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λ-FIXED POINT THEOREM WITH KINDS OF FUNCTIONS OF MIXED MONOTONE OPERATOR

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Abstract Our work is related to the existence and uniqueness of positive solution to the fractional boundary value problem(BVP) with Riemann-Liouville fractional derivative. We employ the fixed point theorem of mixed monotone operator and the attributes of the Green function to consider the following:

$$
\begin{aligned}
&-D^{\nu}_{0+}\mathfrak{u}(t)=\lambda^{-1}(\mathfrak{f}(t,\mathfrak{u}(t),\mathbf{v}(t))+\mathfrak{g}(t,\mathfrak{u}(t))+\mathbf{k}(t,\mathbf{v}(t))),\ 0<t<1,\ 3\leq\nu\leq 4,\\
&\mathfrak{u}(0)=\mathfrak{u}'(0)=\mathfrak{u}''(0)=0,\\
&[D^{\rho}_{0+}\mathfrak{u}(t)]_{t=1}=0,\quad 1\leq\rho\leq 2.
\end{aligned}
$$

λ is a positive number. D^{ν}_{0+} and D^{ρ}_{0+} are the standard Riemann-Liouville fractional derivatives of degree ν and ρ, respectively. In the end, we provide an exemplar to illustrate the outcome. It should also be noted that in this paper we have assumed the variable $\mathbf{v}$ as follows:

$$
\mathbf{v}(t)=1-\frac{\Gamma(2-\rho)}{t^{1-\rho}}D^{\rho}_{0+}\mathfrak{u}(t).
$$

Keywords BVP, positive solution, mixed monotone operator, Green function, fixed point theorem, fractional derivative.

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1. Introduction

Years ago, fractional differential equations was located a lot of writers and is still ongoing. Here are some of the works done by these writers. We want to examine the existence and uniqueness of positive solutions to the fractional BVP, which is a sum of three continuous functions $\mathfrak{f},\mathfrak{g}$ and $\mathbf{k}$. The $\mathfrak{f}$-function is a three-variable function in which the variable $\mathbf{v}$ is a special case of the variable $\mathfrak{u}$. Our problem in as follows:

$$
\begin{aligned}
&-D^{\nu}_{0+}\mathfrak{u}(t)=\lambda^{-1}(\mathfrak{f}(t,\mathfrak{u}(t),\mathbf{v}(t))+\mathfrak{g}(t,\mathfrak{u}(t))+\mathbf{k}(t,\mathbf{v}(t))),\ 0<t<1,\ 3\leq\nu\leq 4,\\
&\mathfrak{u}(0)=\mathfrak{u}'(0)=\mathfrak{u}''(0)=0,\\
&[D^{\rho}_{0+}\mathfrak{u}(t)]_{t=1}=0,\quad 1\leq\rho\leq 2.
\end{aligned}
\tag{1.1}
$$

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λ is a positive number. ν and ρ are real numbers, the functions $\mathfrak{f} : [0,1] \times \Re^+ \times \Re^+ \longrightarrow \Re^+$, $\mathfrak{g} : [0,1] \times \Re^+ \longrightarrow \Re^+$ and $\mathbf{k} : [0,1] \times \Re^+ \longrightarrow \Re^+$ are continuous. D^{ν}_{0+} and D^{ρ}_{0+} are the standard Riemann-Liouville fractional derivatives of degree ν and ρ, respectively. A function $\mathfrak{u} \in \mathbf{C}[0,1]$ is called a positive solution of problem (1.1) if $\mathfrak{u} > 0$ on $(0,1)$, $D^{\rho}\mathfrak{u} \in \mathbf{C}[0,1]$, $D^{\nu}\mathfrak{u} \in \mathbf{L}^1[0,1]$, $\mathfrak{u}$ meets the boundary conditions, and parity (1.1) holds a.e. on $[0,1]$.

Remark 1.1. In this paper, we express the variable $\mathbf{v}$ in the function $\mathfrak{f}$ as follows:

$$\mathbf{v}(t) = 1 - \frac{\Gamma(2-\rho)}{t^{1-\rho}} D^{\rho}_{0+}\mathfrak{u}(t).$$

In [3], D. Min, L. Liu, and Y. Wu shown unique positive solutions to the following fractional differential equation with integral boundary conditions:

$$\begin{aligned}
&D^{\alpha}_{0+}x(t) + \mathfrak{f}(t, x(t), D^{\alpha_1}_{0+}x(t), ..., D^{\alpha_{n-2}}_{0+}x(t)) = 0,\ t \in (0,1),\\
&x(0) = D^{\gamma_1}_{0+}x(0) = ... = D^{\gamma_{n-2}}_{0+}x(0) = 0,\\
&D^{\beta_1}_{0+}x(1) = \int_0^{\eta} \overline{h}(s) D^{\beta_2}_{0+}x(s) d\overline{A}(s) + \int_0^1 \overline{a}(s) D^{\beta_2}_{0+}x(s) d\overline{A}(s),
\end{aligned}$$

where $\overline{a}, \overline{h}$ are continuous functions in the (0,1). The above integrals in the second boundary condition are called Riemann-Stieljes integrals with respect to $\overline{A}$.

In [5], S. Song, and Y. Cui concerned the existence of solutions of the following nonlinear mixed fractional differential equation with the integral BVP:

$$\begin{aligned}
&{}^{C}D^{\alpha}_{1-} D^{\beta}_{0+}u(t) = f(t, u(t), D^{\beta+1}_{0+}u(t), D^{\beta}_{0+}u(t)),\ 0 < t < 1, \qquad (1.2)\\
&u(0) = u'(0) = 0, \quad u(1) = \int_0^1 u(t) dA(t),
\end{aligned}$$

where ${}^{C}D^{\alpha}_{1-}$ is the Capato fractional derivative of degree $\alpha \in (1,2]$, and D^{β}_{0+} is the right Riemann-Liouville fractional derivative of degree $\beta \in (0,1]$. Authors concerned the nonlinear mixed fractional differential equation with a combination of the Capato derivative and the fractional Riemann-Liouville derivative with an integral boundary condition of the type of Riemann-Stieltjes integral. Using Mawhin's coincidence degree theory, they proved the existence of solution of such problems. The authors were able to check the existence of a solution to the boundary value problem (1.2) based on the existence of a solution to a type of operator equation with the Fredholm operator on a Banach space E.

In [6], Y. Sang, H. Luxuan, W.Yanling, R. Yaqi, and S. Na checked the existence and uniqueness of solutions of the following fractional order BVP:

$$\begin{aligned}
&D^{\alpha}_{0+}u(t) + f(t, u(t), D^{\beta}_{0+}u(t)) + g(t, u(t), (Hu)(t)) - b = 0,\ t \in (0,1),\\
&u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\ n-1 < \alpha < n,\ n > 3\ (n \in N), \qquad (1.3)\\
&[D^{\gamma}_{0+}u(t)]_{t=1} = K(u(1)),
\end{aligned}$$

where $b > 0$ is a constant. D^{α}_{0+} is the Riemann-Liouville fractional derivative of degree α. The authors proved the unique solution and construct the corresponding iterative sequence to approximate the unique solution of a class of boundary value problem with derivative term. Using the existence of the unique solution of the mixed monotone operator on the Banach space E with cone $P_{h,e}$ and obtained itrerative sequences, they investigated the existence and uniqueness of the positive solution of the problem (1.3).

In [10], Z. Z. Yue, and Y. Zou studied the uniqueness of solutions of Dirichlet BVP for fractional differential equation given by:

$$
\begin{aligned}
&D_{0+}^{\alpha}u(t)+f(t,u(t),u'(t))=0,\ t\in(0,1),\ 1<\alpha\leqslant 2,\\
&u(0)=u(1)=0,
\end{aligned}
$$

where $D_{0+}^{\alpha}u(t)$ is the standard Riemann-Liouville fractional derivative. Using the Banach contraction mapping principle and a weighted norm in the product space, they investigated the sufficient conditions for the uniqueness of the solutions. They introduced two Green's functions G and G_1,and by introducing the system of inequalities that have solutions, they were able to check the uniqueness of the problem. That is, they proved the existence of a positive solution by using the existence of a fixed point on T on the product Banach space.

In [7], H.Wang, and L.L. Zhang established the following non-singular BVP with a parameter:

$$
\begin{aligned}
&u^{(4)}(t)=\lambda f(t,u(t),(Hu)(t)),\ 0<t<1,\\
&u(0)=u'(0)=0,\\
&u'(1)=0,\ u''''(1)=\lambda g(u(1)),
\end{aligned}
$$

where λ is a positive parameter. H is a individual parameter. By using the fixed point theorems of mixed monotone operator and properties of cone, they checked the non-singular and singular case, respectively, and achievabled the sufficient conditions which guarantee the local existence and uniqueness of increasing positive solutions.

In [11], B. Zhou, L. L. Zhang, E. N. Addai, and N. Zhang studied the existence of multiple positive solutions for BVP of high-order Riemann-Liouville fractional differential equations involving the P-Laplacian operator given by:

$$
\begin{aligned}
&{}_0^{\Re}D_t^{\alpha}(\varphi_P({}_0^{\Re}D_t^{\alpha}u(t)))=f(t,u(t),{}_0^{\Re}D_t^{\alpha}u(t)),\ 0\leqslant t\leqslant 1,\\
&u^{(i)}(0)=0,\ [\varphi_P({}_0^{\Re}D_t^{\alpha}]^{(i)}(0)=0,\ i=0,1,2,\cdots,n-2,\\
&[{}_0^{\Re}D_t^{\beta}u(t)]_{t=1}=0,\ 0<\beta\leqslant\alpha-1,\\
&[{}_0^{\Re}D_t^{\beta}(\varphi_P({}_0^{\Re}D_t^{\alpha}u(t)))]_{t=1}=0,
\end{aligned}
$$

where $n-1<\alpha\leqslant n$, ${}_0^{\Re}D_t^{\alpha}$ is the standard Riemann-Liouville fractional derivative. φ_P is the P-Laplacian operator, $P>1$.

Motivated by above mentioned works, this is our goal to investigate the existence and uniqueness of the positive solution to BVP (1.1), which is different from previous work. In this problem, we consider a different condition for mixed monotone operator with a positive number λ. We check the existence and uniqueness of positive solution.

In second part, we express some Lemmas, Definitions and several attributes of the Green function that we use to prove the main results.

2. Preliminaries

2.1. Definition, Theorem and Lemma

In this section, we express several Definitions and Lemmas. Presuppose that $(\mathsf{E}, \|.\|)$ is a real Banach space. If $\mathcal{P}$ is a convex, closed, non-empty set, and it is valid in the following conditions:

(i) $\mathbf{x} \in \mathcal{P}, \quad \lambda > 0 \quad \Longrightarrow \lambda\mathbf{x} \in \mathcal{P};$

(ii) $\mathbf{x} \in \mathcal{P}, \quad -\mathbf{x} \in \mathcal{P} \quad \Longrightarrow \mathbf{x} = 0.$

then, $\mathcal{P}$ is called a cone. E is partly ordered by a cone $\mathcal{P} \subset \mathsf{E}$, that's mean $\mathbf{x} \leqslant \mathbf{y}$ if and only if $\mathbf{y} - \mathbf{x} \in \mathcal{P}$. If $\mathbf{x} \leqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ thus, we show $\mathbf{x} < \mathbf{y}$. If $\mathcal{P}^0 = \{\mathbf{x} \in \mathcal{P} | \mathbf{x}$ is an interior point of $\mathcal{P}\}$ is non-empty then, cone $\mathcal{P}$ is called solidity. Moreover, if a constant $N > 0$ exists so that for all $\mathbf{x}, \mathbf{y} \in \mathsf{E}$, $0 \leqslant \mathbf{x} \leqslant \mathbf{y}$ implies $\|\mathbf{x}\| \leqslant N\|\mathbf{y}\|$ then, $\mathcal{P}$ is called the normal cone; in this case N is called the normality constant of $\mathcal{P}$. An operator $\mathbf{A} : \mathsf{E} \longrightarrow \mathsf{E}$ is increasing if $\mathbf{x} \leqslant \mathbf{y}$ inferring $\mathbf{Ax} \leqslant \mathbf{Ay}$, and an operator $\mathbf{A} : \mathsf{E} \longrightarrow \mathsf{E}$ is deceasing if $\mathbf{x} \leqslant \mathbf{y}$ inferring $\mathbf{Ax} \geqslant \mathbf{Ay}$.

The following Definitions are derived from references [2] and [13].

Definition 2.1. If $\Omega = [0, b](0 < b < \infty)$ be a finite interval on the real axis $\Re$. Riemann-Liouville fractional integrals $I^{\rho}_{0+}\mathfrak{f}$ of degree $\rho > 0$ is defined by

$$I^{\rho}_{0+}\mathfrak{f}(\mathbf{x}) = \frac{1}{\Gamma(\rho)} \int_0^{\mathbf{x}} \frac{\mathfrak{f}(t)}{(\mathbf{x} - t)^{1-\rho}} dt \quad (\mathbf{x} > 0;\ \rho > 0).$$

Here $\Gamma(\rho)$ is the Gamma function.

