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THE EFFECT ON THE SOLUTION OF THE FITZHUGH-NAGUMO EQUATION BY THE EXTERNAL PARAMETER α USING THE GALERKIN METHOD*

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Abstract The Fitzhugh-Naguno equation is one of the most popular and attractive equation in real life. This equation is applicable in many different areas of physics, biology, population genetics and applied sciences to mention a few. In this paper, we design and analyze a coupled scheme consisting of the non-standard finite difference and the Galerkin methods in both time and space variables respectively. We show analytically by the use of the Galerkin method and the compactness theorem that the solution of this equation exists uniquely in appropriate spaces with the parameter α that determines the main dynamics of the equation, under controlled. We further show numerically that the above scheme is stable and converge optimally in specified norms with its numerical solution replicating the qualitative properties of the exact solution. We finally present numerical experiments with the help of an example and a careful choice of α to validate the theoretical results.

Keywords Fitzhugh-Nagumo equation, non-standard finite difference, Galerkin and compactness methods, stability and optimal rate of convergence.

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1. Introduction

Nonlinear models such as the Fitzhugh-Nagumo equation plays a significant role in most studied models in analytical neuroscience /electro-cardiology which covers some scientific, biological and engineering dynamical systems. The significance in this regard, comes from the presence of the nonlinear term that consists of an external parameter α which determines the main dynamics of the model. If α is small, the output of the model becomes more-likely sinusoidal and on the other hand, if α is large, the model produces oscillations of relaxation. The behavior of α in this manner display several phenomena in the diffusion processes [42]. Such processes include the cardiac/neuron dynamics and the active pulse transmission line simulating a nerve axon [15, 29]. The Fitzhugh-Nagumo equation originated from the Hodgkin-Huxley model in 1952 where Hodgkin and Huxley presented their pioneer

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work based on the ionic mechanisms underlying current conduction and excitation of action potential in nerve [16]. The work in this form then opened doors to a new era of electro-physiological studies. In these series of studies, Fitzhugh in 1961 found out that boundary value problems model can serve as a simply representative of a class of excitable-oscillatory systems including the Hodgkin-Huxley model. This was followed in 1962 by Nagumo et al. [29] who confirmed with experimental evidence what Fitzhugh proposed [15]. The experimental confirmation of Nagumo et al. [29] to the work of Fitzhugh in 1961, gave birth to the Fitzhugh-Nagumo equation. The successes of their collaborative model is not only because of its mathematical simplicity and richness from the point of view of the system dynamics, but also because of its correlation to the Hodgkin-Huxley model. Many variations of the Fitzhugh-Nagumo equation have been derived from the Hodgkin-Huxley model where the conditions on the external parameter α is well controlled within threshold. One of the most commonly used model is stated as followed:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u(u-\alpha)(u-1) = 0, \quad (x,t) \in \Omega \equiv [a,b], t \geq 0. \tag{1.1}$$

$$u(a,t) = u(b,t) = 0 \text{ on } \partial\Omega \ \ t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x) \text{ on } \Omega \ \ t = 0, \tag{1.3}$$

where the interval $[a,b]$ is an open bounded domain and the external parameter α considered to play the dormenant role in the fast dynamics of the model. If $\alpha = -1$ which is not interesting to our study, the equation (1.1) will be reduced to a real Newell-Whitehead equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u^3 - u = 0. \tag{1.4}$$

If $\alpha < 0$, the model could produce electrical wave pulse (in other words, excitable) which is still not of interest to our study but plays an important role to the scientific community as can be seen in [1, 15]. If on the other hand $\alpha > 0$, in which case $\alpha \in (0,1)$, then the model is in refractory mode and external stimulations can not provoke action potential. The model in this condition of α attracts a wide range of application because it keeps its qualitative electro-physiological meaning and hence very interesting to our study. Other advantages of the model are also in the area of circuit theory, biology and population genetics found in [7, 30, 35, 37]

So many interesting contributions from physicist and mathematicians have been made to obtain the solution of this equation. Among these famous contributions, were Kawahara et al. [22] who obtained exact solutions of the equation using Hirota method followed by Nucci et al. [31] who computed some new solutions using Jacobic elliptic functions. This was proceeded by Li and Guo [23] who contributed in obtaining new series of exact solutions using integral method. The contributions continued with Alford [4], who studied the equation using the bifurcation structure of rotating wave solutions and followed in 2010 by Van Gorder [41] who used the variational formulation technique to study the solution of the problem under investigation. Other contributions involving finite or Galerkin approximations were presented by Jackson [18, 19]. In [18], he used the method to find the existence and the regularity of the solution of the problem and in [19] he established estimates for obtaining the solution of the problem.

Instead of the numerous interesting contributions from the above outstanding scientists, we exploit the contribution by Jackson and introduce in this paper a different approach consisting of the non-standard finite difference method in the time and the Galerkin method together with the compactness theorem in the space variables denoted by NSFD-GM. With this method, we show analytically by controlling the parameter α as prescribed by Hodgkin-Huxley with the Galerkin method and the compactness theorem that the solution of the Fitzhugh-Nagumo equation exists and is unique in the space

$$L^\infty\left[(0,T);L^2(\Omega)\right]\cap L^2\left[(0,t);H_0^1(\Omega)\right]\cap L^4\left[(0,T);L^4(\Omega)\right]$$

to be defined as we progress. With this determination, we design a numerical scheme NSFD-GM and show that the designed scheme is stable. We proceed with the stable scheme and show that this scheme converges optimally in the L^2 as well as in H^1-norms. We further, proceed to show that the numerical solution from the scheme replicates the decaying properties of the exact solution. Furthermore, with some numerical experiments conducted with the help of an example, we justify that the theory is indeed valid. Hence, we further show numerically that the theory remained valid, irrespective of the effect on the numerical solution of the problem, caused by the variation in the parameter α. The reason for the introduction of this method is not by gamble but because schemes that emanated in the past from the method has always, preserve all the qualitative properties of the exact solution of the problem and the compactness theorem has also helps in this regard to control the effect of the parameter α. On a more interesting note is the fact that, where this method has been used in the past to solve partial or ordinary differential equations, the method has in most cases, performed better than the traditional Euler method. To the best of this author's knowledge, the method has not been used before to solve the Fitzhugh-Nagumo equation. A similar approach was used for the first time to solve a linear heat equation in a nonsmooth domain [10] and also to obtain the optimal convergence of the solution of the wave equation [9]. The method has recently been extended to solve nonlinear differential equations such as Burgers'-Fisher and the Real Ginzburg-Landau equations [11] and [12] respectively. The NSFD method was initiated by Mickens in 1994 found in [27] and major contributions to the foundation of the NSFD method as seen in Anguelov et al. [5, 6] and Lubuma et al. [25, 26] has been extensively applied to a variety of concrete problems in physics, epidemiology, engineering, business and biological sciences to mention a few. For more on application of the technique see [26–28]. For an overview of the said technique see [32]. As regard the comparison of the standard and nonstandard finite difference methods we refer to [27].

The rest of the paper is organized as follows: In section 2, we briefly outline the notation and preliminaries to be used in our work. Followed by section 3 where we gather essential results necessary in the Galerkin method and the compactness theorem to prove the existence and uniqueness of the solution to our problem. In section 4, we shall design the main technique of the paper which is geared to couple the NSFD-GM methods and show that the scheme is optimally convergent. Section 5 will be devoted to some numerical experiments which will be constructed to serve as evidence to justify our theory. Finally, section 6 will serve as the conclusion and future remarks of our work.

