the second formula of system (4.4) and condition (4.6), it yields

$$\mathcal{G}_{i_\alpha}(t^+_{\alpha_k}) = \mu^T(t_{\alpha_k})B^T(t_{\alpha_k})P_{i_\alpha}B(t_{\alpha_k})\mu(t_{\alpha_k}) \leq b(t_{\alpha_k})\mathcal{G}_{i_\alpha}(t_{\alpha_k}), \quad t = t_{\alpha_k}, \tag{4.10}$$

specially $t = t_\alpha$, the system switches to the i_α-th subsystem from the $i_{\alpha-1}$-th subsystem. Further, we have that

$$\mathcal{G}_{i_\alpha}(t^+_\alpha) = \mu^T(t_\alpha)B^T(t_\alpha)P_{i_{\alpha-1}}B(t_\alpha)\mu(t_\alpha) \leq b(t_\alpha)\mathcal{G}_{i_{\alpha-1}}(t_\alpha). \tag{4.11}$$

Substituting (4.10) into (4.9),

$$\mathcal{G}_{i_\alpha}(t) \leq e^{\int_{t_{\alpha_k}}^{t}\aleph_{i_\alpha}(\nu)d\nu}\mathcal{G}_{i_\alpha}(t^+_{\alpha_k}) \leq e^{\int_{t_{\alpha_k}}^{t}\aleph_{i_\alpha}(\nu)d\nu}b(t_{\alpha_k})\mathcal{G}_{i_\alpha}(t_{\alpha_k}). \tag{4.12}$$

When $t = t_{\alpha+1}$, we get

$$\mathcal{G}_{i_\alpha}(t_{\alpha+1}) \leq e^{\int_{t_{\alpha_k}}^{t_{\alpha+1}}\aleph_{i_\alpha}(\nu)d\nu}\mathcal{G}_{i_\alpha}(t^+_{\alpha_k}) \leq e^{\int_{t_{\alpha_k}}^{t_{\alpha+1}}\aleph_{i_\alpha}(\nu)d\nu}b(t_{\alpha_k})\mathcal{G}_{i_\alpha}(t_{\alpha_k}), \tag{4.13}$$

where $t \in (t_{\alpha_k}, t_{\alpha+1}]$. Similarly, it is easy to get

$$\begin{aligned}
\mathcal{G}_{i_\alpha}(t_{\alpha_k}) &\leq e^{\int_{t_{\alpha_{k-1}}}^{t_{\alpha_k}} \aleph_{i_\alpha}(\nu)d\nu} b(t_{\alpha_{k-1}})\mathcal{G}_{i_\alpha}(t_{\alpha_{k-1}}), \quad t \in (t_{\alpha_{k-1}}, t_{\alpha_k}], \\
&\vdots \\
\mathcal{G}_{i_\alpha}(t_{\alpha_2}) &\leq e^{\int_{t_{\alpha_1}}^{t_{\alpha_2}} \aleph_{i_\alpha}(\nu)d\nu} b(t_{\alpha_1})\mathcal{G}_{i_\alpha}(t_{\alpha_1}), \quad t \in (t_{\alpha_1}, t_{\alpha_2}], \\
\mathcal{G}_{i_\alpha}(t_{\alpha_1}) &\leq e^{\int_{t_{\alpha}}^{t_{\alpha_1}} \aleph_{i_\alpha}(\nu)d\nu} b(t_{\alpha})\mathcal{G}_{i_{\alpha-1}}(t_{\alpha}), \quad t \in (t_{\alpha}, t_{\alpha_1}].
\end{aligned} \tag{4.14}$$

From (4.13)-(4.14) we get that the following formula is satisfied for any $t \in (t_\alpha, t_{\alpha+1}]$,

$$\mathcal{G}_{i_\alpha}(t) \leq e^{\int_{t_\alpha}^{t} \aleph_{i_\alpha}(\nu)d\nu} \prod_{z=1}^{k_\alpha} b(t_{\alpha_z}) b(t_\alpha)\mathcal{G}_{i_{\alpha-1}}(t_\alpha). \tag{4.15}$$

