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LIE SYMMETRY, EXACT SOLUTIONS AND CONSERVATION LAWS OF SOME FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract In this paper, Lie symmetry analysis method is applied to space-time fractional reaction-diffusion equations and diffusion-convection Boussinesq equations. The Lie symmetries for the governing equations are obtained and used to get group generators for reducing the space-time fractional partial differential equations(FPDEs) with Riemann-Liouville fractional derivative to space-time fractional ordinary differential equations(FODEs) with Erdélyi-Kober fractional derivative. Then the Laplace transformation and the power series methods are applied to derive explicit solutions for the reduced equations. Moreover, the conservation theorems and the generalization of the Noether operators are developed to acquire the conservation laws for the equations. Some figures for the obtained explicit solutions are also presented.

Keywords Lie symmetry analysis, space-time fractional differential equation, Riemann-Liouville fractional derivative, Erdélyi-Kober fractional derivative; exact solution, conservation law.

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1. Introduction

As we all know, fractional differential equation has been successfully applied in many aspects of science and technology, recently. At the same time, many related masterpieces emerged, such as S. Samko et al. [16], I. Podlubny [15], R. Hilfer [7], A. Kilbas et al. [13], etc. Some analytic techniques have been developed to deal with fractional differential equations (FDEs). Among them, the Lie symmetry analysis method is an effective technique to derive exact solutions of FDEs [2–5, 8, 10, 14, 21–25] and the systems of FDEs [9, 17–20].

In this paper, we extended the Lie symmetry analysis to the following space-time FPDEs:

$$D_t^{\alpha} u = p(u) D_x^{2\beta} u + q(u), \tag{1.1}$$

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$$D_t^\alpha u = kuD_x^{2\beta}u + l(uu_x)_x, \tag{1.2}$$

where $u = u(t,x)$, $0 < \alpha, \beta \leq 1$, $p(u)$ and $q(u)$ are arbitrary functions, k and l are constants. The corresponding integer-order forms of both the above equations are studied by invariant subspace method in [6].

Eq.(1.1) is a class of quasi-linear heat or reaction-diffusion equations, which are widely used in many areas of mechanics, combustion theory and biology. There are some special cases in Eq.(1.1), such as $p(u) = u^n$ and $q(u) = u^m (m \neq 1)$, i.e.

$$D_t^\alpha u = u^n D_x^{2\beta}u + u^m. \tag{1.3}$$

Eq.(1.2), known as the diffusion-convection Boussinesq equation, occurs in the various fields of petroleum technology and ground water hydrology. The basic nonlinear diffusion operator in such parabolic equation was already derived by J. Boussinesq [1], who, in 1904, studied non-stationary flows of soil water under the presence of free surface, and derived the quadratic porous medium equation(PME)

$$u_t = l(uu_x)_x, \tag{1.4}$$

where l, a positive constant, stands the ratio of the filtration coefficient to the porosity of soil. The function $u = u(t,x)$ is the pressure of the ground water. That is, Eq.(1.2) is time fractional PME with a nonlinear space fractional convection term. If $\beta = 1$, it becomes time fractional diffusion-convection Boussinesq equation as follows:

$$D_t^\alpha u = l(uu_x)_x + kuu_{xx}. \tag{1.5}$$

where D_t^α is the Riemann-Liouville fractional derivative defined by [7, 13, 15, 16]

$$ {}_aD_t^\alpha f(t,x) = D_t^n \; {}_aJ_t^{n-\alpha} f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s,x)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, & n-1 < \alpha < n, n \in \mathbb{N} \\ D_t^n f(t,x), & \alpha = n \in \mathbb{N} \end{cases}$$

for $t > a$. We denote the operator ${}_0D_t^\alpha$ as D_t^α throughout this paper.

The rest of this paper is organized as follows. In Section 2, we study the Lie symmetry analysis of space-time FPDEs. In Sections 3 and 4, we respectively show the applications of Lie symmetry analysis method to Eqs.(1.1)-(1.2). Section 5 gives the conservation laws for Eqs.(1.1)-(1.2). The conclusion is given in the last section.

2. Lie symmetry analysis of space-time FPDE

Consider space-time FPDE as follows:

$$F(t, x, u, D_t^\alpha u, D_x^{2\beta}u, u_x, u_{xx}, \cdots) = 0, \tag{2.1}$$

where $u = u(t,x)$, $0 < \alpha, \beta \leq 1$.

We assume that the FPDE (2.1) is invariant under the one-parameter(ϵ) Lie group of continuous point transformations, i.e.

$$\begin{aligned} t^* &= t + \epsilon\tau(t,x,u) + o(\epsilon), \\ x^* &= x + \epsilon\xi(t,x,u) + o(\epsilon), \\ u^* &= u + \epsilon\eta(t,x,u) + o(\epsilon), \end{aligned}$$

$$
\begin{aligned}
&D^{\alpha}_{t^*}u^* = D^{\alpha}_{t}u + \epsilon\eta^{\alpha,t} + o(\epsilon),\\
&D^{2\beta}_{x^*}u^* = D^{2\beta}_{x}u + \epsilon\eta^{2\beta,x} + o(\epsilon),\\
&D_{x^*}u^* = D_{x}u + \epsilon\eta^{x} + o(\epsilon),\\
&D^{2}_{x^*}u^* = D^{2}_{x}u + \epsilon\eta^{xx} + o(\epsilon), \qquad \cdots\cdots
\end{aligned}
\tag{2.2}
$$

where τ, ξ and η are known as infinitesimals, $\eta^{\alpha,t}$, $\eta^{2\beta,x}$, η^{x} and η^{xx} are the corresponding prolongations of orders α, 2β, 1 and 2, respectively.

According to Lie group theory, the group generator X of the point transformations (2.2) is expressed as

$$
X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2.3}
$$

So the prolongation of the above group generator X has the form

$$
prX = X + \eta^{\alpha,t}\frac{\partial}{\partial u^{\alpha}_{t}} + \eta^{2\beta,x}\frac{\partial}{\partial u^{2\beta}_{x}} + \eta^{x}\frac{\partial}{\partial u_{x}} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \cdots, \tag{2.4}
$$

where

$$
\eta^{\alpha,t} = D^{\alpha}_{t}(\eta) + \xi D^{\alpha}_{t}(u_x) - D^{\alpha}_{t}(\xi u_x) + D^{\alpha}_{t}(uD_t(\tau)) - D^{\alpha+1}_{t}(\tau u) + \tau D^{\alpha+1}_{t}(u), \tag{2.5}
$$

$$
\eta^{2\beta,x} = D^{2\beta}_{x}(\eta) + \tau D^{2\beta}_{x}(u_t) - D^{2\beta}_{x}(\tau u_t) + D^{2\beta}_{x}(uD_x(\xi)) - D^{2\beta+1}_{t}(\xi u) + \xi D^{2\beta+1}_{x}(u), \tag{2.6}
$$

$$
\eta^{x} = D_x(\eta) - u_tD_x(\tau) - u_xD_x(\xi) = \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \tag{2.7}
$$

$$
\begin{aligned}
\eta^{xx} =& D_x(\eta^{x}) - u_{xt}D_t(\tau) - u_{xx}D_x(\xi)\\
=& \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{xu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_xu_t - \xi_{uu}u_x^3\\
& - \tau_{uu}u_x^2u_t + (\eta_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_t - 2\tau_u u_{xt}u_x,
\end{aligned}
\tag{2.8}
$$