2. Notations and preliminaries

Under this section, we assemble some notation and facts that will be very relevant to the analysis of the problem under investigation. These facts include certain fundamental function spaces such as the space of functions which are infinitely differentiable with compact support on Ω denoted by $\mathcal{D}(\Omega)$. Other spaces include the space of distributions on Ω denoted by $\mathcal{D}'(\Omega)$ which is the dual of $\mathcal{D}(\Omega)$. We also need to denote the duality pairing between the spaces $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$ denoted by $\langle \cdot, \cdot \rangle$. We remark at this point that if a function v is locally integrable then v can be identified with distributions by

$$\langle v, \rho \rangle := \int_\Omega v(x)\rho(x)dx, \quad \forall \rho \in \mathcal{D}(\Omega). \tag{2.1}$$

We proceeded to introduce Lebesque spaces spaces denoted by $L^p(\Omega)$ for $1 \leq p \leq \infty$ defined by

$$L^p(\Omega) := \left\{ v : \left(\int_\Omega |v(x)|^p dx \right)^{1/p} < \infty \right\}.$$

This space is a Banach space with the norm defined by

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p dx \right)^{1/p}. \tag{2.2}$$

The above L^p space is followed by the definition of the Sobolev space stated for $m \in \mathbb{N}$ and $p \in \mathbb{R}$ with $1 < p \leq \infty$ by

$$W^{m,p}(\Omega) := \{ v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega), \text{ for all multi index } |\alpha| \leq m \}. \tag{2.3}$$

This is also a Banach space with the norms

$$\|v\|_{m,p,\Omega} = \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^p(\Omega)} \right)^{1/p}, \quad p < \infty. \tag{2.4}$$

and

$$\|v\|_{m,\infty,\Omega} = \sup_{|\alpha| \leq m} \left(\sup_{x \in \Omega} ess |D^\alpha v(x)| \right), \quad p = \infty. \tag{2.5}$$

When $p = 2, W^{m,2}(\Omega)$ is usually denoted by $H^m(\Omega)$ and if there is no ambiguity, we drop the subscript $p = 2$ when referring to its norm and semi-norm. Hence $H^m(\Omega)$ is a Hilbert space for the scalar product

$$\langle w, v \rangle_{m,\Omega} = \sum_{|\alpha| \leq m} \int_\Omega (D^\alpha w, D^\alpha v)\, dx. \tag{2.6}$$

and in particular, we write the scalar product of $L^2(\Omega)$ with no subscript at all. Continuing in the assembling of the relevant tools, we shall most frequently in our problem denote by X the Hilbert space. This according to Lions and Magenes [24] will be more generally use in conjunction with the Sobolev space $H^m[(0,T); X]$

where $m \geq 0$. We will define this as a space of functions in $L^2(\Omega)$ whose distribution derivatives of order up to m are also in $L^2(\Omega)$ with values from $(0,T)$ to X. The norm of this space will be given by

$$\|v\|_{H^m[(0,T);X]} := \left(\sum_{|\alpha|\leq m} \int_0^T \|D^\alpha v\|_X^2 dt\right)^{1/2}. \tag{2.7}$$

In practice, X will either be L^p or $W^{m,p}$ space and in particular $X = L^2, L^4, H_0^1$ in our paper. To conclude this section, it will be good to mention that some important tools like the Hölder, Gronwall's, Young's, Poincaré and Cauchy-Schwarz inequalities to mention a few, will be referred to some standard text books such as [2,13,14,24] and [39] when required. Since we shall also be dealing with a fully-discrete problem, it will also be important that we introduce the discrete framework on which our discrete problem will be analyzed for its solution. To this end, we let $\mathcal{J}_h$ be a regular family of discretization of Ω consisting of compatible intervals $\mathcal{J}$ of sizes $h_{\mathcal{J}} < h$ see [13] for more details. For each mesh size $\mathcal{J}_h$ we associate the finite element space $\mathcal{V}$ of continuous piece-wise linear functions that are zero on the end points defined as following

$$\mathcal{V}_h := \left\{v_h \in C^0(\bar{\Omega}) : v_h|_{\partial\Omega} = 0, v_h|_{\mathcal{J}} \in P_1, \ \ \forall \mathcal{J} \in \mathcal{J}_h\right\} \tag{2.8}$$

where P_1 is the space of polynomial of degree less than or equal to 1. $\mathcal{V}_h$ also will be a finite dimensional space which is contained in the Sobolev space $H_0^m(\Omega)$. If $\{P_j\}_{j=1}^n$ are the interior of end points of $\mathcal{J}_h$, then any function in $\mathcal{V}_h$ is uniquely determined by its values at the point P_j.

3. The Fitzhugh-Nagumo equation

We devote this section to show that the solution of the afore-mentioned equation exists and is unique in the space $L^\infty\left[(0,T); L^2(\Omega)\right] \cap L^2\left[(0,T); H_0^1(\Omega)\right] \cap L^4\left[(0,T); L^4(\Omega)\right]$. We will use the Galerkin method and the compactness theorem to achieve this goal. The process will start by first stating the variational or the weak formulation of equation (1.1)-(1.3) as follows: given the initial solution $u_0 \in H_0^1(\Omega)$, we find $u \in L^\infty\left[(0,T); L^2(\Omega)\right] \cap L^2\left[(0,T); H_0^1(\Omega)\right] \cap L^4\left[(0,T); L^4(\Omega)\right]$ such that for all $t \in (0,T)$ we obtain

$$\left\langle \frac{\partial u}{\partial t}, v\right\rangle + \left\langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right\rangle + \left\langle u^3 - (\alpha+1)u^2 + \alpha u, v\right\rangle = 0, \tag{3.1}$$

$$\langle u(x,0), v\rangle = \langle u_0, v\rangle. \tag{3.2}$$

for all $v \in H_0^1(\Omega)$. The above variational problem (3.1)-(3.2) will be followed by its discrete version, find the discrete solution $u_h : [0,T] \longrightarrow V_h$ such that

$$\left\langle \frac{\partial u_h}{\partial t}, v_h\right\rangle + \left\langle \frac{\partial u_h}{\partial x}, \frac{\partial v_h}{\partial x}\right\rangle + \left\langle (u_h^3 - (\alpha+1)u_h^2 + \alpha u_h), v_h\right\rangle = 0, \tag{3.3}$$

$$\langle u_h(0), v_h\rangle = \langle P_h u_0, v_h\rangle, \ \forall \ v_h \in \mathcal{V}_h, \tag{3.4}$$

where $(\cdot,\cdot)$ denotes the inner product in L^2 and P_h the orthogonal-projection onto $\mathcal{V}_h$. With the above frame-work in place, we are in the position to discuss the error and the convergence of the discrete problem (3.3)-(3.4) to (3.1)-(3.2). To achieve this, we will assume the regularity of the solution u of the problem (3.1)-(3.2). We will also in view of [20] assume that the subspace $\mathcal{V}_h \subset H_0^1(\Omega)$ is such that the corresponding linear elliptic problem admits an $O(h^2)$ error estimate in L^2. More precisely, we assume that the operator P_h with respect to the Dirichlet inner product $(\frac{\partial v}{\partial x}, \frac{\partial w}{\partial x})$, satisfies the inequality

$$\|P_h v - v\| \leq Ch^2 \|v\|_{H^2}, \text{ for } v \in H_0^1 \cap H^2, \tag{3.5}$$

where $\|\cdot\|$ is the usual norm in L^2 and H^2 and some constant C. It is well known in view of [43] that if u is sufficiently smooth on the closed time interval $[0,T]$ and the discrete initial data v_h are suitably chosen, then

$$|u(t) - u_h(t)| \leq C_1(u, C_2, C_3)h^2 \text{ for } t \in [0,T] \tag{3.6}$$

where C_2 is the bound on u and $\frac{\partial u}{\partial x}$ with C_3 the constant in (3.5).

With the above assumptions frame-work in place, we proceed with the variational problem (3.1)-(3.2) by introducing some orthogonal basis of $L^2(\Omega)$ which will be very useful in the approximation of the solution to our problem. These basis will be denoted for $m \in \mathbb{N}$ by $\{e_1, e_2, \cdots, e_m\} \subset H_0^1 \cap H^2(\Omega)$. The basis together with the test function will be spanned by $v \in span\{e_1, e_2, \cdots, e_m\}$ and these will approximate the solution of our problem as follows:

$$u_m = \sum_{i=1}^{m} \gamma_i(t) e_i. \tag{3.7}$$

We will then in view of (3.7) proceed by applying the Galerkin approximation $\{u_m\}$ $m \in \mathbb{N}$ of the Fitzhugh-Nagumo equations (1.1)-(1.3) to satisfy the following equations.

$$\frac{\partial u_m}{\partial t} - \frac{\partial^2 u_m}{\partial x^2} + P_m u_m (u_m - \alpha)(1 - u_m) = 0, \text{ on } \Omega \times (0,T), \tag{3.8}$$

$$u_m(a,t) = P_m u_m(b,t) = 0 \text{ on } \partial\Omega, \tag{3.9}$$

$$u_m(x,0) = P_m u_0(x) \text{ on } \Omega. \tag{3.10}$$

The above equation (3.8)-(3.10) should also be satisfied with $\{u_m\}$ taking values in the finite dimensional subspace $\mathcal{V}_m \subset H_0^1 \cap H^2(\Omega)$ as defined by equation (3.7) and the operator P_m as indicated in equation (3.8)-(3.10) above will denotes the orthogonal projection

$$P_m : H^{-1}(\Omega) \longrightarrow \mathcal{V}_m \subset H^{-1}(\Omega) \tag{3.11}$$

obtained by extending P_m from $L^2(\Omega)$ onto $H^{-1}(\Omega)$ and defined on $H^{-1}(\Omega)$ by

$$P_m\left(\sum_{k \in m} \gamma_m^k(t) u_k\right) = \sum_{k=1}^{m} \gamma_m^k(t) u_k. \tag{3.12}$$

The above connection between the Fitzhugh-Nagumo equation (1.1)-(1.3) and the system of ordinary differential equation (3.8)-(3.10) justifies the fact that the solution of these problems is the same as shown classically in Temam 1997 [40] and Evans 1998 [14].