Repeating the above process, it is easy to sort out

$$\begin{aligned}
\mathcal{G}_{i_{\alpha-1}}(t_\alpha) &\leq e^{\int_{t_{\alpha-1}}^{t_\alpha} \aleph_{i_{\alpha-1}}(\nu)d\nu} \prod_{z=1}^{k_{\alpha-1}} b(t_{\alpha-1_z}) b(t_{\alpha-1})\mathcal{G}_{i_{\alpha-2}}(t_{\alpha-1}), \\
&\vdots \\
\mathcal{G}_{i_1}(t_2) &\leq e^{\int_{t_1}^{t_2} \aleph_{i_1}(\nu)d\nu} \prod_{z=1}^{k_1} b(t_{1_z}) b(t_1)\mathcal{G}_{i_0}(t_1), \\
\mathcal{G}_{i_0}(t_1) &\leq e^{\int_{t_0}^{t_1} \aleph_{i_0}(\nu)d\nu} \prod_{z=1}^{k_0} b(t_{0_z})\mathcal{G}_{i_0}(t_0^+),
\end{aligned} \tag{4.16}$$

combining (4.15) with (4.16),

$$\mathcal{G}(t) \leq e^{\int_{t_0}^{t} \aleph(\nu)d\nu} \prod_{n=0}^{\alpha} \prod_{z=1}^{k_n} b(t_{n_z}) b(t_1)\cdots b(t_\alpha)\mathcal{G}_{i_0}(t_0^+), \quad t \in [t_0, t_0 + \mathcal{T}], \tag{4.17}$$

where satisfies $\int_{t_0}^{t} \aleph(\nu)d\nu = \int_{t_0}^{t_1} \aleph_{i_0}(\nu)d\nu + \cdots + \int_{t_\alpha}^{t} \aleph_{i_\alpha}(\nu)d\nu$. Then, we prove that the $\mathcal{G}(t)$ satisfies condition (a) of Theorem 3.1. On the one hand,

$$\mathcal{G}_{i_0}(t_0^+) = \mu_0^T P_{i_0}\mu_0 \leq a_1\mu_0^T W\mu_0 \leq a_1\beta_1, \tag{4.18}$$

on the other hand,

$$\mathcal{G}(t) = \mu^T(t)P_{\varrho(t)}\mu(t) \geq a_2\mu^T(t)W\mu(t), \tag{4.19}$$

thus, the condition (a) of Theorem 3.1 is established. It is easy to see from conditions (4.7) and (4.17) - (4.19)

$$\mu^T(t)W\mu(t) \leq \beta_2. \tag{4.20}$$

It further shows that condition (4.7) is equivalent to condition (c) of Theorem 3.1. For this reason, system (4.4) is FTS. The proof is accomplished. □

5. Numerical examples

Example 5.1. The scalar system with delay $\rho(t) = \vartheta = 0.15$ is considered as follows:

$$\dot{\mu}(t) = \Big(0.5\cos t - 0.4\Big)\mu(t) + 0.5\mu(t - \rho(t)). \tag{5.1}$$

Select a Lyapunov function as $\mathcal{G}(\mu(t)) = h\mu^2(t) + \frac{q}{\vartheta}\int_{t-\vartheta}^{t}\mu^2(\nu)d\nu$, where $h = 1, q = 0.15$, and then one has

$$\dot{\mathcal{G}}(\mu(t)) \leq \Big(\aleph(t) \vee 0\Big)\mathcal{G}(\mu(t)),$$

where $\aleph(t) = 0.46 + \cos t$. By computation,

$$\begin{pmatrix} 2hA(t) + \frac{1}{\vartheta}q - \aleph(t)h & hZ(t) \\ * & -\frac{1}{\vartheta}q \end{pmatrix} < 0,$$

and

$$\int_0^t \Big(\aleph(\nu) \vee 0\Big)d\nu \leq \ln\frac{\lambda_2\beta_2}{\lambda_1\beta_1}, \quad t \in [0, 6.3],$$

where $\lambda_1 = 1.15, \lambda_2 = 1, \beta_1 = 1, \beta_2 = 44.7230, W = I$.

Hence, the criteria of Corollary 3.1 are satisfied, and according to Corollary 3.1, the system (2.1) is FTS with $\beta_1 = 1, \beta_2 = 44.7230, W = I, \mathcal{T} = 6.3$. The simulation result are demonstrated in Fig. 1 and Fig. 2.

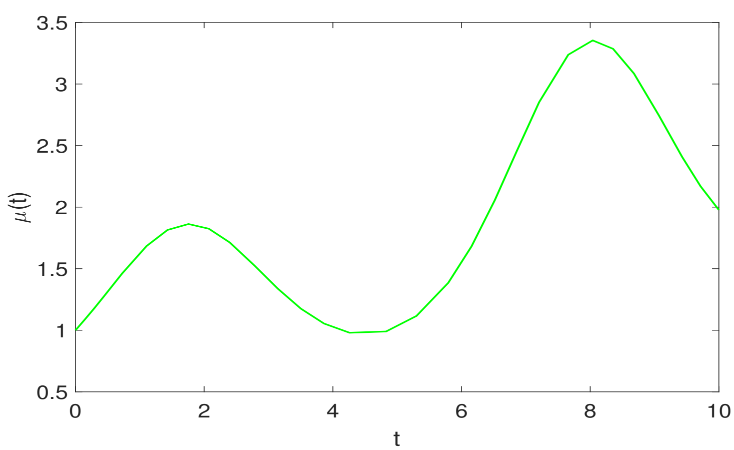

Figure 1. The dynamical behaviors of $\mu(t)$ in Example 5.1.

