Journal of Applied Analysis and Computation
Volume 13, Number 4, August 2023, 1813–1836
Website:http://www.jaac-online.com
DOI:10.11948/20220242

PERIODIC DYNAMICS AND MEAN-SQUARE EXPONENTIAL CONVERGENCE OF NONLOCAL STOCHASTIC FUZZY BIDIRECTIONAL ASSOCIATIVE MEMORY LATTICE NEURAL NETWORKS*

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Abstract This paper firstly establishes the lattice model for nonlocal stochastic fuzzy bidirectional associative memory neural networks with reaction diffusions by employing a mix of the finite difference and Mittag-Leffler time Euler difference techniques. Secondly, the existence of a unique bounded periodic sequence solution in distribution and global mean-square exponential convergence to the achieved difference model are investigated. Some illustrative example is used to show the feasible of the works of the current paper.

Keywords Lattice, reaction diffusion, stochastic, finite difference, period.

MSC(2010) 39A23, 34A33.

1. Introduction

Bidirectional associative memory (BAM) neural network model is a specific type of repetitive neural network that can store bipolar sub pairs. It consists of two layers of neurons and the neurons in a single layer are fully connected to the neurons in the next layer. In real world, BAM neural networks (BAMNNs) provide strong information processing ability and a few excellent applied domains, like information associative memory, image processing, artificial intelligence, etc. On the other hand, Yang and Yang [23] in 1996 proposed a novel fuzzy cellular neural network, which incorporates fuzzy logic into the architecture of a cellular neural network. Fuzzy neural networks possess fuzzy logic for template inputs and/or outputs, in addition to summation of product operations. For the past decades, fuzzy neural networks have received more and more attention because of their superiority in image processing and pattern recognition, see [1–4, 19, 20, 24].

In biologically based neural systems, the concentration of constituents is not uniform, resulting in diffusion of cytoplasm from higher to lower concentrations. It is said to be diffusion. Since neural systems can be build by scaling down and modeling

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* The authors were supported by Scientific Research Fund Project of Education Department of Yunnan Province under Grant No. 2021J0835.

biological neural networks, diffusion ought to be inserted to neural network patterns. Accordingly, reaction diffusion neural networks were formulated and have shown significant prospects for spatio-temporal pattern storage and matching. Recently, stochastic neural networks have been broadly investigated since they are commonly found in people's daily lives. Stochastic perturbations in neural networks not only separate neural networks from deterministic neural networks, but also can bring about substantial modifications in dynamic actions of neural networks. In general, the behavior of stochastic systems is highly reliant on time and spatial dependence. Consequently, reaction diffusion is necessary to be taken into account, and this induces the investigations of stochastic reaction diffusion BAMNNs [5, 15, 17, 18], concretely see the researching topics on the Markovian jumping impulsive models [15] and exponential stability [18], etc.

During the last two decades, a number of real problems are portrayed in terms of fractional-order systems of dynamics [6,16,25,26]. Further, discrete-time neural networks are better fitted for real-time implementations. Firstly, appropriate technology can be used to implement digital controllers rather than analog ones. Secondly, the synthesized controller is directly implemented in a digital processor. Therefore, control methodologies developed for discrete-time nonlinear systems can be implemented in real systems more effectively [7–10,27–29]. Thirdly, many processes have a certain regularity, so the study of periodic sequence has been a significant and interesting topic in the field of difference equations owing to the intensive evolution of the theories of difference equations and the applications in the areas of science and engineering. For all the authors know, up to now, there are few papers focusing on the study of periodic sequences to discrete-time BAMNNs in literatures [11, 14, 30]. However, almost no paper discusses periodic oscillations of discrete-time BAMNNs with stochastic perturbations or reaction diffusions. Therefore, this paper is concentrated on the discussion of nonlocal stochastic fuzzy BAMNNs with reaction diffusions.

By employing a mix of the finite difference methods and Mittag-Leffler Euler time difference techniques, the objective of the current paper is to achieve the discrete-time and discrete-space schemes corresponding to Eqs. (2.1)-(2.2). And on this basis, the existence of a unique bounded periodic sequence solution in distribution and global exponential stability in the mean-square sense are investigated. Compared with the previous literatures, the distinct characteristics of this article are narrated as follows: **1)** Based on the finite difference and Mittag-Leffler Euler difference techniques, a novel stochastic lattice models for Eqs. (2.1)-(2.2) is introduced. **2)** The existence of a unique bounded periodic sequence solution in distribution is discussed. **3)** Global exponential convergence in the mean-square sense is considered. **4)** The research findings in this article extend and complement the works in literatures [10, 11, 14, 28–30].

The organization of the rest is as follows. In Section 2, a stochastic lattice BAMNNs for Eqs. (2.1)-(2.2) is achieved by using the finite difference methods and Mittag-Leffler Euler difference techniques. The existence of a unique bounded periodic sequence solution in distribution and global exponential convergence in the mean-square sense are discussed in Sections 3-4. In Section 5, an illustrative example and some numerical simulations are employed to visually expound the current research findings. The conclusions and future works of this paper are presented in Section 6.

Symbols: $\mathbb{R}^n$ denotes the space of n-dimensional real vectors; $\mathbb{Z}$ is the field of integral numbers; $\mathbb{N}_0 = \{0, 1, 2, \dots\}$; $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$; $\mathbb{N}_a^b = \{a, a+1, \dots, b\}$ for any $a, b \in \mathbb{Z}$; $I_J = I \cap J$, $\forall I, J \subseteq \mathbb{R}$. Let $A_1, A_2, \dots, A_{\mathcal{N}}$ be some sets, $x_1 \in A_1$, $x_2 \in A_2$, $\dots$, $x_{\mathcal{N}} \in A_{\mathcal{N}} \Leftrightarrow (x_1, x_2, \dots, x_{\mathcal{N}}) \in (A_1, A_2, \dots, A_{\mathcal{N}})$.

2. Stochastic lattice networks

In this paper, we deal with nonlocal stochastic fuzzy BAMNNs with reaction diffusions in the shape of

$$\begin{aligned} {}^cD_{t_p}^{\alpha_i} u_i(t,x) = & \sum_{q=1}^{\mathcal{N}} \frac{\partial}{\partial x_q}\left[D_{iq}\frac{\partial u_i(t,x)}{\partial x_q}\right] - c_i(x)u_i(t,x) + \sum_{j=1}^{n} a_{ij}(t,x) \\ & \times f_j(v_j(t,x)) + \bigwedge_{j=1}^{n} \gamma_{ij}(t,x) f_j(v_j(t,x)) + \bigvee_{j=1}^{n} \eta_{ij}(t,x) f_j(v_j(t,x)) \\ & + I_i(t,x) + \sum_{j=1}^{n} \mu_{ij}(t,x) H_{ij}(v_j(t,x)) \frac{\mathrm{d}\mathbb{W}_{1j}(t)}{\mathrm{d}t}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} {}^cD_{t_p}^{\beta_j} v_j(t,x) = & \sum_{q=1}^{\mathcal{N}} \frac{\partial}{\partial x_q}\left[K_{jq}\frac{\partial v_j(t,x)}{\partial x_q}\right] - d_j(x)v_j(t,x) + \sum_{i=1}^{m} e_{ji}(t,x) \\ & \times g_i(u_i(t,x)) + \bigwedge_{i=1}^{m} \varepsilon_{ji}(t,x) g_i(u_i(t,x)) + \bigvee_{i=1}^{m} \vartheta_{ji}(t,x) g_i(u_i(t,x)) \\ & + J_j(t,x) + \sum_{i=1}^{m} \varsigma_{ji}(t,x) L_{ji}(u_i(t,x)) \frac{\mathrm{d}\mathbb{W}_{2i}(t)}{\mathrm{d}t}, \end{aligned} \tag{2.2}$$

where $(t,x) \in (\mathcal{I}_p, \Omega)$, $\mathcal{I}_p = (t_p, t_{p+1}]$, $t_0 = 0$, $\lim_{p\to\pm\infty} t_p = \pm\infty$, $t_p \le t_{p+1}$, $p \in \mathbb{Z}$, $\Omega = \{x = (x_1, x_2, \dots, x_{\mathcal{N}})^T \in \mathbb{R}^{\mathcal{N}} : r_{1q} < x_q < r_{2q}, r_{1q}, r_{2q} \in \mathbb{R}, q = 1, 2, \dots, \mathcal{N}\}$; ${}^cD_*^{\alpha_i}$ and ${}^cD_*^{\beta_j}$ denote Caputo fractional-order derivatives from initial point 0, $\alpha_i, \beta_j \in (0,1]$, u_i and v_j are the neural states, c_i and d_j are the self-inhibitions, a_{ij} and e_{ji} are the synaptic connection strengths, f_j and g_i are the feedback functions, I_i and J_j denote the external inputs; γ_{ij}, ε_{ji}, η_{ij} and ϑ_{ji} are the elements of fuzzy feedback MIN template and fuzzy feedback MAX template, respectively; $D := (D_{iq})_{m\times\mathcal{N}} \ge 0$ and $K := (K_{iq})_{n\times\mathcal{N}} \ge 0$ stand for the transmission diffusion matrixes; $\mathbb{W}_{1j}$ and $\mathbb{W}_{2i}$ denote the Brownian motion on a complete probability space, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

The corresponding initial boundary conditions of Eqs. (2.1)-(2.2) are depicted by

$$\begin{cases} u_i(0,x) = \varphi_i(x), \quad \forall x \in \Omega; \quad u_i(t,x)\Big|_{x\in\partial\Omega} = \phi_i(t,x), \\ v_j(0,x) = \tilde{\varphi}_j(x), \quad \forall x \in \Omega; \quad v_j(t,x)\Big|_{x\in\partial\Omega} = \tilde{\phi}_j(t,x), \end{cases} \tag{2.3}$$

where $t \in \mathbb{R}$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

Let us introduce the relative conception of fractional calculus in literature [12]. The α-order Caputo fractional derivative of $x \in C^n([a,b], \mathbb{R}^n)$ is defined by

$${}^cD_a^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{x^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, \quad \forall t \in [a,b],$$

where $0 < n-1 < \alpha < n, n \in \mathbb{Z}_0$. Let $\alpha > 0$ and $x \in C^1([a,b],\mathbb{R})$. Then

(1) $\lim_{\alpha\to 0^+} {}^cD_a^\alpha x(t) = x(t) - x(a)$.
(2) $\lim_{\alpha\to n} {}^cD_a^\alpha x(t) = x^{(n)}(t)$, $n \in \mathbb{N}$.

The Riemann-Liouville fractional integral of $x \in C([a,b],\mathbb{R})$ is given by

$$I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)\mathrm{d}s, \quad \forall t \in [a,b],$$

where $\alpha > 0$. Let $\alpha \in (n-1,n]$, $n \in \mathbb{N}$. Then

(1) If $x \in C([a,b],\mathbb{R})$, then ${}^cD_a^\alpha\, I_a^\alpha x(t) = x(t)$.
(2) If $x \in C^n([a,b],\mathbb{R})$, then

$$\lim_{\alpha\to n} I_a^\alpha\, {}^cD_a^\alpha x(t) = x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(t-a)^k, \quad \forall t \in [a,b].$$

The Mittag-Leffler functions are described as

$$\mathbf{E}_\alpha(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \mathbf{E}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $z, \beta \in \mathbb{R}$ and $\alpha > 0$.

Lemma 2.1 ([12]). $\frac{d}{dz}[z^\alpha \mathbf{E}_{\alpha,\alpha+1}(\lambda z^\alpha)] = z^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda z^\alpha)$, *where* $\alpha, \lambda, z \in \mathbb{R}$.

Lemma 2.2 ([26]). *If* $\alpha \in (0,1]$ *and* $\lambda \in \mathbb{R}$, *then*

(1) $\mathbf{E}_\alpha(0) = 1$, $\mathbf{E}_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}$.
(2) $\mathbf{E}_\alpha(\lambda t^\alpha) \in (0,1)$ *for* $\lambda < 0$ *and* $\mathbf{E}_\alpha(\lambda t^\alpha) \in (1,+\infty)$ *for* $\lambda > 0$, $\forall t > 0$.
(3) $\mathbf{E}_\alpha(\lambda t_1^\alpha) > \mathbf{E}_\alpha(\lambda t_2^\alpha)$ *and* $\mathbf{E}_{\alpha,\alpha}(\lambda t_1^\alpha) > \mathbf{E}_{\alpha,\alpha}(\lambda t_2^\alpha)$ *for* $\lambda < 0$ *and* $t_1 < t_2$.
(4) $\mathbf{E}_\alpha(\lambda t_1^\alpha) < \mathbf{E}_\alpha(\lambda t_2^\alpha)$ *and* $\mathbf{E}_{\alpha,\alpha}(\lambda t_1^\alpha) < \mathbf{E}_{\alpha,\alpha}(\lambda t_2^\alpha)$ *for* $\lambda > 0$ *and* $t_1 < t_2$.