The above connection provide the frame-work to show that the solution of Fitzhugh-Nagumo exists and is unique. This is achieved thanks to the following Theorem 3.1 of problem (3.8)-(3.10).

Theorem 3.1. *Given the initial solution $u_0 \in H_0^1(\Omega)$, then there exists a unique solution of the Fitzhugh-Nagumo equation (1.1)-(1.3) $u \in L^\infty\left[(0,T);L^2(\Omega)\right] \cap L^2\left[(0,T);H_0^1(\Omega)\right] \cap L^4\left[(0,T);L^4(\Omega)\right]$ and $\frac{\partial u}{\partial t} \in L^2\left[(0,T);H^{-1}(\Omega)\right]$ such that equation (3.1) and (3.2) is satisfied for $\alpha \in (0,1)$.*

Proof. The proof of the above Theorem 3.1 will be done in the following three series of subsections 3.1, 3.2 and 3.3. Subsection 3.1 will show uniform estimates, followed by 3.2 which will address the compactness method and passage to the limit and finally 3.3 where the uniqueness of the solution will be established. □

3.1. Uniform estimates

We proceed under this subsection and replace for the sake of simplicity and notation u_m by u and further take all constants C independent of m. With these organization, we proceed to show that the solution u_m is uniformly bounded in the space $L^\infty\left[(0,T);L^2(\Omega)\right] \cap L^2\left[(0,T);H_0^1(\Omega)\right] \cap L^4\left[(0,T);L^4(\Omega)\right]$. This is achieved by setting $v = u$ in equation (3.1) to have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{L^2}^2 + \int_\Omega (u^3 - (\alpha+1)u^2 + \alpha u)u\,dx = 0. \tag{3.13}$$

When the third term of the left hand side of (3.13) is bounded using the Hölder and Young inequalities for $\epsilon \geq 0$ we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^4}^4 \leq \alpha\|u\|_{L^2}^2 + C(3/2)^{16/3}(\alpha+1)^4 \tag{3.14}$$

after choosing ϵ in such a way that $\frac{3\epsilon}{4} = \frac{1}{2}$. Integrating both sides of (3.14) over the interval $[0,T]$ this yield

$$\begin{aligned}\|u(t)\|_{L^2}^2 + \int_0^T 2\left\|\frac{\partial u(s)}{\partial x}\right\|_{L^2}^2 ds + \int_0^T \|u(s)\|_{L^4}^4 ds \leq& \|u_0\|_{L^2}^2 + \int_0^T \|u(s)\|_{L^2}^2 ds \\ &+ C(\Omega,\alpha)T\end{aligned} \tag{3.15}$$

where $C(\Omega,\alpha) = (\frac{3}{2})^{16/3}(\alpha+1)^4|\Omega|$. Keeping only the term $\|u(t)\|_{L^2}^2$ on the left hand side of (3.15) and applying the Gronwall's inequality, we have

$$\|u(t)\|_{L^2}^2 \leq C\left(\|u_0\|_{L^2}^2 + C(\Omega,\alpha)T\right)e^T \tag{3.16}$$

and hence

$$\int_0^T 2\left\|\frac{\partial u(s)}{\partial x}\right\|_{L^2}^2 ds + \int_0^T \|u(s)\|_{L^4}^4 ds \leq C\left(\|u_0\|_{L^2}^2, C(\Omega,\alpha), T\right), \tag{3.17}$$

after the introduction of (3.16) back into (3.15). Hence, inequalities (3.16) and (3.17) implies that the solution $u(t)$ of equations (3.1)-(3.2) is uniformly bounded in the space $L^\infty\left[(0,T);L^2(\Omega)\right] \cap L^2\left[(0,T);H_0^1(\Omega)\right] \cap L^4\left[(0,T);L^4(\Omega)\right]$. What is left to be proved is the fact that the first term of the left hand side of (3.1) is uniformly bounded as well. This is achieved in view of (3.1) as follows

$$\int_0^T \left|\left\langle\frac{\partial u}{\partial t}, v\right\rangle\right|^2 dx \leq \int_0^T \left|\frac{\partial u}{\partial x}\right|_{L^2}^2 \left|\frac{\partial v}{\partial x}\right|_{L^2}^2 dx$$

$$+\int_0^T \left|\left\langle (u^3-(\alpha+1)u^2+\alpha u, v\right\rangle\right| dx \tag{3.18}$$

from where we have

$$\int_0^T \left|\left\langle \frac{\partial u}{\partial t}, v\right\rangle\right|^2 dx \le \int_0^T \left\|\frac{\partial u}{\partial x}\right\|_{L^2}^2 \left|\frac{\partial v}{\partial x}\right|_{L^2}^2 dx + \alpha \sup_{0\le t\le T} \|u\|_{L^4}^3 \|v\|_{H^1}^2$$
$$+ (\alpha+1) \sup_{0\le t\le T} \|u\|_{L^4}^2 \|v\|_{H^1}^2 + \alpha \sup_{0\le t\le T} \|u\|_{L^2} \|v\|_{H^1}^2 \tag{3.19}$$

after bounding the right hand side of (3.18) by using the Sobolev embedding Theorem on $L^4 \subset H^1$ and taking the supremum on the norms of u and $\frac{\partial u}{\partial x}$. Hence in view of equation (3.19) we conclude that

$$\int_0^T \left\|\frac{\partial u(s)}{\partial t}\right\|_{H^{-1}} ds \le C \tag{3.20}$$

after using the fact that $\|w\|_{H^{-1}} = \sup_{v\in H_0^1} |\langle w, v\rangle|$ with $\|v\|_{H_0^1} \le 1$ and inequality (3.17). With all these analysis above, we can conclude in view of (3.16), (3.17) and (3.20) that the sequence of the solutions $\{u_m\}, m \in \mathbb{N}$ is uniformly bounded in the space

$$L^\infty\left[(0,T); L^2(\Omega)\right] \cap L^2\left[(0,T); H_0^1(\Omega)\right] \cap L^4\left[(0,T); L^4(\Omega)\right].$$

3.2. Compactness method and passage to the limit

This subsection is devoted to show that the approximate sequence of solutions $\{u_m\}, m \in \mathbb{N}$ converges strongly to the solution $u(t)$ instead of the boundedness of the approximate solutions as indicated in subsection 3.1. To this end we proceed by recalling firstly that the approximate solution $u_m(t)$ is obtained and defined on the interval $[0,T]$ as follows:

$$\begin{aligned}
&u_m \text{ is uniformly bounded in } L^\infty\left[(0,T); L^2(\Omega)\right],\\
&u_m \text{ is uniformly bounded in } L^2\left[(0,T); H_0^1(\Omega)\right],\\
&u_m \text{ is uniformly bounded in } L^4\left[(0,T); L^4(\Omega)\right],\\
&\frac{\partial u_m}{\partial t} \text{ is uniformly bounded in } L^2\left[(0,T); H^{-1}.(\Omega)\right].
\end{aligned}$$

Therefore by the following compact embedding

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$$

by Banach-Alaoglu's Theorem found in [] there exists a subsequence of u_m still denoted by u_m such that

$$\begin{aligned}
&u_m \longrightarrow u \text{ weakly star in } L^\infty\left[(0,T); L^2(\Omega)\right],\\
&u_m \longrightarrow u \text{ weakly in } L^2\left[(0,T); H_0^1(\Omega)\right],\\
&u_m \longrightarrow u \text{ weakly in } L^4\left[(0,T);\ L^4(\Omega)\right],\\
&\frac{\partial u_m}{\partial t} \longrightarrow \frac{\partial u}{\partial t} \text{ weakly in } L^2\left[(0,T); H^{-1}.(\Omega)\right].
\end{aligned}$$

In view of the following Theorem 3.2 found in [8] the sequence

$$u_m \longrightarrow u \text{ strongly in } \ L^2\left[(0,T);L^2(\Omega)\right]$$

where $X = H_0^1(\Omega),\ Y = L^2(\Omega)$ and $Z = H^{-1}(\Omega)$.