Definition 2.2. If $\rho > 0$ be. Riemann-Liouville fractional derivative $D^{\rho}_{0+}\mathfrak{g}$ of degree $\rho > 0$ is defined as follows:

$$(D^{\rho}_{0+}\mathfrak{g})(\mathbf{x}) := (\frac{d}{d\mathbf{x}})^{\mathbf{n}} (I^{\mathbf{n}-\rho}_{0+}\mathfrak{g})(\mathbf{x}) = \frac{1}{\Gamma(\mathbf{n} - \rho)} (\frac{d}{d\mathbf{x}})^{\mathbf{n}} \int_0^{\mathbf{x}} \frac{\mathfrak{g}(t)}{(\mathbf{x} - t)^{\rho - \mathbf{n} + 1}} dt,$$
$$(\mathbf{n} = [\rho] + 1;\ \mathbf{x} > 0).$$

Where $[\rho]$ mean the integral part of ρ.

Definition 2.3. $\mathbf{A} : \mathcal{P} \longrightarrow \mathcal{P}$ is called to be a general α-concave operator if it satisfies:

for all $x \in \mathcal{P}$ and $t \in (0, 1)$ there is $0 < \alpha(t) < 1$ such that

$$\mathbf{A}(t\mathbf{x}) \geqslant t^{\alpha(t)}\mathbf{Ax}, \ \forall t \in (0, 1), \ \mathbf{x} \in \mathcal{P}, \tag{2.1}$$

Definition 2.4. An operator $\mathbf{A} : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}$ is called a mixed monotone operator if $\mathbf{A}(\mathfrak{u}, \mathbf{v})$ is increasing in $\mathfrak{u}$ and decreasing in $\mathbf{v}$, that's mean, $\mathfrak{u}_i, \mathbf{v}_i \in \mathcal{P}$ $(i = 1, 2)$, $\mathfrak{u}_1 \leq \mathfrak{u}_2$, $\mathbf{v}_1 \geq \mathbf{v}_2$ imply

$$\mathbf{A}(\mathfrak{u}_1, \mathbf{v}_1) \leq \mathbf{A}(\mathfrak{u}_2, \mathbf{v}_2),$$

the element $\mathbf{x} \in \mathcal{P}$ is called a fixed point of $\mathbf{A}$ if $\mathbf{A}(\mathbf{x}, \mathbf{x}) = \mathbf{x}$.

Theorem 2.1. *Let $\mathbf{y} \in \mathbf{C}[0, 1]$ be given. Thus, the uniquely solution to problem $-D^{\nu}_{0+}\mathfrak{u}(t) = \lambda^{-1}\mathbf{y}(t)$ along with the boundary conditions $\mathfrak{u}(0) = \mathfrak{u}'(0) = \mathfrak{u}''(0) = 0$ and $[D^{\rho}_{0+}\mathfrak{u}(t)]_{t=1} = 0$, where $1 \leqslant \rho \leqslant 2$ is*

$$\mathfrak{u}(t) = \lambda^{-1} \int_0^1 G(t, r)\mathbf{y}(r)dr, \tag{2.2}$$

where

$$G(t,r)=\begin{cases}\dfrac{t^{\nu-1}(1-r)^{\nu-\rho-1}-(t-r)^{\nu-1}}{\Gamma(\nu)}, & 0\le r\le t\le 1,\\ \dfrac{t^{\nu-1}(1-r)^{\nu-\rho-1}}{\Gamma(\nu)}, & 0\le t\le r\le 1,\end{cases} \tag{2.3}$$

is the Green function for this problem.

Proof. The process of proving this theorem is similar to the process of proving theorem 3.1 in reference [1]. □

We now describe several properties of the Green function as in (2.3). The following Lemmas is expressed in references [12] and [14].

Lemma 2.1. *Suppose $G(t,r)$ is the same as in the statement of theorem 2.1. Hence, we have that on set $[0,1]\times[0,1]$, the G function is continuous and positive.*

Lemma 2.2. *Suppose $G(t,r)$ is the same as in the statement of theorem 2.1. Hence, it has after feature:*

$$[1-(1-r)^{\rho}](1-r)^{\nu-\rho-1}t^{\nu-1}\leqslant\Gamma(\nu)G(t,r)\leqslant(1-r)^{\nu-\rho-1}t^{\nu-1},\quad t,r\in[0,1].$$

3. Main results

In this part, we check uniqueness and existence of positive solutions for the fractional BVP (1.1) by means of the FPT of mixed monotone operator. If $\mathsf{E}=\mathbf{C}[0,1]=\{\mathbf{x}|\mathbf{x}:[0,1]\longrightarrow\Re,\mathbf{x}$ is continuous function$\}$be a Banach space with the standard norm $\|\mathbf{x}\|=Sup\{|\mathbf{x}(t)|:t\in[0,1]\}$. Notice that E can be equipped with a regular component given by $\mathbf{x},\mathbf{y}\in\mathbf{C}[0,1]$, $\mathbf{x}\leqslant\mathbf{y}\Leftrightarrow\mathbf{x}(t)\leqslant\mathbf{y}(t)$ for $t\in[0,1]$. For all $\mathbf{x},\mathbf{y}\in\mathsf{E}$, the notational $\mathbf{x}\sim\mathbf{y}$ that's mean there are $\lambda_1>0$ and $\lambda_2>0$ so that $\lambda_1\mathbf{x}\leqslant\mathbf{y}\leqslant\lambda_2\mathbf{x}$. Clearly, $\sim$ is an equivalence relation. Give $\mathbf{h}>0$, we indicate by $\mathcal{P}_{\mathbf{h}}$ the set $\mathcal{P}_{\mathbf{h}}=\{\mathbf{x}\in\mathsf{E}\mid\mathbf{x}\sim\mathbf{h}\}$. It is clear that $\mathcal{P}_{\mathbf{h}}\subset\mathcal{P}$ for $\mathbf{h}\in\mathcal{P}$. Set $\mathcal{P}=\{\mathbf{x}\in\mathbf{C}[0,1]\mid\mathbf{x}(t)\geqslant 0,t\in[0,1]\}$ is the standard cone. It is obvious that $\mathcal{P}$ is a normal cone in $\mathbf{C}[0,1]$ and the normality constant is 1. We take $\mathbf{h}_0\in\mathsf{E},\mathbf{h}_0>0$, and the set $\mathcal{P}_{\mathbf{h}}\subseteq\mathcal{P}$ is expressed as follows:

$$\mathcal{P}_{\mathbf{h}}=\{\mathbf{h}_0\in\mathsf{E}:\exists t_0\in(0,1),\mathbf{h}t_0\le\mathbf{h}_0\le\frac{1}{t_0}\mathbf{h}\},$$

where $\mathbf{h}(t)=t^{\nu-1},t\in(0,1)$.

Lemma 3.1. *Assume that $\mathcal{P}$ be a normal cone in a real Banach space E, $\mathcal{T}:\mathcal{P}\times\mathcal{P}\longrightarrow\mathcal{P}$ mixed monotone operator and satisfies:*

$$\mathcal{T}(t\mathfrak{u},t^{-1}\mathbf{v})\geqslant t^{\beta(t)+\alpha(t)}\mathcal{T}(\mathfrak{u},\mathbf{v}), \tag{3.1}$$

functions $\alpha(t)$, $\beta(t)$ are different and continuous. Also, $0<\alpha(t)<1$, $0<\beta(t)<1$ and $0<\beta(t)+\alpha(t)<1$.

- *(J_1) there is $\mathbf{h}_0\in\mathcal{P}$ with $h_0\neq 0$ so that $\mathcal{T}(\mathbf{h}_0,\mathbf{h}_0)\in\mathcal{P}_{\mathbf{h}}$;*
- *(J_2) for each $\mathfrak{u},\mathbf{v}\in\mathcal{P}$ and $t\in(0,1)$, there is $\varphi(t)\in(t,1]$ so that $\mathcal{T}(t\mathfrak{u},t^{-1}\mathbf{v})\geq\varphi(t)\mathcal{T}(\mathfrak{u},\mathbf{v})$.*

Thus,

(1) $\mathcal{T}:\mathcal{P}_{\mathbf{h}}\times\mathcal{P}_{\mathbf{h}}\longrightarrow\mathcal{P}_{\mathbf{h}}$*;*

(2) there exists $\mathfrak{u}_0,\mathbf{v}_0\in\mathcal{P}_{\mathbf{h}}$ *and* $\mathbf{r}\in(0,1)$ *so that* $\mathbf{r}\mathbf{v}_0\leqslant\mathfrak{u}_0<\mathbf{v}_0$*,* $\mathfrak{u}_0\leqslant\lambda^{-1}\mathcal{T}(\mathfrak{u}_0,\mathbf{v}_0)\leqslant\lambda^{-1}\mathcal{T}(\mathbf{v}_0,\mathfrak{u}_0)\leqslant\mathbf{v}_0$*;*

(3) $\lambda^{-1}\mathcal{T}(\mathbf{x},\mathbf{x})=\mathbf{x}$ *has a uniquely λ-fixed point* $\mathbf{x}^*_\lambda$ *in* $\mathcal{P}_{\mathbf{h}}$*,* $\lambda>0$*;*

(4) for any initial values $\mathbf{x}_0,\mathbf{y}_0\in\mathcal{P}_{\mathbf{h}}$*,* $\lambda\in(0,1)$*, building consecutive the sequences*

$$\mathbf{x_n}=\lambda^{-1}\mathcal{T}(\mathbf{x_{n-1}},\mathbf{y_{n-1}}),\qquad \mathbf{y_n}=\lambda^{-1}\mathcal{T}(\mathbf{y_{n-1}},\mathbf{x_{n-1}}),\quad \mathbf{n}=1,2,...,$$

we have $\mathbf{x_n}\longrightarrow\mathbf{x}^*_\lambda$ *and* $\mathbf{y_n}\longrightarrow\mathbf{y}^*_\lambda$ *as* $\mathbf{n}\longrightarrow\infty$*.*

Proof. Since $\mathcal{T}(\mathbf{h}_0,\mathbf{h}_0)\in\mathcal{P}_{\mathbf{h}}$ there are constants $0<\lambda^{-1}\eta<1$ and $\lambda^{-1}\gamma>1$ so that

$$\gamma\mathbf{h}\leq\mathcal{T}(\mathbf{h}_0,\mathbf{h}_0)\leq\eta\mathbf{h}. \tag{3.2}$$

We also have a high relations by multiplying the constant λ^{-1} on the sides $\lambda^{-1}\gamma\mathbf{h}\leq\lambda^{-1}\mathcal{T}(\mathbf{h}_0,\mathbf{h}_0)\leq\lambda^{-1}\eta\mathbf{h}$. We show that $\mathcal{T}:\mathcal{P}_{\mathbf{h}}\times\mathcal{P}_{\mathbf{h}}\longrightarrow\mathcal{P}_{\mathbf{h}}$. Since $\mathbf{h}_0\in\mathcal{P}_{\mathbf{h}}$, let's select a number $t_0\in(0,1)$ that it is enough small so that $t_0\mathbf{h}\leq\mathbf{h}_0\leq\frac{1}{t_0}\mathbf{h}$. Using the relations (3.1) and (3.2), we have

$$\mathcal{T}(\mathbf{h},\mathbf{h})\geq\mathcal{T}(t_0\mathbf{h}_0,\frac{1}{t_0}\mathbf{h}_0)\geq t_0^{\beta(t_0)+\alpha(t_0)}\mathcal{T}(\mathbf{h}_0,\mathbf{h}_0)\geqslant\gamma\mathbf{h}t_0^{\beta(t_0)+\alpha(t_0)}, \tag{3.3}$$

$$\mathcal{T}(\mathbf{h},\mathbf{h})\leqslant\mathcal{T}(\frac{1}{t_0}\mathbf{h}_0,t_0\mathbf{h}_0)\leqslant\frac{1}{t_0^{\beta(t_0)+\alpha(t_0)}}\mathcal{T}(\mathbf{h}_0,\mathbf{h}_0)\leqslant\frac{\eta\mathbf{h}}{t_0^{\beta(t_0)+\alpha(t_0)}}. \tag{3.4}$$

Nothing that $\gamma\mathbf{h}t_0^{\beta(t_0)+\alpha(t_0)}$, $\frac{\eta\mathbf{h}}{t_0^{\beta(t_0)+\alpha(t_0)}}>0$. We can get $\mathcal{T}(\mathbf{h},\mathbf{h})\in\mathcal{P}_{\mathbf{h}}$, that's mean $\mathcal{T}:\mathcal{P}_{\mathbf{h}}\times\mathcal{P}_{\mathbf{h}}\longrightarrow\mathcal{P}_{\mathbf{h}}$. Also $0<\alpha(t_0)<1, 0<\beta(t_0)<1$ and $0<\beta(t_0)+\alpha(t_0)<1$. We can get a positive integer $\mathbf{m}$ so that

$$\mathbf{m}>\frac{2(\beta(t_0)+\alpha(t_0))}{1-(\beta(t_0)+\alpha(t_0))}. \tag{3.5}$$