$\cdots\cdots$

and D_t, D_x are the total derivative with respect to t, x respectively. With the generalized Leibnitz rule and the generalized chain rule [4], we can get the expansions of $\eta^{\alpha,t}$ and $\eta^{2\beta,x}$ as follows:

$$
\begin{aligned}
\eta^{\alpha,t} =& \frac{\partial^{\alpha}\eta}{\partial t^{\alpha}} + (\eta_u - \alpha D_t(\tau))\frac{\partial^{\alpha}u}{\partial t^{\alpha}} - u\frac{\partial^{\alpha}\eta_u}{\partial t^{\alpha}} - \sum_{n=1}^{\infty}\binom{\alpha}{n}D^{n}_{t}(\xi)D^{\alpha-n}_{t}(u_x)\\
& + \sum_{n=1}^{\infty}\Big[\binom{\alpha}{n}\frac{\partial^{n}\eta_u}{\partial t^{n}} - \binom{\alpha}{n+1}D^{n+1}_{t}(\tau)\Big]D^{\alpha-n}_{t}(u) + \mu_1
\end{aligned}
\tag{2.9}
$$

with

$$
\mu_1 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-u)^{r}}{k!\Gamma(n+1-\alpha)}\frac{\partial^{m}u^{k-r}}{\partial t^{m}}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^{k}},
$$

and

$$
\begin{aligned}
\eta^{2\beta,x} =& \frac{\partial^{2\beta}\eta}{\partial x^{2\beta}} + (\eta_u - 2\beta D_x(\xi))\frac{\partial^{2\beta}u}{\partial x^{2\beta}} - u\frac{\partial^{2\beta}\eta_u}{\partial x^{2\beta}} - \sum_{n=1}^{\infty}\binom{2\beta}{n}D^{n}_{x}(\tau)D^{2\beta-n}_{x}(u_t)\\
& + \sum_{n=1}^{\infty}\Big[\binom{2\beta}{n}\frac{\partial^{n}\eta_u}{\partial x^{n}} - \binom{2\beta}{n+1}D^{n+1}_{x}(\xi)\Big]D^{2\beta-n}_{x}(u) + \mu_2
\end{aligned}
\tag{2.10}
$$

with

$$\mu_2 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{2\beta}{n}\binom{n}{m}\binom{k}{r}\frac{x^{n-2\beta}(-u)^r}{k!\Gamma(n+1-2\beta)}\frac{\partial^m u^{k-r}}{\partial x^m}\frac{\partial^{n-m+k}\eta}{\partial x^{n-m}\partial u^k}.$$

Remark 2.1. The infinitesimal transformations (2.2) should conserve the structure of the Riemann-Liouville fractional derivative operator, of which, the lower limit in the integral is fixed. Therefore, the manifold $t = 0$ should be invariant with respect to such transformations. The invariance condition arrives at

$$\tau(t,x,u)|_{t=0} = 0. \tag{2.11}$$

Remark 2.2. The derivatives $\partial^k\eta/\partial u^k, k \geq 2$ exist in the expression of μ_1 and μ_2. Therefore, if the infinitesimal η be linear with respect to the variable u then $\mu_1 = \mu_2 = 0$, that is,

$$\frac{\partial^2\eta}{\partial u^2} = 0. \tag{2.12}$$

The Lie symmetry transformations (2.2) are admitted by the FPDE (2.1), if the following invariance criterion holds:

$$prX\big(F(t,x,u,D_t^\alpha u, D_x^{2\beta}u, u_x, u_{xx},\cdots)\big)|_{(2.1)} = 0, \tag{2.13}$$

which is known as the determining equation.

Putting $\eta^{\alpha,t}$, $\eta^{2\beta,x}$, η^x, η^{xx}, $\cdots$ into (2.13) and letting coefficients of various derivatives of u to be zero, we can obtain the over-determined system of differential equations about $\tau(t,x,u)$, $\xi(t,x,u)$ and $\eta(t,x,u)$. The solutions τ, ξ and η constitute the group generator X of the FPDE (2.1). Solving the characteristic equation associated with X,

$$\frac{\mathrm{d}t}{\tau(t,x,u)} = \frac{\mathrm{d}x}{\xi(t,x,u)} = \frac{\mathrm{d}u}{\eta(t,x,u)}, \tag{2.14}$$

we can get some similarity variables, which reduce the FPDE (2.1) to some FODEs. Then the solutions of the FODEs can construct the group-invariant solutions of the FPDE (2.1). Next, as examples, we will give the Lie symmetry analysis of Eqs.(1.1) and (1.2).

3. Application to Eq.(1.1)

The determining equation of Eq.(1.1) is

$$prX\big(D_t^\alpha u - p(u)D_x^{2\beta}u - q(u)\big)|_{(1.1)} = 0, \tag{3.1}$$

which can be rewritten as

$$\big(\eta^{\alpha,t} - p(u)\eta^{2\beta,x} - \eta p'(u)D_x^{2\beta}u - \eta q'(u)\big)|_{(1.1)} = 0. \tag{3.2}$$

Putting $\eta^{\alpha,t}$ and $\eta^{2\beta,x}$ into (3.2) and letting coefficients of various derivatives of u, such as $D_x^{2\beta}u$, $D_t^{\alpha-n}(u)$, $D_t^{\alpha-n}(u_x)$, $D_x^{2\beta-n}(u)$ and $D_x^{2\beta-n}(u_t)(n = 1,2,\cdots)$ to be zero, we can obtain the over-determined system of differential equations as follows:

$$\tau_x = \tau_u = \xi_t = \xi_u = 0, \tag{3.3}$$

$$\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad n \in N, \tag{3.4}$$

$$\binom{2\beta}{n}\frac{\partial^n \eta_u}{\partial x^n} - \binom{2\beta}{n+1} D_x^{n+1}(\xi) = 0, \quad n \in N, \tag{3.5}$$

$$(2\beta\xi_x - \alpha\tau_t)p(u) - \eta p'(u) = 0, \tag{3.6}$$

$$\frac{\partial^\alpha \eta}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - p(u)\frac{\partial^{2\beta}\eta}{\partial x^{2\beta}} + up'(u)\frac{\partial^{2\beta}\eta_u}{\partial x^{2\beta}} + (\eta_u - \alpha\tau_t)q(u) - \eta q'(u) = 0. \tag{3.7}$$