Example 5.2. Take the linear system with time-varying delay into consideration, as follows:

$$\begin{cases} \dot{\mu}(t) = A(t)\mu(t) + Z(t)\mu(t - 0.7\sin t), \ t \geq 0, \\ \mu_0(t) = (0.7070, -0.7070), \qquad t \in [-\vartheta, 0], \end{cases} \tag{5.2}$$

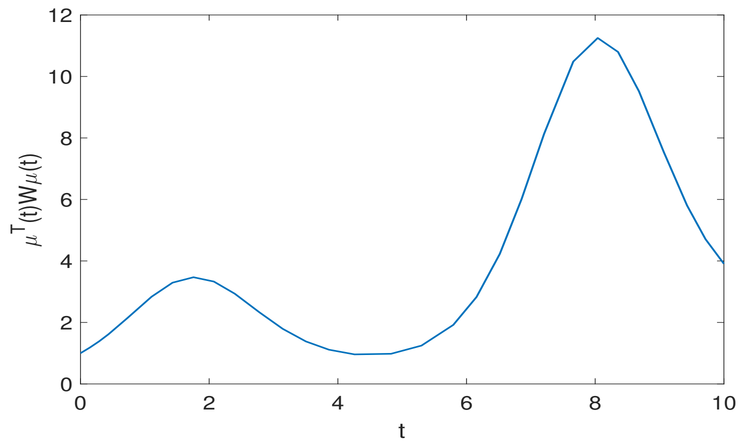

Figure 2. The dynamical behaviors of $\mu^T(t)W\mu(t)$ with $\beta_1 = 1, \beta_2 = 44.7230, W = I, \mathcal{T} = 6.3$ in Example 5.1.

where

$$A(t) = \begin{pmatrix} 0.5\cos(0.5t) + 0.05 & -0.6 \\ 0.6 & 0.5\cos(0.5t) + 0.05 \end{pmatrix}, Z(t) = \begin{pmatrix} 0.01 & 0 \\ -0.01 & -0.01 \end{pmatrix}.$$

Consider the FTS the system with respect to $(\beta_1 = 1, \beta_2 = 34.1083, W = I, \mathcal{T} = 9.2)$, and select Lyapunov function $\mathcal{G}(t) = \mu^T(t)\mu(t) + 0.05\int_{t-0.7\sin t}^{t} \mu^T(\nu)\mu(\nu)d\nu$. By computation, $\omega_1(\nu) = 2\nu,\ \omega_2(\nu) = \nu,\ \aleph(t) = 0.18 + \cos(0.5t)$, and $\int_0^t(\aleph(\nu) \vee 0)d\nu \leq 2.598,\ t \in [0, 9.2]$. Therefore, the sufficient criteria to satisfy Corollary 3.2, and the system is FTS subject to $(\beta_1 = 1, \beta_2 = 34.1083, W = I, \mathcal{T} = 9.2)$. The behaviors of the system are shown in Fig. 3 and Fig. 4.

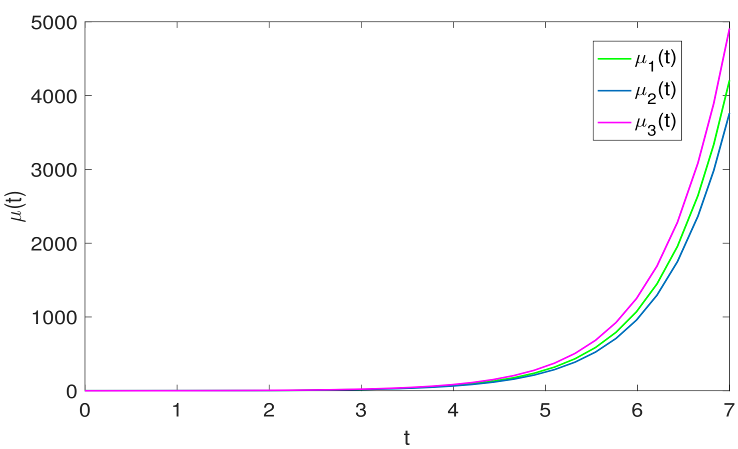

Figure 3. The trajectories of $\mu_1(t), \mu_2(t)$ with $\mu_1(0) = 0.7070, \mu_2(0) = -0.707$.