Put $\mathfrak{u}_0=t_0^{\mathbf{m}}\mathbf{h}$ and $\mathbf{v}_0=\frac{1}{t_0^{\mathbf{m}}}\mathbf{h}$. Evidently, $\mathfrak{u}_0,\mathbf{v}_0\in\mathcal{P}_{\mathbf{h}}$, $\mathfrak{u}_0<\mathbf{v}_0$. Through the mixed monotone operator of $\mathcal{T}$, we have

$$\mathcal{T}(\mathfrak{u}_0,\mathbf{v}_0)\leqslant\mathcal{T}(\mathbf{v}_0,\mathfrak{u}_0).$$

We also have a high relationship by multiplying the constant λ^{-1} on both sides

$$\lambda^{-1}\mathcal{T}(\mathfrak{u}_0,\mathbf{v}_0)\leqslant\lambda^{-1}\mathcal{T}(\mathbf{v}_0,\mathfrak{u}_0).$$

Using the relation (3.5), we have

$$\mathbf{m}-\mathbf{m}(\beta(t_0)+\alpha(t_0))>2(\beta(t_0)+\alpha(t_0)),$$

that's mean $\mathbf{m}>(\mathbf{m}+2)(\beta(t_0)+\alpha(t_0))$. Thus,

$$t_0^{\mathbf{m}}<t_0^{(\mathbf{m}+2)(\beta(t_0)+\alpha(t_0))}. \tag{3.6}$$

Using the relations (3.1), (3.3) and (3.6), we have

$$\begin{aligned}
\lambda^{-1}\mathcal{T}(\mathfrak{u}_0,\mathbf{v}_0) &= \lambda^{-1}\mathcal{T}(t_0^{\mathbf{m}}\mathbf{h},\frac{1}{t_0^{\mathbf{m}}}\mathbf{h})\\
&= \lambda^{-1}\mathcal{T}(t_0t_0^{\mathbf{m}-1}\mathbf{h},t_0^{-1}t_0^{1-\mathbf{m}}\mathbf{h})\\
&\geqslant \lambda^{-1}t_0^{\beta(t_0)+\alpha(t_0)}\mathcal{T}(t_0^{\mathbf{m}-1}\mathbf{h},t_0^{1-\mathbf{m}}\mathbf{h})\\
&= \lambda^{-1}t_0^{\beta(t_0)+\alpha(t_0)}\mathcal{T}(t_0t_0^{\mathbf{m}-2}\mathbf{h},t_0^{-1}t_0^{2-\mathbf{m}}\mathbf{h})\\
&\geqslant \lambda^{-1}t_0^{2(\beta(t_0)+\alpha(t_0))}\mathcal{T}(t_0^{\mathbf{m}-2}\mathbf{h},t_0^{2-\mathbf{m}}\mathbf{h})\\
&\vdots\\
&\geqslant \lambda^{-1}t_0^{\mathbf{m}(\beta(t_0)+\alpha(t_0))}\mathcal{T}(t_0\mathbf{h},t_0^{-1}\mathbf{h})\\
&\geqslant \lambda^{-1}t_0^{(\mathbf{m}+1)(\beta(t_0)+\alpha(t_0))}\mathcal{T}(\mathbf{h},\mathbf{h})\\
&\geqslant \lambda^{-1}\gamma\mathbf{h}t_0^{(\mathbf{m}+2)(\beta(t_0)+\alpha(t_0))}\\
&\geqslant \lambda^{-1}t_0^{\mathbf{m}}\gamma\mathbf{h}\\
&= \lambda^{-1}\gamma\mathfrak{u}_0\\
&\geqslant \mathfrak{u}_0.
\end{aligned}$$

Now, we show that $\lambda^{-1}\mathcal{T}(\mathfrak{u}_0,\mathbf{v}_0)\leqslant\mathbf{v}_0$. Using the relations (3.1), (3.4) and (3.6), we have

$$\begin{aligned}
\lambda^{-1}\mathcal{T}(\mathbf{v}_0,\mathfrak{u}_0) &= \lambda^{-1}\mathcal{T}(\frac{1}{t_0^{\mathbf{m}}}\mathbf{h},t_0^{\mathbf{m}}\mathbf{h})\\
&= \lambda^{-1}\mathcal{T}(t_0^{-1}t_0^{1-\mathbf{m}}\mathbf{h},t_0t_0^{\mathbf{m}-1}\mathbf{h})\\
&\leqslant \frac{\lambda^{-1}}{t_0^{\beta(t_0)+\alpha(t_0)}}\mathcal{T}(t_0^{1-\mathbf{m}}\mathbf{h},t_0^{\mathbf{m}-1}\mathbf{h})\\
&\vdots\\
&\leqslant \frac{\lambda^{-1}}{t_0^{\beta(t_0)+\alpha(t_0)}}\mathcal{T}(t_0^{-1}\mathbf{h},t_0\mathbf{h})\\
&\leqslant \frac{\lambda^{-1}}{t_0^{(\mathbf{m}+1)(\beta(t_0)+\alpha(t_0))}}\mathcal{T}(\mathbf{h},\mathbf{h})\\
&\leqslant \frac{\lambda^{-1}}{t_0^{(\mathbf{m}+2)(\beta(t_0)+\alpha(t_0))}}\eta\mathbf{h}\\
&\leqslant \frac{\lambda^{-1}}{t_0^{\mathbf{m}}}\eta\mathbf{h}\\
&= \lambda^{-1}\eta\mathbf{v}_0\\
&\leqslant \mathbf{v}_0.
\end{aligned}$$

Consequently

$$\mathfrak{u}_0\leqslant\lambda^{-1}\mathcal{T}(\mathfrak{u}_0,\mathbf{v}_0)\leqslant\lambda^{-1}\mathcal{T}(\mathbf{v}_0,\mathfrak{u}_0)\leqslant\mathbf{v}_0. \tag{3.7}$$

Building consecutive the sequence

$$\mathfrak{u}_{\mathbf{n}}=\lambda^{-1}\mathcal{T}(\mathfrak{u}_{\mathbf{n}-1},\mathbf{v}_{\mathbf{n}-1}),$$

$$\mathbf{v_n} = \lambda^{-1}\mathcal{T}(\mathbf{v_{n-1}}, \mathfrak{u}_{\mathbf{n}-1}), \quad \mathbf{n} = 1, 2, \cdots .$$

By the relation (3.7), we have

$$\mathfrak{u}_1 = \lambda^{-1}\mathcal{T}(\mathfrak{u}_0, \mathbf{v}_0) \leqslant \lambda^{-1}\mathcal{T}(\mathbf{v}_0, \mathfrak{u}_0) = \mathbf{v}_1,$$

in general we got that $\mathfrak{u}_\mathbf{n} < \mathbf{v_n}$, $\mathbf{n} = 1, 2, ...$ So, we have

$$\mathfrak{u}_0 \leqslant \mathfrak{u}_1 \leqslant \cdots \leqslant \mathfrak{u}_\mathbf{n} \leqslant \cdots \leqslant \mathbf{v_n} \leqslant \cdots \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_0. \tag{3.8}$$

We can take $\mathfrak{u}_0 = t_0^{2\mathbf{m}}\mathbf{v}_0$ therefore,

$$\mathfrak{u}_\mathbf{n} \geqslant \mathfrak{u}_0 \geqslant t_0^{2\mathbf{m}}\mathbf{v}_0 \geqslant t_0^{2\mathbf{m}}\mathbf{v_n}, \quad \mathbf{n} = 1, 2, \cdots .$$

Suppose

$$t_\mathbf{n} = sup\{t > 0 \mid \mathfrak{u}_\mathbf{n} \geqslant t\mathbf{v_n}\}, \quad \mathbf{n} = 1, 2, \cdots .$$

So, we have:

$$\begin{aligned} &\mathfrak{u}_\mathbf{n} \geqslant t_\mathbf{n}\mathbf{v_n}, \\ &\mathfrak{u}_{\mathbf{n}+1} \geqslant \mathfrak{u}_\mathbf{n} \geqslant t_\mathbf{n}\mathbf{v_n} \geqslant t_\mathbf{n}\mathbf{v_{n+1}}, \quad \mathbf{n} = 1, 2, \cdots . \end{aligned}$$

Thus, $t_{\mathbf{n}+1} \geqslant t_\mathbf{n}$ that's mean $t_\mathbf{n}$ is increasing with $t_\mathbf{n} \subseteq (0, 1]$. We presume $\lim_{\mathbf{n}\to\infty} t_\mathbf{n} = t^*_\lambda$. We show $t^*_\lambda = 1$. Differently $0 < t^*_\lambda < 1$, we check two cases.

Case (1): existence an integer N so that $t_N = t^*_\lambda$ in this case we be aware $t_\mathbf{n} = t^*_\lambda$ for all $\mathbf{n} \geqslant N$. Thus, for each $\mathbf{n} \geqslant N$, we have

$$\begin{aligned} \mathfrak{u}_{\mathbf{n}+1} &= \lambda^{-1}\mathcal{T}(\mathfrak{u}_\mathbf{n}, \mathbf{v_n}) \\ &\geqslant \lambda^{-1}\mathcal{T}(t_\mathbf{n}\mathbf{v_n}, \frac{1}{t_\mathbf{n}}\mathfrak{u}_\mathbf{n}) \\ &= \lambda^{-1}\mathcal{T}(t^*_\lambda\mathbf{v_n}, \frac{1}{t^*_\lambda}\mathfrak{u}_\mathbf{n}) \\ &\geqslant t_\lambda^{*\beta(t^*_\lambda)+\alpha(t^*_\lambda)}\lambda^{-1}\mathcal{T}(\mathbf{v_n}, \mathfrak{u}_\mathbf{n}) \\ &= t_\lambda^{*\beta(t^*_\lambda)+\alpha(t^*_\lambda)}\mathbf{v_{n+1}}. \end{aligned}$$

According to the definition of $t_\mathbf{n}$, we have

$$t_{\mathbf{n}+1} = t^*_\lambda \geqslant t_\lambda^{*(\beta(t^*_\lambda)+\alpha(t^*_\lambda))} > t^*_\lambda,$$

which is against our hypothesis.

Case(2): For each integer $\mathbf{n}$, $t_\mathbf{n} < t^*_\lambda$ thus, we attain:

$$\begin{aligned} \mathfrak{u}_{\mathbf{n}+1} &= \lambda^{-1}\mathcal{T}(\mathfrak{u}_\mathbf{n}, \mathbf{v_n}) \\ &\geqslant \lambda^{-1}\mathcal{T}(t_\mathbf{n}\mathbf{v_n}, \frac{1}{t_\mathbf{n}}\mathfrak{u}_\mathbf{n}) \\ &= \lambda^{-1}\mathcal{T}(\frac{t_\mathbf{n}}{t^*_\lambda} \times t^*_\lambda\mathbf{v_n}, \frac{t^*_\lambda}{t_\mathbf{n}t^*_\lambda}\mathfrak{u}_\mathbf{n}) \\ &\geqslant (\frac{t_\mathbf{n}}{t^*_\lambda})^{\beta(\frac{t_\mathbf{n}}{t^*_\lambda})+\alpha(\frac{t_\mathbf{n}}{z^*_\lambda})}\lambda^{-1}\mathcal{T}(t^*_\lambda\mathbf{v_n}, \frac{1}{t^*_\lambda}\mathfrak{u}_\mathbf{n}) \end{aligned}$$

$$\begin{aligned}
&\geqslant \frac{t_{\mathbf{n}}}{t^*_\lambda} t_\lambda^{*(\beta(t^*_\lambda)+\alpha(t^*_\lambda))}\lambda^{-1}\mathcal{T}(\mathbf{v_n},\mathfrak{u}_\mathbf{n})\\
&= t_{\mathbf{n}} t_\lambda^{*(\beta(t^*_\lambda)+\alpha(t^*_\lambda))-1}\lambda^{-1}\mathcal{T}(\mathbf{v_n},\mathfrak{u}_\mathbf{n})\\
&= t_{\mathbf{n}} t_\lambda^{*(\beta(t^*_\lambda)+\alpha(t^*_\lambda))-1}\mathbf{v}_{\mathbf{n}+1}.
\end{aligned}$$

By the definition of $t_{\mathbf{n}}$, and $t_{\mathbf{n}}$ is increasing

$$t_{\mathbf{n}+1} \geqslant t_{\mathbf{n}} t_\lambda^{*(\beta(t^*_\lambda)+\alpha(t^*_\lambda))-1}.$$

