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BOUNDARY LAYER-PRESERVING METHODS FOR A CLASS OF NONLINEAR SINGULAR PERTURBATION BOUNDARY VALUE PROBLEMS*

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Abstract The aim of this paper is to develop a new uniformly convergent numerical approach for nonlinear singularly perturbed boundary value problems (BVPs). The method combines the advantages of the variation-of-constants formula and the reproducing kernel function approximation. It can preserve the boundary layer structure of the solution to the considered singular perturbation problems. In addition, compared with some existing numerical techniques, the present method has no restriction on the choice of nodes. Three numerical experiments are implemented and the numerical results indicate our new technique is quite promising.

Keywords Reproducing kernel method, boundary layers, nonlinear singularly perturbed problems.

MSC(2010) 35A35, 35B30, 65N45.

1. Introduction

The main theme of this paper is to solve the singularly perturbed problems (SPPs) as follows:

$$\begin{cases} \varepsilon y''(x) + \gamma y'(x) = f(x, y(x)), \ \ x \in [0,1], \\ y(0) = \alpha_0, \ y(1) = \alpha_1, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ and γ is a positive constant. For the solution of (1.1), it is known that there is a left boundary layer at the point $x = 0$.

SPPs arise in several fields such as fluid mechanics, biological sciences and chemical reactions. Solutions of SPPs have boundary or interior layers, in which the derivatives of the solutions grow without bound as $\varepsilon \to 0$. As we all know, the standard numerical methods usually result in disappointed numerical solutions to

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SPPs. It is a challenging work to find efficient numerical schemes for SPPs. In the last decade, there have been many effective numerical methods for SPPs (see, e.g. [15–18, 20–24]).

Reproducing kernel Hilbert space(RKHS) theory is an essential approach in numerical analysis. By employing the RKHS theory, Cui and Geng [7] proposed a approach to solve systems of ordinary differential equations BVPs and it can be extended to numerically dealing with general operator equations. Cui and Lin [6] discussed the nonlinear numerical analysis in RKHS. Over the last decades, the related theory has been used to solve the fractional order integral and differential equations, singular integral equations, fuzzy differential equations, and so on (see, e.g. [1–5, 8–14, 19, 25, 27]).

In this paper, by combining the RKHS theory and the variation-of-constants formula, we will present a boundary layer-preserving technique for solving the nonlinear singular perturbation BVPs.

This paper is organized as follows. As background reading, we summarize RKHS theory in Section 2. In Section 3, we develp boundary layer-preserving technique for singular perturbation BVPs (1.1). Numerical experiments are undertaken in Section 4. The conclusion is given in the last section.

2. Preliminaries to RKHS theory

For the definition and properties of the RKHS and reproducing kernel function (RKF), please refer to [6]. Following we provide the Sobolev RKHS, which will be used in the next section.

Definition 2.1. Let $H^m[0,1]$ be the Sobolev function space, which consists of functions $w(t)$ defined on $[0,1]$. The m-th order derivative of $w(t)$ is absolutely continuous and $w^{(m)}(t) \in L^2[0,1]$. The inner product on $H^m[0,1]$ is

$$(w_1, w_2)_m = \sum_{k=0}^{m-1} w_1^{(k)}(0) w_2^{(k)}(0) + \int_0^1 w_1^{(m)}(t) w_2^{(m)}(t) dt.$$

Theorem 2.1. *Space $H^m[0,1]$ is an RKHS with RKF*

$$K^m(x,s) = \begin{cases} \xi(x,s), & s \leq x, \\ \xi(s,x), & s > x, \end{cases}$$

where $\xi(x,s) = \sum\limits_{i=0}^{m-1} (\frac{s^i}{i!} + (-1)^{m-1-i} \frac{s^{2m-1-i}}{(2m-1-i)!}) \frac{x^i}{i!}$.

The detailed contents of the theorem can be referred to the reference [7].