From (3.4) and (3.5), we get

$$\tau_{tt} = \xi_{xx} = \eta_{ut} = \eta_{ux} = 0. \tag{3.8}$$

Then $\tau = c_1 t + c_2$, $\xi = c_3 x + c_4$, $\eta = c_5 u + \varphi(t, x)$ from (3.3), (3.8) and (2.12). Substituting them into (3.6) and (3.7), we can get $\varphi(t, x) = 0$ and

$$\begin{cases} (2\beta c_3 - \alpha c_1)p(u) - c_5 u p'(u) = 0, \\ (c_5 - \alpha c_1)q(u) - c_5 u q'(u) = 0. \end{cases} \tag{3.9}$$

The general solutions of the above system can be easily obtained by separation of variables as follows:

$$p(u) = Au^{\frac{2\beta c_3 - \alpha c_1}{c_5}}, \quad q(u) = Bu^{\frac{c_5 - \alpha c_1}{c_5}},$$

where A and B are arbitrary constants. That is, for $p(u) = u^n$ and $q(u) = u^m (m \neq 1)$, we have

$$c_5 = \frac{\alpha}{1-m}c_1, \quad c_3 = \frac{\alpha(1-m+n)}{2\beta(1-m)}c_1. \tag{3.10}$$

Therefore, from (3.10) and (2.11), the infinitesimals τ, ξ and η for the special case Eq.(1.3) are obtained as follows:

$$\tau = c_1 t, \quad \xi = \frac{\alpha(1-m+n)}{2\beta(1-m)}c_1 x + c_4, \quad \eta = \frac{\alpha}{1-m}c_1 u. \tag{3.11}$$

So Eq.(1.3) admits the two-dimensional Lie algebra spanned by

$$X_1 = x\frac{\partial}{\partial x}, \quad X_2 = t\frac{\partial}{\partial t} + \frac{\alpha(1-m+n)}{2\beta(1-m)}x\frac{\partial}{\partial t} + \frac{\alpha}{1-m}u\frac{\partial}{\partial u}. \tag{3.12}$$

For group generator X_2, from the characteristic equation (2.14), i.e.

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{\frac{\alpha(1-m+n)}{2\beta(1-m)}x} = \frac{\mathrm{d}u}{\frac{\alpha}{1-m}u}, \tag{3.13}$$

we obtain the similarity variables $xt^{-\frac{\alpha(1-m+n)}{2\beta(1-m)}}$ and $ut^{-\frac{\alpha}{1-m}}$. So we get the following form of group invariant solutions:

$$u(t, x) = t^{\frac{\alpha}{1-m}} f(\omega), \quad \omega = xt^{-\frac{\alpha(1-m+n)}{2\beta(1-m)}}. \tag{3.14}$$

Theorem 3.1. *The similarity transformations* $u(t,x) = t^{\frac{\alpha}{1-m}} f(\omega)$ *with the similarity variable* $\omega = xt^{-\frac{\alpha(1-m+n)}{2\beta(1-m)}}$ *reduce the space-time FPDE (1.3) to the space-time FODE given by*

$$(\mathcal{P}^{1+\frac{m\alpha}{1-m},\alpha}_{\frac{2\beta(1-m)}{\alpha(1-m+n)}} f)(\omega) = \omega^{-2\beta} f^n(\omega)(\mathcal{D}_1^{-2\beta,2\beta} f)(\omega) + f^m(\omega), \tag{3.15}$$

where $(\mathcal{P}_\delta^{\iota,\kappa})$ *is the left-hand Erdélyi-Kober fractional differential operator defined by*

$$(\mathcal{P}_\delta^{\iota,\kappa}\psi)(\omega) := \prod_{j=0}^{m-1}(\iota + j - \frac{1}{\delta}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\mathcal{K}_\delta^{\iota+\kappa,m-\kappa}\psi)(\omega), \quad \omega > 0,\ \delta > 0,\ \kappa > 0, \tag{3.16}$$

$$m = \begin{cases} [\kappa] + 1, & if\ \kappa \notin \mathbb{N}, \\ \kappa, & if\ \kappa \in \mathbb{N}, \end{cases}$$

where

$$(\mathcal{K}_\delta^{\iota,\kappa}\psi)(\omega) := \begin{cases} \frac{1}{\Gamma(\kappa)}\int_1^\infty (s-1)^{\kappa-1}s^{-(\iota+\kappa)}\psi(\omega s^{\frac{1}{\delta}})\mathrm{d}s, & if\ \kappa > 0, \\ \psi(\omega), & if\ \kappa = 0, \end{cases} \tag{3.17}$$

is the left-hand Erdélyi-Kober fractional integral operator. Meanwhile, $(\mathcal{D}_\delta^{\iota,\kappa})$ *is the right-hand Erdélyi-Kober fractional differential operator defined as follows:*

$$(\mathcal{D}_\delta^{\iota,\kappa}\psi)(\omega) := \prod_{j=1}^{m}(\iota + j + \frac{1}{\delta}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\mathcal{I}_\delta^{\iota+\kappa,m-\kappa}\psi)(\omega), \quad \omega > 0,\ \delta > 0,\ \kappa > 0, \tag{3.18}$$

$$m = \begin{cases} [\kappa] + 1, & if\ \kappa \notin \mathbb{N}, \\ \kappa, & if\ \kappa \in \mathbb{N}, \end{cases}$$

where

$$(\mathcal{I}_\delta^{\iota,\kappa}\psi)(\omega) := \begin{cases} \frac{1}{\Gamma(\kappa)}\int_0^1 (1-s)^{\kappa-1}s^{\iota}\psi(\omega s^{\frac{1}{\delta}})\mathrm{d}s, & if\ \kappa > 0, \\ \psi(\omega), & if\ \kappa = 0, \end{cases} \tag{3.19}$$

is the right-hand Erdélyi-Kober fractional integral operator.

Proof. For $0 < \alpha < 1$, the Riemann-Liouville time fractional derivative of $u(t,x)$ can be obtained as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha}(t^{\frac{\alpha}{1-m}} f(\omega)) = \frac{\partial}{\partial t}\Big[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{\frac{\alpha}{1-m}} f(xs^{-\frac{\alpha(1-m+n)}{2\beta(1-m)}})\mathrm{d}s\Big].$$

Assuming $r = \frac{t}{s}$, we have

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial}{\partial t}\Big[\frac{t^{1+\frac{m\alpha}{1-m}}}{\Gamma(1-\alpha)}\int_1^\infty (r-1)^{-\alpha}r^{\frac{m\alpha}{1-m}-2} f(\omega r^{-\frac{\alpha(1-m+n)}{2\beta(1-m)}})\mathrm{d}r\Big] \\ &= \frac{\partial}{\partial t}\Big[t^{1+\frac{m\alpha}{1-m}}(\mathcal{K}^{1+\frac{\alpha}{1-m},1-\alpha}_{\frac{2\beta(1-m)}{\alpha(1-m+n)}} f)(\omega)\Big]. \end{aligned}$$