By taking the limit from the parties as $\mathbf{n} \longrightarrow \infty$, we have

$$\begin{aligned}
t^*_\lambda &\geqslant t^*_\lambda t_\lambda^{*(\beta(t^*_\lambda)+\alpha(t^*_\lambda))-1}\\
&= t_\lambda^{*(\beta(t^*_\lambda)+\alpha(t^*_\lambda))}\\
&> t^*_\lambda,
\end{aligned}$$

which is against our hypothesis. So, $\lim_{\mathbf{n}\to\infty} t_{\mathbf{n}} = 1$. For each natural number $\mathcal{P}$, we have:

$$\begin{aligned}
&0 \leqslant \mathfrak{u}_{\mathbf{n}+\mathcal{P}} - \mathfrak{u}_\mathbf{n} \leqslant \mathbf{v_n} - \mathfrak{u}_\mathbf{n} \leqslant \mathbf{v_n} - t_\mathbf{n}\mathbf{v_n} = (1-t_\mathbf{n})\mathbf{v_n} \leqslant (1-t_\mathbf{n})\mathbf{v}_0,\\
&0 \leqslant \mathbf{v_n} - \mathbf{v}_{\mathbf{n}+\mathcal{P}} \leqslant \mathbf{v_n} - \mathfrak{u}_\mathbf{n} \leqslant \mathbf{v_n} - t_\mathbf{n}\mathbf{v_n} = (1-t_\mathbf{n})\mathbf{v_n} \leqslant (1-t_\mathbf{n})\mathbf{v}_0.
\end{aligned}$$

Since $\mathcal{P}$ is normal, we have

$$\begin{aligned}
&\|\mathfrak{u}_{\mathbf{n}+\mathcal{P}} - \mathfrak{u}_\mathbf{n}\| \leqslant \mathcal{N}(1-t_\mathbf{n})\|\mathbf{v}_0\| \to 0, \quad (\text{as } \mathbf{n}\to\infty),\\
&\|\mathbf{v_n} - \mathbf{v}_{\mathbf{n}+\mathcal{P}}\| \leqslant \mathcal{N}(1-t_\mathbf{n})\|\mathbf{v}_0\| \to 0, \quad (\text{as } \mathbf{n}\to\infty),
\end{aligned}$$

where $\mathcal{N}$ is a normal constant. Thus, $\mathfrak{u}_\mathbf{n}$ and $\mathbf{v_n}$ are Cauchy sequences. Because E is complete, there are $\mathfrak{u}^*_\lambda$, $\mathbf{v}^*_\lambda$ so that $\mathfrak{u}_\mathbf{n} \to \mathfrak{u}^*_\lambda$ and $\mathbf{v_n} \to \mathbf{v}^*_\lambda$ as $\mathbf{n}\to\infty$. By (3.8), we know that

$$\mathfrak{u}_\mathbf{n} \leqslant \mathfrak{u}^*_\lambda \leqslant \mathbf{v}^*_\lambda \leqslant \mathbf{v_n},$$

and

$$0 \leqslant \mathbf{v}^*_\lambda - \mathfrak{u}^*_\lambda \leqslant (1-t_\mathbf{n})\mathbf{v}_0,$$

therefore,

$$\|\mathbf{v}^*_\lambda - \mathfrak{u}^*_\lambda\| \leqslant \mathcal{N}(1-t_\mathbf{n})\|\mathbf{v}_0\| \to 0, \quad (\text{as } \mathbf{n}\to\infty).$$

That's mean $\mathfrak{u}^*_\lambda = \mathbf{v}^*_\lambda$. Let $\mathbf{x}^*_\lambda := \mathfrak{u}^*_\lambda = \mathbf{v}^*_\lambda$ thus, we attain

$$\mathfrak{u}_{\mathbf{n}+1} = \lambda^{-1}\mathcal{T}(\mathfrak{u}_\mathbf{n},\mathbf{v_n}) \leqslant \lambda^{-1}\mathcal{T}(\mathbf{x}^*_\lambda,\mathbf{x}^*_\lambda) \leqslant \lambda^{-1}\mathcal{T}(\mathbf{v_n},\mathfrak{u}_\mathbf{n}) = \mathbf{v}_{\mathbf{n}+1}.$$

If $\mathbf{n}\to\infty$ we receive $\mathbf{x}^*_\lambda = \lambda^{-1}\mathcal{T}(\mathbf{x}^*_\lambda,\mathbf{x}^*_\lambda)$. That's mean $\mathbf{x}^*_\lambda$ is the λ-fixed point of $\mathcal{T}$ in $\mathcal{P}_\mathbf{h}$. We showed that $\mathbf{x}^*_\lambda$ is the λ-fixed point of $\mathcal{T}$. Now we show that $\mathbf{x}^*_\lambda$ is the uniquely λ-fixed point of $\mathcal{T}$ in $\mathcal{P}_\mathbf{h}$. Assume that $\overline{\mathbf{x}}_\lambda$ is the λ-fixed point of $\mathcal{T}$ in $\mathcal{P}_\mathbf{h}$. There are $e_1, \frac{1}{e_1} > 0$ so that

$$e_1\overline{\mathbf{x}}_\lambda \leqslant \mathbf{x}^*_\lambda \leqslant \frac{1}{e_1}\overline{\mathbf{x}}_\lambda.$$

Let

$$e_2 = \sup\{e_1 > 0 \mid e_1\overline{\mathbf{x}}_\lambda \leqslant \mathbf{x}^*_\lambda \leqslant \frac{1}{e_1}\overline{\mathbf{x}}_\lambda\},$$

we prove that $e_2 = 1$. Differently $0 < e_2 < 1$. Thus,

$$\begin{aligned}
\mathbf{x}^*_\lambda &= \lambda^{-1}\mathcal{T}(\mathbf{x}^*_\lambda, \mathbf{x}^*_\lambda) \\
&\geqslant \lambda^{-1}\mathcal{T}(e_2\overline{\mathbf{x}}_\lambda, \frac{1}{e_2}\overline{\mathbf{x}}_\lambda) \\
&\geqslant e_2^{\beta(e_2)+\alpha(e_2)}\lambda^{-1}\mathcal{T}(\overline{\mathbf{x}}_\lambda, \overline{\mathbf{x}}_\lambda) \\
&= e_2^{\beta(e_2)+\alpha(e_2)}\overline{\mathbf{x}}_\lambda,
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}^*_\lambda &= \lambda^{-1}\mathcal{T}(\mathbf{x}^*_\lambda, \mathbf{x}^*_\lambda) \\
&\leqslant \lambda^{-1}\mathcal{T}(\frac{1}{e_2}\overline{\mathbf{x}}_\lambda, e_2\overline{\mathbf{x}}_\lambda) \\
&\leqslant \frac{1}{e_2^{\beta(e_2)+\alpha(e_2)}}\lambda^{-1}\mathcal{T}(\overline{\mathbf{x}}_\lambda, \overline{\mathbf{x}}_\lambda) \\
&= \frac{1}{e_2^{\beta(e_2)+\alpha(e_2)}}\overline{\mathbf{x}}_\lambda.
\end{aligned}$$

Because $e_2^{\beta(e_2)+\alpha(e_2)} > e_2$ the above relationship is against our hypothesis. Thus, $e_2 = 1$. Therefore, $\mathbf{x}^*_\lambda = \overline{\mathbf{x}}_\lambda$. Thus, $\mathbf{x}^*_\lambda$ is the uniquely λ-fixed point of $\mathcal{T}$ in $\mathcal{P}_\mathbf{h}$. For each $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{P}_\mathbf{h}$, we can select a small number $e_3 \in (0, 1)$ so that

$$e_3\mathbf{h} \leqslant \mathbf{y}_0 \leqslant \frac{1}{e_3}\mathbf{h}, \quad e_3\mathbf{h} \leqslant \mathbf{x}_0 \leqslant \frac{1}{e_3}\mathbf{h}.$$

Let $\overline{\mathrm{u}}_0 = e_3\mathbf{h}$, $\overline{\mathbf{v}}_0 = \frac{1}{e_3}\mathbf{h}$, we look that $\overline{\mathrm{u}}_0, \overline{\mathbf{v}}_0 \in \mathcal{P}_\mathbf{h}$ and $\overline{\mathrm{u}}_0 \leqslant \mathbf{y}_0 \leqslant \overline{\mathbf{v}}_0$, $\overline{\mathrm{u}}_0 \leqslant \mathbf{x}_0 \leqslant \overline{\mathbf{v}}_0$. We put

$$\begin{aligned}
\overline{\mathrm{u}}_\mathbf{n} &= \lambda^{-1}\mathcal{T}(\overline{\mathrm{u}}_{\mathbf{n}-1}, \overline{\mathbf{v}}_{\mathbf{n}-1}), \\
\overline{\mathbf{v}}_\mathbf{n} &= \lambda^{-1}\mathcal{T}(\overline{\mathbf{v}}_{\mathbf{n}-1}, \overline{\mathrm{u}}_{\mathbf{n}-1}), \\
\mathbf{x}_\mathbf{n} &= \lambda^{-1}\mathcal{T}(\mathbf{x}_{\mathbf{n}-1}, \mathbf{y}_{\mathbf{n}-1}), \\
\mathbf{y}_\mathbf{n} &= \lambda^{-1}\mathcal{T}(\mathbf{y}_{\mathbf{n}-1}, \mathbf{x}_{\mathbf{n}-1}).
\end{aligned}$$

Similarly to the previous part, there exists a $\mathbf{y}^*_\lambda \in \mathcal{P}_\mathbf{h}$ so that $\lambda^{-1}\mathcal{T}(\mathbf{y}^*_\lambda, \mathbf{y}^*_\lambda) = \mathbf{y}^*_\lambda$, $\lim_{\mathbf{n}\to\infty} \overline{\mathrm{u}}_\mathbf{n} = \mathbf{y}^*_\lambda$ and $\lim_{\mathbf{n}\to\infty} \overline{\mathbf{v}}_\mathbf{n} = \mathbf{y}^*_\lambda$. Given the uniqueness of the constant point of the operator $\mathcal{T}$ in $\mathcal{P}_\mathbf{h}$, we take $\mathbf{x}^*_\lambda = \mathbf{y}^*_\lambda$. We have by analysis

$$\begin{aligned}
\overline{\mathrm{u}}_\mathbf{n} &\leqslant \mathbf{x}_\mathbf{n} \leqslant \overline{\mathbf{v}}_\mathbf{n}, \\
\overline{\mathrm{u}}_\mathbf{n} &\leqslant \mathbf{y}_\mathbf{n} \leqslant \overline{\mathbf{v}}_\mathbf{n},
\end{aligned}$$

because $\mathcal{P}$ is normal, we have

$$\begin{aligned}
\lim_{\mathbf{n}\to\infty} \mathbf{x}_\mathbf{n} &= \mathbf{x}^*_\lambda, \\
\lim_{\mathbf{n}\to\infty} \mathbf{y}_\mathbf{n} &= \mathbf{y}^*_\lambda.
\end{aligned}$$

□

Theorem 3.1. *Assume that $\alpha(t), \beta(t)$ are two different and continuous functions, $0 < \alpha(t) < 1$, $0 < \beta(t) < 1$, $0 < \beta(t) + \alpha(t) < 1$, and $\mathfrak{A} : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}$ is a mixed monotone operator. $\mathfrak{B} : \mathcal{P} \longrightarrow \mathcal{P}$ is a increasing $\sigma(t)$-concave operator, and $\mathfrak{C} : \mathcal{P} \longrightarrow \mathcal{P}$ is a decreasing $\sigma(t)$-concave operator. Where $\sigma(t)$ is a continuous function. They satisfy the following conditions:*

$$\mathfrak{A}(t\mathbf{x}, t^{-1}\mathbf{y}) \geqslant t^{\beta(t)+\alpha(t)}\mathfrak{A}(\mathbf{x}, \mathbf{y}), \quad \forall t \in (0,1), \quad \mathbf{x}, \mathbf{y} \in \mathcal{P}, \tag{3.9}$$

$$\mathfrak{B}(t\mathbf{x}) \geqslant t^{\sigma(t)}\mathfrak{B}(\mathbf{x}), \quad \forall t \in (0,1), \quad \mathbf{x} \in \mathcal{P}, \tag{3.10}$$

$$\mathfrak{C}(t^{-1}\mathbf{y}) \geqslant t^{\sigma(t)}\mathfrak{C}(\mathbf{y}), \quad \forall t \in (0,1), \quad \mathbf{y} \in \mathcal{P}. \tag{3.11}$$

Assume that J_3 and J_4 hold:

(J_3) there exist $\mathbf{h}_0 \in \mathcal{P}_\mathbf{h}$ so that $\mathfrak{A}(\mathbf{h}_0, \mathbf{h}_0) \in \mathcal{P}_\mathbf{h}$, $\mathfrak{B}(\mathbf{h}_0) \in \mathcal{P}_\mathbf{h}$, $\mathfrak{C}(\mathbf{h}_0) \in \mathcal{P}_\mathbf{h}$;