Let $h_q = \frac{r_{2q}-r_{1q}}{N_q}$ for some $N_q \in \mathbb{N}$, $q = 1,2,\ldots,\mathcal{N}$. Define $x_q^{\ell_q} = r_{1q} + \ell_q h_q$ for all $(\ell_q, q) \in (\mathbb{N}_0^{N_q}, \mathbb{N}_1^{\mathcal{N}})$. Set $\partial\Omega_\mathrm{d} = \bar{\Omega}_\mathrm{d}\backslash\Omega_\mathrm{d}$, where

$$\bar{\Omega}_\mathrm{d} = \left\{x^\ell = (x_1^{\ell_1},\ldots,x_{\mathcal{N}}^{\ell_{\mathcal{N}}})^T \in \mathbb{R}^{\mathcal{N}} : x_q^{\ell_q} = r_{1q} + \ell_q h_q, (\ell_q,q) \in (\mathbb{N}_0^{N_q}, \mathbb{N}_1^{\mathcal{N}})\right\},$$
$$\Omega_\mathrm{d} = \left\{x^\ell = (x_1^{\ell_1},\ldots,x_{\mathcal{N}}^{\ell_{\mathcal{N}}})^T \in \mathbb{R}^{\mathcal{N}} : x_q^{\ell_q} = r_{1q} + \ell_q h_q, (\ell_q,q) \in (\mathbb{N}_1^{N_q-1}, \mathbb{N}_1^{\mathcal{N}})\right\}.$$

Based on the derivation in Appendix A, it obtains the lattice equations of Eqs. (2.1)-(2.2) below

$$u_i^{x^\ell}(k+1) = \lambda_{\alpha_i}^{x^\ell}(k)u_i^{x^\ell}(l) + \sum_{l=\mu_k}^{k}\theta_{\alpha_i}^{x^\ell}(k-l,k)\Big[F_i(u_i^{x^\ell}(l)) + \sum_{j=1}^{n}a_{ij}^{x^\ell}(l)$$
$$\times f_j(v_j^{x^\ell}(l)) + \bigwedge_{j=1}^{n}\gamma_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l)) + \bigvee_{j=1}^{n}\eta_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))$$

$$+I_i^{x^\ell}(l)+\sum_{j=1}^{n}\mu_{ij}^{x^\ell}(l)H_{ij}(v_j^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{1j}(l)\Bigg], \tag{2.4}$$

$$\begin{aligned} v_j^{x^\ell}(k+1) \;=\; & \lambda_{\beta_j}^{x^\ell}(k)v_j^{x^\ell}(l)+\sum_{l=\mu_k}^{k}\theta_{\beta_j}^{x^\ell}(k-l,k)\Bigg[G_j(v_j^{x^\ell}(l))+\sum_{i=1}^{m}e_{ji}^{x^\ell}(l) \\ & \times g_i(u_i^{x^\ell}(l))+\bigwedge_{i=1}^{m}\varepsilon_{ji}^{x^\ell}(l)g_i(u_i^{x^\ell}(l))+\bigvee_{i=1}^{m}\vartheta_{ji}^{x^\ell}(l)g_i(u_i^{x^\ell}(l))+ \\ & +J_j^{x^\ell}(l)+\sum_{i=1}^{m}\varsigma_{ji}^{x^\ell}(l)L_{ji}(u_i^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{2i}(l)\Bigg] \end{aligned} \tag{2.5}$$

with boundary conditions

$$u_i^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\rm d}}=\phi_{i,k}^{x^\ell}:=\phi_i(kh,x^\ell),\quad v_j^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\rm d}}=\tilde\phi_{j,k}^{x^\ell}:=\tilde\phi_j(kh,x^\ell),$$

where $\mu_k=\max_{p\in\mathbb{Z}}\{Q_p:Q_p\le k\}$, $\nu_k=k-\mu_k$,

$$\begin{aligned} & u_i^{x^\ell}(k)=u_i(kh,x^\ell),\quad v_j^{x^\ell}(k)=v_j(kh,x^\ell),\quad F_i(u_i^{x^\ell}(k))=F_i(kh,u_i^{x^\ell}(k)),\\ & G_j(v_j^{x^\ell}(k))=G_j(kh,v_j^{x^\ell}(k)),\quad a_{ij}^{x^\ell}(k)=a_{ij}(kh,x^\ell),\quad \gamma_{ij}^{x^\ell}(k)=\gamma_{ij}(kh,x^\ell),\\ & \eta_{ij}^{x^\ell}(k)=\eta_{ij}(kh,x^\ell),\quad I_i^{x^\ell}(k)=I_i(kh),\quad \mu_{ij}^{x^\ell}(k)=\mu_{ij}(kh,x^\ell),\\ & \mathbb{W}_{1j}(k)=\mathbb{W}_{1j}(kh),\quad \varepsilon_{ji}^{x^\ell}(k)=\varepsilon_{ji}(kh,x^\ell),\\ & \vartheta_{ji}^{x^\ell}(k)=\vartheta_{ji}(kh,x^\ell),\quad J_j^{x^\ell}(k)=J_j(kh,x^\ell),\quad e_{ji}^{x^\ell}(k)=e_{ji}(kh,x^\ell),\\ & \varsigma_{ji}^{x^\ell}(k)=\varsigma_{ji}(kh,x^\ell),\quad \mathbb{W}_{2i}(k)=\mathbb{W}_{2i}(kh),\quad w_{\alpha_i}^{x^\ell}(-1)=w_{\beta_j}^{x^\ell}(-1)=0,\\ & \lambda_{\alpha_i}^{x^\ell}(k)=\frac{\mathbf{E}_{\alpha_i}[-c_i^*(x^\ell)(\nu_kh+h)^{\alpha_i}]}{\mathbf{E}_{\alpha_i}[-c_i^*(x^\ell)(\nu_kh)^{\alpha_i}]},\quad \lambda_{\beta_j}^{x^\ell}(k)=\frac{\mathbf{E}_{\beta_j}[-d_j^*(x^\ell)(\nu_kh+h)^{\beta_j}]}{\mathbf{E}_{\beta_j}[-d_j^*(x^\ell)(\nu_kh)^{\beta_j}]},\\ & \theta_{\alpha_i}^{x^\ell}(l^+,k)=\frac{1}{c_i^*(x^\ell)}\left[w_{\alpha_i}^{x^\ell}(l^+)-\lambda_{\alpha_i}^{x^\ell}(k)w_{\alpha_i}^{x^\ell}(l^+-1)\right],\\ & \theta_{\beta_j}^{x^\ell}(l^+,k)=\frac{1}{d_j^*(x^\ell)}\left[w_{\beta_j}^{x^\ell}(l^+)-\lambda_{\beta_j}^{x^\ell}(k)w_{\beta_j}^{x^\ell}(l^+-1)\right],\\ & w_{\alpha_i}^{x^\ell}(l^+)=\mathbf{E}_{\alpha_i}[-c_i^*(x^\ell)(l^+h)^{\alpha_i}]-\mathbf{E}_{\alpha_i}[-c_i^*(x^\ell)(l^+h+h)^{\alpha_i}],\\ & w_{\beta_j}^{x^\ell}(l^+)=\mathbf{E}_{\beta_j}[-d_j^*(x^\ell)(l^+h)^{\beta_j}]-\mathbf{E}_{\beta_j}[-d_j^*(x^\ell)(l^+h+h)^{\beta_j}] \end{aligned}$$

for all $(x^\ell,l^+,k,i,j)\in(\bar\Omega_{\rm d},\mathbb{N}_0,\mathbb{Z},\mathbb{N}_1^m,\mathbb{N}_1^n)$. The initial conditions of Eqs. (2.4)-(2.5) can be rewritten as

$$u_i^{x^\ell}(0)=\varphi_i(x^\ell):=\varphi_i^{x^\ell},\ v_j^{x^\ell}(0)=\tilde\varphi_j(x^\ell):=\tilde\varphi_j^{x^\ell},\ \forall(x^\ell,i,j)\in(\Omega_{\rm d},\mathbb{N}_1^m,\mathbb{N}_1^n). \tag{2.6}$$

Remark 2.1. Let $\alpha_i=\beta_j=1$, $D_{iq}=K_{jq}=0$ for $(i,j,q)\in(\mathbb{N}_1^m,\mathbb{N}_1^n,\mathbb{N}_1^{\mathcal{N}})$, and removing the space variable x^ℓ in Eqs. (2.4)-(2.5), it is reduced into the exponential difference models in literature [10] of the form

$$u_i(k+1)\;=\;e^{-c_i(k)}u_i(k)+\frac{1-e^{-c_i(k)}}{c_i(k)}\Bigg[\sum_{j=1}^{n}a_{ij}(k)f_j(v_j(k))+\bigwedge_{j=1}^{n}\gamma_{ij}(k)f_j(v_j(k))$$

$$
\begin{aligned}
&+\bigvee_{j=1}^{n}\eta_{ij}(k)f_j(v_j(k))+I_i(k)+\sum_{j=1}^{n}\mu_{ij}(k)H_{ij}(v_j(k))h^{-1}\Delta_h\mathbb{W}_{1j}(k)\Bigg],\\
v_j(k+1)\;=\;& e^{-d_j(k)}v_j(k)+\frac{1-e^{-d_j(k)}}{d_j(k)}\Bigg[\sum_{i=1}^{m}e_{ji}(k)g_i(u_i(k))+\bigwedge_{i=1}^{m}\varepsilon_{ji}(k)g_i(u_i(k))\\
&+\bigvee_{i=1}^{m}\vartheta_{ji}(k)g_i(u_i(k))+J_j(k)+\sum_{i=1}^{m}\varsigma_{ji}(k)L_{ji}(u_i(k))\Delta_h\mathbb{W}_{2j}(k)\Bigg]
\end{aligned}
$$

for all $(k,i,j)\in(\mathbb{Z},\mathbb{N}_1^m,\mathbb{N}_1^n)$. Obviously, Eqs. (2.4)-(2.5) is an expansion of discrete analogues in paper [10]. Thus the current work extends the works in paper [10].

Remark 2.2. Huang et al. [8,9] researched the almost periodic sequence solution of a class of general neural networks

$$
\begin{aligned}
u_i(k+1)\;=\;& e^{-c_i(k)}u_i(k)+\frac{1-e^{-c_i(k)}}{c_i(k)}\Bigg[\sum_{j=1}^{n}a_{ij}(k)f_j(v_j(k))+I_i(k)\Bigg],\\
v_j(k+1)\;=\;& e^{-d_j(k)}v_j(k)+\frac{1-e^{-d_j(k)}}{d_j(k)}\Bigg[\sum_{i=1}^{m}e_{ji}(k)g_i(u_i(k))+J_j(k)\Bigg]
\end{aligned}
$$

for all $(k,i,j)\in(\mathbb{Z},\mathbb{N}_1^m,\mathbb{N}_1^n)$. The above equations are simple versions of Eqs. (2.4)-(2.5) and the work in this paper is an extension of articles [8,9].

From Lemma 2.3 in literature [29], the lemma below holds.

Lemma 2.3. *Eqs. (2.4)-(2.5) can be transformed into*

$$
\begin{aligned}
u_i^{x^\ell}(k)\;=\;&\prod_{s=k_0}^{k-1}\lambda_{\alpha_i}^{x^\ell}(s)u_i^{x^\ell}(k_0)+\sum_{q=k_0}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\alpha_i}^{x^\ell}(s)\sum_{l=\mu_q}^{q}\theta_{\alpha_i}^{x^\ell}(q-l,q)\Bigg[F_i(u_i^{x^\ell}(l))\\
&+\sum_{j=1}^{n}a_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))+\bigwedge_{j=1}^{n}\gamma_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))+\bigvee_{j=1}^{n}\eta_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))\\
&+I_i^{x^\ell}(l)+\sum_{j=1}^{n}\mu_{ij}^{x^\ell}(l)H_{ij}(v_j^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{1j}(l)\Bigg], \qquad (2.7)\\
v_j^{x^\ell}(k)\;=\;&\prod_{s=k_0}^{k-1}\lambda_{\beta_j}^{x^\ell}(s)v_j^{x^\ell}(k_0)+\sum_{q=k_0}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\beta_j}^{x^\ell}(s)\sum_{l=\mu_q}^{q}\theta_{\beta_j}^{x^\ell}(q-l,q)\Bigg[G_j(v_j^{x^\ell}(l))\\
&+\sum_{i=1}^{m}e_{ji}^{x^\ell}(l)g_i(u_i^{x^\ell}(l))+\bigwedge_{i=1}^{m}\varepsilon_{ji}^{x^\ell}(l)g_i(u_i^{x^\ell}(l))+\bigvee_{i=1}^{m}\vartheta_{ji}^{x^\ell}(l)g_i(u_i^{x^\ell}(l))\\
&+J_j^{x^\ell}(l)+\sum_{i=1}^{m}\varsigma_{ji}^{x^\ell}(l)L_{ji}(u_i^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{2i}(l)\Bigg], \quad \forall x^\ell\in\Omega_{\rm d}, \qquad (2.8)
\end{aligned}
$$

where $k_0\in\mathbb{Z}$, $u_i^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\rm d}}=\phi_{i,k}^{x^\ell}$, $v_j^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\rm d}}=\tilde{\phi}_{j,k}^{x^\ell}$, $\forall(k,i,j)\in([k_0,+\infty)_{\mathbb{Z}},\mathbb{N}_1^m,\mathbb{N}_1^n)$.

