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STABILITY AND EXISTENCE OF SOLUTIONS FOR A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS*

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Abstract In this paper, we study a coupled system of Caputo type fractional differential equations with integral boundary conditions. By Leray-Schauder alternative theorem, the existence of solutions for the factional differential system are obtained. The Hyers-Ulam stability of solutions is discussed and sufficient conditions for the stability are developed. The main results are well illustrated with examples and numerical simulation graphs. The interesting point of this article is that it not only gives approximate graphs of solution by using the iterative methods, but also verifies the Hyers-Ulam stability of the coupling system by numerical simulation.

Keywords Fractional differential equation, fixed point theorem, Hyers-Ulam stability, numerical simulation.

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1. Introduction

In recent years, fractional differential equations have attracted much attention because of their wide application in many fields, including fractal theory, potential theory, biology, chemistry and diffusion [1, 3–7, 21, 35]. For example, fractional diffusion differential operators have been used to describe the diffusion in fractal geometric media [21]. Considering that fractional differential equations can be applied to various practical problems, the study of fractional differential systems has important value and significance. Fractional differential systems are more suitable for describing the physical phenomena possessing memory and genetic characteristics. Such as distributed-order dynamical systems [16], synchronization of coupled fractional-order chaotic systems [11, 12].

Solvability and stability are important research directions in the theory of fractional differential systems. The Hyers-Ulam stability of differential equations is the criterion for the existence of exact solutions near the approximate solutions of

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differential equations. There are a lot of theoretical research results on the stability of fractional differential equations [2, 9, 13, 18, 19, 29, 34].

Bashir Ahmad [2] studied a coupled system of nonlinear fractional differential equations

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t), y(t), D^{\gamma}y(t)), \ t \in [0, T], \ 1 < \alpha \leq 2, \ 0 < \gamma < 1, \\ D^{\beta}y(t) = g(t, x(t), D^{\delta}x(t), y(t)), \ t \in [0, T], \ 1 < \beta \leq 2, \ 0 < \delta < 1, \end{cases}$$

with coupled nonlocal and integral boundary conditions of the form

$$\begin{cases} x(0) = h(y), \displaystyle\int_0^T y(s) = \mu_1 x(\eta), \ \eta \in [0, T], \\ y(0) = \phi(x), \displaystyle\int_0^T x(s) = \mu_2 y(\xi), \ \xi \in [0, T], \end{cases}$$

where $D^i, \ i = \alpha, \beta, \gamma, \delta$ are Caputo fractional derivatives, μ_1, μ_2 are real constants, and f, g, h, ϕ are given continuous functions. The existence of solutions for the coupled system of nonlinear fractional differential equations with coupled nonlocal and integral boundary conditions are obtained via contraction mapping principle and Leray-Schauder alternative theorem.

In [19], the authors discussed a new coupled system of fractional differential equations with derivative boundary conditions

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t), y(t)), \ t \in [0, T], \ 1 < \alpha \leq 2, \\ D^{\beta}y(t) = g(t, x(t), y(t)), \ t \in [0, T], \ 1 < \beta \leq 2, \\ x(0) = 0, x(T) = \eta y'(\rho), \ \rho \in [0, T], \\ y(0) = 0, y(T) = \zeta x'(\mu), \ \mu \in [0, T], \end{cases}$$

where $D^{\alpha}, \ D^{\beta}$ are usual Caputo fractional derivatives, ζ, η are real constants, and $f, g \in C([0, T] \times \mathbb{R}^2, \mathbb{R})$. Based upon Leray-Schauder¡¯s alternative and contraction mapping principle, the authors established the existence and uniqueness of solutions for the new coupled system dependent on two constants, and proved that the equations are Hyers-Ulam stable under some conditions.

In the literature mentioned above, although some authors have well studied solutions of differential equations using fixed point theorem [10, 25–28, 30, 31, 33], numerical simulation have been rarely studied for the solutions of coupled systems as well as Hyers-Ulam stablility [20, 24, 32]. Therefore, it is worth studying to draw the approximate graphs of the solution and verify the stability of the system by numerical simulation. So the current paper studies a new coupled system of fractional differential equations and consider the boundary value problem for the system with integral boundary conditions. Namely, we investigate the following problem:

$$\begin{cases} {}^{c}D^{\alpha}x(t) = f(t, x(t), y(t), x'(t), y'(t)), \quad t \in [0, T], \ 1 < \alpha \leq 2, \\ {}^{c}D^{\beta}y(t) = g(t, x(t), y(t), x'(t), y'(t)), \quad t \in [0, T], \ 1 < \beta \leq 2, \end{cases} \tag{1.1}$$

with coupled integral boundary conditions, respectively, given by

$$x(T) = \eta \int_0^T y(\tau)d\tau, \quad y(T) = \delta \int_0^T x(\tau)d\tau, \quad x(0) = y(0) = 0,$$

where ${}^cD^\alpha, {}^cD^\beta$ are Caputo fractional derivative, η, δ are real constants and $\delta\eta T^2 - 4 \neq 0$. $f, g \in C([0,1] \times \mathbb{R}^4, \mathbb{R})$ are given continuous functions. We obtain the result that there is at least one solution through the Leray-Schauder alternative theorem, and prove that the fractional differential equations are Hyers-Ulam stable by definition. In addition, two examples are given to prove the conclusion. It is worth noting that the approximate graphs of the solutions are given by iterative method, and the stability of the equations is verified by numerical simulation.