(J_4) there are two constants $\delta_1, \delta_2 > 0$ so that

$$\mathfrak{A}(\mathbf{x}, \mathbf{y}) \geqslant \delta_1\mathfrak{B}(\mathbf{x}) + \delta_2\mathfrak{C}(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{P}. \tag{3.12}$$

Thus,

(1) $\mathfrak{A} : \mathcal{P}_\mathbf{h} \times \mathcal{P}_\mathbf{h} \longrightarrow \mathcal{P}_\mathbf{h}$, $\mathfrak{B} : \mathcal{P}_\mathbf{h} \longrightarrow \mathcal{P}_\mathbf{h}$, $\mathfrak{C} : \mathcal{P}_\mathbf{h} \longrightarrow \mathcal{P}_\mathbf{h}$;

(2) there are $\mathfrak{u}_0, \mathbf{v}_0 \in \mathcal{P}_\mathbf{h}$ and $\mathbf{r} \in (0,1)$ so that

$$\begin{aligned} &\mathbf{r}\mathbf{v}_0 \leqslant \mathfrak{u}_0 < \mathbf{v}_0, \\ &\mathfrak{u}_0 \leqslant \lambda^{-1}\mathcal{T}(\mathfrak{u}_0, \mathbf{v}_0) = \mathfrak{A}(\mathfrak{u}_0, \mathbf{v}_0) + \mathfrak{B}(\mathfrak{u}_0) + \mathfrak{C}(\mathbf{v}_0) \\ &\quad \leqslant \lambda^{-1}\mathcal{T}(\mathbf{v}_0, \mathfrak{u}_0) = \mathfrak{A}(\mathbf{v}_0, \mathfrak{u}_0) + \mathfrak{B}(\mathbf{v}_0) + \mathfrak{C}(\mathfrak{u}_0) \leqslant \mathbf{v}_0; \end{aligned}$$

(3) the operator equation $\lambda^{-1}\mathcal{T}(\mathbf{x}, \mathbf{y}) = \mathfrak{A}(\mathbf{x}, \mathbf{y}) + \mathfrak{B}(\mathbf{x}) + \mathfrak{C}(\mathbf{y}) = \mathbf{x}$ has a uniquely solution $\mathbf{x}_\lambda^*$ in $\mathcal{P}_\mathbf{h}$;

(4) for each $\lambda > 0$ and initial values $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{P}_\mathbf{h}$, building consecutive the sequences

$$\begin{aligned} \mathbf{x}_\mathbf{n} &= \lambda^{-1}\mathcal{T}(\mathbf{x}_\mathbf{n}, \mathbf{y}_\mathbf{n}) = \mathfrak{A}(\mathbf{x}_{\mathbf{n}-1}, \mathbf{y}_{\mathbf{n}-1}) + \mathfrak{B}(\mathbf{x}_{\mathbf{n}-1}) + \mathfrak{C}(\mathbf{y}_{\mathbf{n}-1}), \ \mathbf{n} = 1, 2, \cdots, \\ \mathbf{y}_\mathbf{n} &= \lambda^{-1}\mathcal{T}(\mathbf{y}_\mathbf{n}, \mathbf{x}_\mathbf{n}) = \mathfrak{A}(\mathbf{y}_{\mathbf{n}-1}, \mathbf{x}_{\mathbf{n}-1}) + \mathfrak{B}(\mathbf{y}_{\mathbf{n}-1}) + \mathfrak{C}(\mathbf{x}_{\mathbf{n}-1}), \ \mathbf{n} = 1, 2, \cdots. \end{aligned}$$

We have $\mathbf{x}_\mathbf{n} \longrightarrow \mathbf{x}_\lambda^*$ and $\mathbf{y}_\mathbf{n} \longrightarrow \mathbf{y}_\lambda^*$ as $\mathbf{n} \longrightarrow \infty$.

Proof. Using of relations (3.9), (3.10) and (3.11), we have

$$\mathfrak{A}(t^{-1}\mathbf{x}, t\mathbf{y}) \leqslant \frac{1}{t^{\beta(t)+\alpha(t)}}\mathfrak{A}(\mathbf{x}, \mathbf{y}), \quad t \in (0,1), \quad \mathbf{x}, \mathbf{y} \in \mathcal{P}, \tag{3.13}$$

$$\mathfrak{B}(t^{-1}\mathbf{x}) \leqslant \frac{1}{t^{\sigma(t)}}\mathfrak{B}(\mathbf{x}), \quad t \in (0,1), \quad \mathbf{x} \in \mathcal{P}, \tag{3.14}$$

$$\mathfrak{C}(t\mathbf{y}) \leqslant \frac{1}{t^{\sigma(t)}}\mathfrak{C}(\mathbf{y}), \ t \in (0,1), \ \mathbf{x} \in \mathcal{P}. \tag{3.15}$$

We show that $\mathfrak{A} : \mathcal{P}_\mathbf{h} \times \mathcal{P}_\mathbf{h} \longrightarrow \mathcal{P}_\mathbf{h}$, $\mathfrak{B} : \mathcal{P}_\mathbf{h} \longrightarrow \mathcal{P}_\mathbf{h}$ and $\mathfrak{C} : \mathcal{P}_\mathbf{h} \longrightarrow \mathcal{P}_\mathbf{h}$. Because $\mathfrak{A}(\mathbf{h}_0, \mathbf{h}_0) \in \mathcal{P}_\mathbf{h}$, $\mathfrak{B}(\mathbf{h}_0) \in \mathcal{P}_\mathbf{h}$ and $\mathfrak{C}(\mathbf{h}_0) \in \mathcal{P}_\mathbf{h}$, there are constants $\gamma_1, \gamma_2, \gamma_3, \eta_1, \eta_2$ and $\eta_3 > 0$, where $0 < \lambda^{-1}\eta_i < 1$, $\lambda^{-1}\gamma_i > 1$, so that

$$\lambda^{-1}\gamma_1\mathbf{h} \leqslant \mathfrak{A}(\mathbf{h}_0, \mathbf{h}_0) \leqslant \lambda^{-1}\eta_1\mathbf{h}, \tag{3.16}$$

$$\lambda^{-1}\gamma_2\mathbf{h} \leqslant \mathfrak{B}(\mathbf{h}_0) \leqslant \lambda^{-1}\eta_2\mathbf{h}, \tag{3.17}$$

$$\lambda^{-1}\gamma_3\mathbf{h} \leqslant \mathfrak{C}(\mathbf{h}_0) \leqslant \lambda^{-1}\eta_3\mathbf{h}. \tag{3.18}$$

From $\mathbf{h}_0 \in \mathcal{P}_{\mathbf{h}}$, there is a constant $t_0 \in (0,1)$ so that $t_0\mathbf{h} \leqslant \mathbf{h}_0 \leqslant \frac{1}{t_0}\mathbf{h}$. Using the relations (3.14), (3.15), (3.17) and (3.18), we have

$$\mathfrak{B}(\mathbf{h}) \leqslant \mathfrak{B}(\frac{1}{t_0}\mathbf{h}_0) \leqslant \frac{1}{t_0^{\sigma(t_0)}}\mathfrak{B}(\mathbf{h}_0) \leqslant \frac{\lambda^{-1}}{t_0^{\sigma(t_0)}}\eta_2\mathbf{h}, \tag{3.19}$$

$$\mathfrak{B}(\mathbf{h}) \geqslant \mathfrak{B}(t_0\mathbf{h}_0) \geqslant t_0^{\sigma(t_0)}\mathfrak{B}(\mathbf{h}_0) \geqslant \lambda^{-1}t_0^{\sigma(t_0)}\gamma_2\mathbf{h}, \tag{3.20}$$

$$\mathfrak{C}(\mathbf{h}) \leqslant \mathfrak{C}(t_0\mathbf{h}_0) \leqslant \frac{1}{t_0^{\sigma(t_0)}}\mathfrak{C}(\mathbf{h}_0) \leqslant \frac{\lambda^{-1}}{t_0^{\sigma(t_0)}}\eta_3 h, \tag{3.21}$$

$$\mathfrak{C}(\mathbf{h}) \geqslant \mathfrak{C}(\frac{1}{t_0}\mathbf{h}_0) \geqslant t_0^{\sigma(t_0)}\mathfrak{C}(\mathbf{h}_0) \geqslant \lambda^{-1}t_0^{\sigma(t_0)}\gamma_3\mathbf{h}. \tag{3.22}$$

Also using the relations (3.13) and (3.16), we have

$$\mathfrak{A}(\mathbf{h},\mathbf{h}) \leqslant \mathfrak{A}(\frac{1}{t_0}\mathbf{h}_0, t_0\mathbf{h}_0) \leqslant \frac{1}{t_0^{\beta(t_0)+\alpha(t_0)}}\mathfrak{A}(\mathbf{h}_0,\mathbf{h}_0) \leqslant \frac{\lambda^{-1}\eta_1}{t_0^{\beta(t_0)+\alpha(t_0)}}\mathbf{h}, \tag{3.23}$$

$$\mathfrak{A}(\mathbf{h},\mathbf{h}) \geqslant \mathfrak{A}(t_0\mathbf{h}_0, \frac{1}{t_0}\mathbf{h}_0) \geqslant t_0^{\beta(t_0)+\alpha(t_0)}\mathfrak{A}(\mathbf{h}_0,\mathbf{h}_0) \geqslant \lambda^{-1}t_0^{\beta(t_0)+\alpha(t_0)}\gamma_1\mathbf{h}. \tag{3.24}$$

Noting that $\frac{\lambda^{-1}\eta_1}{t_0^{\beta(t_0)+\alpha(t_0)}}$, $\lambda^{-1}t_0^{\beta(t_0)+\alpha(t_0)}\gamma_1 > 0$, we can get $\mathfrak{A}(\mathbf{h},\mathbf{h}) \in \mathcal{P}_{\mathbf{h}}$. For each $\mathbf{x},\mathbf{y} \in \mathcal{P}_{\mathbf{h}}$, a enough small number $\zeta \in (0,1)$ choose so that $\zeta\mathbf{h} \leqslant \mathbf{x},\mathbf{y} \leqslant \frac{1}{\zeta}\mathbf{h}$. Using the relations (3.17), (3.18), (3.19), (3.20), (3.21) and (3.22), we have

$$\begin{aligned}
\mathfrak{B}(\mathbf{x}) &\leqslant \mathfrak{B}(\frac{1}{\zeta}\mathbf{h}) \leqslant \frac{1}{\zeta^{\sigma(\zeta)}} \times \frac{\lambda^{-1}}{t_0^{\sigma(t_0)}}\eta_2\mathbf{h},\\
\mathfrak{B}(\mathbf{x}) &\geqslant \mathfrak{B}(\zeta\mathbf{h}) \geqslant \zeta^{\sigma(\zeta)}t_0^{\sigma(t_0)}\lambda^{-1}\gamma_2\mathbf{h},\\
\mathfrak{C}(\mathbf{y}) &\leqslant \mathfrak{C}(\zeta\mathbf{h}) \leqslant \frac{1}{\zeta^{\sigma(\zeta)}} \times \frac{\lambda^{-1}}{t_0^{\sigma(t_0)}}\eta_3\mathbf{h},\\
\mathfrak{C}(\mathbf{y}) &\geqslant \mathfrak{C}(\frac{1}{\zeta}\mathbf{h}) \geqslant \zeta^{\sigma(\zeta)}t_0^{\sigma(t_0)}\lambda^{-1}\gamma_3\mathbf{h}.
\end{aligned}$$

Evidently, $\frac{\lambda^{-1}}{\zeta^{\sigma(\zeta)}t_0^{\sigma(t_0)}}\eta_2, \frac{\lambda^{-1}}{\zeta^{\sigma(\zeta)}t_0^{\sigma(t_0)}}\eta_3, \zeta^{\sigma(\zeta)}t_0^{\sigma(t_0)}\lambda^{-1}\gamma_2$ and $\zeta^{\sigma(\zeta)}t_0^{\sigma(t_0)}\lambda^{-1}\gamma_3 > 0$. Thus, $\mathfrak{B}(\mathbf{x}) \in \mathcal{P}_{\mathbf{h}}$, $\mathfrak{C}(\mathbf{y}) \in \mathcal{P}_{\mathbf{h}}$; that's mean, $\mathfrak{B} : \mathcal{P}_{\mathbf{h}} \longrightarrow \mathcal{P}_{\mathbf{h}}$, $\mathfrak{C} : \mathcal{P}_{\mathbf{h}} \longrightarrow \mathcal{P}_{\mathbf{h}}$. Using the relations (3.16), (3.23) and (3.24) we have:

$$\begin{aligned}
A(x,y) &\leq A(\frac{1}{\zeta}h, \zeta h) \leq \frac{1}{\zeta^{\beta(\zeta)+\alpha(\zeta)}} \times \frac{\lambda^{-1}\eta_1}{t_0^{\beta(t_0)+\alpha(t_0)}}h,\\
A(x,y) &\geq A(\zeta h, \frac{1}{\zeta}h) \geq \zeta^{\beta(\zeta)+\alpha(\zeta)}\lambda^{-1}t_0^{\beta(t_0)+\alpha(t_0)}h.
\end{aligned}$$