Because of $\omega = xt^{-\frac{\alpha(1-m+n)}{2\beta(1-m)}}$, the following relation holds:

$$t\frac{\partial}{\partial t}\psi(\omega) = tx\Big(-\frac{\alpha(1-m+n)}{2\beta(1-m)}\Big)t^{-\frac{\alpha(1-m+n)}{2\beta(1-m)}-1}\psi'(\omega) = -\frac{\alpha(1-m+n)}{2\beta(1-m)}\omega\frac{\mathrm{d}}{\mathrm{d}\omega}\psi(\omega).$$

Hence, we arrive at

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= t^{\frac{m\alpha}{1-m}}\Big[\Big(1+\frac{m\alpha}{1-m}-\frac{\alpha(1-m+n)}{2\beta(1-m)}\omega\frac{\mathrm{d}}{\mathrm{d}\omega}\Big)\Big(\mathcal{K}^{1+\frac{\alpha}{1-m},1-\alpha}_{\frac{2\beta(1-m)}{\alpha(1-m+n)}}f\Big)(\omega)\Big]\\ &= t^{\frac{m\alpha}{1-m}}\Big(\mathcal{P}^{1+\frac{m\alpha}{1-m},\alpha}_{\frac{2\beta(1-m)}{\alpha(1-m+n)}}f\Big)(\omega).\end{aligned}$$

For $0<\beta<1$ and $\beta\neq\frac{1}{2}$, the Riemann-Liouville space fractional derivative of $u(t,x)$ becomes

$$\begin{aligned}&\frac{\partial^{2\beta}u}{\partial x^{2\beta}} = \frac{\partial^n}{\partial x^n}\Big[\frac{1}{\Gamma(n-2\beta)}\int_0^x (x-s)^{n-2\beta-1}t^{\frac{\alpha}{1-m}}f\big(st^{-\frac{\alpha(1-m+n)}{2\beta(1-m)}}\big)\mathrm{d}s\Big],\\ &n-1<2\beta<n\ (n=1,2).\end{aligned}$$

Let $r=\frac{s}{x}$, we get

$$\begin{aligned}\frac{\partial^{2\beta}u}{\partial x^{2\beta}} &= t^{\frac{\alpha}{1-m}}\frac{\partial^n}{\partial x^n}\Big[\frac{x^{n-2\beta}}{\Gamma(n-2\beta)}\int_0^1(1-r)^{n-2\beta-1}f(\omega r)\mathrm{d}r\Big]\\ &= t^{\frac{\alpha}{1-m}}\frac{\partial^n}{\partial x^n}\Big[x^{n-2\beta}\big(\mathcal{I}_1^{0,n-2\beta}f\big)(\omega)\Big].\end{aligned}$$

Because of $(\mathcal{I}_1^{\iota,0}\psi)(\omega)=\psi(\omega)$, if $2\beta=n=1,2$, the above equation still holds. Taking into account the relation

$$x\frac{\partial}{\partial x}\psi(\omega) = xt^{-\frac{\alpha(1-m+n)}{2\beta(1-m)}}\psi'(\omega) = \omega\frac{\mathrm{d}}{\mathrm{d}\omega}\psi(\omega),$$

we arrive at

$$\begin{aligned}\frac{\partial^{2\beta}u}{\partial x^{2\beta}} =& t^{\frac{\alpha}{1-m}}\frac{\partial^{n-1}}{\partial x^{n-1}}\Big[x^{n-2\beta-1}\Big(n-2\beta+\omega\frac{\mathrm{d}}{\mathrm{d}\omega}\Big)\big(\mathcal{I}_1^{0,n-2\beta}f\big)(\omega)\Big]\\ =& \cdots = t^{\frac{\alpha}{1-m}}x^{-2\beta}\prod_{j=1}^{n}\Big[\Big(-2\beta+j+\omega\frac{\mathrm{d}}{\mathrm{d}\omega}\Big)\big(\mathcal{I}_1^{0,n-2\beta}f\big)(\omega)\Big]\\ =& t^{\frac{\alpha}{1-m}}x^{-2\beta}\big(\mathcal{D}_1^{-2\beta,2\beta}f\big)(\omega) = t^{\frac{\alpha(m-n)}{1-m}}\omega^{-2\beta}\big(\mathcal{D}_1^{-2\beta,2\beta}f\big)(\omega).\end{aligned}$$

This completes the proof. □

From group generator X_2, if $m=1+n$, then it becomes

$$X_2 = t\frac{\partial}{\partial t}-\frac{\alpha}{n}u\frac{\partial}{\partial u}. \tag{3.20}$$

At the same time, the group invariant solutions (3.14) become $u(t,x)=t^{-\frac{\alpha}{n}}f(x)$. Therefore, for special case $n=1, m=1+n=2$, Eq.(1.3) becomes

$$D_t^\alpha u = uD_x^{2\beta}u+u^2, \tag{3.21}$$

whose group invariant solution is $u(t,x) = t^{-\alpha} f(x)$. In this case, the space-time FPDE (3.21) is reduced to the following time FODE:

$$D_x^{2\beta} f(x) + f(x) = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}. \tag{3.22}$$

Next we will give some exact solutions of the reduced equation (3.22) by Laplace transform. Firstly, the Laplace transform of Riemann-Liouville fractional derivative is

$$L\{D_t^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k f^{(\alpha-k-1)}(0), \quad n-1 < \alpha \leq n, n \in N. \tag{3.23}$$

If $0 < 2\beta < 1$, the Laplace transform of Eq.(3.22) is

$$s^{2\beta} F(s) - f^{(2\beta-1)}(0) + F(s) = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} s^{-1}, \tag{3.24}$$

which is rewritten as

$$F(s) = \frac{\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} s^{-1} + f^{(2\beta-1)}(0)}{s^{2\beta}+1}. \tag{3.25}$$

From the inverse Laplace transform of the above equation, we can get the explicit solutions as follows:

$$f(x) = k_1 x^{2\beta} E_{2\beta,2\beta+1}(-x^{2\beta}) + k_2 x^{2\beta-1} E_{2\beta,2\beta}(-x^{2\beta}), \tag{3.26}$$

where $k_1 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}$, $k_2 = f^{(2\beta-1)}(0)$, and $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}$ is the Mittag-Leffler function. Therefore,

$$u(t,x) = t^{-\alpha} x^{2\beta} \big(k_1 E_{2\beta,2\beta+1}(-x^{2\beta}) + k_2 x^{-1} E_{2\beta,2\beta}(-x^{2\beta})\big). \tag{3.27}$$

The graphs of the solutions (3.27) are plotted in Fig.1 for some different values of fractional orders α and β.