Theorem 3.2. *Suppose that $X \hookrightarrow Y \hookrightarrow Z$ are Banach spaces where X, Z are reflexive and X is compactly embedding in Y. Let $1 < p < \infty$. If the functions $w_N : (0,T) \longrightarrow X$ are such that $\{w_N\}$ is uniformly bounded in $L^2[(0,T);X]$ and $\left\{\frac{\partial w_N}{\partial t}\right\}$ is uniformly bounded in $L^p[(0,T);Z]$, then there is a subsequence that converges strongly in $L^2[(0,T);Y]$.*

What is left under this subsection, is to show that the solution $u(t)$ satisfies equation (3.1) and (3.2) in the distributional sense. To this end, we introduce another function $\psi(t)$ which is continuously differentiable on $[0,T]$ with values $\psi(0) = 1$ and $\psi(T) = 0$. Taking the variational formulation (3.8)-(3.10) we have by the use of $\psi(t)$

$$\left\langle \frac{\partial u_m}{\partial t}, v\right\rangle \psi(t) + \left\langle \frac{\partial u_m}{\partial x}, \frac{\partial v}{\partial x}\right\rangle \psi(t) + \left\langle u_m^3 - (\alpha+1)u_m^2 + \alpha u_m, v\right\rangle \psi(t) = 0. \quad (3.21)$$

Integrating (3.21) by part over the interval $[0,T]$ using the boundary conditions (3.9) yield

$$\begin{aligned} &-\int_0^T \left\langle u_m, \frac{\partial \psi(t)}{\partial t}\right\rangle v dt + \int_0^T \left\langle \frac{\partial u_m}{\partial x}, \frac{\partial v\psi(t)}{\partial x}\right\rangle dt \\ &+ \int_0^T \left\langle (u_m^3 - (\alpha+1)u_m^2 + \alpha u_m), v\psi(t)\right\rangle dt \\ &= \langle u(0), v\rangle \psi(t). \end{aligned} \quad (3.22)$$

In view of Theorem 3.2, $u_m(t)$ was uniformly bounded which then passing to the limit, we have from (3.22)

$$\begin{aligned} &\int_0^T \left\langle u, \frac{\partial \psi(t)}{\partial t}\right\rangle v dt + \int_0^T \left\langle \frac{\partial u}{\partial x}, \frac{\partial v\psi(t)}{\partial x}\right\rangle dt \\ &+ \int_0^T \left\langle (u^3 - (\alpha+1)u^2 + \alpha u), v\psi(t)\right\rangle dt \\ &= \langle u(0), v\rangle \psi(0) \end{aligned} \quad (3.23)$$

which in particular holds for $\psi(t) \in \mathcal{D}(0,T)$ meaning therefore that u in equation (3.1) is satisfied in the distributional sense. Comparing equations (3.22) and (3.23) we have

$$\langle u(0) - u_0, v\rangle \psi(0) = 0$$

and since $\psi(0) = 1$ then this yields

$$\langle u(0) - u_0, v\rangle = 0 \ \ \forall v \in H_0^1(\Omega)$$

which shows that $u(t)$ satisfies equation (3.2) as required.

3.3. Uniqueness of the solution

In this subsection, we will focus on the proving of the uniqueness of the solution of the Fitzhugh-Nagumo equation (1.1)-(1.3). To this end, we will proceed by letting u_1 and u_2 to be the solution of (1.1)-(1.3) such that $u := u_1 - u_2$. Since the solution u satisfy equation (1.1) and (1.3) where $u|_{\partial\Omega} = 0$, then $u(0) = u_1(0) - u_2(0) = 0$. In view of this, we proceed using equation (1.1) as follows:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + (u_1^3 - (\alpha+1)u_1^2 + \alpha u_1) - (u_2^3 - (\alpha+1)u_2^2 + \alpha u_2) = 0$$

from where we obtain

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u(u_1^2 + u_1u_2 + u_2^2 - (\alpha+1)(u_1+u_2)) + \alpha u = 0$$

and on multiplying this by u with integration over t we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{L^2}^2 =& - \int_\Omega u^2\left(u_1^2 + u_1u_2 + u_2^2 - (\alpha+1)(u_1+u_2)\right)dx \\ &- \alpha\int_\Omega u^2 dx. \end{aligned} \tag{3.24}$$

Estimating the right hand side of (3.24) we obtain

$$\begin{aligned}&\int_\Omega \left|u^2\left(u_1^2 + u_1u_2 + u_1^2\right) - u^2\right| dx \\ \leq& \int_\Omega |u|^2|u_1^2 + u_1u_2 + u_2^2 + (\alpha+1)(u_1+u_2)|dx + \alpha\int_\Omega |u|^2 \\ \leq& \|u\|_{L^2}^2\left(|u_1|_{H^1}^2 + |u_1|_{H^1}|u|_{H^1} + |u_1|_{H^1}^2 + (\alpha+1)(|u_1|_{H^1} + |u_2|_{H^1}) + \alpha\right)\end{aligned} \tag{3.25}$$

after using Cauchy-Schwartz inequality on the right hand side of (3.25) and also the fact that $H^1 \subset L^\infty$. Re-introducing the inequality (3.25) into (3.24) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{L^2}^2 \\ \leq& \|u\|_{L^2}^2\left(|u_1|_{H^1}^2 + |u_1|_{H^1}|u|_{H^1} + |u_1|_{H^1}^2 + (\alpha+1)(|u_1|_{H^1} + |u_2|_{H^1}) + \alpha\right)\end{aligned}$$

from where we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{L^2}^2 \leq C\|u\|_{L^2}^2\mathcal{Y} \tag{3.26}$$

where $\mathcal{Y} = |u_1|_{H^1}^2 + |u_1|_{H^1}|u|_{H^1} + |u_1|_{H^1}^2 + (\alpha+1)(|u_1|_{H^1} + |u_2|_{H^1}) + \alpha$. Integrating (3.26) over the time interval $(0, T)$ keeping only the term $\|u\|_{L^2}^2$ on the left hand side we obtain

$$\|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2 e^{\int_0^T \mathcal{Y}(t)dt} = 0, \quad \forall t \geq 0$$

after applying the Gronwall inequality. Hence this proves the uniqueness of the solution of the problem.

4. The design of the NSFD-GM scheme

Instead of the Galerkin method and the compactness theorem used in solving the afore mentioned problem (1.1)-(1.3) in section 3, we exploit and present in this section, a reliable scheme consisting of a non-standard finite difference method in the time and the Galerkin method in the space variables abbreviated by (NSFD-GM). We show that this numerical scheme is stable. The stability of the scheme will lead us to show also that the solution attains optimal convergence in both the L^2 as well as the H^1-norms. The analysis to achieve these goals will be proceeded by considering the discretization on $(0,T)$ and letting the step size in this interval be $t_n = n\Delta t$ for $n = 0, 1, 2, \cdots N$. With this, we find the NSFD-GM approximation $\{U_h^n\}$ such that $U_h^n \approx u_h^n$ at each discrete time t_n in the space $\mathcal{V}_h$ for sufficiently smooth functions. This approximation frame-work define the NSFD-GM scheme to consists of one which finds the fully discrete solution of the Fitzhugh-Nagumo equation $U_h^n \in \mathcal{V}_h$ for $v_h \in \mathcal{V}_h$ such that for all $v_h \in \mathcal{V}_h \subset H_0^1(\Omega)$ we have

$$\langle \delta_n U_h^n(t), v_h\rangle + \left\langle \frac{\partial U_h^n}{\partial x}, \frac{\partial v_h}{\partial x} \right\rangle = \left\langle \left(U_h^{3n} - (\alpha+1)U_h^{2n} + \alpha U_h^n\right), v_h \right\rangle = 0, \tag{4.1}$$

$$\langle U_h^n, v_h\rangle = \langle P_h u_0, v_h\rangle, \tag{4.2}$$

which is satisfied with

$$\delta_n U_h^n = \frac{U_h^n - U_h^{n-1}}{\phi(\Delta t)} \quad \text{and} \quad \phi(\Delta t) = \frac{e^{\lambda\Delta t} - 1}{\lambda}. \tag{4.3}$$

We clarify at this point that the function $\phi(\Delta t)$ is special and complicated is restricted between 0 and 1 and is also in such a way that

$$0 < \phi(\Delta t) < 1 \quad \text{for} \quad n = 1, 2, 3, \cdots, N. \tag{4.4}$$

If the nonlinear term on the left hand side of (4.1) is made very small so much so that the effect is negligible or even zero, then the scheme (4.1) will coincide with the exact scheme

$$\langle \delta_n U_h^n(t), v_h\rangle + \left\langle \frac{\partial}{\partial x} U_h^n, \frac{\partial v_h}{\partial x} \right\rangle = 0 \tag{4.5}$$

which according to Mickens [27], preserves the decay to zero which are the main features of the exact solution (1.1)-(1.3).