3. Boundary layer-preserving method for (1.1)

(1.1) can be written as

$$\begin{cases} y''(x) + \omega y'(x) = F(x, y(x)), & x \in [0,1], \\ y(0) = \alpha_0, \ y(1) = \alpha_1, \end{cases} \tag{3.1}$$

where $\omega = \frac{1}{\varepsilon}$ and $F(x, y) = \frac{f(x,y)}{\varepsilon}$.

Theorem 3.1. *If $f(x, y)$ is continuous, then the solution of (3.1) satisfies*

$$\begin{aligned} y(x) =& \alpha_0(1 - \frac{1 - e^{-\omega x}}{1 - e^{-\omega}}) + \alpha_1(\frac{1 - e^{-\omega x}}{1 - e^{-\omega}}) + \frac{1}{\omega}\int_0^x (1 - e^{-\omega(x-t)})F(t, y(t))dt \\ & - (\frac{1 - e^{-\omega x}}{1 - e^{-\omega}})\frac{1}{\omega}\int_0^1 (1 - e^{-\omega(1-t)})F(t, y(t))dt. \end{aligned} \tag{3.2}$$

Putting $g(t) = F(t, u(t))$, $H(x) = \frac{1-e^{-\omega x}}{1-e^{-\omega}}$ and $G(x) = \frac{1}{\omega}\int_0^x (1 - e^{-\omega(x-t)})g(t)dt$, (3.2) reduces to

$$y(x) = \alpha_0(1 - H(x)) + \alpha_1 H(x) - H(x)G(1) + G(x). \tag{3.3}$$

In (3.3), the most important issue is how to deal with the integral $G(x)$. We will propose a new method for handling the integral $G(x)$ based on the RKF approximation.

We approximate $g(t)$ appearing in the integral $G(x)$ by an RKF interpolation, which is a kind of piecewise polynomials. Then, the corresponding moments can be computed explicitly. Moreover, the powerful interpolation can avoid the Runge phenomenon of equidistant-node polynomial interpolations.

Given a finite set $X = \{x_1, x_2, \ldots, x_N\} \subseteq I = [0, 1]$. For each $g(x) \in H^m$, the interpolation function is represented by $g_N(x) = \sum\limits_{i=1}^{N} \beta_i K^m(x, x_i)$, where β_i are coefficients to be determined.

Theorem 3.2. *The interpolation linear system*

$$\sum_{i=1}^{N} \beta_i K^m(x_j, x_i) = g(x_j), 1 \le j \le N$$

with kernel matrix

$$A = (K^m(x_j, x_i))_{j,i=1}^N$$

is solvable [26].

Theorem 3.3. *If the kernel function $K^m(x, y)$ is strictly positive definite, then A is nonsingular [26].*

Theorem 3.4. *If $g(x) \in C^{2m}[0, 1]$ and g_N is the obtained interpolation function in RKHS $H^m[0, 1]$, then [19]*

$$\| g_N(x) - g(x) \| = \max_{x\in[a,b]} | g(x) - g_N(x) | \le d_1\, h^{2m}, \tag{3.4}$$

where $d_1 > 0$ is a real number and $h = \max\limits_{1\le i\le N-1} | x_i - x_{i+1} |$.

It follows from replacing $g(t)$ by $g_N(t)$ in $G(x)$ that

$$\begin{aligned} G_N(x) &= \frac{1}{\omega}\int_0^x (1 - e^{-\omega(x-t)})g_N(t)dt \\ &= \frac{1}{\omega}\sum_{i=1}^{N} \beta_i \int_0^x (1 - e^{-\omega(x-t)})K^m(t, x_i)dt \end{aligned}$$

$$= \frac{1}{\omega}\sum_{i=1}^{N}\beta_i\mu_i(x) \tag{3.5}$$

where $\mu_i = \int_0^x (1-e^{-\omega(x-t)})K^m(t,x_i)dt$ can be calculated explicitly. $G_N(x)$ can be used for the approximation of $G(x)$.