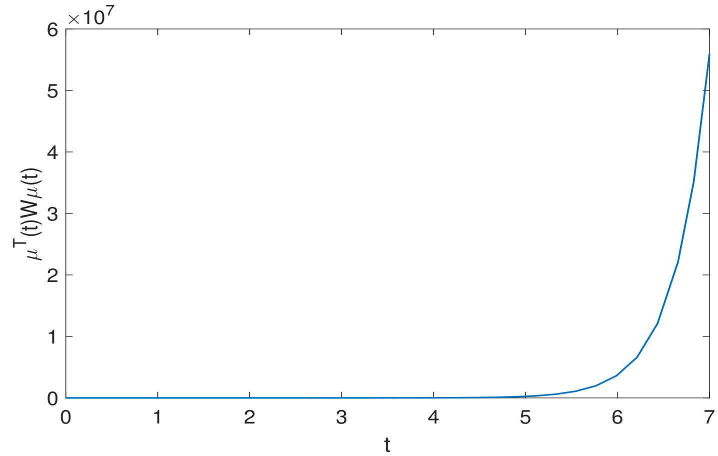

Figure 4. The trajectories of $\mu^T(t)W\mu(t)$ change with time t.

Example 5.3. Consider system (4.1) with constant delays $\vartheta = 0.2$,

$$A = \begin{pmatrix} -0.7 & 1.7 & 0 \\ 1.3 & -0.5 & 0.7 \\ 0.7 & 1 & -0.6 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1.5 & -1.0 & 0.2 \\ -1.3 & 1 & -0.3 \\ -0.7 & 1 & 0.6 \end{pmatrix},$$

$$S = \begin{pmatrix} 0.6717 & 0.4486 & -0.5402 \\ 0.0460 & 0.4922 & 0.3269 \\ -0.0134 & -0.0134 & 0.6471 \end{pmatrix}, \tag{5.3}$$

the simulation result about the behavior of system (4.1) without controller are depicted in Fig.5 and Fig.6.

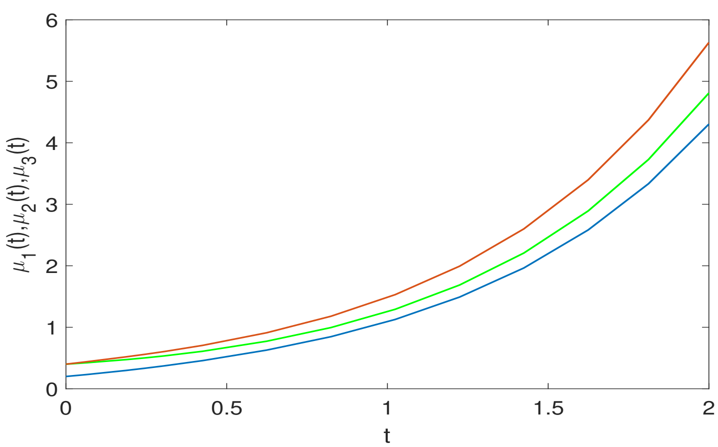

Figure 5. The dynamical behaviors of $\mu(t)$ with $\mu_1(0) = 0.4, \mu_2(0) = 0.2, \mu_3(0) = 0.4.$ in Example 5.3.

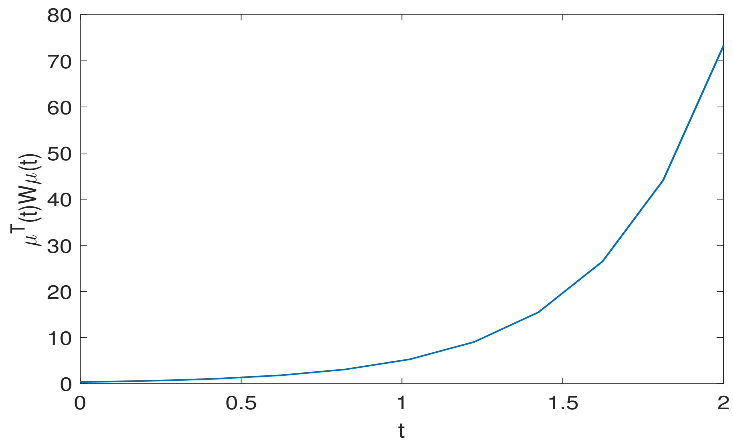

Figure 6. The dynamical behaviors of $\mu^T(t)W\mu(t)$ with $\beta_1 = 0.36, \beta_2 = 3, W = I, \mathcal{T} = 7$ in Example 5.3.