With the above frame-work in place, we proceed to show in the next subsection that the scheme (4.1)-(4.2) is stable. This will be shown thanks to the adapted result from [36] and for details see [12].

Lemma 4.1. *Let a^n, b^n be two positive series satisfying*

$$\frac{a^{n+1} - a^n}{\phi(\Delta t)} + \alpha a^{n+1} < b^n$$

where $b^n < b$, $\forall\, n \geq 0$ and $0 < \phi(\Delta t) < 1$ for each Δt. Then

$$a^n \leq \frac{1}{1 + \phi(\Delta t\ \alpha)^n} a^0 + \frac{1 + \phi(\Delta t)\ \alpha}{\alpha} \left(1 - \frac{1}{(1 + \phi(\Delta t)\ \alpha)^{n+1}}\right) b, \quad \forall\, n \geq 0$$

provided $\phi(\Delta t), 1 + \phi(\Delta t) > 0$.

With the above results in place we proceed to study the stability of the scheme as follows:

4.1. Stability of the NSFD-GM scheme

This subsection is preserved for the stability result of the scheme (4.1)-(4.2). That is, we show that the numerical solution of the NSFD-GM scheme is uniformly bounded in the following Theorem 4.1

Theorem 4.1. *Assume that the solution of the Fitzhugh-Nagumo equation u in equation (3.1)-(3.2) is regular in $H^2(\Omega)$. Then given $U_h^0 \in \mathcal{V}_h$, we show that the solution $U_h^n(t)$ of the NSFD-GM scheme (4.1)-(4.2) remain bounded in the following sense*

$$|U_h^n|^2 \leq |U_h^0|^2 + 4\phi(\Delta t)C(\alpha, \Omega), \tag{4.6}$$

$$\sum_{n=1}^{N} |U^n - U_h^{n-1}|^2 \leq |U_h^0|^2 + 4\phi(\Delta t)C(\alpha, \Omega). \tag{4.7}$$

Proof. The proof of the above Theorem 4.1 is proceeded by setting $v_h = U_h^n(t)$ in inequality (4.1) to yield

$$\begin{aligned}&\left\langle U_h^n(t) - U_h^{n-1}(t), U_h^n(t)\right\rangle + \phi(\Delta t)\left\|\frac{\partial U_h^n}{\partial x}\right\|_{L^2}^2 + \phi(\Delta t)\|U_h^n\|_{L^4}^4\\ \leq&\phi(\Delta t)\|U_h^n\|_{L^2}^2 + \phi(\Delta t)\left(\frac{3}{2}\right)^{16/3}(\alpha+1)^4|\Omega|\end{aligned}$$

from where we have thanks to the inequalities (3.15), (4.3) and $C(\Omega, \alpha) = \left(\frac{3}{2}\right)^{16/3}(\alpha+1)^4|\Omega|$.

$$\begin{aligned}&\left\langle U_h^n(t) - U_h^{n-1}(t), U_h^n(t)\right\rangle + 2\phi(\Delta t)\left\|\frac{\partial U_h^n}{\partial x}\right\|_{L^2}^2 + \phi(\Delta t)\|U_h^n\|_{L^4}^4\\ \leq&2\alpha\phi(\Delta t)\|U_h^n\|_{L^2}^2 + 2\phi(\Delta t)C(\alpha, \Omega).\end{aligned} \tag{4.8}$$

It is well known in view of (4.8) that the first term of the left hand side is given by

$$\left\langle U_h^n(t) - U_h^{n-1}(t), U_h^n(t)\right\rangle = \frac{1}{2}|U_h^n|^2 - \frac{1}{2}|U_h^{n-1}|^2 + \frac{1}{2}|U_h^n - U_h^{n-1}|^2.$$

and re-introducing this back into (4.8) yield

$$\begin{aligned}&|U_h^n|^2 - |U_h^{n-1}|^2 + |U_h^n - U_h^{n-1}|^2 + 4\phi(\Delta t)\left\|\frac{\partial U_h^n}{\partial x}\right\|_{L^2}^2 + 2\phi(\Delta t)\|U_h^n\|_{L^4}^4\\ \leq& 4\phi(\Delta t)\|U_h^n\|_{L^2}^2 + 4\phi(\Delta t)C(\alpha, \Omega).\end{aligned} \tag{4.9}$$

The summing of the above inequality (4.9) for $n = 1, 2, 3, \cdots, N$ we obtain

$$\begin{aligned}&|U_h^n|^2 + \sum_{n=1}^{N}|U_h^n - U_h^{n-1}|^2 + 4\phi(\Delta t)\sum_{n=1}^{N}\left\|\frac{\partial U_h^n}{\partial x}\right\|_{L^2}^2 + 2\phi(\Delta t)\sum_{n=1}^{N}\|U_h^n\|_{L^4}^4\\ \leq&4\alpha\phi(\Delta t)\sum_{n=1}^{N}\|U_h^n\|_{L^2}^2 + |U_h^0|^2 + 4\phi(\Delta t)C(\alpha, \Omega).\end{aligned} \tag{4.10}$$

Hence in view of inequalities (3.16) and (3.17) we can obtain the results (4.6) and (4.7) directly from (4.10) as required. □

4.2. Convergence of the NSFD-GM scheme

Unlike the stability of the Fitzhugh-Nagumo scheme presented in subsection 4.1, we in this subsection show that the convergence rate of the afore-mentioned scheme is optimal in the L^2 as well as the H^1-norm. Furthermore, we show that the numerical solution from the scheme preserves or replicates the decaying properties of the exact solution. This is achieved by first stating without proof some important result needed to prove the main result. For more on this result see Shen [36].

Lemma 4.2. *Let Δt, γ and a_k, b_k, d_k, γ_k for the integer $k \geq 0$ be non-negative numbers such that*

$$a_J + \sum_{k=0}^{J} b_k \Delta t \leq \sum_{k=0}^{J} d_k a_J \Delta t + \sum_{k=0}^{J} \gamma_k \Delta t + \gamma, \quad \forall\ J \geq 0. \tag{4.11}$$

Suppose that

$$d_k \Delta t < 1 \ \ \textit{and set} \ \ \sigma_k = (1 - d_k \Delta t)^{-1}, \ \forall\ k \geq 0. \tag{4.12}$$

Then we have

$$a_J + \sum_{k=0}^{J} b_k \Delta t \leq \exp\left(\sum_{k=0}^{J} d_k \Delta t\right)\left(\sum_{k=0}^{J} \gamma_k \Delta t + \gamma\right) \ \forall\ J \geq 0. \tag{4.13}$$

With the above Lemma 4.2 and NSFD-GM frame-work in mind, we can then state and prove the error estimate in the next theorem 4.2

Theorem 4.2. *Assume that Φ_k be a non-negative number and the continuous and discrete solution of the Fitzhugh-Nagumo equation (3.1)-(3.2) and (4.1)-(4.2) respectively exists and are unique together with $\frac{\partial^2 u}{\partial t^2} \in L^2\left[(0,T); H^{-1}(\Omega)\right]$ satisfying*

$$\Phi_k \phi(\Delta t) < 1 \ \ \textit{and} \ \ \sigma_k = (1 - \Phi_k \phi(\Delta t))^{-1}, \ \forall\ k \geq 0.$$

Then we have

$$\|u(t_J) - U_h(t_J)\| + \phi(\Delta t) \sum_{k=0}^{J} \left|\frac{\partial}{\partial x}(u(t_J) - U_h(t_J))\right|^2 \leq C(t_J)(\phi(\Delta t))^2, \ \ \forall J \geq 0. \tag{4.14}$$

Proof. To prove this above theorem, we use the implicit non-standard finite difference in time as follows:

$$\frac{U_{n+1} - U_n}{\phi(\Delta t)} = \frac{\partial^2 U_{n+1}}{\partial x^2} - \left(U_{n+1}^3 - (\alpha+1)U_{n+1}^2 + \alpha U_{n+1}\right). \tag{4.15}$$