Replacing $G(x)$ by $G_N(x)$ in (3.3), we have the approximate solution to (3.1)

$$y_N(x) = \alpha_0(1-H(x)) + \alpha_1 H(x) - H(x)G_N(1) + G_N(x). \tag{3.6}$$

Theorem 3.5. *If $g(x) \in C^{2m}[0,1]$ and G_N is the approximate solution obtained in RKHS $H^m[0,1]$, then*

$$\parallel G(x) - G_N(x) \parallel \le d_1 \frac{h^{2m}}{\omega}.$$

Proof. It follows from Theorem 3.4 that

$$\parallel g_N(x) - g(x) \parallel \le d_1\, h^{2m}.$$

We then have

$$\begin{aligned} \mid G(x) - G_N(x) \mid &= \left|\tfrac{1}{\omega}\int_0^x (1-e^{-\omega(x-t)})(g(t)-g_N(t))dt\right| \\ &\le \tfrac{1}{\omega}\int_0^x \mid (1-e^{-\omega(x-t)}) \mid\mid (g(t)-g_N(t)) \mid dt \\ &\le d_1\, \tfrac{h^{2m}}{\omega}, \end{aligned}$$

which means that $\parallel G(x) - G_N(x) \parallel \le d_1 \frac{h^{2m}}{\omega}$. The proof is complete. □

Theorem 3.6. *Let $d > 0$ be a constant. If $g(x) \in C^{2m}[0,1]$ and G_N is approximated in RKHS $H^m[0,1]$, then we have*

$$\parallel y(x) - y_N(x) \parallel \le d\, \frac{h^{2m}}{\omega}.$$

Proof. The application of Theorem 3.5 yields

$$\parallel G_N(x) - G(x) \parallel \le d_1 \frac{h^{2m}}{\omega},$$

and hence

$$\begin{aligned} \parallel y(x) - y_N(x) \parallel &= \parallel H(x)(G(1) - G_N(1)) + (G(x) - G_N(x)) \parallel \\ &\le \mid (H(x)+1)d_1\, \tfrac{h^{2m}}{\omega} \mid \\ &\le d\, \tfrac{h^{2m}}{\omega}, \end{aligned}$$

where $d = 2\ d_1$. This completes the proof. □

If $f(x,y)$ is dependent on y, the approximation to solution of (3.1) is is established by using the iterative approach.

We select a proper initial approximation $y_0(x)$:

$$y_0(x) = \alpha_0(1-H(x)) + \alpha_1 H(x).$$

We construct the following iterative procedure:

$$\begin{aligned} y_{k+1}(x) =& y_0(x) - H(x)\int_0^1 \frac{1}{\omega}(1-e^{-\omega(1-t)})F(t,y_k(t))dt \\ &+ \int_0^x \frac{1}{\omega}(1-e^{-\omega(x-t)})F(t,y_k(t))dt,\ \ k\geq 0. \end{aligned} \tag{3.7}$$

Putting $g_k(t)=F(t,y_k(t)),(k=0,1,2\cdots)$ and $G_k(x)=\int_0^x \frac{1}{\omega}(1-e^{-\omega(x-t)})g_k(t)dt$, (3.7) reduces to

$$y_{k+1}(x) = y_0(x) - H(x)G_k(1) + G_k(x), k\geq 0. \tag{3.8}$$

Using the RKF interpolation method stated above, the approximation of $g_0(t)$ is given by

$$g_{0,N}(t) = \sum_{i=1}^N \beta_{0,i}K^m(t,x_i),$$

where $\beta_{0,i}$ are determined by the functions values $g_0(x_i)$ for $i=1,2,\ldots,N$. Then the approximation of integral $G_0(x)$ can be obtained

$$\begin{aligned} \overline{G}_{0,N}(x) &= \frac{1}{\omega}\int_0^x (1-e^{-\omega(x-t)})g_{0,N}(t)dt \\ &= \frac{1}{\omega}\sum_{i=1}^N \beta_{0,i}\int_0^x (1-e^{-\omega(x-t)})K^m(t,x_i)dt \\ &= \frac{1}{\omega}\sum_{i=1}^N \beta_{0,i}\mu_i(x), \end{aligned} \tag{3.9}$$

where $\mu_i = \int_0^x (1-e^{-\omega(x-t)})K^m(t,x_i)dt$ can be calculated explicitly.