If $1 < 2\beta < 2$, similarly, we can get the following exact solutions of Eq.(3.21):

$$u(t,x) = t^{-\alpha} x^{2\beta} \big(k_1 E_{2\beta,2\beta+1}(-x^{2\beta}) + k_2 x^{-1} E_{2\beta,2\beta}(-x^{2\beta}) + k_3 x^{-2} E_{2\beta,2\beta-1}(-x^{2\beta})\big), \tag{3.28}$$

where $k_1 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}$, $k_2 = f^{(2\beta-1)}(0)$ and $k_3 = f^{(2\beta-2)}(0)$.

The graphs of the solutions (3.28) are plotted in Fig.2 for some different values of fractional orders α and β.

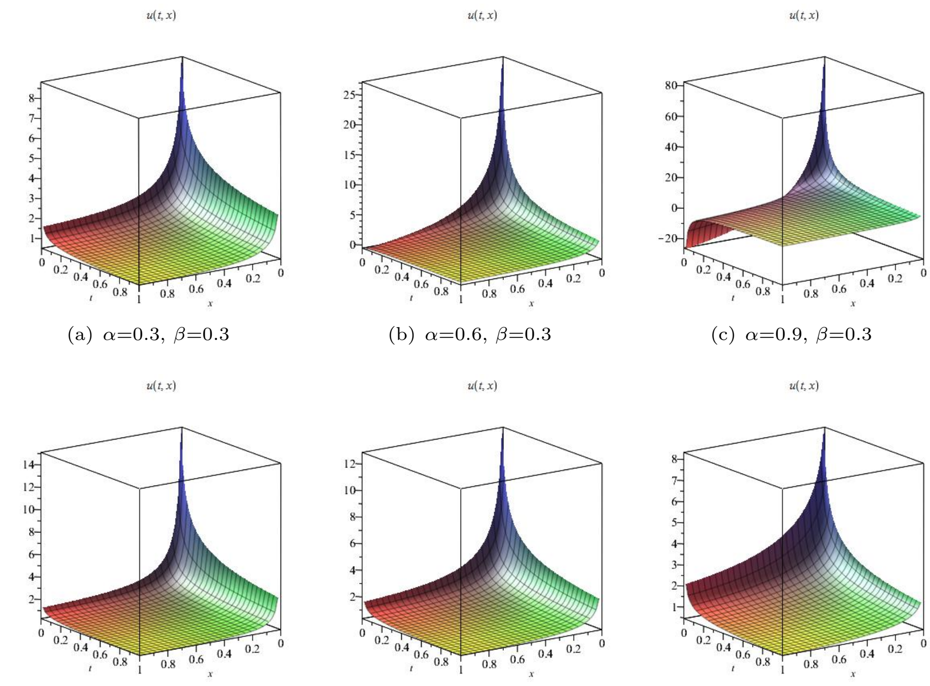
4. Application to Eq.(1.2)

The determining equation of Eq.(1.2) is

$$prX\big(D_t^\alpha u - kuD_x^{2\beta} u - l(uu_x)_x\big)|_{(1.2)} = 0, \tag{4.1}$$

which can be rewritten as

$$(\eta^{\alpha,t} - ku\eta^{2\beta,x} - kD_x^{2\beta} u\eta - 2lu_x\eta^x - lu\eta^{xx} - lu_{xx}\eta)|_{(1.2)} = 0. \tag{4.2}$$



(a) α=0.3, β=0.3 (b) α=0.6, β=0.3 (c) α=0.9, β=0.3

(d) α=0.4, β=0.2 (e) α=0.4, β=0.3 (f) α=0.4, β=0.4

Figure 1. Graphs of solutions (3.27) with $k_1 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}$ and $k_2 = 1$.

Putting $\eta^{\alpha,t}$, $\eta^{2\beta,x}$, η^x and η^{xx} into (4.2) and letting coefficients of various derivatives of u, such as u_t, u_x, u_x^2, $u_{xx}, \cdots$, $D_x^{2\beta}u$, $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x$, $D_x^{2\beta-n}u$ and $D_x^{2\beta-n}u_t(n = 1, 2, \cdots)$ to be zero, we can obtain the following over-determined system:

$$\tau_x = \tau_u = \xi_t = \xi_u = 0, \tag{4.3}$$

$$\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau) = 0, \quad n \in N, \tag{4.4}$$

$$\binom{2\beta}{n}\frac{\partial^n \eta_u}{\partial x^n} - \binom{2\beta}{n+1}D_x^{n+1}(\xi) = 0, \quad n \in N, \tag{4.5}$$

$$(2\beta\xi_x - \alpha\tau_t)u - \eta = 0, \tag{4.6}$$

$$\eta_u - \alpha\tau_t - 2(\eta_u - \xi_x) = 0, \tag{4.7}$$

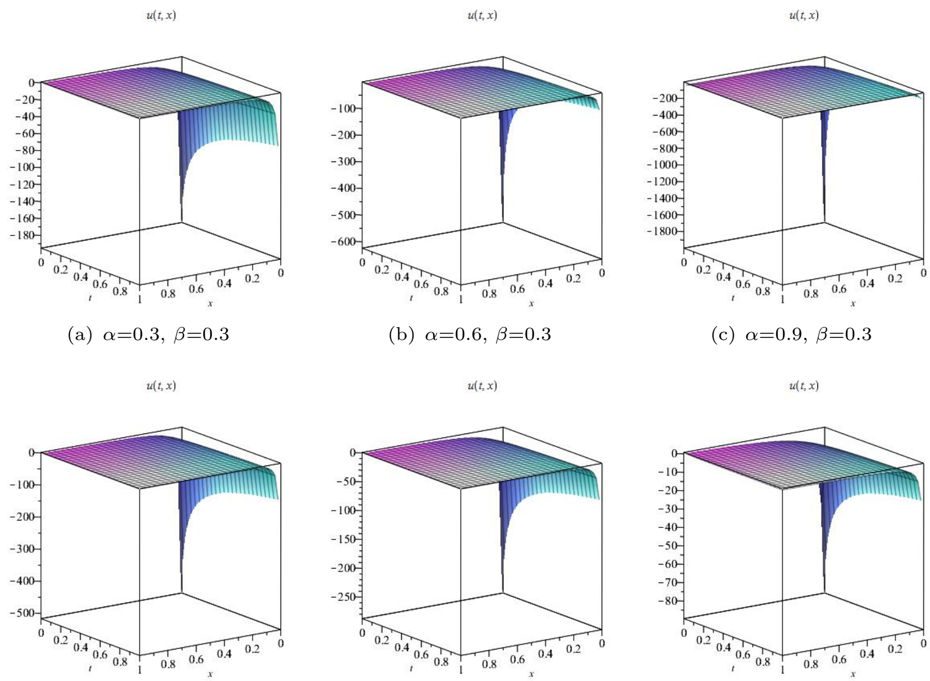
$$\frac{\partial^\alpha \eta}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - ku\frac{\partial^{2\beta}\eta}{\partial x^{2\beta}} + ku^2\frac{\partial^{2\beta}\eta_u}{\partial x^{2\beta}} - lu\eta_{xx} = 0. \tag{4.8}$$

Similarly as Lie symmetry analysis of Eq.(1.1), from (4.6) and (4.7), we get

$$c_5 = 2\beta c_3 - \alpha c_1, \quad c_5 = 2c_3 - \alpha c_1. \tag{4.9}$$

So only if $\beta = 1$, Eq.(1.2) has the infinitesimals

$$\tau = c_1 t, \quad \xi = c_3 x + c_4, \quad \eta = (2c_3 - \alpha c_1)u. \tag{4.10}$$



(a) α=0.3, β=0.3 (b) α=0.6, β=0.3 (c) α=0.9, β=0.3

(d) α=0.4, β=0.2 (e) α=0.4, β=0.3 (f) α=0.4, β=0.4

Figure 2. Graphs of solutions (3.28) with $k_1 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}$ and $k_2 = k_3 = 1$.