Let $Q^*=\sup_{p\in\mathbb{Z}}(Q_{p+1}-Q_p)$,

$$
\bar{\lambda}_{\alpha_i}=\sup_{k\in\mathbb{Z},x^\ell\in\Omega_{\rm d}}\lambda_{\alpha_i}^{x^\ell}(k),\quad \bar{\lambda}_{\beta_j}=\sup_{k\in\mathbb{Z},x^\ell\in\Omega_{\rm d}}\lambda_{\beta_j}^{x^\ell}(k),
$$

$$\underline{c}_i^* = \inf_{x^\ell \in \Omega_{\rm d}} c_i^*(x^\ell), \quad \underline{d}_j^* = \inf_{x^\ell \in \Omega_{\rm d}} d_j^*(x^\ell), \quad \forall (i,j) \in (\mathbb{N}_1^m, \mathbb{N}_1^n).$$

It easily gets the lemma below.

Lemma 2.4. *If $Q^* < +\infty$, $\alpha_i, \beta_j \in (0,1]$ and $\underline{c}_i^*, \underline{d}_j^* > 0$, one has*

$$0 < \lambda_{\alpha_i}^{x^\ell}(k) \le \bar{\lambda}_{\alpha_i} < 1, \quad 0 < \lambda_{\beta_j}^{x^\ell}(k) \le \bar{\lambda}_{\beta_j} < 1,$$
$$\sum_{l=0}^{Q^*} |\theta_{\alpha_i}^{x^\ell}(l,k)| \le \frac{2}{\underline{c}_i^*}, \quad \sum_{l=0}^{Q^*} |\theta_{\beta j}^{x^\ell}(l,k)| \le \frac{2}{\underline{d}_j^*},$$

for all $(x^\ell, k, i, j) \in (\bar{\Omega}_{\rm d}, \mathbb{Z}, \mathbb{N}_1^m, \mathbb{N}_1^n)$.

3. Periodic sequence

Let $\mathbb{E}(\cdot)$ be the expectation under a complete probability space $(\Lambda, \mathcal{F}, \mathbb{P})$ and $\mathbb{W} = (\mathbb{W}_{11}, \dots, \mathbb{W}_{1m}, \mathbb{W}_{21}, \dots, \mathbb{W}_{2n})^T$ be a two-sided standard $(m+n)$-dimensional Brownian motion defined on $(\Lambda, \mathcal{F}, \mathbb{P})$. Set $\mathcal{F}_t = \sigma\{\mathbb{W}(s) : s \le t\}$ for $t \in \mathbb{R}$. Further, $L^2(\Lambda, \mathbb{R}^{m+n})$ denotes the family of all square integrable $\mathbb{R}^{m+n}$-valued random variables and $\mathbb{X} = \mathbb{B}(\mathbb{Z} \times \bar{\Omega}_{\rm d}, L^2(\Lambda, \mathbb{R}^{m+n}))$ stands for the set of all functions from $\mathbb{Z} \times \bar{\Omega}_{\rm d}$ to $L^2(\Lambda, \mathbb{R}^{m+n})$ endowed with the norm

$$\|\mathcal{Z}\|_{\mathbb{X}} = \sup_{k \in \mathbb{Z}} \max_{1 \le i \le m, 1 \le i \le n, x^\ell \in \bar{\Omega}_{\rm d}} \max \left\{ \left[\mathbb{E} \left| u_i^{x^\ell}(k) \right|^2 \right]^{1/2}, \left[\mathbb{E} \left| v_j^{x^\ell}(k) \right|^2 \right]^{1/2} \right\},$$

where $\mathcal{Z} = (u_1, \dots, u_m, v_1, \dots, v_n)^T \in \mathbb{X}$. Obviously, $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ becomes a Banach space. In the whole paper, $\varphi_i^{x^\ell}, \tilde{\varphi}_i^{x^\ell}$ are $\mathcal{F}_0$-adapted and $\phi_i^{x^\ell}(k), \tilde{\phi}_i^{x^\ell}(k)$ are $\mathcal{F}_k$-adapted, $\forall (x^\ell, k, i, j) \in (\bar{\Omega}_{\rm d}, \mathbb{Z}, \mathbb{N}_1^m, \mathbb{N}_1^n)$.

Definition 3.1. A discrete-time stochastic process $\mathcal{Z} = (u_1, \dots, u_m, v_1, \dots, v_n)^T \in \mathbb{X}$ is called the solution of Eqs. (2.4)-(2.5) if it is $\mathcal{F}_t$-adapted and meets Eqs. (2.7)-(2.8).

Definition 3.2. Let $\omega \in \mathbb{R}$. A discrete-time stochastic process $\mathcal{Z} \in \mathbb{X}$ is called ω-periodic in distribution if the law of $\mathcal{Z}(\cdot + \omega)$ is the same as that of $\mathcal{Z}(\cdot)$.

Lemma 3.1. ([13]) (Minkowski inequality) *If $f, g \in L^2(\Lambda, \mathbb{R})$, then*

$$\left[\mathbb{E}(f+g)^2\right]^{\frac{1}{2}} \le \left(\mathbb{E}f^2\right)^{\frac{1}{2}} + \left(\mathbb{E}g^2\right)^{\frac{1}{2}}.$$

Lemma 3.2. ([13]) (Hölder inequality) *Let $p > 1$ and $a_k, b_k : \mathbb{Z} \to \mathbb{R}$. Then*

$$\sum_k |a_k b_k| \le \left[\sum_k |a_k|\right]^{1-1/p} \left[\sum_k |a_k| |b_k|^p\right]^{1/p}.$$

Lemma 3.3. ([29]) *Let $\{f(k)\}_{k \in \mathbb{Z}} \subseteq L^2(\Lambda, \mathbb{R})$ and $\{\mathbb{W}(t)\}_{t \in \mathbb{R}}$ be a two-sided standard one dimensional Brownian motion. Then*

$$\mathbb{E}\left|f(k)\Delta_h \mathbb{W}(k)\right|^2 \le 4h\mathbb{E}|f(k)|^2,$$

where $\Delta_h \mathbb{W}(k) = \mathbb{W}(kh+h) - \mathbb{W}(kh)$, $\forall k \in \mathbb{Z}$.

Here, we needs the following assumptions.

(H_1) $\{\nu_k\}$ is a ω-periodic sequence, $\omega \in \mathbb{Z}$, i.e., $\nu_{k+\omega} = \nu_k$, $\forall k \in \mathbb{Z}$.

(H_2) $a_{ij}^{x^\ell}(k)$, $\gamma_{ij}^{x^\ell}(k)$, $\eta_{ij}^{x^\ell}(k)$, $I_i^{x^\ell}(k)$, $\mu_{ij}^{x^\ell}(k)$, $e_{ji}^{x^\ell}(k)$, $\varepsilon_{ji}^{x^\ell}(k)$, $\vartheta_{ji}^{x^\ell}(k)$, $J_j^{x^\ell}(k)$ and $\varsigma_{ji}^{x^\ell}(k)$ are ω-periodic sequences with respect to variable $k \in \mathbb{Z}$, $(x^\ell, i, j) \in (\Omega_{\mathrm{d}}, \mathbb{N}_1^m, \mathbb{N}_1^n)$.

(H_3) It exists positive numbers $\mathcal{L}_i^F$, $\mathcal{L}_j^f$, $\mathcal{L}_{ij}^H$, $\mathcal{L}_j^G$, $\mathcal{L}_i^g$, $\mathcal{L}_{ji}^L$ such that

$$
\begin{aligned}
&|F_i(x) - F_i(y)| \le \mathcal{L}_i^F |x-y|, \quad |f_j(x) - f_j(y)| \le \mathcal{L}_j^f |x-y|,\\
&|H_{ij}(x) - H_{ij}(y)| \le \mathcal{L}_{ij}^H |x-y|, \quad |G_j(x) - G_j(y)| \le \mathcal{L}_j^G |x-y|,\\
&|g_i(x) - g_i(y)| \le \mathcal{L}_i^g |x-y|, \quad |L_{ji}(x) - L_{ji}(y)| \le \mathcal{L}_{ji}^L |x-y|
\end{aligned}
$$

for any $x, y \in \mathbb{R}$, $(i,j) \in (\mathbb{N}_1^m, \mathbb{N}_1^n)$.

Define $\bar{f} = \sup_{k\in\mathbb{Z}, s\in\Omega_{\mathrm{d}}} |f(k,s)|$, which $f := \mathbb{Z} \times \Omega_{\mathrm{d}} \to \mathbb{R}$ is a sequence. Let $D_i^* = 2\sum_{q=1}^{\mathcal{N}} \frac{D_{iq}}{h_q^2}$, $K_j^* = 2\sum_{q=1}^{\mathcal{N}} \frac{K_{jq}}{h_q^2}$, $(i,j) \in (\mathbb{N}_1^m, \mathbb{N}_1^n)$; $\sigma_0 = \frac{\sigma_2}{1-\sigma_1}$, where

$$
\begin{aligned}
\sigma_1 = \max_{1\le i\le m, 1\le j\le n} \Bigg\{ & \frac{2}{\underline{c}_i^*(1-\bar{\lambda}_{\alpha_i})} \left[D_i^* + \sum_{j=1}^n (\bar{a}_{ij} + \bar{\gamma}_{ij} + \bar{\eta}_{ij}) \mathcal{L}_j^f + 2h^{-\frac{1}{2}} \sum_{j=1}^n \bar{\mu}_{ij} \mathcal{L}_{ij}^H \right],\\
& \frac{2}{\underline{d}_j^*(1-\bar{\lambda}_{\beta_j})} \left[K_j^* + \sum_{i=1}^m (\bar{e}_{ji} + \bar{\varepsilon}_{ji} + \bar{\vartheta}_{ji}) \mathcal{L}_i^g + 2h^{-\frac{1}{2}} \sum_{i=1}^m \bar{\varsigma}_{ji} \mathcal{L}_{ji}^L \right] \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_2 = \sup_{k\in\mathbb{Z}} \max_{1\le i\le m, 1\le j\le n} \Bigg\{ & \left[E|\varphi_{i,k}^{x^\ell}|^2 \right]^{\frac{1}{2}}, \left[E|\tilde{\varphi}_{j,k}^{x^\ell}|^2 \right]^{\frac{1}{2}},\\
& \frac{2}{\underline{c}_i^*(1-\bar{\lambda}_{\alpha_i})} \left[\left(\sum_{j=1}^n \bar{a}_{ij} + \bigwedge_{j=1}^n \bar{\gamma}_{ij} + \bigvee_{j=1}^n \bar{\eta}_{ij} \right) |f_j(0)| + \bar{I}_i + 2h^{-\frac{1}{2}} \sum_{j=1}^n \bar{\mu}_{ij} |H_{ij}(0)| \right],\\
& \frac{2}{\underline{d}_j^*(1-\bar{\lambda}_{\beta_j})} \left[\left(\sum_{i=1}^m \bar{e}_{ji} + \bigwedge_{i=1}^m \bar{\varepsilon}_{ji} + \bigvee_{i=1}^m \bar{\vartheta}_{ji} \right) |g_i(0)| + \bar{J}_j + 2h^{-\frac{1}{2}} \sum_{i=1}^m \bar{\varsigma}_{ji} |L_{ji}(0)| \right] \Bigg\}.
\end{aligned}
$$

Let $\mathcal{Z}_0 = \{\mathcal{Z} \in \mathbb{X} : \|\mathcal{Z}\|_{\mathbb{X}} \le \sigma_0\}$. In accordance with Eqs. (2.7), define a mapping $\Psi : \mathcal{Z}_0 \to \mathbb{X}$ as

$$
\Psi\mathcal{Z} = \Big((\Psi u)_1, (\Psi u)_2, \ldots, (\Psi u)_m, (\Psi v)_1, (\Psi v)_2, \ldots, (\Psi v)_n \Big)^T
$$

where

$$
\begin{aligned}
(\Psi u)_{i,k}^{x^\ell} = \sum_{q=-\infty}^{k-1} \prod_{s=q+1}^{k-1} \lambda_{\alpha_i}^{x^\ell}(s) \sum_{l=\mu_q}^{q} \theta_{\alpha_i}^{x^\ell}(q-l,q) \Bigg[& F_i(u_i^{x^\ell}(l)) + \sum_{j=1}^n a_{ij}^{x^\ell}(l) f_j(v_j^{x^\ell}(l))\\
& + \bigwedge_{j=1}^n \gamma_{ij}^{x^\ell}(l) f_j(v_j^{x^\ell}(l)) + \bigvee_{j=1}^n \eta_{ij}^{x^\ell}(l) f_j(v_j^{x^\ell}(l)) + I_i^{x^\ell}(l) + \sum_{j=1}^n \mu_{ij}^{x^\ell}(l)\\
& \times H_{ij}(v_j^{x^\ell}(l)) h^{-1} \Delta_h \mathbb{W}_{1j}(l) \Bigg],
\end{aligned} \tag{3.1}
$$

$$
(\Psi v)_{j,k}^{x^\ell} = \sum_{q=-\infty}^{k-1} \prod_{s=q+1}^{k-1} \lambda_{\beta_j}^{x^\ell}(s) \sum_{l=\mu_q}^{q} \theta_{\beta_j}^{x^\ell}(q-l,q) \Bigg[G_j(v_j^{x^\ell}(l)) + \sum_{i=1}^m e_{ji}^{x^\ell}(l) g_i(u_i^{x^\ell}(l))
$$

$$+\bigwedge_{i=1}^{m}\varepsilon_{ji}^{x^\ell}(l)g_i(u_i^{x^\ell}(l))+\bigvee_{i=1}^{m}\vartheta_{ji}^{x^\ell}(l)g_i(u_i^{x^\ell}(l))+J_j^{x^\ell}(l)+\sum_{i=1}^{m}\varsigma_{ji}^{x^\ell}(l)$$
$$\times L_{ji}(u_i^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{2i}(l)\Bigg],\quad x^\ell\in\Omega_{\rm d}; \tag{3.2}$$
$$(\Psi u)_{i,k}^{x^\ell}\Big|_{x^\ell\in\partial\Omega_{\rm d}}=\varphi_i^{x^\ell}(k),(\Psi v)_{j,k}^{x^\ell}\Big|_{x^\ell\in\partial\Omega_{\rm d}}=\tilde{\varphi}_j^{x^\ell}(k),\forall(k,i,j)\in(\mathbb{Z},\mathbb{N}_1^m,\mathbb{N}_1^n).$$

Proposition 3.1. *Ψ is well defined and maps $\mathcal{Z}_0$ to $\mathcal{Z}_0$ if (H_3) and (H_4) below hold.*

(H_4) $\sigma_1<1$.