Combining (3.8) and (3.9), we obtain the approximation to $y_1(x)$

$$y_{1,N}(x) = y_0(x) - H(x)\overline{G}_{0,N}(1) + \overline{G}_{0,N}(x). \tag{3.10}$$

Let $\overline{g}_k(t) = F(t,y_{k,N}(t))$ and $\overline{G}_k(x) = \int_0^x \frac{1}{\omega}(1-e^{-\omega(x-t)})\overline{g}_k(t)dt$ for $k\geq 1$.

Likewise, we have the approximation to $\overline{G}_k(x)$

$$\begin{aligned} \overline{G}_{k,N}(x) &= \frac{1}{\omega}\int_0^x (1-e^{-\omega(x-t)})\overline{g}_{k,N}(t)dt \\ &= \frac{1}{\omega}\sum_{i=1}^N \beta_{k,i}\int_0^x (1-e^{-\omega(x-t)})K^m(t,x_i)dt \\ &= \frac{1}{\omega}\sum_{i=1}^N \beta_{k,i}\mu_i(x). \end{aligned} \tag{3.11}$$

Replacing $\overline{G}_k(x)$ by $\overline{G}_{k,N}(x)$ in (3.8), we obtain the following iterative formula

$$y_{k+1,N}(x) = y_0(x) - H(x)\overline{G}_{k,N}(1) + \overline{G}_{k,N}(x), k\geq 1. \tag{3.12}$$

We denote

$$\lambda(y) = y_0(x) - H(x)\int_0^1 \frac{1}{\omega}(1-e^{-\omega(x-t)})F(t,y(t))dt + \int_0^x \frac{1}{\omega}(1-e^{-\omega(x-t)})F(t,y(t))dt.$$

Clearly, $y_n(x) = \lambda(y_{n-1}(x))$. It is east to see that

$$\|\lambda(y) - \lambda(z)\| \leq \frac{2}{\omega} \parallel f(t, y(t)) - f(t, zv(t)) \parallel .$$

Theorem 3.7. *Suppose that* $\parallel f(t,\mu) - f(t,\nu) \parallel \leq L \parallel \mu - \nu \parallel$ *with Lipschitz constant* L. *Let* $\rho = \frac{2L}{\omega}$. *If* $\rho < 1$, *then* $y_{n,N}(x)$ *converges to* $y(x)$.

Proof. In view of $\|\lambda(u) - \lambda(v)\| \leq \rho\|u - v\|$, we have

$$\|y_n(x) - y(x)\| = \|\lambda(y_{n-1}) - \lambda(y)\| \leq \rho\|y_{n-1} - y\| \tag{3.13}$$

and

$$\|y_n(x) - y_{n-1}(x)\| = \|\lambda(y_{n-1}) - \lambda(y_{n-2})\| \leq \rho\|y_{n-1} - y_{n-2}\|. \tag{3.14}$$

By (3.13) and (3.14), we find that

$$\begin{aligned} \|y_n(x) - y(x)\| &\leq \rho\|y_{n-1} - y\| \\ &= \rho\|y_n - y - (y_n - y_{n-1})\| \\ &\leq \rho\|y_n - y\| + \rho\|y_n - y_{n-1}\|. \end{aligned} \tag{3.15}$$