This is proceeded by the use of the non-standard Taylor's integral Theorem on the discrete equation of (1.1) as follows:

$$\begin{aligned}\frac{u(t_{n+1}) - u(t_n)}{\phi(\Delta t)} &= \frac{\partial u(t_{n+1})}{\partial t} - \frac{1}{2}\int_{t_n}^{t_{n+1}} \frac{\partial^2 u(t)}{\partial t^2}(t_{n+1} - t)dt, \\ &= \frac{\partial^2 u(t_{n+1})}{\partial x^2} - \left(u^3(t_{n+1}) - (\alpha+1)u^2(t_{n+1}) + \alpha u(t_{n+1})\right)\end{aligned}$$

$$-\frac{1}{2}\int_{t_n}^{t_{n+1}}\frac{\partial^2 u(t)}{\partial t^2}(t_{n+1}-t)dt. \tag{4.16}$$

Combining (4.15) and (4.16) for $\Theta_n = u(t_n) - U_n$ we have

$$\begin{aligned}
&\frac{1}{\phi(\Delta t)}\left[\Theta_{n+1}-\Theta_n,\Theta_{n+1}\right]\\
&=\Big\langle\Big(u^{3(n+1)}-(\alpha+1)u^{2(n+1)}+\alpha u^{n+1}\Big)-\big(U_{n+1}^3-(\alpha+1)U_{n+1}^2+\alpha U_{n+1}\big)\,,\Theta_{n+1}\Big\rangle\\
&\quad-\left\|\frac{\partial\Theta_{n+1}}{\partial x}\right\|_{L^2}^2-\frac{1}{2}\int_{t_n}^{t_{n+1}}\left\langle\frac{\partial^2 u(t)}{\partial t^2},\Theta_{n+1}\right\rangle(t-t_{n+1})dt
\end{aligned} \tag{4.17}$$

after setting $u^{n+1} = u(t_{n+1})$ and multiplying equation (4.15) by Θ_{n+1}. Estimating the first term of the right hand side of (4.17) yield

$$\begin{aligned}
&\int_\Omega\Big|\Big(\Big(u^{3(n+1)}-(\alpha+1)u^{2(n+1)}+\alpha u^{n+1}\Big)-\big(U_{n+1}^3-(\alpha+1)U_{n+1}^2+\alpha U_{n+1}\big)\Big)\,,\Theta_{n+1}\Big|dx\\
&\le\int_\Omega\left|\Theta_{n+1}^2\left(u^{2(n+1)}+u^{n+1}U_{n+1}+U_{n+1}^2+1+(\alpha+1)u^{n+1}+U_{n+1}+\alpha\right)\right|dx\\
&\le\|\Theta_{n+1}\|^2\big(|u^{n+1}|_{H^1}^2+|u^{n+1}|_{H^1}|U_{n+1}|_{H^1}+|U_{n+1}|_{H^1}^2\\
&\quad+(\alpha+1)(|u^{n+1}|_{H^1}+|U_{n+1}|_{H^1})+\alpha\big)
\end{aligned} \tag{4.18}$$

after using the Cauchy-Schwartz inequality on the right hand side of (4.18) and also the fact that $H^1 \subset L^\infty$ and $u^{n+1}, U_{n+1} \in L^2[(0,T), H_0^1]$. Estimating the third term of the right hand side of equation (4.17) we have

$$\begin{aligned}
&\left|\frac{1}{2\phi(\Delta t)}\int_{t_n}^{t_{n+1}}\left(\frac{\partial^2 u(t)}{\partial t^2},\Theta_{n+1}\right)(t-t_{n+1})\,dt\right|\\
&\le\frac{C}{2\phi(\Delta t)}\left|\frac{\partial\Theta_{n+1}}{\partial x}\right|_{L^2}\int_{t_n}^{t_{n+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|_{H^{-1}}|t-t_{n+1}|dt
\end{aligned} \tag{4.19}$$

because

$$\left\langle\frac{\partial^2 u}{\partial t^2},\Theta_{n+1}\right\rangle_{H^{-1}}\le\left|\frac{\partial^2 u}{\partial t^2}\right|_{H^{-1}}|\Theta_{n+1}|_{H_0^1}$$

and also in view of Poincaré inequality, $|\Theta_{n+1}|_{H_0^1} \le C\left|\frac{\partial\Theta_{n+1}}{\partial x}\right|_{L^2}$. Using Cauchy-Schwartz or Hölder's inequality on (4.19) we have

$$\begin{aligned}
&\left|\frac{1}{2\phi(\Delta t)}\int_{t_n}^{t_{n+1}}\left(\frac{\partial^2 u(t)}{\partial t^2},\Theta_{n+1}\right)(t-t_{n+1})\,dt\right|\\
&\le\frac{C}{2\phi(\Delta t)}\left|\frac{\partial\Theta_{n+1}}{\partial x}\right|_{L^2}\left(\int_{t_n}^{t_{n+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2dt\right)^{1/2}\left(\int_{t_n}^{t_{n+1}}|t-t_{n+1}|^2\,dt\right)^{1/2}.
\end{aligned} \tag{4.20}$$

But we have for $t_n < t < t_{n+1}$ that there exists a $\phi(t_n) < \phi(t) < \phi(t_{n+1})$ such that $|\phi(t) - \phi(t_{n+1})| = \phi(\Delta t) = |t - t_{n+1}|\Delta t$. Hence,

$$\left(\int_{t_n}^{t_{n+1}}|t-t_{n+1}|\right)^{1/2}\le\phi(\Delta t)(t-t_{n+1})^{1/2}\le\phi(\Delta t)^{3/2}$$

and re-introducing it back into the estimate in (4.20) yield

$$\left|\frac{1}{2\phi(\Delta t)}\int_{t_n}^{t_{n+1}}\left(\frac{\partial^2 u(t)}{\partial t^2},\Theta_{n+1}\right)(t-t_{n+1})\,dt\right|$$
$$\leq C\phi(\Delta t)^{1/2}\left(\int_{t_n}^{t_{n+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dt\right)^{1/2}\left|\frac{\partial\Theta_{n+1}}{\partial x}\right|_{L^2}. \tag{4.21}$$

Using Cauchy-Schwart inequality on the right hand side of inequality (4.21), we obtain

$$\left|\frac{1}{\phi(\Delta t)}\int_{t_n}^{t_{n+1}}\left(\frac{\partial^2 u(t)}{\partial t^2},\Theta_{n+1}\right)(t-t_{n+1})\,dt\right|$$
$$\leq \frac{1}{2}\left|\frac{\partial\Theta_{n+1}}{\partial x}\right|_{L^2}^2+\frac{C}{2}\phi(\Delta t)\int_{t_n}^{t_{n+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|_{H^{-1}}^2 dt. \tag{4.22}$$

Re-introducing (4.18) and (4.22) into (4.17) and the fact that

$$\langle\Theta_{n+1}-\Theta_n,\Theta_{n+1}\rangle=\frac{1}{2}\left[|\Theta_{n+1}|_{L^2}^2-|\Theta_n|_{L^2}^2+|\Theta_{n+1}-\Theta_n|_{L^2}^2\right],$$

we have after some manipulations

$$\frac{1}{\phi(\Delta t)}\left[|\Theta_{n+1}|_{L^2}^2-|\Theta_n|_{L^2}^2+|\Theta_{n+1}-\Theta_n|_{L^2}^2\right]+\frac{1}{2}\left\|\frac{\partial\Theta_{n+1}}{\partial x}\right\|_{L^2}^2$$
$$\leq\|\Theta_{n+1}\|_{L^2}^2\,\Psi_{n+1}+\Phi_{n+1}C \tag{4.23}$$

where

$$\Psi_{n+1}=|u^{n+1}|_{H^1}^2+|u^{n+1}|_{H^1}|U_{n+1}|_{H^1}+|U_{n+1}|_{H^1}^2+(\alpha+1)(|u^{n+1}|_{H^1}+|U_{n+1}|_{H^1})+\alpha,$$
$$\Phi_{n+1}=\phi(\Delta t)\int_{t_n}^{t_{n+1}}\left|\frac{\partial^2 u}{\partial t^2}|\right|_{H^{-1}}^2 dt.$$

Multiplying all through (4.23) by $\phi(\Delta t)$ we obtain

$$|\Theta_{n+1}|_{L^2}^2-|\Theta_n|_{L^2}^2+\phi(\Delta t)\left\|\frac{\partial\Theta_{n+1}}{\partial x}\right\|_{L^2}^2$$
$$\leq\phi(\Delta t)\,\|\Theta_{n+1}\|_{L^2}^2\,\Psi_{n+1}+C\phi(\Delta t)\Phi_{n+1}. \tag{4.24}$$