Proof. Suppose that $\mathcal{Z}=(u_1,\dots,u_m,v_1,\dots,v_n)^T\in\mathcal{Z}_0$. Based on Eqs. (3.1), Lemmas 3.1, 3.2 and 3.3, it gets

$$\begin{aligned}
&\left[\mathbb{E}\left|(\Psi u)_{i,k}^{x^\ell}\right|^2\right]^{\frac{1}{2}}\\
&=\Bigg\{\mathbb{E}\Bigg[\sum_{q=-\infty}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\alpha_i}^{x^\ell}(s)\sum_{l=\mu_q}^{q}\theta_{\alpha_i}^{x^\ell}(q-l,q)\Bigg(F_i(u_i^{x^\ell}(l))\\
&\quad+\sum_{j=1}^{n}a_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))+\bigwedge_{j=1}^{n}\gamma_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))+\bigvee_{j=1}^{n}\eta_{ij}^{x^\ell}(l)\\
&\quad\times f_j(v_j^{x^\ell}(l))+I_i^{x^\ell}(l)+\sum_{j=1}^{n}\mu_{ij}^{x^\ell}(l)H_{ij}(v_j^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{1j}(l)\Bigg)\Bigg]^2\Bigg\}^{\frac{1}{2}}\\
&\le\Bigg\{\mathbb{E}\Bigg\{\Bigg[\sum_{q=-\infty}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\alpha_i}^{x^\ell}(s)\Bigg]^{\frac{1}{2}}\Bigg\{\sum_{q=-\infty}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\alpha_i}^{x^\ell}(s)\\
&\quad\times\Bigg[\sum_{l=\mu_q}^{q}\theta_{\alpha_i}^{x^\ell}(q-l,q)\Bigg(F_i(u_i^{x^\ell}(l))+\sum_{j=1}^{n}a_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))\\
&\quad+\bigwedge_{j=1}^{n}\gamma_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))+\bigvee_{j=1}^{n}\eta_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))+I_i^{x^\ell}(l)\\
&\quad+\sum_{j=1}^{n}\mu_{ij}^{x^\ell}(l)H_{ij}(v_j^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{1j}(l)\Bigg)\Bigg]^2\Bigg\}^{\frac{1}{2}}\Bigg\}^2\Bigg\}^{\frac{1}{2}}\\
&\le\sum_{q=-\infty}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\alpha_i}^{x^\ell}(s)\Bigg\{\mathbb{E}\Bigg[\sum_{l=\mu_q}^{q}\theta_{\alpha_i}^{x^\ell}(q-l,q)\Bigg(F_i(u_i^{x^\ell}(l))\\
&\quad+\sum_{j=1}^{n}a_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))+\bigwedge_{j=1}^{n}\gamma_{ij}^{x^\ell}(l)f_j(v_j^{x^\ell}(l))+\bigvee_{j=1}^{n}\eta_{ij}^{x^\ell}(l)\\
&\quad\times f_j(v_j^{x^\ell}(l))+I_i^{x^\ell}(l)+\sum_{j=1}^{n}\mu_{ij}^{x^\ell}(l)H_{ij}(v_j^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{1j}(l)\Bigg)\Bigg]^2\Bigg\}^{\frac{1}{2}}\\
&\le\frac{2}{\underline{c}_i^*(1-\bar{\lambda}_{\alpha_i})}\Bigg\{\Bigg[D_i^*+\sum_{j=1}^{n}(\bar{a}_{ij}+\bar{\gamma}_{ij}+\bar{\eta}_{ij})\mathcal{L}_j^f\Bigg]\|\mathcal{Z}\|_{\mathbb{X}}
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\sum_{j=1}^{n}\bar{a}_{ij}+\bigwedge_{j=1}^{n}\bar{\gamma}_{ij}+\bigvee_{j=1}^{n}\bar{\eta}_{ij}\Big)|f_j(0)|+\bar{I}_i+2h^{-\frac{1}{2}}\sum_{j=1}^{n}\bar{\mu}_{ij}\\
&\times|H_{ij}(0)|+\sum_{j=1}^{n}\bar{\mu}_{ij}\mathcal{L}_{ij}^{H}\left[\mathbb{E}\left|v_j^{x^\ell}(l)h^{-1}\Delta_h\mathbb{W}_{1j}(l)\right|^2\right]^{\frac{1}{2}}\Bigg\}\\
\le\ & \frac{2}{\underline{c}_i^*(1-\bar{\lambda}_{\alpha_i})}\Big[D_i^*+\sum_{j=1}^{n}(\bar{a}_{ij}+\bar{\gamma}_{ij}+\bar{\eta}_{ij})\mathcal{L}_j^f+2h^{-\frac{1}{2}}\sum_{j=1}^{n}\bar{\mu}_{ij}\mathcal{L}_{ij}^H\Big]\\
&\times\|\mathcal{Z}\|_{\mathbb{X}}+\frac{2}{\underline{c}_i^*(1-\bar{\lambda}_{\alpha_i})}\Big[\Big(\sum_{j=1}^{n}\bar{a}_{ij}+\bigwedge_{j=1}^{n}\bar{\gamma}_{ij}+\bigvee_{j=1}^{n}\bar{\eta}_{ij}\Big)|f_j(0)|\\
&+\bar{I}_i+2h^{-\frac{1}{2}}\sum_{j=1}^{n}\bar{\mu}_{ij}|H_{ij}(0)|\Big]\\
\le\ & \sigma_1\sigma_0+\sigma_2,\quad \forall(\mathcal{Z},x^\ell,k,i,j)\in(\mathcal{Z}_0,\Omega_{\mathrm{d}},\mathbb{Z},\mathbb{N}_1^m,\mathbb{N}_1^n).
\end{aligned}
$$

Similarly, it yields from (3.2) that

$$
\begin{aligned}
&\left[\mathbb{E}\left|(\Psi v)_{j,k}^{x^\ell}\right|^2\right]^{\frac{1}{2}}\\
=\ &\Bigg\{\mathbb{E}\Big[\sum_{q=-\infty}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\beta_j}^{x^\ell}(s)\sum_{l=\mu_q}^{q}\theta_{\beta_j}^{x^\ell}(q-l,q)\Big(G_j(v_j^{x^\ell}(l))\\
&+\sum_{i=1}^{m}e_{ji}^{x^\ell}(l)g_i(u_i^{x^\ell}(l))+\bigwedge_{i=1}^{m}\varepsilon_{ji}^{x^\ell}(l)g_i(u_i^{x^\ell}(l))+\bigvee_{i=1}^{m}\vartheta_{ji}^{x^\ell}(l)\\
&\times g_i(u_i^{x^\ell}(l))+J_j^{x^\ell}(l)+\sum_{i=1}^{m}\varsigma_{ji}^{x^\ell}(l)L_{ji}(u_i^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{2i}(l)\Big)\Big]^2\Bigg\}^{\frac{1}{2}}\\
\le\ & \frac{2}{\underline{d}_j^*(1-\bar{\lambda}_{\beta_j})}\Big[K_j^*+\sum_{i=1}^{m}(\bar{e}_{ji}+\bar{\varepsilon}_{ji}+\bar{\vartheta}_{ji})\mathcal{L}_i^g+2h^{-\frac{1}{2}}\sum_{i=1}^{m}\bar{\varsigma}_{ji}\mathcal{L}_{ji}^L\Big]\\
&\times\|\mathcal{Z}\|_{\mathbb{X}}+\frac{2}{\underline{d}_j^*(1-\bar{\lambda}_{\beta_j})}\Big[\Big(\sum_{i=1}^{m}\bar{e}_{ji}+\bigwedge_{i=1}^{m}\bar{\varepsilon}_{ji}+\bigvee_{i=1}^{m}\bar{\vartheta}_{ji}\Big)|g_i(0)|\\
&+\bar{J}_j+2h^{-\frac{1}{2}}\sum_{i=1}^{m}\bar{\varsigma}_{ji}|L_{ji}(0)|\Big]\\
\le\ & \sigma_1\sigma_0+\sigma_2,\quad \forall(\mathcal{Z},x^\ell,k,i,j)\in(\mathcal{Z}_0,\Omega_{\mathrm{d}},\mathbb{Z},\mathbb{N}_1^m,\mathbb{N}_1^n).
\end{aligned}
$$

Summarizing the above analyses, it leads to $\|\Psi\mathcal{Z}\|_{\mathbb{X}}\le\sigma_1\sigma_0+\sigma_2=\sigma_0$, $\forall\mathcal{Z}\in\mathcal{Z}_0$. Therefore, Ψ is well defined and maps $\mathcal{Z}_0$ to $\mathcal{Z}_0$. The proof is achieved. □

Remark 3.1. In order to ensure the validity of condition (H_4) in Proposition 3.1, we should pay attention to the following aspects in application.

- **(i)** As for the coefficients of BAMNNs (2.1)-(2.2), except for c_i and d_j, it is better to choose smaller constants, inversely, c_i, d_j should be chosen the larger positive constants for any $(i,j)\in(\mathbb{N}_1^m,\mathbb{N}_1^n)$.
- **(ii)** It is best to choose small positive constants for the time-space discrete step lengths h and h_q.

(iii) The activation functions F_i, f_j, H_{ij}, G_j, g_i and L_{ji} of BAMNNs (2.1)-(2.2) are best to select some small enough positive constants $\mathcal{L}_i^F$, $\mathcal{L}_j^f$, $\mathcal{L}_{ij}^H$, $\mathcal{L}_j^G$, $\mathcal{L}_i^g$, $\mathcal{L}_{ji}^L$ for any $(i,j) \in (\mathbb{N}_1^m, \mathbb{N}_1^n)$.

Proposition 3.2. *Eqs. (2.4)-(2.5) admits a unique solution in* $\mathcal{Z}_0$ *if* (H_3)-(H_4) *hold.*

Proof. By Proposition 3.1, $\Psi : \mathcal{Z}_0 \to \mathcal{Z}_0$. Assume that

$$\mathcal{Z} = (u_1, \ldots, u_m, v_1, \ldots, v_n)^T, \tilde{\mathcal{Z}} = (\tilde{u}_1, \ldots, \tilde{u}_m, \tilde{v}_1, \ldots, \tilde{v}_n)^T \in \mathcal{Z}_0,$$

it derives from Eqs. (3.1)-(3.2) that

$$\begin{aligned}
&\left[\mathbb{E}\left|(\Psi u)_{i,k}^{x^\ell} - (\Psi \tilde{u})_{i,k}^{x^\ell}\right|^2\right]^{\frac{1}{2}} \\
&\le \Bigg\{\mathbb{E}\Bigg|\sum_{q=-\infty}^{k-1} \prod_{s=q+1}^{k-1} \lambda_{\alpha_i}^{x^\ell}(s) \sum_{l=\mu_q}^{q} \theta_{\alpha_i}^{x^\ell}(q-l,q)\Bigg[\left|F_i(u_i^{x^\ell}(l)) - F_i(\tilde{u}_i^{x^\ell}(l))\right| \\
&\quad + \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{\gamma}_{ij} + \bar{\eta}_{ij}) \left|f_j(v_j^{x^\ell}(l)) - f_j(\tilde{v}_j^{x^\ell}(l))\right| \\
&\quad + \sum_{j=1}^{n} \bar{\mu}_{ij} h^{-1} |\Delta_h \mathbb{W}_{1j}(l)| \left|H_{ij}(v_j^{x^\ell}(l)) - H_{ij}(\tilde{v}_j^{x^\ell}(l))\right|\Bigg]\Bigg|^2\Bigg\}^{\frac{1}{2}} \\
&\le \sum_{q=-\infty}^{k-1} \prod_{s=q+1}^{k-1} \lambda_{\alpha_i}^{x^\ell}(s) \Bigg\{\mathbb{E}\Bigg[\sum_{l=\mu_q}^{q} \left|\theta_{\alpha_i}^{x^\ell}(q-l,q)\right| \Bigg(\left|F_i(u_i^{x^\ell}(l)) - F_i(\tilde{u}_i^{x^\ell}(l))\right| \\
&\quad + \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{\gamma}_{ij} + \bar{\eta}_{ij}) \left|f_j(v_j^{x^\ell}(l)) - f_j(\tilde{v}_j^{x^\ell}(l))\right| \\
&\quad + \sum_{j=1}^{n} \bar{\mu}_{ij} h^{-1} |\Delta_h \mathbb{W}_{1j}(l)| \left|H_{ij}(v_j^{x^\ell}(l)) - H_{ij}(\tilde{v}_j^{x^\ell}(l))\right|\Bigg)\Bigg]^2\Bigg\}^{\frac{1}{2}} \\
&\le \frac{2}{\underline{c}_i^*(1-\bar{\lambda}_{\alpha_i})}\Bigg[D_i^* + \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{\gamma}_{ij} + \bar{\eta}_{ij}) \mathcal{L}_j^f + 2h^{-\frac{1}{2}} \sum_{j=1}^{m} \bar{\mu}_{ij} \mathcal{L}_{ij}^H\Bigg] \|\mathcal{Z} - \tilde{\mathcal{Z}}\|_{\mathbb{X}} \\
&\le \sigma_1 \|\mathcal{Z} - \tilde{\mathcal{Z}}\|_{\mathbb{X}}, \quad \forall (x^\ell, k, i, j) \in (\bar{\Omega}_{\mathrm{d}}, \mathbb{Z}, \mathbb{N}_1^m, \mathbb{N}_1^n).
\end{aligned}$$