From Fig.6, it is obvious that the system (4.1) without controller is not FTS with $\beta_1 = 0.36, \beta_2 = 3, W = I, \mathcal{T} = 7$.

By computing LMI (4.3), we have $\aleph = 0.2$,

$$R = \begin{pmatrix} 21.7948 & 2.5260 & 1.4054 \\ 2.5260 & 22.1948 & -1.2161 \\ 1.4054 & -1.2161 & 22.4758 \end{pmatrix}, \ E = \begin{pmatrix} 9.1317 & -2.5311 & -1.3010 \\ -2.5311 & 8.7449 & 1.1005 \\ -1.3010 & 1.1005 & 8.7134 \end{pmatrix},$$

$$M = \begin{pmatrix} -36.4397 & -25.8121 & -5.1372 \\ -25.8121 & -37.3960 & -34.0367 \\ -5.1372 & -34.0367 & -26.2394 \end{pmatrix}, Q = \begin{pmatrix} 0.0067 & -0.0016 & -0.0009 \\ -0.0016 & 0.0064 & 0.0008 \\ -0.0009 & 0.0008 & 0.0061 \end{pmatrix}.$$

Then, the gain matrix is given as

$$U = MR^{-1} = \begin{pmatrix} -1.5444 & -0.9974 & -0.1860 \\ -0.8911 & -1.6684 & -1.5489 \\ 0.0314 & -1.6060 & -1.2563 \end{pmatrix},$$

$\lambda_1 = 0.0538 + 0.0087 = 0.0625$, $\lambda_2 = 0.0408$, $Y_1 = H^{-1}\lambda_1 H^{-1} = 0.0625$, $Y_2 = H^{-1}\lambda_2 H^{-1} = 0.0408$, and the condition (b) of Corollary 4.1 hold, therefore, the system (4.1) with controller is FTS subject to $(\beta_1 = 0.36, \beta_2 = 3, W = I, \mathcal{T} = 7)$. The simulation of the dynamical behaviors of the system (4.1) with controller are shown in Fig.7 and Fig.8.

Example 5.4. Let us consider system (4.4) with matrices $A_{i_\alpha}(t), B(t)$,

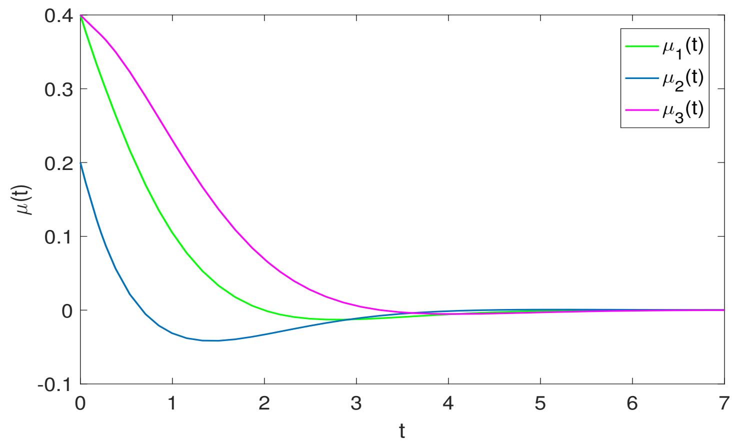

Figure 7. The dynamical behaviors of $\mu(t)$ with controller in Example 5.3.

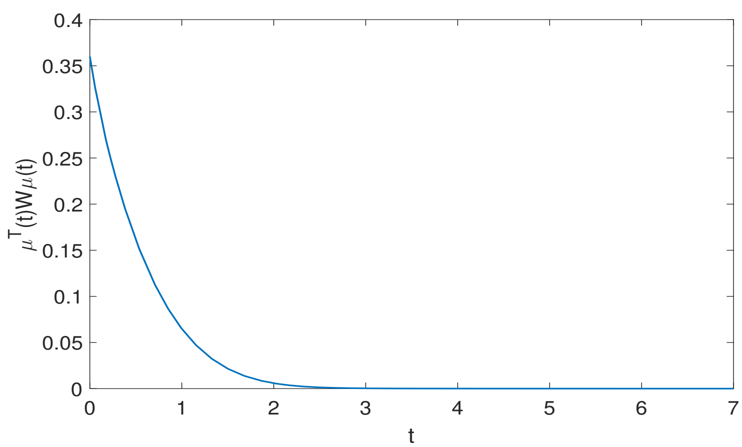

Figure 8. The dynamical behaviors of $\mu^T(t)W\mu(t)$ with $\beta_1 = 0.36, \beta_2 = 3, W = I, \mathcal{T} = 7$ in Example 5.3.