Thus $A(x,y) \in \mathcal{P}_{\mathbf{h}}$; that's mean, $\mathfrak{A} : \mathcal{P}_{\mathbf{h}} \times \mathcal{P}_{\mathbf{h}} \longrightarrow \mathcal{P}_{\mathbf{h}}$. Using Lemma 3.1, we conclude that $\mathfrak{A} : \mathcal{P}_{\mathbf{h}} \times \mathcal{P}_{\mathbf{h}} \longrightarrow \mathcal{P}_{\mathbf{h}}$ and $\mathfrak{B}, \mathfrak{C} : \mathcal{P}_{\mathbf{h}} \longrightarrow \mathcal{P}_{\mathbf{h}}$. So, the conclusion (1) is true. Now we have to prove that the results (2)-(4) are true. Now we describe an operator $\lambda^{-1}\mathcal{T} = \mathfrak{A} + \mathfrak{B} + \mathfrak{C}$ by

$$\lambda^{-1}\mathcal{T}(\mathbf{x},\mathbf{y}) = \mathfrak{A}(\mathbf{x},\mathbf{y}) + \mathfrak{B}(\mathbf{x}) + \mathfrak{C}(\mathbf{y}). \tag{3.25}$$

Thus, $\mathcal{T} : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}$ is a mixed monotone operator and $\mathcal{T}(\mathbf{h}, \mathbf{h}) \in \mathcal{P}_{\mathbf{h}}$. Now we demonstrate that there is $\varphi(t) \in (t, 1]$ so that

$$\mathcal{T}(t\mathbf{x}, t^{-1}\mathbf{y}) \geqslant \varphi(t)\mathcal{T}(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{P}.$$

For each $\mathbf{x}, \mathbf{y} \in \mathcal{P}$, using J_4 there are two constants δ_1, δ_2 that if $\delta = \min\{\delta_1, \delta_2\}$, we demonstrate

$$\mathfrak{A}(\mathbf{x}, \mathbf{y}) + \delta\mathfrak{A}(\mathbf{x}, \mathbf{y}) \geqslant \delta(\mathfrak{B}(\mathbf{x}) + \mathfrak{C}(\mathbf{y})) + \delta\mathfrak{A}(\mathbf{x}, \mathbf{y}). \tag{3.26}$$

It results that

$$\mathfrak{A}(\mathbf{x}, \mathbf{y}) \geqslant \frac{\mathfrak{A}(\mathbf{x}, \mathbf{y}) + \mathfrak{B}(\mathbf{x}) + \mathfrak{C}(\mathbf{y})}{1 + \delta^{-1}} = \frac{\lambda^{-1}\mathcal{T}(\mathbf{x}, \mathbf{y})}{1 + \delta^{-1}}, \forall \mathbf{x}, \mathbf{y} \in \mathcal{P}. \tag{3.27}$$

Also using the relations (3.13), (3.14), (3.15) and (3.27), we have

$$\begin{aligned}
\lambda^{-1}\mathcal{T}(t\mathbf{x}, t^{-1}\mathbf{y}) - t^{\sigma(t)}\lambda^{-1}\mathcal{T}(\mathbf{x}, \mathbf{y}) &= \mathfrak{A}(t\mathbf{x}, t^{-1}\mathbf{y}) + \mathfrak{B}(t\mathbf{x}) + \mathfrak{C}(t^{-1}\mathbf{y}) \\
&\quad - t^{\sigma(t)}\mathfrak{A}(\mathbf{x}, \mathbf{y}) - t^{\sigma(t)}\mathfrak{B}(\mathbf{x}) - t^{\sigma(t)}\mathfrak{C}(\mathbf{y}) \\
&\geqslant (t^{\beta(t)+\alpha(t)} - t^{\sigma(t)})\mathfrak{A}(\mathbf{x}, \mathbf{y}) \\
&\geqslant \frac{t^{\beta(t)+\alpha(t)} - t^{\sigma(t)}}{1 + \delta^{-1}}\lambda^{-1}\mathcal{T}(\mathbf{x}, \mathbf{y}), \\
&\forall \mathbf{x}, \mathbf{y} \in \mathcal{P}, t \in (0, 1).
\end{aligned}$$

Hence, for each $\mathbf{x}, \mathbf{y} \in \mathcal{P}$, $t \in (0, 1)$,

$$\begin{aligned}
\mathcal{T}(t\mathbf{x}, t^{-1}\mathbf{y}) &\geqslant t^{\sigma(t)}\mathcal{T}(\mathbf{x}, \mathbf{y}) + \frac{t^{\beta(t)+\alpha(t)} - t^{\sigma(t)}}{1 + \delta^{-1}}\mathcal{T}(\mathbf{x}, \mathbf{y}) \\
&= (t^{\sigma(t)} + \frac{t^{\beta(t)+\alpha(t)} - t^{\sigma(t)}}{1 + \delta^{-1}})\mathcal{T}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Let

$$\varphi(t) = t^{\sigma(t)} + \frac{t^{\beta(t)+\alpha(t)} - t^{\sigma(t)}}{1 + \delta^{-1}}, \quad t \in (0, 1).$$

Thus, $\varphi(t) \in (t, 1]$ and $\mathcal{T}(t\mathbf{x}, t^{-1}\mathbf{y}) \geqslant \varphi(t)\mathcal{T}(\mathbf{x}, \mathbf{y})$ for each $t \in (0, 1)$ and $\mathbf{x}, \mathbf{y} \in \mathcal{P}$. Hence, the condition (J_2) in the Lemma 3.1 is satisfied. Using Lemma 3.1 inferring:

(I) there are $\mathfrak{u}_0, \mathbf{v}_0 \in \mathcal{P}_{\mathbf{h}}$ and $\mathbf{h} \in (0, 1)$ so that

$$\mathbf{r}\mathbf{v}_0 \leqslant \mathfrak{u}_0 < \mathbf{v}_0, \quad \mathfrak{u}_0 \leqslant \lambda^{-1}\mathcal{T}(\mathfrak{u}_0, \mathbf{v}_0) \leqslant \lambda^{-1}\mathcal{T}(\mathbf{v}_0, \mathfrak{u}_0) \leqslant \mathbf{v}_0;$$

(II) the operator $\lambda^{-1}\mathcal{T}$ has a unique λ-fixed point x_λ^* in $\mathcal{P}_{\mathbf{h}}$;

(III) for each initial values $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{P}_{\mathbf{h}}$ and $\lambda > 0$,

building consecutive the sequences

$$\begin{aligned}
\mathbf{x}_{\mathbf{n}} &= \lambda^{-1}\mathcal{T}(\mathbf{x}_{\mathbf{n}-1}, \mathbf{y}_{\mathbf{n}-1}), \quad \mathbf{n} = 1, 2, ..., \\
\mathbf{y}_{\mathbf{n}} &= \lambda^{-1}\mathcal{T}(\mathbf{y}_{\mathbf{n}-1}, \mathbf{x}_{\mathbf{n}-1}), \quad \mathbf{n} = 1, 2, ...
\end{aligned}$$

We have $\mathbf{x}_{\mathbf{n}} \longrightarrow x_\lambda^*$ and $\mathbf{y}_{\mathbf{n}} \longrightarrow x_\lambda^*$ as $\mathbf{n} \longrightarrow \infty$. that's mean, the results (2)-(4) hold. The prove is finished. □

Theorem 3.2. *Let (J_1') - (J_4') be hold in* (1.1)*:*

(J_1') $\mathfrak{f} : [0,1] \times \Re^+ \times \Re^+ \longrightarrow \Re^+$, $\mathfrak{g} : [0,1] \times \Re^+ \longrightarrow \Re^+$ *and* $\mathbf{k} : [0,1] \times \Re^+ \longrightarrow \Re^+$ *are continuous, with* $\mathfrak{f}(t,0,1) \not\equiv 0$, $\mathbf{k}(t,1) \not\equiv 0$, $\mathfrak{g}(t,0) \not\equiv 0$, $t \in [0,1]$*;*

(J_2') *for each fixed* $t \in [0,1]$ *and* $\mathbf{v} \in [0,\infty)$, $\mathfrak{f}(t,\mathfrak{u},\mathbf{v})$ *is increasing in* $\mathfrak{u} \in [0,\infty)$*; for each fixed* $t \in [0,1]$ *and* $\mathfrak{u} \in [0,\infty)$, $\mathbf{f}(t,\mathfrak{u},\mathbf{v})$ *is decreasing in* $\mathbf{v} \in [0,\infty)$*. For each fixed* $t \in [0,1]$, $\mathfrak{g}(t,\mathfrak{u})$ *is increasing in* $\mathfrak{u} \in [0,\infty)$*, and* $\mathbf{k}(t,\mathbf{v})$ *is decreasing in* $\mathbf{v} \in [0,\infty)$*;*

(J_3') *for each* $\iota \in (0,1)$, $t \in [0,1]$, $\mathfrak{u},\mathbf{v} \in [0,\infty)$,

$$\mathfrak{g}(t,\iota\mathfrak{u}) \geqslant \iota^{\sigma(\iota)}\mathfrak{g}(t,\mathfrak{u}),$$
$$\mathbf{k}(t,\iota^{-1}\mathbf{v}) \geqslant \iota^{\sigma(\iota)}\mathbf{k}(t,\mathbf{v}),$$

where $\sigma(\iota)$ *is a continuous function. There are two different and continuous functions* $0 < \alpha(\iota) < 1$, $0 < \beta(\iota) < 1$, $0 < \beta(\iota) + \alpha(\iota) < 1$ *so that*

$$\mathfrak{f}(t,\iota\mathfrak{u},\iota^{-1}\mathbf{v}) \geqslant \iota^{\beta(\iota)+\alpha(\iota)}\mathfrak{f}(t,\mathfrak{u},\mathbf{v});$$

(J_4') *there are two constants* δ_1, $\delta_2 > 0$ *so that*

$$\delta_1\mathfrak{g}(t,\mathfrak{u}) + \delta_2\mathbf{k}(t,\mathbf{v})) \leqslant \mathfrak{f}(t,\mathfrak{u},\mathbf{v}), \quad \forall t \in [0,1], \quad \mathfrak{u},\mathbf{v} \in [0,\infty).$$

Thus,

(1) *there are* $\mathfrak{u}_0,\mathbf{v}_0 \in \mathcal{P}_{\mathbf{h}}$ *and* $\mathbf{r} \in (0,1)$ *so that* $\mathbf{r}\mathbf{v}_0 \leqslant \mathfrak{u}_0 < \mathbf{v}_0$ *and*

$$\mathfrak{u}_0(t) \leqslant \lambda^{-1}\int_0^1 G(t,r)[\mathfrak{f}(r,\mathfrak{u}_0,\mathbf{v}_0) + \mathfrak{g}(r,\mathfrak{u}_0) + \mathbf{k}(r,\mathbf{v}_0)]dr, \quad t \in [0,1],$$
$$\mathbf{v}_0(t) \geqslant \lambda^{-1}\int_0^1 G(t,r)[\mathfrak{f}(r,\mathbf{v}_0,\mathfrak{u}_0) + \mathfrak{g}(r,\mathbf{v}_0) + \mathbf{k}(r,\mathfrak{u}_0)]dr, \quad t \in [0,1],$$

where $\mathbf{h}(t) = t^{\nu-1}$, $t \in [0,1]$*;*

(2) *the problem (1.1) has a uniquely positive solution* $\mathfrak{u}_\lambda^*$ *in* $\mathcal{P}_{\mathbf{h}}$*;*

(3) *for each* $\mathbf{x}_0,\mathbf{y}_0 \in \mathcal{P}_{\mathbf{h}}$*, building consecutive the sequences*

$$\mathbf{x}_{\mathbf{n}+1} = \lambda^{-1}\int_0^1 G(t,r)[\mathfrak{f}(r,\mathbf{x}_\mathbf{n}(r),\mathbf{y}_\mathbf{n}(r)) + \mathfrak{g}(r,\mathbf{x}_\mathbf{n}(r)) + \mathbf{k}(r,\mathbf{y}_\mathbf{n}(r))]dr,$$
$$\mathbf{y}_{\mathbf{n}+1} = \lambda^{-1}\int_0^1 G(t,r)[\mathfrak{f}(r,\mathbf{y}_\mathbf{n}(r),\mathbf{x}_\mathbf{n}(r)) + \mathfrak{g}(r,\mathbf{y}_\mathbf{n}(r)) + \mathbf{k}(r,\mathbf{x}_\mathbf{n}(r))]dr,$$
$$\mathbf{n} = 0,1,2,...,$$

we have $\|\mathbf{x}_\mathbf{n} - \mathfrak{u}_\lambda^*\| \longrightarrow 0$ *and* $\|\mathbf{y}_\mathbf{n} - \mathfrak{u}_\lambda^*\| \longrightarrow 0$*, as* $\mathbf{n} \longrightarrow \infty$*.*