Setting some terms in the inequality (4.24) as $a_k=|\Theta_{n+1}|_{L^2}^2$ and $b_k=\left\|\frac{\partial\Theta_{n+1}}{\partial x}\right\|_{L^2}^2$ and summing for $k=0,1,\cdots,n-1$ and also using the fact that $a_0=\Theta_0=u_0-U_0=0$, we obtain

$$a_n+\sum_{k=0}^n b_k\phi(\Delta t)\leq\sum_{k=0}^n\Psi_k a_k\phi(\Delta t)+\sum_{k=0}^n\Phi_k\phi(\Delta t). \tag{4.25}$$

Applying Lemma 4.2 to inequality (4.25)

$$a_n+\sum_{k=0}^n b_k\phi(\Delta t)\leq\exp\left(\sum_{k=0}^n\sigma_k\Psi_k\phi(\Delta t)\right)\left(\sum_{k=0}^n\Phi_k\phi(\Delta t)\right),\quad\forall\, n\geq 0 \tag{4.26}$$

provided

$$\Psi_k\phi(\Delta t) < 1 \text{ and } \sigma_k = (1-\Psi_k\phi(\Delta t))^{-1} \quad \forall\, k \geq 0.$$

Since $a_n,\ b_k,\ \Psi_k$ and Φ_k are all positive series, then in view of Lemma 4.2

$$a_n + \sum_{k=0}^{n} b_k\phi(\Delta t) \leq C(t)\phi(\Delta t)^2.$$

and hence the proof of Theorem 4.2 is completed. □

With the above error estimate in place, we can in the next Theorem 4.3 state the main theoretical result of the paper.

Theorem 4.3. *Under the assumption of Theorem 4.2, the numerical solution of the Fitzhugh-Nagumo equation (4.1)-(4.2) using the $NSFD-GM$ method attains the following optimal rate of convergence*

$$\|u(t)-U_h(t)\|_{L^2} \leq C(t)(h^2+\phi(\Delta t)), \quad \forall t \geq 0 \tag{4.27}$$

where $C(t)$ depends on t. Furthermore, the discrete solution $U_h(t)$ preserves all the qualitative properties of the exact solution of the nonlinear Fitzhugh-Nagumo equation under investigation.

Proof. We use the following error decomposition to prove the above theorem

$$\begin{aligned}\|u(t_n)-U_h(t_n)\|_{L^2} &= \|u(t_n)-P_hu(t_n)+P_hu(t_n)-U_h(t_n)\|_{L^2}\\ &\leq \|u(t_n)-P_hu(t_n)\|_{L^2} + \|P_hu(t_n)-U_h(t_n)\|_{L^2}\\ &\leq \|\xi_n\|_{L^2} + \|\eta_n\|_{L^2}\,,\end{aligned} \tag{4.28}$$

where $\xi_n = u(t_n)-P_hu(t_n)$ is the representation of the error inherent in the Galerkin approximation of the linearized Fitzhugh-Nagumo equation and $\eta_n = P_hu(t_n) - U_h(t_n)$ the error caused by the nonlinearity in the problem. Hence in view of the equation (3.6) and Theorem 4.2, we have from inequality (4.28)

$$\begin{aligned}\|u(t_n)-U_h(t_n)\|_{L^2} &\leq C(t_{n+1})h^2 + \sup_{t\in[t_n,t_{n+1}]} \|P_hu(t_{n+1})-U_h(t_{n+1})\|_{L^2}\\ &\leq C(t_{n+1})h^2 + C(t_{n+1})\phi(\Delta t)^2, \quad \forall\ t\in[t_n,t_{n+1}].\end{aligned} \tag{4.29}$$

In view of inequality (4.29) we can conclude without any difficulties that inequality (4.27) is completed.

As for the replication of the decaying properties of the exact solution, we proceed by first high-lighting from Mickens [27] that the above scheme was designed for

$$\phi(\Delta t) = \frac{e^{\lambda\Delta t}-1}{\lambda} \approx \Delta t + O(\Delta t)^2.$$

Based on this high-light, we can observe that as $\Delta t \longrightarrow 0,\ \phi(\Delta t) \approx \Delta t$. In view of this fundamental principle of the above scheme, we deduce that the numerical solution of the NSFD-GM scheme $U_h^n \in \mathcal{V}_h \subset H_0^1(\Omega)$ converges point-wise in $H_0^1(\Omega)$ to u as $\Delta t \longrightarrow 0$ by the compactness Theorem 3.1. This is justified as follows: If we choose the data of our scheme in equation (4.1) to be $\mathbf{F} \in L^2\left[(0,T);H^{-1}(\Omega)\right]$, then we have

$$\langle \delta_n U_h^n(t), v_h\rangle + \left\langle \frac{\partial}{\partial x}U_h^n, \frac{\partial v_h}{\partial x}\right\rangle + \left\langle \left(U_h^{3n}-(\alpha+1)U_h^{2n}+\alpha U_h^n\right), v_h\right\rangle = \mathbf{F}. \tag{4.30}$$

If we in addition, let the support of $\mathbf{F}$ be very small that the test function $v_h = 1$ far inside the support say $\Omega_1 \subset \Omega$ and $\mathbf{F}$ being regular, then integrating equation (4.30) over Ω will culminate to

$$\int_\Omega \mathbf{F} v_h dx = \mathbf{F}(a) \text{ measure over the } supp(v_h), \quad a \in \Omega_1.$$

Thus, the uniform convergence of the solution U_h^n over Ω is equivalent to the pointwise convergence of the scheme (4.30). For more on such analysis see [2]. Hence, $U_h^n(a)$ is the NSFD-GM solution converging to u and possessing all the qualities of u in (4.5). This justification therefore complete the second part of the proof of Theorem 4.3 which is the main result of the paper. □

5. Numerical experiments

This section is reserved for numerical experiments to justify the analysis of the theory presented in subsection 4.2 above. These experiments will be carried out using the computer software MATLAB 7.10.0($R2014a$). The software will be used after we first design codes to implement the algorithm that are geared toward solving equation (1.1)-(1.3) by adopting the NSFD-GM scheme (4.1)-(4.2). With the scheme, we evaluate the discrete solution at every discrete points of the domain $\Omega = (-8, 8)$ where Ω is a regular partition of mesh size h in the space variables and Δt in the time variable $(0, T)$ as well. With these discretization process in place, we proceed to compute the discrete solution of the scheme by considering the maximum time $T = 0.098$ and $\Delta t = 0.01$. The complicated non-standard function $\phi(\Delta t)$ is evaluated in such a way that $\lambda = 5$ and $h = \frac{1}{M}$ where M denotes the number of nodes in the discretization and $\alpha = 1/3$. The initial solution is taken to be

$$u(x, 0) = 1/2 + 1/2 \tanh(\sqrt{2}x/4). \tag{5.1}$$

This is followed by choosing the exact solution as in [3]:

$$u(x, t) = 1/2 + 1/2 \tanh\left(\frac{\sqrt{2}x + (1 - 2\alpha)t}{4}\right). \tag{5.2}$$

and introducing this solution into the left hand side of equation (1.1) we obtain the data function f. The choice of $u(x, t)$ above leads us to compute the NSFD-GM approximate solution in the scheme (4.1).

The same experiment is repeated but this time by replacing the complicated denominator $\phi(\Delta t)$ with Δt giving the SFD-GM which is the standard finite difference and Galerkin method. All these set of experiments lead to the following results illustrated on the figures below:

Figure 1 illustrates the exact solution of both experiments and Figure 2, the computed solutions of both NSFD-GM and SFD-GM. Figure 3 consists of eight figures, which demonstrate the effect on the computed solutions of the parameter α in and out of the interval (0,1) as prescribed by Hodgkin-Huxley. With the above display of the effect of the parameter α on the computed solution of the problem, together with illustration of the exact and computed solutions resulting from the afore-mentioned experiments, we compute both errors and rate of convergence in L^2 as well as H^1-norms. The latter results are displayed on the two tables below.

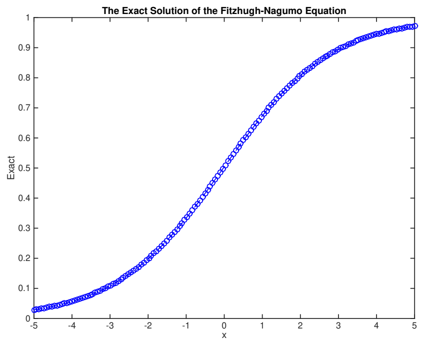


Figure 1. Exact solution of the NSFD and SFD-GM Schemes

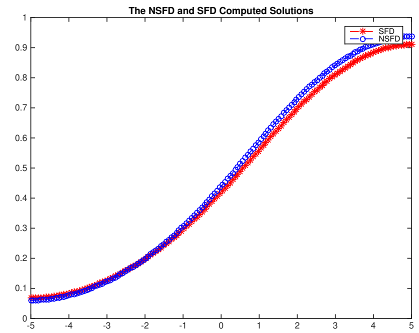


Figure 2. Approximate solution for NSFD and SFD- GM Schemes

Table 1 display the NSFD-GM errors and rate of convergence in both the L^2 and H^1-norms and Table 2 the SFD-GM errors and rate of convergence in both L^2 as well as H^1-norms.