$i_\alpha = 1, 2$, where

$$A_1(t) = \begin{pmatrix} \frac{1-\sin 2t}{2} & -1.1t \\ 1.1t & \frac{1-\sin 2t}{2} \end{pmatrix}, \quad A_2(t) = \begin{pmatrix} \frac{1-\sin 2t}{2} & 1.1t \\ -1.1t & \frac{1-\sin 2t}{2} \end{pmatrix},$$

$B(t) = (1 - 0.3\sin t)I_2$. Let $\mu_0 = [0.8\ 0.6]$, $t_0 = 0$, $\alpha = 1$, $k = 1$, $P_1 = P_2 = I_2$, and $W = I$, it is easy to get $a_1 = 1$, $a_2 = 1$. Then we have

$$\begin{aligned}
&A_1^T(t)P_1 + P_1A_1(t) \le (1 - \sin 2t)P_1, && i_0 = 1. \\
&A_2^T(t)P_2 + P_2A_2(t) \le (1 - \sin 2t)P_2, && i_1 = 2. \\
&B^T(t)P_1B(t) \le (1 - 0.3\sin t)^2 I_2, && i_0 = 1, t = t_{0_1}, t_1. \\
&B^T(t)P_2B(t) \le (1 - 0.3\sin t)^2 I_2, && i_1 = 2, t = t_{1_1}.
\end{aligned}$$

Let $\aleph_1(t) = \aleph_2(t) = 1 - \sin 2t, \quad b(t) = (1 - 0.3\sin t)^2$, then (4.5) and (4.6) are satisfied.

For a given finite time interval $[0, 4]$, switching period is 2. $t_{0_1},\ t_{1_1}$ is the moment when only the impulsive occurs, and t_1 is the moment when impulsive and switching happen at the same time. Given $t_{0_1} = 0.9,\ t_1 = 2,\ t_{1_1} = 2.3,\ \beta_1 = 1,\ \beta_2 = 8.213$, then

$$b(t_{0_1})b(t_{1_1})b(t_1)e^{\int_0^t \aleph(\nu)d\nu} \leq \frac{a_2\beta_2}{a_1\beta_1}, \quad t \in [0, 4],$$

thus, the conditions (4.5)-(4.7) of Theorem 4.1 are satisfied.

From Fig.9, it is obvious that system (4.4) without impulsive is not FTS with $\beta_1 = 1, \beta_2 = 8.213, W = I, t_0 = 0, T = 4$. As shown in Fig.10, system (4.4) with impulsive is FTS subject to $(\beta_1 = 1, \beta_2 = 8.213, W = I, t_0 = 0, \mathcal{T} = 4)$. Therefore, when the impulsive and switching occur at the same time, the minimum value of β_2 can be obtained by ensuring FTS of system (4.4).

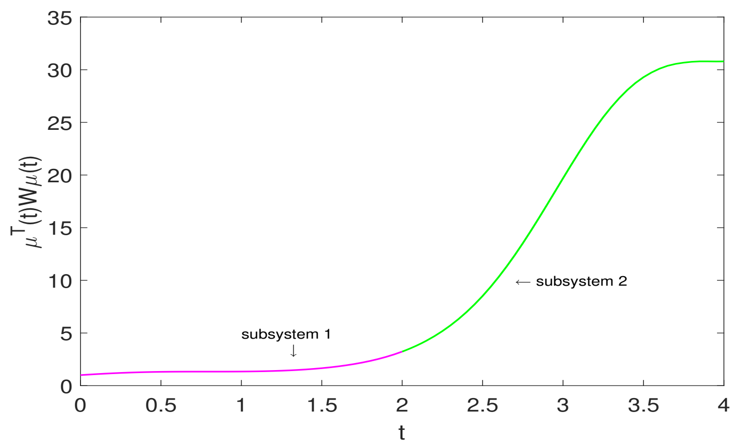

Figure 9. The dynamical behaviors of $\mu^T(t)W\mu(t)$ without impulsive in Example 5.4.