That is, it admits the three-dimensional Lie algebra spanned by

$$X_1 = x\frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}, \quad X_3 = t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u}. \tag{4.11}$$

It's worth noting that these results are the same as Lie symmetries of Eq.(1.5) obtained by using Lie symmetry analysis method directly.
Case 1: X_2**.** From the characteristic equation of group generator X_2

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}u}{2u}, \tag{4.12}$$

we obtain the similarity variables t and ux^{-2}. So we get the following form of group invariant solutions:

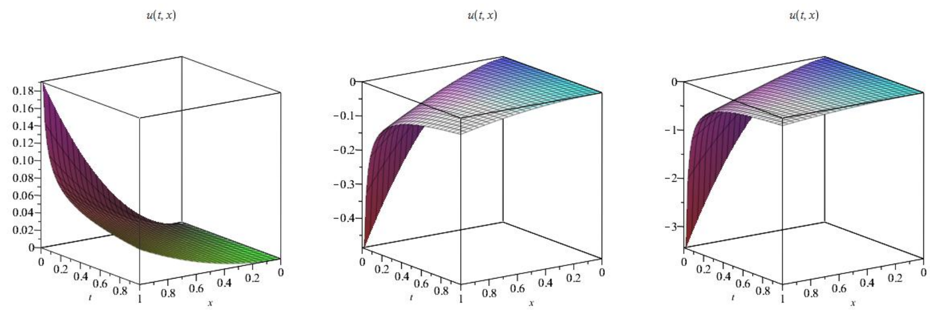
$$u(t,x) = x^2 f(t), \tag{4.13}$$

which reduce the FPDE (1.5) to the following FODE:

$$D_t^\alpha f(t) = (2k+6l)f^2(t). \tag{4.14}$$

Analytical solutions of the above equation can be obtained as $f(t) = \frac{\Gamma(1-\alpha)}{(2k+6l)\Gamma(1-2\alpha)}t^{-\alpha}$. Therefore, when $\beta = 1$, Eq.(1.2) has the following explicit solution:

$$u(t,x) = \frac{\Gamma(1-\alpha)}{(2k+6l)\Gamma(1-2\alpha)}t^{-\alpha}x^2. \tag{4.15}$$



(a) α=0.4 (b) α=0.6 (c) α=0.8

Figure 3. Graphs of solutions (4.15) with $k = l = 1$.

In Fig. 3, we have illustrated the physical features for the explicit solutions (4.15) with different fractional orders.

Case 2: X_3**.** From the characteristic equation of group generator X_3

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{-\alpha u}, \tag{4.16}$$

we obtain the similarity variables x and ut^{α}. So we get the following form of group invariant solutions:

$$u(t,x) = t^{-\alpha} f(x), \tag{4.17}$$

which reduce the FPDE (1.2) to the following two-order nonlinear ODE:

$$(k+l)f(x)f''(x) + l(f'(x))^2 = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} f(x). \tag{4.18}$$

Next we will derive the implicit solutions and power series solutions of the reduced equation (4.18) by order reduction method and power series method, respectively.

Method 1: Assuming $f(x) = s, f'(x) = y$, we can get $f''(x) = y\frac{\mathrm{d}y}{\mathrm{d}s}$. So (4.18) is reduced to the following one-order nonlinear ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}s} = -\frac{l}{(k+l)s} y + \frac{\Gamma(1-\alpha)}{(k+l)\Gamma(1-2\alpha)} y^{-1}, \tag{4.19}$$

which is known as Bernoulli equation. It can be rewritten as

$$y\frac{\mathrm{d}y}{\mathrm{d}s} = -\frac{l}{(k+l)s} y^2 + \frac{\Gamma(1-\alpha)}{(k+l)\Gamma(1-2\alpha)}. \tag{4.20}$$

Set $z = y^2$, then $\frac{\mathrm{d}z}{\mathrm{d}s} = 2y\frac{\mathrm{d}y}{\mathrm{d}s}$. So the above equation becomes the following one-order linear ODE:

$$\frac{\mathrm{d}z}{\mathrm{d}s} = -\frac{2l}{(k+l)s} z + \frac{\Gamma(1-\alpha)}{(k+l)\Gamma(1-2\alpha)}. \tag{4.21}$$

For (4.21), we can easily get the general solutions as

$$z = \frac{\Gamma(1-\alpha)}{(k+3l)\Gamma(1-2\alpha)} s + c_1 s^{-\frac{2l}{k+l}}. \tag{4.22}$$

At last, (4.18) is changed into

$$(f'(x))^2 = \frac{\Gamma(1-\alpha)}{(k+3l)\Gamma(1-2\alpha)} f(x) + c_1 f^{\frac{-2l}{k+l}}(x). \tag{4.23}$$

Thus we have the implicit solutions as

$$f(x) = \pm \int \sqrt{\frac{\Gamma(1-\alpha)}{(k+3l)\Gamma(1-2\alpha)} f(x) + c_1 f^{\frac{-2l}{k+l}}(x)} \mathrm{d}x + c_2, \tag{4.24}$$

where c_1 and c_1 are arbitrary constants. That is, Eq.(1.5) has the implicit solutions $u(t,x) = t^{-\alpha} f(x)$, where $f(x)$ is defined by (4.24).