Proof. From Theorem 2.1, the problem (1.1) has the following solution

$$\mathfrak{u}(t) = \lambda^{-1}\int_0^1 G(t,r)[\mathfrak{f}(r,\mathfrak{u}(r),\mathbf{v}(r)) + \mathfrak{g}(r,\mathfrak{u}(r)) + \mathbf{k}(r,\mathbf{v}(r))]dr,$$

where $G(t,r)$ is given as in (2.3). We prove that $\mathfrak{u}$ is the solution of the problem (1.1) if and only if $\mathfrak{u} = \lambda^{-1}\mathcal{T}(\mathfrak{u},\mathbf{v}) = \mathfrak{A}(\mathfrak{u},\mathbf{v}) + \mathfrak{B}(\mathfrak{u}) + \mathfrak{C}(\mathbf{v})$. We define three operators $\mathfrak{A} : \mathcal{P}\times\mathcal{P} \longrightarrow \mathsf{E}$, $\mathfrak{B},\mathfrak{C} : \mathcal{P} \longrightarrow \mathsf{E}$ as follows:

$$\begin{aligned}
\mathfrak{A}(\mathfrak{u},\mathbf{v})(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathfrak{f}(r,\mathfrak{u}(r),\mathbf{v}(r))dr,\\
\mathfrak{B}(\mathfrak{u})(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathfrak{g}(r,\mathfrak{u}(r))dr,\\
\mathfrak{C}(\mathbf{v})(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathbf{k}(r,\mathbf{v}(r))dr.
\end{aligned}$$

We show that $\mathfrak{A}$ is a mixed monotone operator, $\mathfrak{B}$ is an increasing σ-concave operator, and $\mathfrak{C}$ is a decreasing σ-concave operator. For $\mathfrak{u}_i,\mathbf{v}_i \in \mathcal{P}$, $i = 1,2$ with $\mathfrak{u}_1 \geqslant \mathfrak{u}_2$, $\mathbf{v}_1 \leqslant \mathbf{v}_2$, we know that $\mathfrak{u}_1(t) \geqslant \mathfrak{u}_2(t)$, $\mathbf{v}_1(t) \leqslant \mathbf{v}_2(t)$, $t \in [0,1]$ and by (J_2') we have

$$\begin{aligned}
\mathfrak{A}(\mathfrak{u}_1,\mathbf{v}_1)(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathfrak{f}(r,\mathfrak{u}_1(r),\mathbf{v}_1(r))dr\\
&\geqslant \lambda^{-1}\int_0^1 G(t,r)\mathfrak{f}(r,\mathfrak{u}_2(r),\mathbf{v}_2(r))dr\\
&= \mathfrak{A}(\mathfrak{u}_2,\mathbf{v}_2)(t).
\end{aligned}$$

That's mean, $\mathfrak{A}(\mathfrak{u}_1,\mathbf{v}_1)(t) \geqslant \mathfrak{A}(\mathfrak{u}_2,\mathbf{v}_2)(t)$. Now we show that the relation (3.13) is true for $\iota \in (0,1)$, $0 < \alpha(\iota) < 1$, $0 < \beta(\iota) < 1$, $0 < \beta(\iota)+\alpha(\iota) < 1$ and $\mathfrak{u},\mathbf{v} \in [0,\infty)$. Where $\alpha(\iota)$, $\beta(\iota)$ are different and continuous functions. Using J_3', we have

$$\begin{aligned}
\mathfrak{A}(\iota\mathfrak{u},\iota^{-1}\mathbf{v})(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathfrak{f}(r,\iota\mathfrak{u}(r),\iota^{-1}\mathbf{v}(r))dr\\
&\geqslant \iota^{\beta(\iota)+\alpha(\iota)}\lambda^{-1}\int_0^1 G(t,r)\mathfrak{f}(r,\mathfrak{u}(r),\mathbf{v}(r))dr\\
&= \iota^{\beta(\iota)+\alpha(\iota)}\mathfrak{A}(\mathfrak{u},\mathbf{v})(t).
\end{aligned}$$

That's mean, $\mathfrak{A}(\iota\mathfrak{u},\iota^{-1}\mathbf{v}) \geqslant \iota^{\beta(\iota)+\alpha(\iota)}\mathfrak{A}(\mathfrak{u},\mathbf{v})$. Which holds at (3.13). Using J_2', we show that $\mathfrak{B}$ is an increasing operator, and $\mathfrak{C}$ is decreasing operator.

$$\begin{aligned}
\mathfrak{B}(\mathfrak{u}_1)(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathfrak{g}(r,\mathfrak{u}_1(r))dr\\
&\geqslant \lambda^{-1}\int_0^1 G(t,r)\mathfrak{g}(r,\mathfrak{u}_2(r))dr\\
&= \mathfrak{B}(\mathfrak{u}_2)(t).
\end{aligned}$$

That's mean, $\mathfrak{B}(\mathfrak{u}_1)(t) \geqslant \mathfrak{B}(\mathfrak{u}_2)(t)$.

$$\begin{aligned}
\mathfrak{C}(\mathbf{v}_1)(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathbf{k}(r,\mathbf{v}_1(r))dr\\
&\geqslant \lambda^{-1}\int_0^1 G(t,r)\mathbf{k}(r,\mathbf{v}_2(r))dr\\
&= \mathfrak{C}(\mathbf{v}_2)(t).
\end{aligned}$$

That's mean, $\mathfrak{C}(\mathbf{v}_1)(t) \geqslant \mathfrak{C}(\mathbf{v}_2)(t)$. Using J_3' for $\iota \in (0,1)$, $t \in [0,1]$, $\mathfrak{u}, \mathbf{v} \in [0,\infty)$ and continuous function $0 < \sigma(\iota) < 1$, we have

$$\begin{aligned}
\mathfrak{B}(\iota\mathfrak{u})(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathfrak{g}(r,\iota\mathfrak{u}(r))dr \\
&\geqslant \iota^{\sigma(\iota)}\lambda^{-1}\int_0^1 G(t,r)\mathfrak{g}(r,\mathfrak{u}(r))dr \\
&= \iota^{\sigma(\iota)}\mathfrak{B}(\mathfrak{u})(t). \\
\mathfrak{C}(\iota^{-1}\mathbf{v})(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathbf{k}(r,\iota^{-1}\mathbf{v}(r))dr \\
&\geqslant \iota^{\sigma(\iota)}\lambda^{-1}\int_0^1 G(t,r)\mathbf{k}(r,\mathbf{v}(r))dr \\
&= \iota^{\sigma(\iota)}\mathfrak{C}(\mathbf{v})(t).
\end{aligned}$$

That's mean, $\mathfrak{B}(\iota\mathfrak{u})(t) \geqslant \iota^{\sigma(\iota)}\mathfrak{B}(\mathfrak{u})(t)$ and $\mathfrak{C}(\iota^{-1}\mathbf{v})(t) \geqslant \iota^{\sigma(\iota)}\mathfrak{C}(\mathbf{v})(t)$, which satisfy respectively at (3.14) and (3.15). Therefore, $\mathfrak{B}$ is a increasing $\sigma(\iota)$-concave operator, and $\mathfrak{C}$ is a decreasing $\sigma(\iota)$-concave operator. Now we show that $\mathfrak{A}(\mathbf{h},\mathbf{h}) \in \mathcal{P}_{\mathbf{h}}$, $\mathfrak{B}(\mathbf{h}) \in \mathcal{P}_{\mathbf{h}}$ and $\mathfrak{C}(\mathbf{h}) \in \mathcal{P}_{\mathbf{h}}$. By using J_1', J_2' and Lemma 2.2 for each $t \in [0,1]$, we know that

$$\begin{aligned}
\mathfrak{A}(\mathbf{h},\mathbf{h})(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathfrak{f}(r,\mathbf{h}(r),\mathbf{h}(r))dr \\
&\leqslant \frac{\lambda^{-1}}{\Gamma(\nu)}\mathbf{h}(t)\int_0^1 (1-r)^{\nu-\alpha-1}\mathfrak{f}(r,1,0), \\
\mathfrak{A}(\mathbf{h},\mathbf{h})(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathfrak{f}(r,\mathbf{h}(r),\mathbf{h}(r))dr \\
&\geqslant \frac{\lambda^{-1}}{\Gamma(\nu)}\mathbf{h}(t)\int_0^1 [1-(1-r)^{\alpha}](1-r)^{\nu-\alpha-1}\mathfrak{f}(r,0,1).
\end{aligned}$$

From (J_2'), we have

$$\mathfrak{f}(r,1,0) \geqslant \mathfrak{f}(r,0,1) \geqslant 0.$$

Because $\mathfrak{f}(t,0,1) \not\equiv 0$, we give

$$\int_0^1 \mathfrak{f}(r,1,0)dr \geqslant \int_0^1 \mathfrak{f}(r,0,1)dr > 0,$$

and in consequence

$$\begin{aligned}
L_1 &:= \frac{\lambda^{-1}}{\Gamma(\nu)}\int_0^1 (1-r)^{\nu-\alpha-1}\mathfrak{f}(r,1,0)dr, \\
L_2 &:= \frac{\lambda^{-1}}{\Gamma(\nu)}\int_0^1 [1-(1-r)^{\alpha}](1-r)^{\nu-\alpha-1}\mathfrak{f}(r,0,1)dr.
\end{aligned}$$

So $L_2\mathbf{h}(t) \leqslant \mathfrak{A}(\mathbf{h},\mathbf{h})(t) \leqslant L_1\mathbf{h}(t)$, $t \in [0,1]$ and hence, we have $\mathfrak{A}(\mathbf{h},\mathbf{h}) \in \mathcal{P}_{\mathbf{h}}$. Similarly, since $\mathfrak{g}(t,0) \not\equiv 0$ and $\mathbf{k}(t,1) \not\equiv 0$, we have

$$\frac{\lambda^{-1}}{\Gamma(\nu)}\mathbf{h}(t)\int_0^1 [1-(1-r)^{\alpha}](1-r)^{\nu-\alpha-1}\mathfrak{g}(r,0)dr$$

$$\leqslant \mathfrak{B}(\mathbf{h})(t) \leqslant \frac{\lambda^{-1}}{\Gamma(\nu)}\mathbf{h}(t)\int_0^1 (1-r)^{\nu-\alpha-1}\mathfrak{g}(r,1)dr,$$

$$\frac{\lambda^{-1}}{\Gamma(\nu)}\mathbf{h}(t)\int_0^1 [1-(1-r)^{\alpha}](1-r)^{\nu-\alpha-1}\mathbf{k}(r,1)dr$$

$$\leqslant \mathfrak{C}(\mathbf{h})(t) \leqslant \frac{\lambda^{-1}}{\Gamma(\nu)}\mathbf{h}(t)\int_0^1 (1-r)^{\nu-\alpha-1}\mathbf{k}(r,0)dr.$$

That's mean $\mathfrak{B}(\mathbf{h}) \in \mathcal{P}_{\mathbf{h}}$ and $\mathfrak{C}(\mathbf{h}) \in \mathcal{P}_{\mathbf{h}}$. For $\mathfrak{u}, \mathbf{v} \in \mathcal{P}$ and each $t \in [0,1]$ by (J_4'), there are two constants δ_1, δ_2 that if $\delta = \min\{\delta_1, \delta_2\}$,

$$\begin{aligned}
\mathfrak{A}(\mathfrak{u},\mathbf{v})(t) &= \lambda^{-1}\int_0^1 G(t,r)\mathfrak{f}(r,\mathfrak{u}(r),\mathbf{v}(r))dr \\
&\geqslant \delta_1\lambda^{-1}\int_0^1 G(t,r)\mathfrak{g}(r,\mathfrak{u}(r)) + \delta_2\lambda^{-1}\int_0^1 G(t,r)\mathbf{k}(r,\mathbf{v}(r))]dr \\
&= \delta_1\mathfrak{B}(\mathfrak{u})(t) + \delta_2\mathfrak{C}(\mathbf{v})(t) \\
&\geqslant \delta[\mathfrak{B}(\mathfrak{u})(t) + \mathfrak{C}(\mathbf{v})(t)].
\end{aligned}$$

Thus, we take $\mathfrak{A}(\mathfrak{u},\mathbf{v}) \geqslant \delta(\mathfrak{B}(\mathfrak{u}) + \mathfrak{C}(\mathbf{v}))$, for $\mathfrak{u}, \mathbf{v} \in \mathcal{P}$.