Similarly,

$$\left[\mathbb{E}\left|(\Psi v)_{j,k}^{x^\ell} - (\Psi \tilde{v})_{j,k}^{x^\ell}\right|^2\right]^{\frac{1}{2}} \le \sigma_1 \|\mathcal{Z} - \tilde{\mathcal{Z}}\|_{\mathbb{X}}, \quad \forall (x^\ell, k, i, j) \in (\bar{\Omega}_{\mathrm{d}}, \mathbb{Z}, \mathbb{N}_1^m, \mathbb{N}_1^n).$$

So $\|\Psi\mathcal{Z} - \Psi\tilde{\mathcal{Z}}\|_{\mathbb{X}} \le \sigma_1 \|\mathcal{Z} - \tilde{\mathcal{Z}}\|_{\mathbb{X}}$, $\forall \mathcal{Z}, \tilde{\mathcal{Z}} \in \mathcal{Z}_0$. By (H_4), Ψ is contractive and Ψ has a unique fixed point $\mathtt{Z} = \Psi\mathtt{Z} \in \mathcal{Z}_0 \subseteq \mathbb{X}$ solving Eqs. (2.4)-(2.5). This finishes the proof. □

Theorem 3.1. *A unique* ω*-periodic sequence in distribution solves Eqs. (2.4)-(2.5) if* (H_1)-(H_4) *hold.*

Proof. Eqs. (2.4)-(2.5) possesses a unique solution $\mathtt{Z}$ in $\mathcal{Z}_0$ on the basis of Proposition 3.2. By Proposition 3.2, the unique solution $\mathtt{Z} = (\mathtt{u}_1, \ldots, \mathtt{u}_m, \mathtt{v}_1, \ldots, \mathtt{v}_n)^T \in \mathcal{Z}_0$

of Eqs. (2.4)-(2.5) meets

$$\begin{aligned}\mathtt{u}_i^{x^\ell}(k+\omega) \;=\; &\sum_{q=-\infty}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\alpha_i}^{x^\ell}(s)\sum_{l=\mu_q}^{q}\theta_{\alpha_i}^{x^\ell}(q-l,q)\Big[F_i(\mathtt{u}_i^{x^\ell}(l+\omega))+\sum_{j=1}^{n}a_{ij}^{x^\ell}(l)\\&\times f_j(\mathtt{v}_j^{x^\ell}(l+\omega))+\bigwedge_{j=1}^{n}\gamma_{ij}^{x^\ell}(l)f_j(\mathtt{v}_j^{x^\ell}(l+\omega))+\bigvee_{j=1}^{n}\eta_{ij}^{x^\ell}(l)f_j(\mathtt{v}_j^{x^\ell}(l+\omega))\\&+I_i^{x^\ell}(l)+\sum_{j=1}^{n}\mu_{ij}^{x^\ell}(l)H_{ij}(\mathtt{v}_j^{x^\ell}(l+\omega))h^{-1}\Delta_h\mathbb{W}_{1j}(l+\omega)\Big],\\ \mathtt{v}_j^{x^\ell}(k+\omega) \;=\; &\sum_{q=-\infty}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\beta_j}^{x^\ell}(s)\sum_{l=\mu_q}^{q}\theta_{\beta_j}^{x^\ell}(q-l,q)\Big[G_j(\mathtt{v}_j^{x^\ell}(l+\omega))+\sum_{i=1}^{m}e_{ji}^{x^\ell}(l)\\&\times g_i(\mathtt{u}_i^{x^\ell}(l+\omega))+\bigwedge_{i=1}^{m}\varepsilon_{ji}^{x^\ell}(l)g_i(\mathtt{u}_i^{x^\ell}(l+\omega))+\bigvee_{i=1}^{m}\vartheta_{ji}^{x^\ell}(l)g_i(\mathtt{u}_i^{x^\ell}(l+\omega))\\&+J_j^{x^\ell}(l)+\sum_{i=1}^{m}\varsigma_{ji}^{x^\ell}(l)L_{ji}(\mathtt{u}_i^{x^\ell}(l+\omega))h^{-1}\Delta_h\mathbb{W}_{2i}(l+\omega)\Big],\quad \forall x^\ell\in\Omega_{\mathrm{d}},\end{aligned}$$

$$\mathtt{u}_i^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\mathrm{d}}}=\phi_i^{x^\ell}(k),\ \mathtt{v}_j^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\mathrm{d}}}=\tilde{\phi}_j^{x^\ell}(k),\ \forall(k,i,j)\in(\mathbb{Z},\mathbb{N}_1^m,\mathbb{N}_1^n).$$

Let us discuss the stochastic process $\mathtt{Z}_\omega=(\mathtt{u}_{1,\omega},\ldots,\mathtt{u}_{m,\omega},\mathtt{v}_{1,\omega},\ldots,\mathtt{v}_{n,\omega})^T$ below

$$\begin{aligned}\mathtt{u}_{i,\omega}^{x^\ell}(k) = &\sum_{q=-\infty}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\alpha_i}^{x^\ell}(s)\sum_{l=\mu_q}^{q}\theta_{\alpha_i}^{x^\ell}(q-l,q)\Big[F_i(\mathtt{u}_{i,\omega}^{x^\ell}(l))+\sum_{j=1}^{n}a_{ij}^{x^\ell}(l)\\&\times f_j(\mathtt{v}_{j,\omega}^{x^\ell}(l))+\bigwedge_{j=1}^{n}\gamma_{ij}^{x^\ell}(l)f_j(\mathtt{v}_{j,\omega}^{x^\ell}(l))+\bigvee_{j=1}^{n}\eta_{ij}^{x^\ell}(l)f_j(\mathtt{v}_{j,\omega}^{x^\ell}(l))\\&+I_i^{x^\ell}(l)+\sum_{j=1}^{n}\mu_{ij}^{x^\ell}(l)H_{ij}(\mathtt{v}_{j,\omega}^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{1j}(l)\Big],\\ \mathtt{v}_{j,\omega}^{x^\ell}(k) = &\sum_{q=-\infty}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\beta_j}^{x^\ell}(s)\sum_{l=\mu_q}^{q}\theta_{\beta_j}^{x^\ell}(q-l,q)\Big[G_j(\mathtt{v}_{j,\omega}^{x^\ell}(l))+\sum_{i=1}^{m}e_{ji}^{x^\ell}(l)\\&\times g_i(\mathtt{u}_{i,\omega}^{x^\ell}(l))+\bigwedge_{i=1}^{m}\varepsilon_{ji}^{x^\ell}(l)g_i(\mathtt{u}_{i,\omega}^{x^\ell}(l))+\bigvee_{i=1}^{m}\vartheta_{ji}^{x^\ell}(l)g_i(\mathtt{u}_{i,\omega}^{x^\ell}(l))\\&+J_j^{x^\ell}(l)+\sum_{i=1}^{m}\varsigma_{ji}^{x^\ell}(l)L_{ji}(\mathtt{u}_{i,\omega}^{x^\ell}(l))h^{-1}\Delta_h\mathbb{W}_{2i}(l)\Big],\quad \forall x^\ell\in\Omega_{\mathrm{d}},\\ \mathtt{u}_{i,\omega}^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\mathrm{d}}} &= \phi_i^{x^\ell}(k),\mathtt{v}_{j,\omega}^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\mathrm{d}}}=\tilde{\phi}_j^{x^\ell}(k),\forall(k,i,j)\in(\mathbb{Z},\mathbb{N}_1^m,\mathbb{N}_1^n).\end{aligned}$$

Similar to $\mathtt{Z}$, $\mathtt{Z}_\omega$ is unique and bounded in $\mathcal{Z}_0$. Noting that $\Delta_h\mathbb{W}_{1j}(k+\omega)$ and $\Delta_h\mathbb{W}_{2i}(k+\omega)$ have the same laws as $\Delta_h\mathbb{W}_{1j}(k)$ and $\Delta_h\mathbb{W}_{2i}(k)$, respectively, $\forall(k,i,j)\in(\mathbb{Z},\mathbb{N}_1^m,\mathbb{N}_1^n)$. Then $\mathtt{Z}_\omega(\cdot)$ has the same distribution as $\mathtt{Z}(\cdot+\omega)$.

Resembling the derivation in Proposition 3.2, it calculates

$$\left[\mathbb{E}\left|\mathtt{u}_{i,\omega}^{x^\ell}(k)-\mathtt{u}_i^{x^\ell}(k)\right|^2\right]^{\frac{1}{2}}\le\sigma_1\|\mathtt{Z}_\omega-\mathtt{Z}\|_{\mathbb{X}},\ \ \left[\mathbb{E}\left|\mathtt{v}_{j,\omega}^{x^\ell}(k)-\mathtt{v}_j^{x^\ell}(k)\right|^2\right]^{\frac{1}{2}}\le\sigma_1\|\mathtt{Z}_\omega-\mathtt{Z}\|_{\mathbb{X}},$$

where $(x^\ell, k, i, j) \in (\bar{\Omega}_{\rm d}, \mathbb{Z}, \mathbb{N}_1^m, \mathbb{N}_1^n)$. Together with assumption (H_4), it leads to $\|\mathtt{Z}_\omega - \mathtt{Z}\|_{\mathbb{X}} = 0$. So the law of $\mathtt{Z}_\omega$ is equal to that of $\mathtt{Z}$. Recalling that $\mathtt{Z}_\omega(\cdot)$ has the same distribution as $\mathtt{Z}(\cdot + \omega)$. Then $\mathtt{Z}(\cdot + \omega)$ has the same distribution as $\mathtt{Z}(\cdot)$. This finishes the proof. □

Remark 3.2. In literatures [11, 14, 30], the authors discussed the periodic oscillations to various discrete-time BAMNNs. However, almost no paper regards the study of periodic sequences of discrete-time stochastic lattice BAMNNs. Thus the current work extends the works in papers [11, 14, 30].

4. Global mean-square λ-exponential convergence

Let $\mathcal{Z} = (u_1, \dots, u_m, v_1, \dots, v_n)^T$ and $\tilde{\mathcal{Z}} = (\tilde{u}_1, \dots, \tilde{u}_m, \tilde{v}_1, \dots, \tilde{v}_n)^T$ be any two solutions of Eqs. (2.4)-(2.5) with initial boundary conditions

$$u_i^{x^\ell}(0)=\psi_i^{x^\ell}, \quad \tilde{u}_i^{x^\ell}(0)=\tilde{\psi}_i^{x^\ell}, \ \forall x^\ell \in \Omega_{\rm d}, \ u_i^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\rm d}} = \tilde{u}_i^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\rm d}} = \phi_i^{x^\ell}(k),$$
$$v_j^{x^\ell}(0) = \varphi_j^{x^\ell}, \quad \tilde{v}_j^{x^\ell}(0) = \tilde{\varphi}_j^{x^\ell}, \ \forall x^\ell \in \Omega_{\rm d}, \ v_j^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\rm d}} = \tilde{v}_j^{x^\ell}(k)\Big|_{x^\ell\in\partial\Omega_{\rm d}} = \tilde{\phi}_j^{x^\ell}(k)$$

for all $(k, i, j) \in (\mathbb{N}_0, \mathbb{N}_1^m, \mathbb{N}_1^n)$.