Method 2: Let us assume that the solutions of (4.18) have the following form:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{4.25}$$

where a_n and b_n are constants to be known later. Then

$$f'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \quad f''(x) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n. \tag{4.26}$$

Substituting (4.25)-(4.26) into (4.18) arrives at the following equation:

$$(k+l) \sum_{i+j=n} (j+2)(j+1) a_i a_{j+2} x^n + l \sum_{i+j=n} (i+1)(j+1) a_{i+1} a_{j+1} x^n = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \sum_{n=0}^{\infty} a_n x^n. \tag{4.27}$$

In what follows, we equate the coefficients of different powers of x to obtain the explicit expressions of a_n and b_n. For $n = 0$, we have

$$a_2 = \frac{-l}{2(k+l)} \frac{a_1^2}{a_0} + \frac{\Gamma(1-\alpha)}{2(k+l)\Gamma(1-2\alpha)}. \tag{4.28}$$

For $n = 1$, we have

$$a_3 = -\frac{k+3l}{3(k+l)} \frac{a_1 a_2}{a_0} + \frac{\Gamma(1-\alpha)}{6(k+l)\Gamma(1-2\alpha)} \frac{a_1}{a_0}. \tag{4.29}$$

For $n \geq 2$, we have

$$(k+l) \sum_{i+j=n} (j+2)(j+1) a_i a_{j+2} + l \sum_{i+j=n} (i+1)(j+1) a_{i+1} a_{j+1} = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \sum_{n=0}^{\infty} a_n. \tag{4.30}$$

Therefore, the power series solutions of Eq.(1.5) are

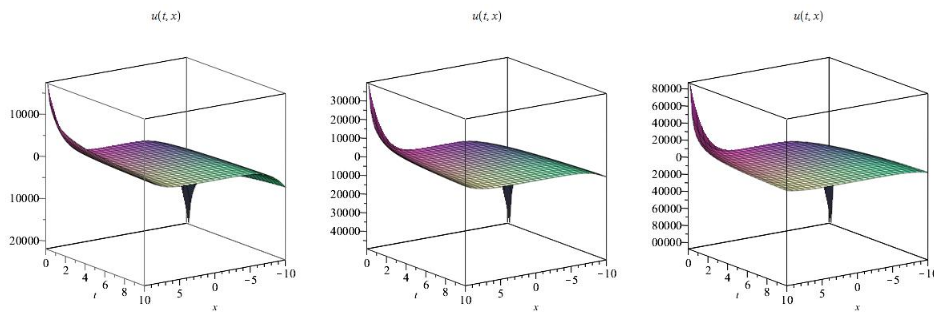
$$u(t,x) = t^{-\alpha} \sum_{n=0}^{\infty} a_n x^n, \tag{4.31}$$

where a_n are defined by (4.28)-(4.30) with arbitrary constants a_0 and a_1.

Tabs.1-2 show some values of a_n and α, while Figs.4-5 illustrate the physical features for the power series solutions (4.31) with different parameter values.

Table 1. Some of a_n with $a_0 = a_1 = 1$ for different fractional orders

	a_0	a_1	a_2	a_3	a_4	a_5
$\alpha = 0.4$	1	1	-0.1689042177	0.1396347392	-0.1165185026	0.1037811394
$\alpha = 0.6$	1	1	-0.3452629674	0.1984209892	-0.1830694517	0.1719683120
$\alpha = 0.8$	1	1	-0.5604495160	0.2701498387	-0.2783150522	0.2748273916

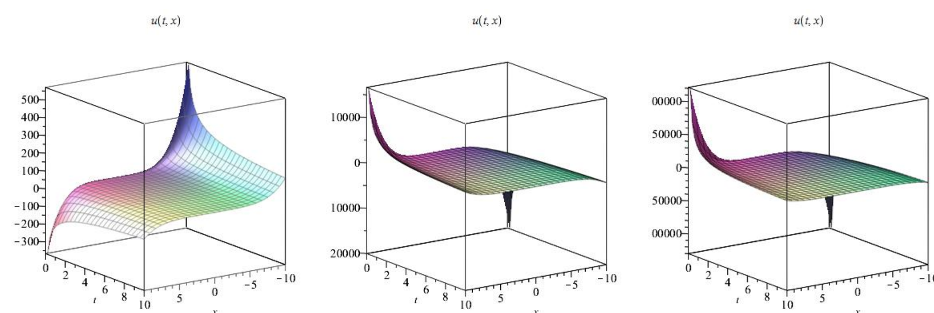


(a) α=0.4 (b) α=0.6 (c) α=0.8

Figure 4. Numerical simulation of the power series solutions (4.31) with $k = l = a_0 = a_1 = 1$.

Table 2. Some of a_n with $a_0 = a_1 = 0.1$ for different fractional orders

	a_0	a_1	a_2	a_3	a_4	a_5
$\alpha = 0.4$	0.1	0.1	0.05609578230	-0.01036526077	0.00486670852	-0.002408156100
$\alpha = 0.6$	0.1	0.1	-0.1202629674	0.04842098915	-0.06543200106	0.07102588110
$\alpha = 0.8$	0.1	0.1	-0.3354495160	0.1201498387	-0.2916334052	0.3572230859



(a) α=0.4 (b) α=0.6 (c) α=0.8

Figure 5. Numerical simulation of the power series solutions (4.31) with $k = l = 1$ and $a_0 = a_1 = 0.1$.

5. Conservation laws of Eqs.(1.1) and (1.2)

In this section, we will construct conservation laws of Eqs.(1.1) and (1.2) by using the generalization of the Noether operators and the new conservation theorem [11, 12].

5.1. Conservation laws of Eq.(1.1)

For Eq.(1.1), if $p(u) = u^n$ and $q(u) = u^m (m \neq 1)$, then a formal Lagrangian for Eq.(1.3)

$$F = D_t^{\alpha} u - u^n D_x^{2\beta} u - u^m = 0, \tag{5.1}$$

is given by

$$\mathcal{L} = v(t,x)F = v(t,x)(D_t^{\alpha} u - u^n D_x^{2\beta} u - u^m), \tag{5.2}$$

where $v(t,x)$ is a new dependent variable. The Euler-Lagrange operator [12] is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^{\alpha})^* \frac{\partial}{\partial (D_t^{\alpha} u)} + (D_x^{2\beta})^* \frac{\partial}{\partial (D_x^{2\beta} u)} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}}, \tag{5.3}$$

where $(D_t^{\alpha})^*$ and $(D_x^{2\beta})^*$ are the adjoint operators of D_t^{α} and $D_x^{2\beta}$, respectively. They are defined by

$$(D_t^{\alpha})^* = (-1)^n \ {}_tJ_T^{n-\alpha}(D_t^n) \equiv {}_t^cD_T^{\alpha}, \ \ (D_x^{2\beta})^* = (-1)^n \ {}_xJ_X^{n-2\beta}(D_x^n) \equiv {}_x^cD_X^{2\beta}, \tag{5.4}$$

where ${}_t^cD_T^{\alpha}$ and ${}_x^cD_X^{2\beta}$ are the right-sided of Caputo fractional derivative. The adjoint equation of Eq.(1.3) is given by

$$F^* = \frac{\delta \mathcal{L}}{\delta u} = (D_t^{\alpha})^* v - (D_x^{2\beta})^* (u^n v) - n u^{n-1} v D_x^{2\beta} u - m u^{m-1} v = 0. \tag{5.5}$$

Next we will use the above adjoint equation and the new conservation theorem to construct conservation laws of Eq.(1.3). From the classical definition of the conservation laws, a vector $C = (C^t, C^x)$ is called a conserved vector for the governing equation if it satisfies the conservation equation $[D_t C^t + D_x C^x]_{F=0} = 0$. By using Noether theorem the components of conserved vector can be obtained.