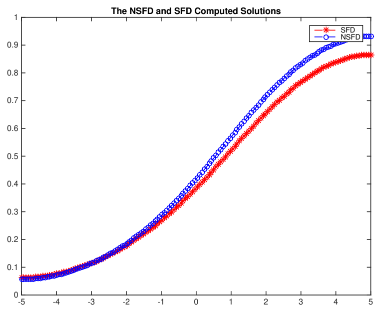
Table 1. NSFD-GM Error in both L^2 and H^1-norms

Nodes	L^2-Errors	Rate L^2-Error	H^1-Errors	Rate H^1-Errors
50	4.4615E-05		1.2201E-02	
100	1.4515E-05	1.62	7.4587E-03	0.71
150	7.3449E-06	1.68	5.3925E-03	0.80
200	4.2277E-06	1.92	4.2106E-03	0.86
250	2.7057E-06	2.0	3.3987E-03	0.96

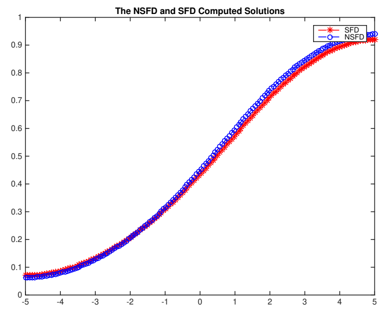
Table 2. SFD-GM Error in both L^2 and H^1-norms

Nodes	L^2-Errors	Rate L^2-Error	H^1-Errors	Rate H^1-Errors
50	1.4814E-04		1.2300E-03	
100	4.8868E-05	1.60	7.6772E-04	0.68
150	2.4829E-05	1.67	5.5957E-04	0.78
200	1.4922E-05	1.77	4.4453E-04	0.80
250	9.5928E-06	1.98	3.6203E-04	0.92

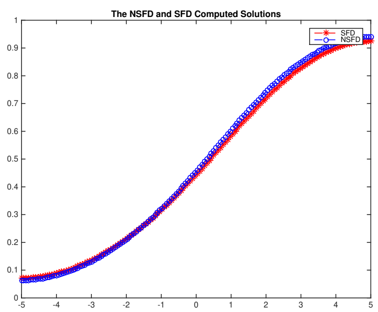
Observations.
Figure 2 shows that the results of the NSFD-GM very close to SFD-GM. As regard Figures 3(a) down to 3(g), the NSFD-GM and the SFD-GM remain very comparable to the exact solution with the variation of parameter α decreasing from $\alpha = 0.25$ to 0.11. Furthermore, when $\alpha = 1$ both the NSFD-GM and SFD-GM become very unstable as can be seen in Figure 3(h). This is simply from the fact that the prescription of Hodgkin-Huxley requires α to lie within the interval (0,1). When $\alpha = 0$ and below, the solution becomes instead that of a different problem called Newell-Whitehead equation as mentioned earlier in section 1.



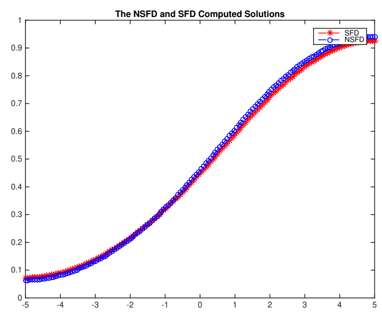
(a) Computed solution for $\alpha = 0.50$



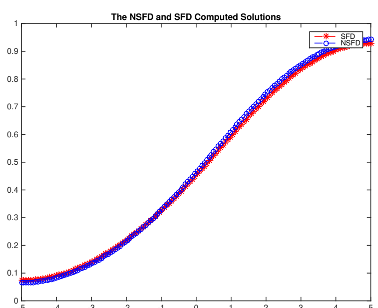
(b) Computed solution for $\alpha = 0.25$



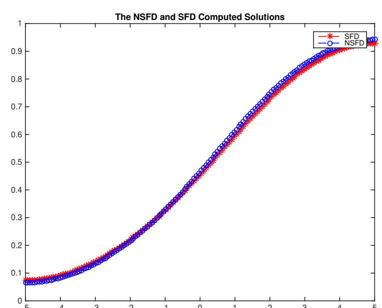
(c) Computed solution for $\alpha = 0.20$



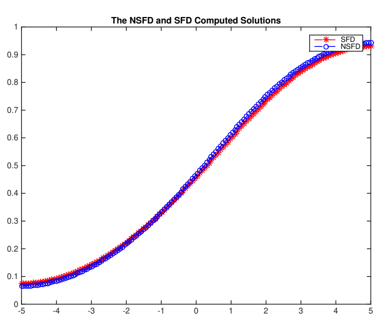
(d) Computed solution for $\alpha = 0.17$



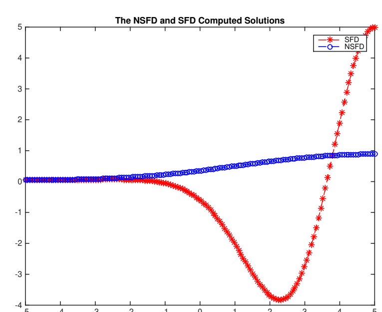
(e) Computed solution for $\alpha = 0.14$



(f) Computed solution for $\alpha = 0.13$



(g) Computed solution for $\alpha = 0.11$



(h) Computed solution for $\alpha = 1$

Figure 3. The Effect of α to the Computed Solutions of the SFD and NSFD

Our expectations were that the rate of convergence in the L^2-norm will be approximately 2 and that of H^1-norm will be approximately 1 using both the NSFD-GM and SFD-GM schemes. Based on the results displayed on the above tables, we observe that the rates of convergence in both schemes seems to exhibit some closeness with the NSFD-GM performing better than the SFD-GM in L^2 as well as H^1-norms. These performances are not surprising for this might have come from the fact that the NSFD-GM scheme always show some qualities of efficiency and viability that comes from its preserving of the qualitative properties of the exact solution. In light of these extra differences, we are compelled to favor the NSFD-GM scheme to be an alternative to the more traditional SFD-GM scheme.

6. Conclusion and future remarks

The work was intended to study the Fitzhugh-Nagumo equation using theoretically the Galerkin method and the compactness theorem. Thanks to these two methods, we showed analytically by controlling the parameter α as prescribed by Hodgkin-Huxley that the solution of the problem under investigation exists uniquely in the space

$$L^{\infty}\left[(0,T),L^2(\Omega)\right]\cap L^2\left[(0,T),H_0^1(\Omega)\right]\cap L^4\left[(0,T),L^4(\Omega)\right].$$

We then proceeded numerically by designing an efficient reliable scheme consisting of the Non-standard finite difference method in the time variable and the Galerkin method in the space variables and showed using this method that the designed scheme was stable. The stability of the scheme was immediately followed by proving that the numerical solution obtained from the designed scheme converges with a rate which is optimal in both the L^2 and H^1-norms. In addition, we showed that this numerical solution replicates the qualitative properties of the exact solution of the problem. Furthermore, the numerical experiments with the help of an example and a careful choice of the parameter α were used as a justification to validate the theoretical results presented in section 4. We went forward to take decreasing random values of the parameter α within the threshold prescribed by Hodgkin-Huxley and showed that the stability of the numerical solution of the scheme NSFD-GM as compared to that of SFD-GM continue to dominate in the process. The instability of both schemes were justified for $\alpha = 1$. We then proceeded with the above results as demonstrated by the experiments, to conclude that the proposed scheme was very efficient, accurate and viable. Based on these reasons, the scheme could act as a fair alternative to the most traditional scheme SFD-GM used in solving similar problems such as the Fitzhugh-Nagumo equation.

In future, we would ike to extend the study to handle nonlinear hyperbolic problems where continuity of the solution and its derivative with respect to the time space is tricky as seen in [21]. In addition, we will also like to apply it on systems of nonlinear equations with meaning in real life related to Fitzhugh-Nagumo equation as can be seen in [38, 44]. Furthermore, other types of studies could be carried out using other methods of Galerkin such as in [34]. Besides, we could work with a variation in the boundary conditions as seen in [17]. To conclude with this future studies, we can add that other studies could be with the view to focusing attention on some comparison of different types of schemes.

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