Set $\mathbf{Y} = (\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{v}_1, \dots, \mathbf{v}_n)^T$ with $\mathbf{u}_i^{x^\ell}(k) = u_i^{x^\ell}(k) - \tilde{u}_i^{x^\ell}(k)$ and $\mathbf{v}_j^{x^\ell}(k) = v_j^{x^\ell}(k) - \tilde{v}_j^{x^\ell}(k)$ for all $(x^\ell, k, i, j) \in (\Omega_{\rm d}, \mathbb{Z}, \mathbb{N}_1^m, \mathbb{N}_1^n)$. Eqs. (2.4)-(2.5) is said to be globally mean-square λ-exponential convergent if it exists $M > 0$ and $0 < \kappa < 1$ such that

$$\|\mathbf{Y}(k)\|_{\Omega_{\rm d}} \le M\|\psi\|_{\Omega_{\rm d}}\lambda^{\kappa k}, \quad \forall k \ge 0, \quad \lambda = \max_{1\le i\le m, 1\le j\le n}\{\bar{\lambda}_{\alpha_i}, \bar{\lambda}_{\beta_j}\} \in (0, 1),$$

where

$$\|\mathbf{Y}(k)\|_{\Omega_{\rm d}} = \max_{1\le i\le m, 1\le j\le n, x^\ell\in\Omega_{\rm d}} \Big\{\|\mathbf{u}_i^{x^\ell}(k)\|_{\Omega_{\rm d}}, \|\mathbf{v}_j^{x^\ell}(k)\|_{\Omega_{\rm d}}\Big\},$$
$$\|\mathbf{u}_i^{x^\ell}(k)\|_{\Omega_{\rm d}} = \Big[\mathbb{E}\big|\mathbf{u}_i^{x^\ell}(k)\big|^2\Big]^{\frac{1}{2}}, \quad \|\mathbf{v}_j^{x^\ell}(k)\|_{\Omega_{\rm d}} = \Big[\mathbb{E}\big|\mathbf{v}_j^{x^\ell}(k)\big|^2\Big]^{\frac{1}{2}},$$

$(i, j) \in (\mathbb{N}_1^m, \mathbb{N}_1^n)$, and

$$\|\psi\|_{\Omega_{\rm d}} = \max_{1\le i\le m, 1\le j\le n, x^\ell\in\Omega_{\rm d}} \Big\{\Big[\mathbb{E}|\psi_i^{x^\ell} - \tilde{\psi}_i^{x^\ell}|^2\Big]^{\frac{1}{2}}, \Big[\mathbb{E}|\varphi_j^{x^\ell} - \tilde{\varphi}_j^{x^\ell}|^2\Big]^{\frac{1}{2}}\Big\}.$$

Here, κ is called the convergent rate of Eqs. (2.4)-(2.5).

Theorem 4.1. *Eqs. (2.4)-(2.5) is globally mean-square λ-exponentially convergent if (H_3)-(H_4) hold.*

Proof. By Eqs. (2.7)-(2.8), it achieves

$$|\mathbf{u}_i^{x^\ell}(k)| \le \prod_{s=0}^{k-1}\lambda_{\alpha_i}^{x^\ell}(s)|\psi_i^{x^\ell} - \tilde{\psi}_i^{x^\ell}| + \sum_{q=0}^{k-1}\prod_{s=q+1}^{k-1}\lambda_{\alpha_i}^{x^\ell}(s)$$
$$\times\sum_{l=0}^{\nu_q}|\theta_{\alpha_i}^{x^\ell}(l, q)|\Big[\Big|F_i(u_i^{x^\ell}(q-l)) - F_i(\tilde{u}_i^{x^\ell}(q-l))\Big|$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}(\bar{a}_{ij}+\bar{\gamma}_{ij}+\bar{\eta}_{ij})\left|f_j(v_j^{x^\ell}(q-l))-f_j(\tilde{v}_j^{x^\ell}(q-l))\right| \\
&+\sum_{j=1}^{n}h^{-1}\bar{\mu}_{ij}\left|H_{ij}(v_j^{x^\ell}(q-l))-H_{ij}(\tilde{v}_j^{x^\ell}(q-l))\right||\Delta_h\mathbb{W}_{1j}(q-l)|\Bigg] \\
\le{}& \bar{\lambda}_{\alpha_i}^{k}|\psi_i^{x^\ell}-\tilde{\psi}_i^{x^\ell}|+\sum_{q=0}^{k-1}\bar{\lambda}_{\alpha_i}^{k-q-1}\sum_{l=0}^{\nu_q}|\theta_{\alpha_i}^{x^\ell}(l,q)| \\
&\times\Bigg[D_i^*|\mathbf{u}_i^{x^\ell}(q-l)|+\sum_{j=1}^{n}(\bar{a}_{ij}+\bar{\gamma}_{ij}+\bar{\eta}_{ij})\mathcal{L}_j^f|\mathbf{v}_j^{x^\ell}(q-l)| \\
&+\sum_{j=1}^{m}\bar{\mu}_{ij}\mathcal{L}_{ij}^{H}h^{-1}|\mathbf{v}_j^{x^\ell}(q-l)||\Delta_h\mathbb{W}_{1j}(q-l)|\Bigg]
\end{aligned}
$$

for all $(x^\ell,k,i,j)\in(\Omega_{\mathrm{d}},\mathbb{N}_0,\mathbb{N}_1^m,\mathbb{N}_1^n)$. Thus, it derives

$$
\begin{aligned}
\|\mathbf{u}_i^{x^\ell}(k)\|_{\Omega_{\mathrm{d}}}\le{}&\bar{\lambda}^{k}\|\psi\|_{\Omega_{\mathrm{d}}}+\sum_{q=0}^{k-1}\bar{\lambda}_{\alpha_i}^{k-q-1}\sum_{l=0}^{\nu_q}|\theta_{\alpha_i}^{x^\ell}(l,q)| \\
&\times\left[D_i^*+\sum_{j=1}^{n}(\bar{a}_{ij}+\bar{\gamma}_{ij}+\bar{\eta}_{ij})\mathcal{L}_j^f+2h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\mu}_{ij}\mathcal{L}_{ij}^{H}\right]\|\mathbf{Y}(q-l)\|_{\Omega_{\mathrm{d}}}, \quad (4.1)
\end{aligned}
$$

where $(x^\ell,k,i,j)\in(\Omega_{\mathrm{d}},\mathbb{N}_0,\mathbb{N}_1^m,\mathbb{N}_1^n)$.

Similarly, it gets

$$
\begin{aligned}
\|\mathbf{v}_j^{x^\ell}(k)\|_{\Omega_{\mathrm{d}}}\le{}&\bar{\lambda}^{k}\|\psi\|_{\Omega_{\mathrm{d}}}+\sum_{q=0}^{k-1}\bar{\lambda}_{\beta_j}^{k-q-1}\sum_{l=0}^{\nu_q}|\theta_{\beta_j}^{x^\ell}(l,q)| \\
&\times\left[K_j^*+\sum_{i=1}^{m}(\bar{e}_{ji}+\bar{\varepsilon}_{ji}+\bar{\vartheta}_{ji})\mathcal{L}_i^g+2h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\varsigma}_{ji}\mathcal{L}_{ji}^{L}\right]\|\mathbf{Y}(q-l)\|_{\Omega_{\mathrm{d}}}, \quad (4.2)
\end{aligned}
$$

where $(x^\ell,k,i,j)\in(\Omega_{\mathrm{d}},\mathbb{N}_0,\mathbb{N}_1^m,\mathbb{N}_1^n)$.

Owing to (H_4), it has $M>1$ and $0<\kappa<1$ ensuring

$$
\begin{cases}
\dfrac{1}{M}+\max\limits_{1\le i\le m}\dfrac{2\bar{\lambda}_{\alpha_i}^{-\kappa(Q^*+1)}}{\underline{c}_i^*(1-\bar{\lambda}_{\alpha_i}^{1-\kappa})}\left[D_i^*+\sum\limits_{j=1}^{n}(\bar{a}_{ij}+\bar{\gamma}_{ij}+\bar{\eta}_{ij})\mathcal{L}_j^f+2h^{-\frac{1}{2}}\sum\limits_{j=1}^{m}\bar{\mu}_{ij}\mathcal{L}_{ij}^{H}\right]<1, \\
\dfrac{1}{M}+\max\limits_{1\le j\le n}\dfrac{2\bar{\lambda}_{\beta_j}^{-\kappa(Q^*+1)}}{\underline{d}_j^*(1-\bar{\lambda}_{\beta_j}^{1-\kappa})}\left[K_j^*+\sum\limits_{i=1}^{m}(\bar{e}_{ji}+\bar{\varepsilon}_{ji}+\bar{\vartheta}_{ji})\mathcal{L}_i^g+2h^{-\frac{1}{2}}\sum\limits_{j=1}^{m}\bar{\varsigma}_{ji}\mathcal{L}_{ji}^{L}\right]<1.
\end{cases}
$$

Supposing that $\|\mathbf{Y}(k)\|_{\Omega_{\mathrm{d}}}\le M\|\psi\|_{\Omega_{\mathrm{d}}}\bar{\lambda}^{\kappa k}$ for all $k\in\mathbb{N}_0$. If not, there must be one of the following cases holds.

(1) It exists $i_0\in\mathbb{N}_1^m$ and $k_0\in\mathbb{N}$ causing

$$
\|\mathbf{u}_{i_0}^{x^\ell}(k_0)\|_{\Omega_{\mathrm{d}}}>M\|\psi\|_{\Omega_{\mathrm{d}}}\bar{\lambda}^{\kappa k_0};\quad \|\mathbf{Y}(k)\|_{\Omega_{\mathrm{d}}}\le M\|\psi\|_{\Omega_{\mathrm{d}}}\bar{\lambda}^{\kappa k},\quad \forall k\in[0,k_0)_{\mathbb{Z}}.
$$

(2) It exists $j_0\in\mathbb{N}_1^n$ and $k_1\in\mathbb{N}$ causing

$$
\|\mathbf{v}_{j_0}^{x^\ell}(k_1)\|_{\Omega_{\mathrm{d}}}>M\|\psi\|_{\Omega_{\mathrm{d}}}\bar{\lambda}^{\kappa k_1};\quad \|\mathbf{Y}(k)\|_{\Omega_{\mathrm{d}}}\le M\|\psi\|_{\Omega_{\mathrm{d}}}\bar{\lambda}^{\kappa k},\quad \forall k\in[0,k_1)_{\mathbb{Z}}.
$$

In the light of (4.1), it results in

$$
\begin{aligned}
&\|\mathbf{u}_{i_0}^{x^\ell}(k_0)\|_{\Omega_{\rm d}} \\
&\le \bar{\lambda}^{k_0}\|\psi\|_{\Omega_{\rm d}} + \sum_{q=0}^{k_0-1}\bar{\lambda}_{\alpha_{i_0}}^{k_0-q-1}\sum_{l=0}^{\nu_q}|\theta_{\alpha_{i_0}}^{x^\ell}(l,q)| \\
&\quad\times\Bigg[D_{i_0}^* + \sum_{j=1}^{n}(\bar{a}_{i_0j}+\bar{\gamma}_{i_0j}+\bar{\eta}_{i_0j})\mathcal{L}_j^f + 2h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\mu}_{i_0j}\mathcal{L}_{i_0j}^H\Bigg]\|\mathbf{Y}(q-l)\|_{\Omega_{\rm d}} \\
&\le \bar{\lambda}^{k_0}\|\psi\|_{\Omega_{\rm d}} + \sum_{q=0}^{k_0-1}\bar{\lambda}_{\alpha_{i_0}}^{(1-\kappa)(k_0-q-1)}\sum_{l=0}^{\nu_q}|\theta_{\alpha_{i_0}}^{x^\ell}(l,q)| \\
&\quad\times\Bigg[D_{i_0}^* + \sum_{j=1}^{n}(\bar{a}_{i_0j}+\bar{\gamma}_{i_0j}+\bar{\eta}_{i_0j})\mathcal{L}_j^f + 2h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\mu}_{i_0j}\mathcal{L}_{i_0j}^H\Bigg]M\|\psi\|_{\Omega_{\rm d}}\bar{\lambda}^{\kappa(q-l)} \\
&\le \bar{\lambda}^{k_0}\|\psi\|_{\Omega_{\rm d}} + \frac{2\bar{\lambda}_{\alpha_{i_0}}^{-\kappa(Q^*+1)}}{\underline{c}_{i_0}^*(1-\bar{\lambda}_{\alpha_{i_0}}^{1-\kappa})} \\
&\quad\times\Bigg[D_{i_0}^* + \sum_{j=1}^{n}(\bar{a}_{i_0j}+\bar{\gamma}_{i_0j}+\bar{\eta}_{i_0j})\mathcal{L}_j^f + 2h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\mu}_{i_0j}\mathcal{L}_{i_0j}^H\Bigg]M\|\psi\|_{\Omega_{\rm d}}\bar{\lambda}^{\kappa k_0} \\
&\le \Bigg\{\frac{1}{M}\bar{\lambda}^{(1-\kappa)k_0} + \frac{2\bar{\lambda}_{\alpha_{i_0}}^{-\kappa(Q^*+1)}}{\underline{c}_{i_0}^*(1-\bar{\lambda}_{\alpha_{i_0}}^{1-\kappa})} \\
&\quad\times\Bigg[D_{i_0}^* + \sum_{j=1}^{n}(\bar{a}_{i_0j}+\bar{\gamma}_{i_0j}+\bar{\eta}_{i_0j})\mathcal{L}_j^f + 2h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\mu}_{i_0j}\mathcal{L}_{i_0j}^H\Bigg]\Bigg\}M\|\psi\|_{\Omega_{\rm d}}\bar{\lambda}^{\kappa k_0} \\
&\le M\|\psi\|_{\Omega_{\rm d}}\bar{\lambda}^{\kappa k_0}.
\end{aligned}
$$

This induces a confliction with (1).