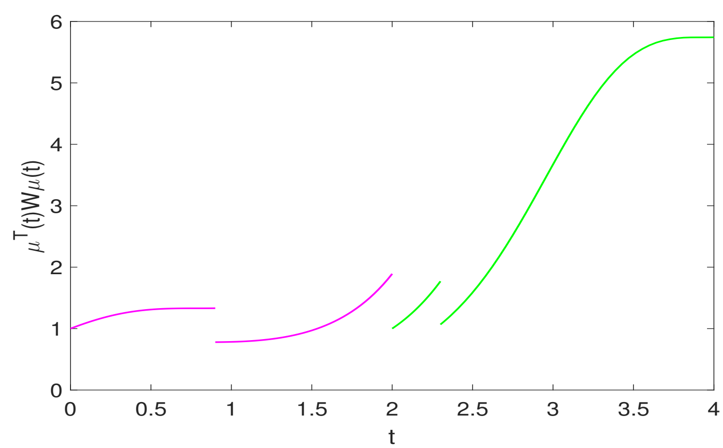

Figure 10. The dynamical behaviors of $\mu^T(t)W\mu(t)$ with impulsive in Example 5.4.

6. Conclusion

By using the Lyapunov method and combining with the technique of matrix inequality, the sufficient criteria for the FTS of time-delay systems are proposed. The algebraic judgment does not need the Lyapunov-Razumikhin condition and the judgment M matrix, and reduces the cost of verification to certain extent. After that, combining the designed feedback controller with Corollary 3.3, a criterion for finite time stabilization is put forwarded. In addition, the conclusion of Theorem 3.1 is extended to the stability study of impulsive switched nonautonomous time-varying systems, and it is concluded that the application of impulsive control in a finite time can make the switching system stable. Ultimately, the effectiveness of proposed approach was verified by four examples.

References

[1] F. Amato, M. Ariola and P. Dorato, *Finite time control of linear systems subject to parametric uncertainties and disturbances*, Automatica, 2001, 37(9), 1459–1463.

[2] F. Amato, M. Ariola and C. Cosentino, *Finite-time stabilization via dynamic output feedback*, Automatica, 2006, 42(2), 337–342.

[3] F. Amato, M. Ariola and C. Cosentino, *Finite-time stability of linear time-varying systems: analysis and controller design*, IEEE Transactions on Automatic Control, 2010, 55(4), 1003–1008.

[4] F. Amato, M. Ariola and C. Cosentino, *Finite-time control of discrete-time linear systems: analysis and design*, Automatica, 2010, 46(5), 919–924.

[5] F. Amato, C. Cosentino and A. Merola, *Sufficient conditions for finite-time stability and stabilization of nonlinear quadratic systems*, IEEE Transactions on Automatic Control, 2010, 55(2), 430–434.

[6] R. Ambrosino, C. Cosentino, F. Arnato and M. Ariola, *Finite-time stability of linear time-varying systems with jumps*, Automatica, 2009, 45(5), 1354–1358.

[7] R. Ambrosino, F. Calabrese, C. Cosentino and G. D. Tommasi, *Sufficient conditions for finite-time stability of impulsive dynamical systems*, IEEE Transactions on Automatic Control, 2009, 54(4), 861–865.

[8] F. Amato, R. Ambrosino, C. Cosentino and G. D. Tommasi, *Finite-time stabilization of impulsive dynamical linear systems*, Nonlinear Analysis, 2011, 5(1), 89–101.

[9] F. Amato, R. Ambrosino, M. Ariola and G. D. Tommasi, *Robust finite-time stability of impulsive dynamical linear systems subject to norm-bounded uncertainties*, International Journal of Robust and Nonlinear Control, 2011, 21, 1080–1082.

[10] G. Arthi, N. Brindha and Y. Ma, *Finite-time stability of multiterm fractional nonlinear systems with multistate time delay*, Advances in Difference Equations, 2021, 2021, 1–15.

[11] G. Chen and Y. Yang, *New sufficient conditions for finite time stability of nonlinear time delay systems*, Asian Journal of Control, 2019, 21, 2321–2329.

[12] M. Chen and J. Sun, *Delay-dependent conditions for finite time stability of linear time-varying systems with delay*, Asian Journal of Control, 2020, 22(2), 924–933.

[13] Z. Cai, L. Huang and Z. Wang, *Finite-fixed-time stability of nonautonomous functional differential inclusion: Lyapunov approach involving indefinite derivative*, IEEE transactions on neural networks and learning systems, 2021, 1–12.

[14] G. Chen and Y. Yang, *New necessary and sufficient conditions for finite-time stability of impulsive switched linear time-varying systems*, IET Control Theory and Applications, 2018, 12, 140–148.

[15] H. Du, X. Lin and S. Li, *Finite-time boundedness and stabilization of switched linear systems*, Kybernetika, 2010, 46(5), 870–889.

[16] G. Garcia, S. Tarbouriech and J. Bernussou, *Finite-time stabilization of linear time-varying continuous systems*, IEEE Transactions on Automatic Control, 2009, 54(2), 364–369.