Firstly, from the fundamental operator identity, i.e.

$$prX + D_t \tau \cdot \mathcal{I} + D_x \xi \cdot \mathcal{I} = W \cdot \frac{\delta}{\delta u} + D_t \mathcal{N}^t + D_x \mathcal{N}^x, \tag{5.6}$$

where prX is mentioned in (2.4), $\mathcal{I}$ is the identity operator and $W = \eta - \tau u_t - \xi u_x$ is the characteristic for group generator X, we can get the Noether operators as follows:

$$\mathcal{N}^t = \tau \mathcal{I} + \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k}(W) D_t^k \frac{\partial}{\partial (D_t^{\alpha} u)} - (-1)^n J_1(W, D_t^n \frac{\partial}{\partial (D_t^{\alpha} u)}), \tag{5.7}$$

$$\mathcal{N}^x = \xi \mathcal{I} + \sum_{k=0}^{m-1} (-1)^k D_x^{2\beta-1-k}(W) D_x^k \frac{\partial}{\partial (D_x^{2\beta} u)} - (-1)^m J_2(W, D_x^m \frac{\partial}{\partial (D_x^{2\beta} u)}), \tag{5.8}$$

where $n = [\alpha] + 1$, $m = [2\beta] + 1$, and J_1, J_2 are given by

$$J_1(f,g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{f(\tau,x) g(\theta,x)}{(\theta-\tau)^{\alpha+1-n}} \mathrm{d}\theta \mathrm{d}\tau, \tag{5.9}$$

$$J_2(f,g) = \frac{1}{\Gamma(m-2\beta)} \int_0^x \int_x^X \frac{f(t,\xi) g(t,\theta)}{(\theta-\xi)^{2\beta+1-m}} \mathrm{d}\theta \mathrm{d}\xi. \tag{5.10}$$

The components of conserved vector are defined by $C^t = \mathcal{N}^t \mathcal{L}$, $C^x = \mathcal{N}^x \mathcal{L}$.

Then based on the group generator of Lie symmetry transformation

$$X_2 = t\frac{\partial}{\partial t} + \frac{\alpha(1-m+n)}{2\beta(1-m)} x\frac{\partial}{\partial t} + \frac{\alpha}{1-m} u\frac{\partial}{\partial u},$$

we obtain the corresponding Lie characteristic function

$$W = \frac{\alpha}{1-m} u - tu_t - \frac{\alpha(1-m+n)}{2\beta(1-m)} xu_x. \tag{5.11}$$

Therefore, for $0 < \alpha < 1$,

$$C^t = D_t^{\alpha-1}(W) D_t \frac{\partial \mathcal{L}}{\partial(D_t^\alpha u)} + J_1(W, D_t \frac{\partial \mathcal{L}}{\partial(D_t^\alpha u)}) = vD_t^{\alpha-1}(W) + J_1(W, v_t). \tag{5.12}$$

For $0 < 2\beta < 1$,

$$C^x = D_x^{2\beta-1}(W) D_x \frac{\partial \mathcal{L}}{\partial(D_x^{2\beta} u)} + J_2(W, D_x \frac{\partial \mathcal{L}}{\partial(D_x^{2\beta} u)}) = u^n v D_x^{2\beta-1}(W) + J_2(W, (u^n v)_x). \tag{5.13}$$

For $1 < 2\beta < 2$,

$$\begin{aligned} C^x =& D_x^{2\beta-1}(W) D_x \frac{\partial \mathcal{L}}{\partial(D_x^{2\beta} u)} - D_x^{2\beta-2}(W) D_x \frac{\partial \mathcal{L}}{\partial(D_x^{2\beta} u)} + J_2(W, D_x \frac{\partial \mathcal{L}}{\partial(D_x^{2\beta} u)}) \\ =& u^n v D_x^{2\beta-1}(W) - (u^n v)_x D_x^{2\beta-2}(W) + J_2(W, (u^n v)_{xx}). \end{aligned} \tag{5.14}$$

5.2. Conservation laws of Eq.(1.2)

For Eq.(1.2), if $\beta = 1$, then a formal Lagrangian for Eq.(1.5)

$$F = D_t^\alpha u - kuu_{xx} - l(uu_x)_x = 0 \tag{5.15}$$

is given by

$$\mathcal{L} = v(t,x)F = v(t,x)(D_t^\alpha u - kuu_{xx} - l(uu_x)_x). \tag{5.16}$$

So the adjoint equation of Eq.(1.5) is given by

$$F^* = \frac{\delta \mathcal{L}}{\delta u} = (D_t^\alpha)^* v - 2kvu_{xx} - 2ku_x v_x - (k+l)uv_{xx} = 0. \tag{5.17}$$

Similarly as the discussion for Eq.(1.3), the components of conserved vector for Eq.(1.5) can be obtained as follows:

$$C^t = D_t^{\alpha-1}(W) D_t \frac{\partial \mathcal{L}}{\partial(D_t^\alpha u)} + J_1(W, D_t \frac{\partial \mathcal{L}}{\partial(D_t^\alpha u)}), \tag{5.18}$$

$$C^x = W(\frac{\partial \mathcal{L}}{\partial u_x} - D_x \frac{\partial \mathcal{L}}{\partial u_{xx}}) + D_x(W)(\frac{\partial \mathcal{L}}{\partial u_{xx}}). \tag{5.19}$$

Case 1: For the group generator X_2 admitted by Eq.(1.5)

$$X_2 = x\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u},$$

we obtain the corresponding Lie characteristic function

$$W_1 = 2u - xu_x. \tag{5.20}$$

Therefore, for $0 < \alpha < 1$,

$$C^t = vD_t^{\alpha-1}(W_1) + J_1(W_1, v_t), \tag{5.21}$$

$$C^x = 2(k+l)u^2v_x - (k+l)xuu_xv_x + (k-3l)uvu_x - (k-l)xvu_x^2 + (k+l)xuvu_{xx}. \tag{5.22}$$

Case 2: For the group generator X_3 admitted by Eq.(1.5)

$$X_3 = t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u},$$

we obtain the corresponding Lie characteristic function

$$W_2 = -\alpha u - tu_t. \tag{5.23}$$

Therefore, for $0 < \alpha < 1$,

$$C^t = vD_t^{\alpha-1}(W_2) + J_1(W_2, v_t), \tag{5.24}$$

$$C^x = -\alpha(k+l)u^2v_x - (k+l)tuu_tv_x - 2\alpha kuvu_x - (k-l)tvu_tu_x - (k+l)tuvu_{tx}. \tag{5.25}$$

6. Conclusion

This paper shows that Lie symmetry analysis method is effective to study the space-time FPDEs. We obtain the Lie symmetries for the governing equations and use them to reduce the FPDEs to FODEs. With Laplace transformation and the power series methods, some explicit solutions are obtained and presented with figures. Furthermore, we construct the conservation laws for the FPDEs using a new conservation theorem.

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