Formula (3.15) implies that

$$\begin{aligned} \|y_n(x) - y(x)\| &\leq \frac{\rho}{1-\rho}\|y_n - y_{n-1}\| \\ &\leq \frac{\rho^n}{1-\rho}\|y_1 - y_0\|. \end{aligned} \tag{3.16}$$

(3.16) shows that $\parallel y_n(x) - y(x) \parallel \to 0, \quad n \to \infty$. Applying Theorem 3.6 yields

$$\parallel y_{1,N}(x) - y_1(x) \parallel \to 0, \quad N \to \infty.$$

Using the fact that

$$\begin{aligned} \parallel y_{2,N}(x) - y_2(x) \parallel &= \parallel y_{2,N}(x) - \lambda(y_{1,N}(x)) + \lambda(y_{1,N}(x)) - \lambda(y_1(x)) \parallel \\ &\leq \parallel y_{2,N}(x) - \lambda(y_{1,N}(x)) \parallel + \rho \parallel y_{1,N}(x) - y_1(x) \parallel, \end{aligned} \tag{3.17}$$

we obtain

$$\parallel y_{2,N}(x) - y_2(x) \parallel \to 0, \quad N \to \infty.$$

For $n > 2$, in like manner, we have

$$\parallel y_{n,N}(x) - y_n(x) \parallel \to 0, \quad N \to \infty.$$

Note that

$$\begin{aligned} \parallel y_{n,N}(x) - y(x) \parallel &= \parallel y_{n,N}(x) - y_n(x) + y_n(x) - y(x) \parallel \\ &\leq \parallel y_{n,N}(x) - y_n(x) \parallel + \parallel y_n(x) - y(x) \parallel . \end{aligned}$$

Therefore,

$$\parallel y_{n,N}(x) - y(x) \parallel \to 0, \quad N \to \infty, \quad n \to \infty.$$

This completes the proof. □

Theorem 3.8. *Under the hypothesis of Theorem 3.7, if $f(x,u) \in C^{2m}[0,1]$ and $\overline{G}_{k,N}$ are approximated in RKHS $H^m[0,1]$, then*

$$\| y_{n,N}(x) - y(x) \| \leq c\,\frac{h^{2m}}{\omega} + d\,\rho^n,$$

where $c > 0$ and $d > 0$ are constants.

Proof. By employing Theorem 3.7, we have

$$\| y_n(x) - y(x) \| \leq \frac{\rho^n}{1-\rho}\|y_1 - y_0\| = d\ \rho^n,$$

where $d > 0$ is a constant.

According to Theorem 3.5 and (3.17), one obtains

$$\| y_{n,N}(x) - y_n(x) \| \leq c\ \frac{h^{2m}}{\omega}.$$

Hence,

$$\begin{aligned}\| y_{n,N}(x) - y(x) \| &= \| u_{y,N}(x) - y_n(x) + y_n(x) - y(x) \| \\ &\leq \| y_{n,N}(x) - y_n(x) \| + \| y_n(x) - y(x) \| \leq c\ \tfrac{h^{2m}}{\omega} + d\ \rho^n.\end{aligned}$$

□

4. Numerical experiments

Three experiments are illustrated to show the applicability and efficiency of the mentioned approach. All computations associated with the experiments are performed via Mathematica 11.0.

Problem 4.1. *Consider the following singular perturbation BVPs from [15, 16]*

$$\begin{cases} \varepsilon y''(x) + y'(x) = 1 + 2x, \quad x \in (0,1), \\ y(0) = 0, y(1) = 1, \end{cases} \tag{4.1}$$