[17] Y. Hong, J. Huang and Y. Xu, *On an output feedback finite time stabilization problem*, IEEE Transactions on Automatic Control, 2001, 46(2), 305–309.

[18] W. Haddad and J. Lee, *Finite-time stability of discrete autonomous systems*, Automatica, 2020, 122, 109282.

[19] I. Hien, *An explicit criterion for finite-time stability of linear nonautonomous systems with delays*, Applied Mathematics Letters, 2014, 30, 12–18.

[20] I. Hien and D. Son, *Finite-time stability of a class of non-autonomous neural networks with heterogeneous proportional delays*, Applied Mathematics and Computation, 2015, 251, 14–23.

[21] X. He, X. Li and J. Juan, *Finite-time stability and stabilization for time-varying systems*, Chaos, Solitons and Fractals, 2021, 111076.

[22] G. V. Kamenkov, *On stability of motion over a finite interval of time*, Journal of Applied Mathematics and Mechanics, 1953, 17, 529–540.

[23] V. Kumar, M. Djemaš[a], M. Defoort and M. Malik, *Finite-time stability and stabilization results for switched impulsive dynamical systems on time scales*, Journal of the Franklin Institute, 2020, 358, 674–698.

[24] X. Lin, K. Liang and H. Li, *Finite-time stability and stabilization for continuous systems with additive time-varying delays*, Circuits, Systems, and Signal Processing, 2017, 36, 2971–2990.

[25] X. Luan, F. Liu and P. Shi, *Finite-time filtering for non-linear stochastic systems with partially known transition jump rates*, IET Control Theory and Applications, 2010, 4(5), 735–745.

[26] X. Li, X. Yang and S. Song, *Lyapunov conditions for finite-time stability of time-varying time-delay systems*, Automatica, 2019, 103, 135–140.

[27] J. Puangmalai, J. Tongkum and T. Rojsiraphisal, *Finite-time stability criteria of linear system with non-differentiable time-varying delay via new integral inequality*, Mathematica and Computers in Simulation, 2020, 171, 170–186.

[28] Y. Shen, *Finite-time control of linear parameter-varying systems with norm-bounded exogenous disturbance*, Journal of Control Theory and Applications, 2008, 6(2), 184–188.

[29] Y. Tian, Y. Cai, Y. Sun and H. Gao, *Finite-time stability for impulsive switched delay systems with nonlinear disturbances*, Journal of the Franklin Institute, 2016, 353, 3578–3594.

[30] X. Wang, S. Huang, W. Zou and Z. Xiang, *Finite-time stabilization for a class of nonlinear systems with time-varying delay*, International Journal of Robust and Nonlinear Control, 2020, 30, 3164–3178.

[31] Y. Wang, X. Shi, Z. Zuo, M. Chen and Y. Shao, *On finite-time stability for nonlinear impulsive switched systems*, Nonlinear Analysis: Real World Applications, 2013, 14(1), 807–814.

[32] J. Xu and J. Sun, *Finite-time stability of linear time-varying singular impulsive systems*, IET Control Theory and Applications, 2010, 4(10), 2239–2244.

[33] W. Xiang and J. Xiao, *H_∞ finite-time control for switched nonlinear discrete-time systems with norm-bounded disturbance*, Journal of the Franklin Institute, 2011, 348(2), 331–352.

[34] Y. Yang, J. Li and G. Chen, *Finite-time stability and stabilization of nonlinear stochastic hybrid systems*, Journal of Mathematical Analysis and Applications, 2009, 356, 338–345.

[35] X. Yang and X. Li, *Finite-time stability of linear non-autonomous systems with time-varying delays*, Advances in Difference Equations, 2018, 2018, 101–110.

[36] S. Zhao, J. Sun and L. Liu, *Finite-time stability of linear time-varying singular systems with impulsive effects*, International Journal of Control, 2008, 81(11), 1824–1829.

[37] B. Zhou, *Finite-time stability analysis and stabilization by bounded linear time-varying feedback*, Automatica, 2020, 121, 109191.

[38] K. Zimenko, D. Efimov, A. Polyakov and A. Kremlev, *On necessary and sufficient conditions for output finite-time stability*, Automatica, 2021, 125, 109427.

[39] B. Zhou, *Finite-time stabilization of linear systems by bounded linear time-varying feedback*, Automatica, 2020, 108760.

[40] G. Zong, X. Wang and H. Zhao, *Robust finite-time guaranteed cost control for impulsive switched systems with time-varying delay*, International Journal of Control, Automation, and Systems, 2017, 15, 113–121.