Finally, by means Theorem 3.1 we calculate that exist $\mathfrak{u}_0, \mathbf{v}_0 \in \mathcal{P}_{\mathbf{h}}$, $\mathbf{r} \in (0,1)$ so that

$$\begin{aligned}
&\mathbf{r}\mathbf{v}_0 \leqslant \mathfrak{u}_0 < \mathbf{v}_0, \\
&\mathfrak{u}_0 \leqslant \lambda^{-1}\mathcal{T}(\mathfrak{u}_0,\mathbf{v}_0) = \mathfrak{A}(\mathfrak{u}_0,\mathbf{v}_0) + \mathfrak{B}(\mathfrak{u}_0) + \mathfrak{C}(\mathbf{v}_0) \\
&\quad \leqslant \lambda^{-1}\mathcal{T}(\mathbf{v}_0,\mathfrak{u}_0) = \mathfrak{A}(\mathbf{v}_0,\mathfrak{u}_0) + \mathfrak{B}(\mathbf{v}_0) + \mathfrak{C}(\mathfrak{u}_0) \leqslant \mathbf{v}_0,
\end{aligned}$$

equation operator $\lambda^{-1}\mathcal{T} = \mathfrak{A}(\mathfrak{u},\mathbf{v}) + \mathfrak{B}(\mathfrak{u}) + \mathfrak{C}(\mathbf{v}) = \mathfrak{u}$ has an uniquely solution $\mathfrak{u}_\lambda^*$ in $\mathcal{P}_{\mathbf{h}}$. For each initial values $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{P}_{\mathbf{h}}$, building consecutive sequences

$$\begin{aligned}
\mathbf{x}_{\mathbf{n}} &= \lambda^{-1}\mathcal{T}(\mathbf{x}_{\mathbf{n}-1},\mathbf{y}_{\mathbf{n}-1}) = \mathfrak{A}(\mathbf{x}_{\mathbf{n-1}},\mathbf{y}_{\mathbf{n}-1}) + \mathfrak{B}(\mathbf{x}_{\mathbf{n}-1}) + \mathfrak{C}(\mathbf{y}_{\mathbf{n}-1}), \quad \mathbf{n} = 1,2,\cdots, \\
\mathbf{y}_{\mathbf{n}} &= \lambda^{-1}\mathcal{T}(\mathbf{y}_{\mathbf{n}-1},\mathbf{x}_{\mathbf{n}-1}) = \mathfrak{A}(\mathbf{y}_{\mathbf{n-1}},\mathbf{x}_{\mathbf{n}-1}) + \mathfrak{B}(\mathbf{y}_{\mathbf{n}-1}) + \mathfrak{C}(\mathbf{x}_{\mathbf{n}-1}), \quad \mathbf{n} = 1,2,\cdots,
\end{aligned}$$

we have $\mathbf{x}_{\mathbf{n}} \longrightarrow \mathfrak{u}_\lambda^*$ and $\mathbf{y}_{\mathbf{n}} \longrightarrow \mathfrak{u}_\lambda^*$ as $\mathbf{n} \longrightarrow \infty$. That's mean

$$\begin{aligned}
\mathfrak{u}_0(t) &\leqslant \lambda^{-1}\int_0^1 G(t,r)[\mathfrak{f}(r,\mathfrak{u}_0(r),\mathbf{v}_0(r)) + \mathfrak{g}(r,\mathfrak{u}_0(r)) + \mathbf{k}(r,\mathbf{v}_0(r))]dr, \\
\mathbf{v}_0(t) &\geqslant \lambda^{-1}\int_0^1 G(t,r)[\mathfrak{f}(r,\mathbf{v}_0(r),\mathfrak{u}_0(r)) + \mathfrak{g}(r,\mathbf{v}_0(r)) + \mathbf{k}(r,\mathfrak{u}_0(r))]dr.
\end{aligned}$$

Therefore, BVP (1.1) has a uniquely solution $\mathfrak{u}_\lambda^*$ in $\mathcal{P}_{\mathbf{h}}$. For $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{P}_{\mathbf{h}}$ building consecutive sequences:

$$\begin{aligned}
&\mathbf{x}_{\mathbf{n}+1}(t) = \lambda^{-1}\int_0^1 G(t,r)[\mathfrak{f}(r,\mathbf{x}_{\mathbf{n}}(r),\mathbf{y}_{\mathbf{n}}(r)) + \mathfrak{g}(r,\mathbf{x}_{\mathbf{n}}(r)) + \mathbf{k}(r,\mathbf{y}_{\mathbf{n}}(r))]dr, \\
&\mathbf{y}_{\mathbf{n}+1}(t) = \lambda^{-1}\int_0^1 G(t,r)[\mathfrak{f}(r,\mathbf{y}_{\mathbf{n}}(r),\mathbf{x}_{\mathbf{n}}(r)) + \mathfrak{g}(r,\mathbf{y}_{\mathbf{n}}(r)) + \mathbf{k}(r,\mathbf{x}_{\mathbf{n}}(r))]dr, \mathbf{n} = 0,1,2,\cdots.
\end{aligned}$$

Thus, $\|\mathbf{x}_{\mathbf{n}} - \mathfrak{u}_\lambda^*\| \longrightarrow 0$ and $\|\mathbf{y}_{\mathbf{n}} - \mathfrak{u}_\lambda^*\| \longrightarrow 0$ as $\mathbf{n} \longrightarrow \infty$. So, the conclusions of Theorem 3.2 follow from Theorem 3.1, and the Proof of Theorem 3.2 is finished. □

4. Example

In this section, we present the following example for the correctness of our theory. We check BVP:

$$-D_{0^+}^{\frac{7}{2}}\mathfrak{u}(t)=\frac{1}{2}(\mathfrak{u}^{\frac{1}{10}\sin(t)}(t)+(1-\frac{\sqrt{\pi}}{t^{\frac{1}{2}}}D_{0^+}^{\frac{3}{2}}\mathfrak{u}(t))^{\frac{-1}{10}\cos(t)} + \mathfrak{u}^{\frac{1}{10}\cos(t)}(t)+\mathbf{v}^{\frac{-1}{10}\cos(t)}(t)+2t^2+2t^3), \tag{4.1}$$

with conditions boundary

$$\mathfrak{u}(0)=\mathfrak{u}'(0)=\mathfrak{u}''(0)=0, \quad [D_{0^+}^{\frac{3}{2}}\mathfrak{u}(t)]_{t=1}=0.$$

In this example, we have $\nu=\frac{7}{2}$. We assume

$$\begin{aligned}&1-\frac{\sqrt{\pi}}{t^{\frac{1}{2}}}D_{0^+}^{\frac{3}{2}}\mathfrak{u}(t)=\mathbf{v}(t),\\ &\beta(t)=\frac{1}{10}\sin(t), \quad \alpha(t)=\sigma(t)=\frac{-1}{10}\cos(t)\\ &\lambda=2\end{aligned}$$

We have

$$\begin{aligned}&\mathfrak{f}(t,\mathfrak{u},\mathbf{v})=\frac{1}{2}\mathfrak{u}^{\frac{1}{10}\sin(t)}(t)+\frac{1}{2}(1-\frac{\sqrt{\pi}}{t^{\frac{1}{2}}}D_{0^+}^{\frac{3}{2}}\mathfrak{u}(t))^{\frac{-1}{10}\cos(t)}+t^2,\\ &\mathfrak{g}(t,\mathfrak{u})=\frac{1}{2}\mathfrak{u}^{\frac{1}{10}\cos(t)}(t)+\frac{1}{2}t^3,\\ &\mathbf{k}(t,\mathbf{v})=\frac{1}{2}\mathbf{v}^{\frac{-1}{10}\cos(t)}(t)+\frac{1}{2}t^3,\\ &\rho=\frac{3}{2}.\end{aligned}$$

Obviously, the functions $\mathfrak{f}:[0,1]\times\Re^+\times\Re^+\longrightarrow\Re^+$, $\mathfrak{g},\mathbf{k}:[0,1]\times\Re^+\longrightarrow\Re^+$ are continuous with $\mathfrak{f}(t,0,1)\not\equiv 0$, $\mathfrak{g}(t,0)\not\equiv 0$ and $\mathbf{k}(t,1)\not\equiv 0$. $\mathfrak{f}(t,\mathfrak{u},\mathbf{v})$ for fixed $t\in[0,1]$ and $\mathbf{v}\in[0,\infty)$ are increasing in $\mathfrak{u}\in[0,\infty)$; $\mathfrak{f}(t,\mathfrak{u},\mathbf{v})$ for fixed $t\in[0,1]$ and $\mathfrak{u}\in[0,\infty)$ are decreasing in $\mathbf{v}\in[0,\infty)$, and $\mathfrak{g}(t,\mathfrak{u})$ is increasing $\sigma(t)$-concave in $\mathfrak{u}\in[0,\infty)$ for fixed $t\in[0,1]$, and $\mathbf{k}(t,\mathbf{v})$ is decreasing in $\mathbf{v}\in[0,\infty)$ for fixed $t\in[0,1]$. For each $\iota\in(0,1)$, $t\in[0,1]$, $\mathfrak{u},\mathbf{v}\geqslant 0$, we have

$$\begin{aligned}\mathfrak{f}(t,\iota\mathfrak{u},\iota^{-1}\mathbf{v})&=\frac{1}{2}\iota^{\frac{1}{10}\sin(t)}\mathfrak{u}^{\frac{1}{10}\sin(t)}(t)+\frac{1}{2}\iota^{\frac{1}{10}\cos(t)}\mathbf{v}^{\frac{-1}{10}\cos(t)}(t)+t^2\\ &\geqslant(\iota^{\frac{1}{10}\sin(t)+\frac{1}{10}\cos(t)})(\frac{1}{2}\mathfrak{u}^{\frac{1}{10}\sin(t)}(t)+\frac{1}{2}\mathbf{v}^{\frac{-1}{10}\cos(t)}(t)+t^2)\\ &=\iota^{\frac{1}{10}\sin(t)+\frac{1}{10}\cos(t)}\mathfrak{f}(t,\mathfrak{u},\mathbf{v}).\end{aligned}$$

For each $\iota\in(0,1)$, $t\in[0,1]$, and $\mathfrak{u},\mathbf{v}\geqslant 0$, we have

$$\begin{aligned}\mathfrak{g}(t,\iota\mathfrak{u})&=\frac{1}{2}\iota^{\frac{1}{10}\cos(t)}\mathfrak{u}^{\frac{1}{10}\cos(t)}(t)+\frac{1}{2}t^3\\ &\geqslant\iota^{\frac{1}{10}\cos(t)}(\frac{1}{2}\mathfrak{u}^{\frac{1}{10}\cos(t)}(t)+\frac{1}{2}t^3)\end{aligned}$$

$$
\begin{aligned}
&= \iota^{\frac{1}{10}\cos(t)}\mathfrak{g}(t,\mathfrak{u}),\\
\mathbf{k}(t,\iota^{-1}\mathbf{v}) &= \iota^{\frac{1}{10}\cos(t)}\mathbf{v}^{\frac{-1}{10}\cos(t)}(t)+\frac{1}{2}t^3\\
&\geqslant \iota^{\frac{1}{10}\cos(t)}(\mathbf{v}^{\frac{-1}{10}\cos(t)}(t)+\frac{1}{2}t^3)\\
&= \iota^{\frac{1}{10}\cos(t)}\mathbf{k}(t,\mathbf{v}).
\end{aligned}
$$

For $\mathfrak{u}, \mathbf{v} \geqslant 0$, we assume that $\delta_1 = \frac{1}{3}, \delta_2 = \frac{1}{6}$, so that $\delta = min\{\frac{1}{3}, \frac{1}{6}\} = \frac{1}{6} \in (0,1]$,

$$
\begin{aligned}
\mathfrak{f}(t,\mathfrak{u},\mathbf{v}) &= \frac{1}{2}\mathfrak{u}^{\frac{1}{10}sin(t)}(t)+\frac{1}{2}\mathbf{v}^{\frac{-1}{10}\cos(t)}(t)+t^2\\
&\geqslant [\frac{1}{2}\mathfrak{u}^{\frac{1}{10}\cos(t)}(t)+\frac{1}{2}t^3+\frac{1}{2}\mathbf{v}^{\frac{-1}{10}\cos(t)}(t)+\frac{1}{2}t^3]\\
&\geqslant \frac{1}{3}(\frac{1}{2}\mathfrak{u}^{\frac{1}{10}\cos(t)}(t)+\frac{1}{2}t^3)+\frac{1}{6}(\mathbf{v}^{\frac{-1}{10}\cos(t)}(t)+\frac{1}{2}t^3)\\
&\geqslant \frac{1}{6}(\frac{1}{2}\mathfrak{u}^{\frac{1}{10}\cos(t)}(t)+\frac{1}{2}\mathbf{v}^{\frac{-1}{10}\cos(t)}(t)+t^3)\\
&= \frac{1}{6}[\mathfrak{g}(t,\mathfrak{u})+\mathbf{k}(t,\mathbf{v})].
\end{aligned}
$$

Therefor, all the conditions of Theorem 3.2 are true. Hence, problem (4.1) has a uniquely positive solution in $\mathcal{P}_{\mathbf{h}}$, where $\mathbf{h}(t) = t^{\nu-1} = t^{\frac{5}{2}}$, $t \in [0,1]$.

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