Similarly, by (4.2), it acquires

$$
\begin{aligned}
\|\mathbf{v}_{j_0}^{x^\ell}(k_1)\|_{\Omega_{\rm d}} &\le \Bigg\{\frac{1}{M}\bar{\lambda}^{(1-\kappa)k_1} + \frac{2\bar{\lambda}_{\beta_{j_0}}^{-\kappa(Q^*+1)}}{\underline{d}_{j_0}^*(1-\bar{\lambda}_{\beta_{j_0}}^{1-\kappa})} \\
&\quad\times\Bigg[K_j^* + \sum_{i=1}^{m}(\bar{e}_{ji}+\bar{\varepsilon}_{ji}+\bar{\vartheta}_{ji})\mathcal{L}_i^g + 2h^{-\frac{1}{2}}\sum_{j=1}^{m}\bar{\varsigma}_{ji}\mathcal{L}_{ji}^L\Bigg]\Bigg\}M\|\psi\|_{\Omega_{\rm d}}\bar{\lambda}^{\kappa k_1} \\
&\le M\|\psi\|_{\Omega_{\rm d}}\bar{\lambda}^{\kappa k_1}.
\end{aligned}
$$

It is a confliction with (2) and $\|\mathbf{Y}(k)\|_{\Omega_{\rm d}} \le M\|\psi\|_{\Omega_{\rm d}}\bar{\lambda}^{\kappa k}$ for all $k \in \mathbb{N}_0$. That is, Eqs. (2.4)-(2.5) is globally mean-square λ-exponentially convergent. The proof is finished. □

Remark 4.1. By employing time (exponential) Euler differences, literatures [10, 11, 14, 30] discussed exponential convergence of discrete-time (stochastic) BAMNNs without diffusions. Apparently, the achieved difference models in literatures [10, 11, 14, 30] can be viewed as a special case of model (2.4)-(2.5) to some extent. Thus, the work of this article supplements and extends the corresponding results in [10, 11, 14, 30].

5. Illustrative example

Considering the following nonlocal stochastic BAMNNs with reaction diffusions in the form of

$$ {}^{c}D_{5p}^{0.6}u_1(t,x) = 0.12\frac{\partial^2 u_1(t,x)}{\partial x^2} - 16u_1(t,x) + 0.3\sin(\iota t+\sqrt{7}x)f_1(v(t,x)) + 0.2\cos(\iota t + x) + 5H_{11}(v(t,x))\frac{\mathrm{d}\mathbb{W}_{11}(t)}{\mathrm{d}t}, \tag{5.1} $$

$$ {}^{c}D_{5p}^{0.4}u_2(t,x) = 0.08\frac{\partial^2 u_2(t,x)}{\partial x^2} - 21u_2(t,x) + 0.5\cos(\iota t+\sqrt{5}x)f_1(v(t,x)) + 0.1\sin(\iota t + x) + 2H_{21}(v(t,x))\frac{\mathrm{d}\mathbb{W}_{12}(t)}{\mathrm{d}t}, \tag{5.2} $$

$$ \begin{aligned} {}^{c}D_{5p}^{0.7}v(t,x) = {} & 0.15\frac{\partial^2 v(t,x)}{\partial x^2} - 19v(t,x) \\ & +0.2\sin(\iota t+\sqrt{7}x)\bigwedge\Big\{g_1(u_1(t,x)), g_2(u_2(t,x))\Big\} \\ & +0.3\cos(\iota t+\sqrt{3}x)\bigvee\Big\{g_1(u_1(t,x)), g_2(u_2(t,x))\Big\} \\ & +0.2\sin(\iota t+\sqrt{11}x) + 2L_{11}(u_1(t,x))\frac{\mathrm{d}\mathbb{W}_{21}(t)}{\mathrm{d}t}, \end{aligned} \tag{5.3} $$

where $(t,x)\in((5p,5p+5],(0,5))$, $p\in\mathbb{Z}_0$, $\iota=\frac{2}{5}\pi$, $f_1(x)=H_{11}(x)=H_{21}(x)=0.01|\sin x|$, $g_1(x)=g_2(x)=L_{11}(x)=\frac{0.01|x|}{1+|x|}$ for $x\in\mathbb{R}$. The corresponding initial boundary conditions are depicted as

$$ \begin{cases} u_1(0,x)=u_2(0,x)=0.5\sin(x(x-5)),\ v(0,x)=0.1\sin(x(x-5)),\ \forall x\in(0,5), \\ u_1(t,0)=u_1(t,5)=u_2(t,0)=u_2(t,5)=v(t,0)=v(t,5)=0, \quad t\in\mathbb{R}_0. \end{cases} $$

Taking $h=1$ and $h_1=0.5$. It obtains the lattice equations corresponding to Eqs. (5.1)-(5.3) in the shape of

$$ u_1^{\zeta}(k+1) = \lambda_{0.6}^{\zeta}(k)u_1^{\zeta}(l) + \sum_{l=\mu_k}^{k}\theta_{0.6}^{\zeta}(k-l,k)\Big[F_1(u_1^{\zeta}(l)) + 0.3\sin(\iota l+\sqrt{7}\zeta) \times f_1(v(l,\zeta)) + 0.2\cos(\iota l+\zeta) + 5H_{11}(v(l,\zeta))h^{-1}\Delta_h\mathbb{W}_{11}(\zeta), \tag{5.4} $$

$$ u_2^{\zeta}(k+1) = \lambda_{0.4}^{\zeta}(k)u_2^{\zeta}(l) + \sum_{l=\mu_k}^{k}\theta_{0.4}^{\zeta}(k-l,k)\Big[F_2(u_2^{\zeta}(l)) + 0.5\cos(\iota l+\sqrt{5}\zeta) \times f_1(v(\iota,\zeta)) + 0.1\sin(\iota l+\zeta) + 2H_{21}(v(l,\zeta))h^{-1}\Delta_h\mathbb{W}_{12}(\zeta), \tag{5.5} $$

$$ \begin{aligned} v^{x^{\ell}}(k+1) = {} & \lambda_{0.7}^{\zeta}(k)v^{\zeta}(l) + \sum_{l=\mu_k}^{k}\theta_{0.7}^{\zeta}(k-l,k)\Big[G_1(v^{\zeta}(l)) \\ & + 0.2\sin(\iota l+\sqrt{7}\zeta)\bigwedge\Big\{g_1(u_1(l,\zeta)), g_2(u_2(l,\zeta))\Big\} \\ & + 0.3\cos(\iota l+\sqrt{3}\zeta)\bigvee\Big\{g_1(u_1(l,\zeta)), g_2(u_2(l,\zeta))\Big\} \end{aligned} $$

$$+ 0.2\sin(\iota l + \sqrt{11}\zeta) + 2L_{11}(u_1(l,\zeta))h^{-1}\Delta_h \mathbb{W}_{21}(\zeta)\Big] \tag{5.6}$$

with initial condition

$$u_1^\zeta(0) = u_2^\zeta(0) = 0.5\sin(\zeta(\zeta-5)), \quad v^\zeta(0) = 0.1\sin(\zeta(\zeta-5))$$

for $\zeta = 0.5\ell$, $\ell = 0, 1, \ldots, 10$, and boundary condition given by

$$u_1^\zeta(k)\Big|_{\zeta\in\{0,5\}} = u_2^\zeta(k)\Big|_{\zeta\in\{0,5\}} = v^\zeta(k)\Big|_{\zeta\in\{0,5\}} = 0,$$

$\nu_k = k - \mu_k$, $\mu_k = \max_{p\in\mathbb{Z}_0}\{5p : 5p \le k\}$, $\lambda_{0.6}^\zeta$, $\theta_{0.6}^\zeta$, $\lambda_{0.4}^\zeta$, $\theta_{0.4}^\zeta$, $\lambda_{0.7}^\zeta$ and $\theta_{0.7}^\zeta$ are defined as those in (2.4) with $c_1^* = 16.96$, $c_2^* = 21.64$ and $d_1^* = 20.2$, respectively;

$$\begin{aligned}
F_1(u_1^\zeta(l)) &= \frac{0.12}{0.25}\left[u_1^{\zeta+0.5}(l) + u_1^{\zeta-0.5}(l)\right],\\
F_2(u_2^\zeta(l)) &= \frac{0.08}{0.25}\left[u_2^{\zeta+0.5}(l) + u_2^{\zeta-0.5}(l)\right],\\
G_1(v^\zeta(l)) &= \frac{0.15}{0.25}\left[v^{\zeta+0.5}(l) + v^{\zeta-0.5}(l)\right], \quad \forall l,k\in\mathbb{Z}_0, \zeta = 0.5\ell, \ell = 1,\ldots,9.
\end{aligned}$$

By a calculation, it gets

$$\sigma_1 = 0.9406 < 1, \quad \sigma_2 = 0.8402, \quad \mathscr{Z}_0 = 14.1448.$$

Therefore, (H_1)-(H_4) hold and Eqs. (5.4)-(5.6) admits a unique 5-periodic sequence solution, which is globally mean-square λ-exponentially convergent, see Figures 1-9.

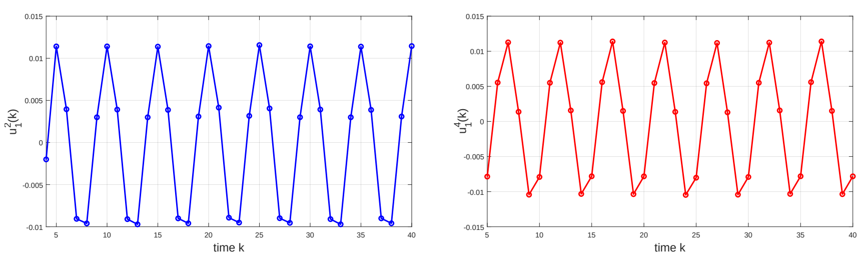

Figure 1. Periodic motion of $u_1^\zeta(\cdot)\,(\zeta = 2, 4)$ to Eqs. (5.4)-(5.6)

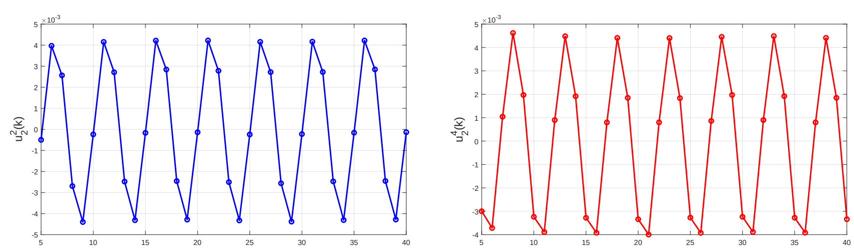

Figure 2. Periodic motion of $u_2^\zeta(\cdot)\,(\zeta = 2, 4)$ to Eqs. (5.4)-(5.6)

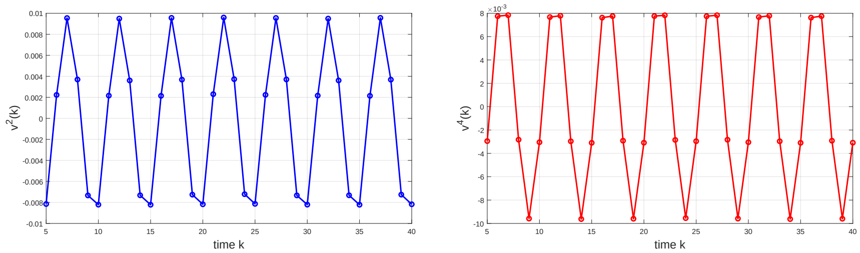

Figure 3. Periodic motion of $v^{\zeta}(\cdot)\,(\zeta = 2, 4)$ to Eqs. (5.4)-(5.6)

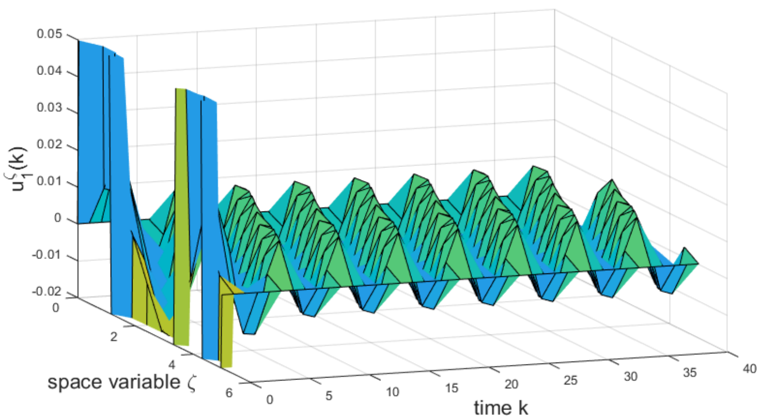

Figure 4. Periodic motion of $u_1^{\zeta}(\cdot)\,(\zeta = 0, 0.5, \ldots, 5)$ to Eqs. (5.4)-(5.6)