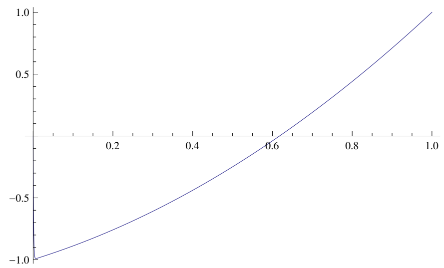
whose true solution is $y(x) = (2\varepsilon - 1)\frac{1-e^{-\frac{x}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}} + x(1 + x - 2\varepsilon)$. Choosing $x_i = \frac{i-1}{N-1}$, we employ the present approach in RKHS H^3 to Problem 4.1. The obtained absolute errors by employing the present method (PM) are compared with the methods in [15, 16] in Table 1. The exact solution and absolute error obtained by our new method with $N = 40$ for $\varepsilon = 10^{-3}$ are shown in Figure 1. It is indicated from the numerical results that our technique has very high accuracy.

Problem 4.2. *Consider the following nonlinear singular perturbation BVPs from [15, 17]*

$$\begin{cases} \varepsilon y''(x) + 2y'(x) + e^y = 0, \quad x \in (0,1), \\ y(0) = y(1) = 0. \end{cases} \tag{4.2}$$

Table 1. Comparison of maximum absolute errors with the methods in [15, 16] for Problem 4.1.

ε	PM(N=16)	PM(N=32)	[15](N=16)	[15](N=32)	[16](N=16)	[16](N=32)
2^{-12}	1.27×10^{-8}	3.38×10^{-10}	4.57×10^{-4}	4.73×10^{-4}	5.81×10^{-2}	2.98×10^{-2}
2^{-20}	1.27×10^{-8}	3.44×10^{-10}	1.78×10^{-6}	1.84×10^{-6}	5.85×10^{-2}	3.02×10^{-2}
2^{-25}	1.27×10^{-8}	3.44×10^{-10}	5.58×10^{-8}	5.77×10^{-8}	5.85×10^{-2}	3.02×10^{-2}
2^{-30}	1.27×10^{-8}	3.44×10^{-10}	1.74×10^{-9}	1.80×10^{-9}	5.85×10^{-2}	3.02×10^{-2}



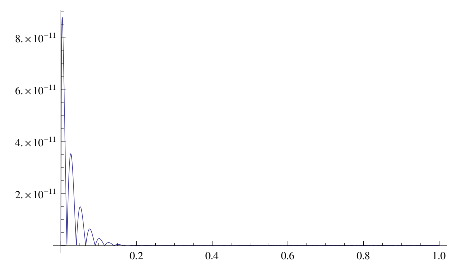


Figure 1. Exact solution (left) and absolute error (right) for $\varepsilon = 10^{-3}$

Its uniformly valid asymptotic solution is $y(x) = \ln(\frac{2}{x+1}) + \ln(2)e^{\frac{-2x}{\varepsilon}}$*, which is used for numerical comparison. Choosing* $x_i = \frac{i-1}{N-1}$ *and iterative step number* $n = 10$*, we employ our technique in RKHS* H^3 *to Problem 4.2. Table 2 lists the comparison of absolute errors with the approach in [17]. Taking* $N = 20$ *and* $N = 40$*, the absolute error obtained for* $\varepsilon = 10^{-9}$ *are shown in Figure 2. It is illustrated that the the method is very promising.*

Table 2. Maximum absolute errors compared with the methods in [17] for Problem 4.2.

ε	PM(N=20)	PM(N=40)	Method in [17](N=20)
10^{-3}	4.55×10^{-7}	1.91×10^{-8}	7.88×10^{-4}
10^{-5}	4.55×10^{-7}	3.48×10^{-8}	7.90×10^{-6}
10^{-7}	5.30×10^{-7}	3.48×10^{-8}	7.90×10^{-8}