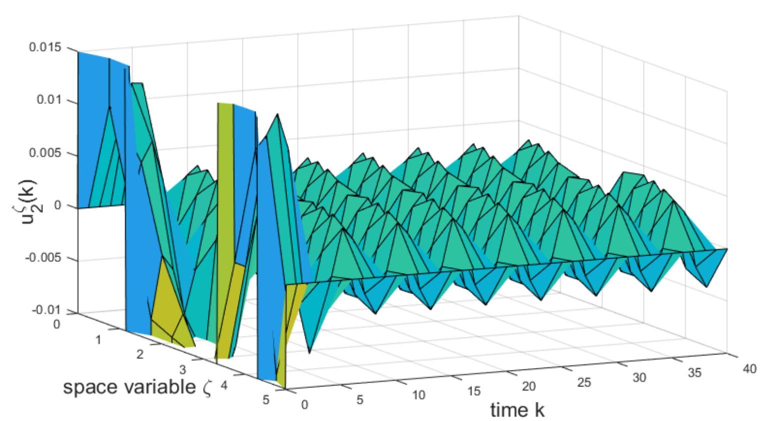

Figure 5. Periodic motion of $u_2^{\zeta}(\cdot)\,(\zeta = 0, 0.5, \ldots, 5)$ to Eqs. (5.4)-(5.6)

6. Conclusions and perspectives

By employing the finite difference and Mittag-Leffler time Euler difference techniques, a novel lattice model for nonlocal fuzzy stochastic BAMNNs with reaction

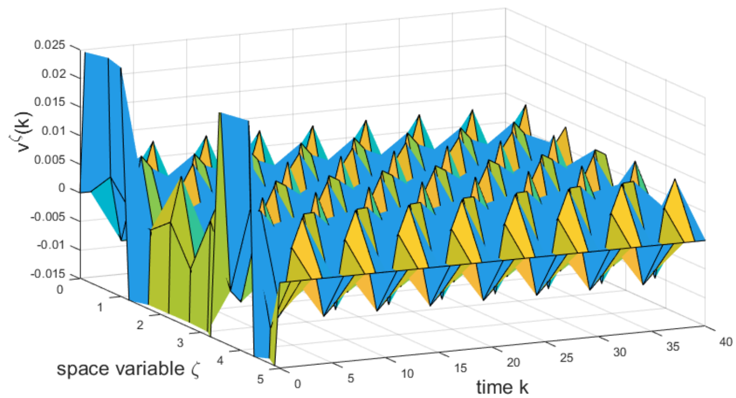

Figure 6. Periodic motion of $v^{\zeta}(\cdot)\,(\zeta = 0, 0.5, \dots, 5)$ to Eqs. (5.4)-(5.6)

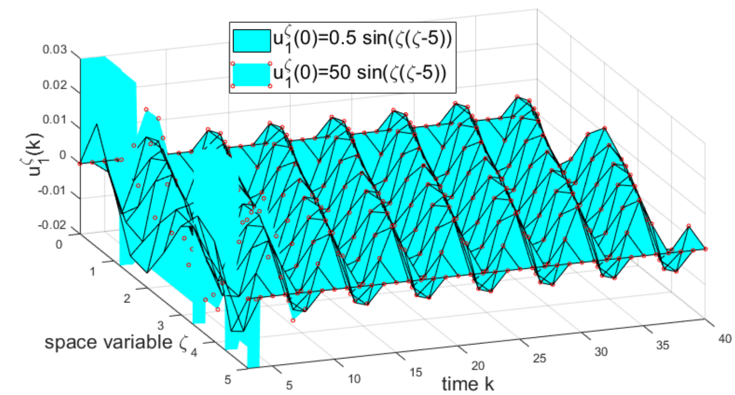

Figure 7. λ-exponential convergence of $u_1^{\zeta}(\cdot)\,(\zeta = 0, 0.5, \dots, 5)$ to Eqs. (5.4)-(5.6)

diffusions are set up. Furthermore, the existence of a unique global bounded periodic sequence solution in distribution and global exponential convergence in the mean-square sense have been investigated for the achieved stochastic lattice model. Besides, in the discussions of this article, several important inequalities, such as Minkowski inequality and Hölder inequality, are indispensable. Remarkably, the work in this literature will open up the researches of periodic dynamics and global mean-square exponential convergence of nonlocal stochastic fuzzy bidirectional associative memory lattice neural networks and it will lay both theoretical and practical foundations for the future work in this field.

According to the works in this literature, it will be many problems worthy of further discussion.

1. This paper only discusses $\alpha_i, \beta_j \in (0, 1]\,(i, j) \in (\mathbb{N}_1^m, \mathbb{N}_1^n)$ and other cases are supposed to be studied.
2. The models with time delays should be considered in the future, see refs. [21, 22].

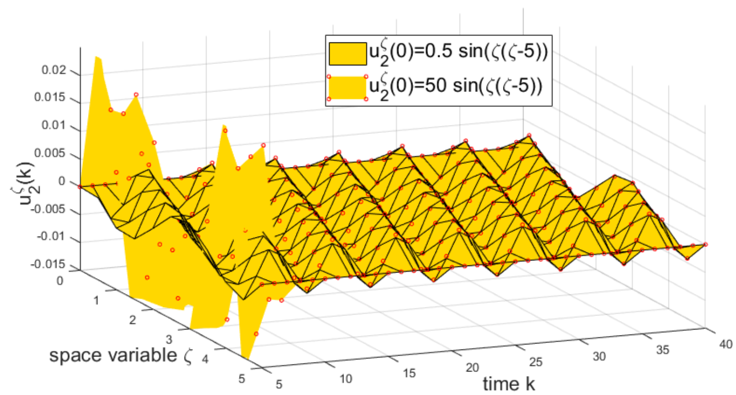

Figure 8. λ-exponential convergence of $u_2^{\zeta}(\cdot)\,(\zeta = 0, 0.5, \ldots, 5)$ to Eqs. (5.4)-(5.6)

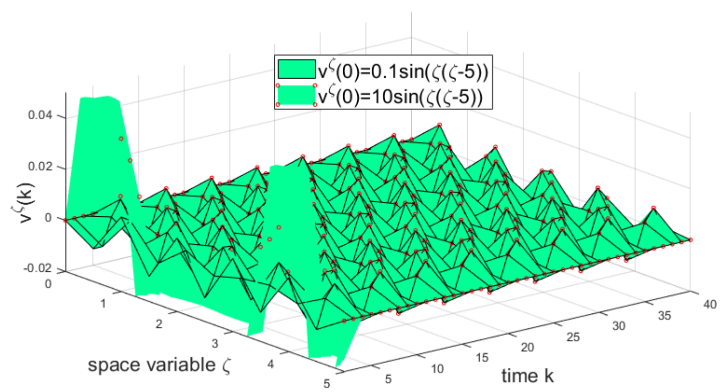

Figure 9. λ-exponential convergence of $v^{\zeta}(\cdot)\,(\zeta = 0, 0.5, \ldots, 5)$ to Eqs. (5.4)-(5.6)

3. Reimann-Liouville derivatives should be studied in the further.
4. Other dynamics ought to be discussed, e.g., bifurcation, chaos and control, etc.

A. Appendix: Derivation of Eqs. (2.4)-(2.5)

By employing the finite difference methods, it gets

$$\begin{cases} \left.\dfrac{\partial^2 u_i(t,x)}{\partial x_q^2}\right|_{x=x^{\ell}} \approx \dfrac{u_i(t,x^{\ell}+h_q\mathbf{e}_q) - 2u_i(t,x^{\ell}) + u_i(t,x^{\ell}-h_q\mathbf{e}_q)}{h_q^2}, \\ \left.\dfrac{\partial^2 v_j(t,x)}{\partial x_q^2}\right|_{x=x^{\ell}} \approx \dfrac{v_j(t,x_q^{\ell}+h_q\mathbf{e}_q) - 2v_j(t,x^{\ell}) + v_j(t,x_q^{\ell}-h_q\mathbf{e}_q)}{h_q^2}, \end{cases} \tag{A.1}$$

where $(t, x^\ell, i, j) \in (\mathbb{R}, \Omega_\mathrm{d}, \mathbb{N}_1^m, \mathbb{N}_1^n)$ and $q \in \mathbb{N}_1^{\mathcal{N}}$, $\{\mathbf{e}_q : q = 1, 2, \ldots, \mathcal{N}\}$ is an orthonormal basis of $\mathbb{R}^{\mathcal{N}}$ denoted by

$$\mathbf{e}_1 = (1, 0, \ldots, 0)^T, \quad \mathbf{e}_2 = (0, 1, \ldots, 0)^T, \quad \mathbf{e}_{\mathcal{N}} = (0, 0, \ldots, 1)^T.$$

Making use of $(A.1)$, Eqs. (2.1)-(2.2) is calculated approximately by

$$\begin{aligned} {}^cD_{t_p}^{\alpha_i} u_i(t, x^\ell) \approx\ & -c_i^*(x^\ell) u_i(t, x^\ell) + F_i(t, u_i) + \sum_{j=1}^{n} a_{ij}(t, x^\ell) f_j(v_j(t, x^\ell)) \\ & + \bigwedge_{j=1}^{n} \gamma_{ij}(t, x^\ell) f_j(v_j(t, x^\ell)) + \bigvee_{j=1}^{n} \eta_{ij}(t, x^\ell) f_j(v_j(t, x^\ell)) \\ & + I_i(t, x^\ell) + \sum_{j=1}^{n} \mu_{ij}(t, x^\ell) H_{ij}(v_j(t, x^\ell)) \frac{\mathrm{d}\mathbb{W}_{1j}(t)}{\mathrm{d}t}, \end{aligned} \tag{A.2}$$

$$\begin{aligned} {}^cD_{t_p}^{\beta_j} v_j(t, x^\ell) \approx\ & -d_j^*(x^\ell) v_j(t, x^\ell) + G_j(t, v_j) + \sum_{i=1}^{m} e_{ji}(t, x^\ell) g_i(u_i(t, x^\ell)) \\ & + \bigwedge_{i=1}^{m} \varepsilon_{ji}(t, x^\ell) g_i(u_i(t, x^\ell)) + \bigvee_{i=1}^{m} \vartheta_{ji}(t, x^\ell) g_i(u_i(t, x^\ell)) \\ & + J_j(t, x^\ell) + \sum_{i=1}^{m} \varsigma_{ji}(t, x^\ell) L_{ji}(u_i(t, x^\ell)) \frac{\mathrm{d}\mathbb{W}_{2i}(t)}{\mathrm{d}t}, \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} & c_i^*(x^\ell) = c_i(x^\ell) + \sum_{q=1}^{\mathcal{N}} \frac{2D_{iq}}{h_q^2}, \quad d_j^*(x^\ell) = d_j(x^\ell) + \sum_{q=1}^{\mathcal{N}} \frac{2K_{jq}}{h_q^2}, \\ & F_i(t, u_i) = \sum_{q=1}^{\mathcal{N}} \frac{D_{iq}}{h_q^2} \big[u_i(t, x^\ell + h_q \mathbf{e}_q) + u_i(t, x^\ell - h_q \mathbf{e}_q)\big], \\ & G_j(t, v_j) = \sum_{q=1}^{\mathcal{N}} \frac{K_{jq}}{h_q^2} \big[v_j(t, x^\ell + h_q \mathbf{e}_q) + v_j(t, x^\ell - h_q \mathbf{e}_q)\big] \end{aligned}$$

for all $(t, x^\ell, p, i, j) \in (\mathcal{I}_p, \Omega_\mathrm{d}, \mathbb{Z}, \mathbb{N}_1^m, \mathbb{N}_1^n)$. Eqs. (2.3) are approximately estimated by

$$\begin{cases} u_i(0, x^\ell) = \varphi_i(x^\ell), \quad \forall x^\ell \in \Omega_\mathrm{d}; \quad u_i(t, x^\ell)\Big|_{x^\ell \in \partial\Omega_\mathrm{d}} = \phi_i(t, x^\ell), \\ v_j(0, x^\ell) = \tilde{\varphi}_j(x^\ell), \quad \forall x^\ell \in \Omega_\mathrm{d}; \quad v_j(t, x^\ell)\Big|_{x^\ell \in \partial\Omega_\mathrm{d}} = \tilde{\phi}_j(t, x^\ell), \end{cases} \tag{A.4}$$

where $(t, i, j) \in (\mathcal{I}_p, \mathbb{N}_1^m, \mathbb{N}_1^n)$.

Assume that it exists a set $\mathcal{Q} = \{Q_p \in \mathbb{Z} : Q_0 = 0, Q_p < Q_{p+1}, p \in \mathbb{Z}\}$ ensuring $\frac{t_{p+1} - t_p}{Q_{p+1} - Q_p} \equiv h$ for all $p \in \mathbb{Z}$. Based on $(A.2)$-$(A.4)$ and by using the exponential time difference techniques in Ref. [7], it obtains the lattice equations (2.4)-(2.5) of Eqs. (2.1)-(2.2).

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