Problem 4.3. *Solve the nonlinear singular perturbation BVPs*

$$\begin{cases} \varepsilon y''(x) + 4y'(x) + y^2 = f(x), \quad x \in (0,1), \\ y(0) = 3, y(1) = 1 + e, \end{cases} \tag{4.3}$$

where $f(x)$ *is chosen such that the true solution of (4.3) is* $y(x) = e^{\frac{-4x}{\varepsilon}} + e^x + 1$*. Choosing* $x_i = \frac{i-1}{N-1}$ *and iterative step number* $n = 10$*, we apply our new method in RKHS* H^3 *to Problem 4.3. The obtained absolute errors are for* $\varepsilon = 10^{-5}$*,* 10^{-7} *and* 10^{-9} *are shown in Figures 3,4. It can be observed from Figure 3 that the absolute error is decreased as the number of nodes increases. It can be observed from Figure 4 that our technique is uniformly effective for any sufficiently small* $\varepsilon > 0$*.*

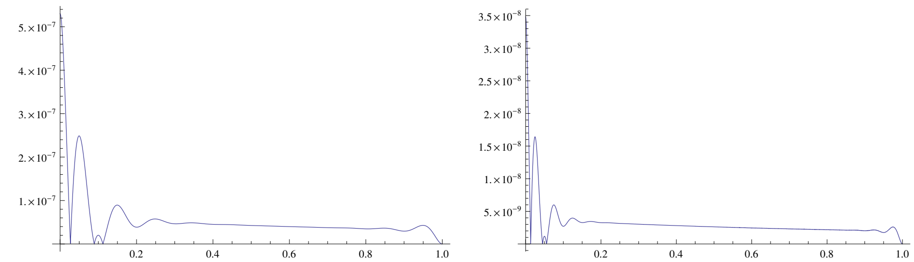


Figure 2. Absolute errors for $\varepsilon = 10^{-9}$ with $N = 20$ (left) and $N = 40$ (right)

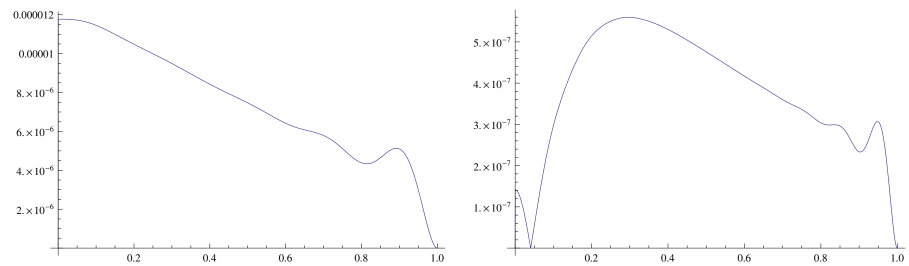


Figure 3. Absolute errors for $\varepsilon = 10^{-5}$ with $N = 10$ (left) and $N = 20$ (right)

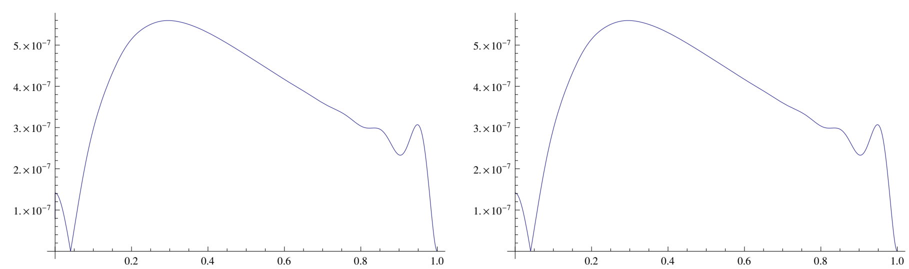


Figure 4. Absolute errors for $\varepsilon = 10^{-7}$ (left) and $\varepsilon = 10^{-9}$ (right) with $N = 20$

5. Conclusion

A new numerical scheme was presented for nonlinear singular perturbation BVPs. One advantage of our technique is that there is no restriction on the choice of nodes. Another advantage is that it can preserve the boundary layer structure of the solution to singular perturbation BVPs. The results of three numerical tests show that our new approach proposed in this paper has higher accuracy.

Acknowledgments

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