The outline of this paper is as follows. In Section 2, we present some definitions from fractional calculus and present an auxiliary lemma. Then in Section 3, the existence of solutions for the coupled system of nonlinear fractional differential equations with coupled integral boundary conditions are obtained by Leray-Schauder alternative theorem. Section 4 discusses the Hyers-Ulam stability of solutions and presents sufficient conditions for the stability. Some examples and numerical simulations are given in the last section.

2. Preliminaries

In this section, we will present some preliminary concepts of fractional calculus and an auxiliary lemma which will be used in this paper later.

Definition 2.1 ([22]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([17]). The Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$, where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Lemma 2.1. *Let $u, v \in C([0,T], \mathbb{R})$ then the unique solution for the problem*

$$\begin{cases} {}^cD^\alpha x(t) = u(t), \quad t \in [0,T], \quad 1 < \alpha \le 2, \\ {}^cD^\beta y(t) = v(t), \quad t \in [0,T], \quad 1 < \beta \le 2, \\ x(T) = \eta \int_0^T y(\tau)d\tau, \ y(T) = \delta \int_0^T x(\tau)d\tau, \ x(0) = y(0) = 0 \end{cases} \tag{2.1}$$

is

$$x(t) = \frac{4t}{T\Delta\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} u(s)ds - \frac{4t\eta}{T\Delta\beta\Gamma(\beta)} \int_0^T (T-s)^\beta v(s)ds$$

$$+\frac{2\eta t}{\Delta\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}v(s)ds - \frac{2\eta\delta t}{\Delta\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}u(s)ds \qquad (2.2)$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

and

$$y(t) = \frac{2\delta t}{\Delta\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}u(s)ds - \frac{2\delta\eta t}{\Delta\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}v(s)ds$$
$$+\frac{4t}{\Delta T\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}v(s)ds - \frac{4\delta t}{\Delta T\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}u(s)ds \quad (2.3)$$
$$+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}v(s)ds,$$

where $\Delta = \delta\eta T^2 - 4 \neq 0$.

Proof. The general solution of the system (2.1) can be written as

$$x(t) = at + b + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds, \qquad (2.4)$$
$$y(t) = ct + d + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}v(s)ds, \qquad (2.5)$$

where a, b, c, d are unknown arbitrary constants.

Using the conditions $x(0) = y(0) = 0$ given by (2.1), we get $b = d = 0$, so

$$x(T) = aT + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}u(s)ds, \qquad (2.6)$$
$$y(T) = cT + \frac{1}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}v(s)ds. \qquad (2.7)$$

Considering another boundary conditions

$$x(T) = \eta\int_0^T y(\tau)d\tau, \quad y(T) = \delta\int_0^T x(\tau)d\tau,$$

we get

$$x(T) = \eta\int_0^T \left(c\tau + \frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-s)^{\beta-1}v(s)ds\right)d\tau, \qquad (2.8)$$
$$y(T) = \delta\int_0^T \left(a\tau + \frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau-s)^{\alpha-1}u(s)ds\right)d\tau. \qquad (2.9)$$

By calculating the equation (2.6) and (2.8), we obtain

$$a = \frac{4}{T\Delta\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}u(s)ds - \frac{4\eta}{T\Delta\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}v(s)ds$$
$$+\frac{2\eta}{\Delta\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}u(s)ds - \frac{2\delta\eta}{\Delta\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}v(s)ds,$$

$$c = \frac{2\delta}{\Delta\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}u(s)ds - \frac{2\eta\delta}{\beta\Delta\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}v(s)ds + \frac{4}{\Delta T\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}u(s)ds - \frac{4\delta}{\Delta T\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}v(s)ds.$$

Substituting the values of a and c in (2.4), we obtain the solution (2.2) and (2.3). This completes the proof. □

3. Main results

In this section, let $E = \{x : x \in C([0,T]),\ x' \in C'([0,T])\}$ be the space respectively equipped with the norm $\|x\| = \sup_{t\in[0,T]}|x| + \sup_{t\in[0,T]}|x'|$. Consequently, the product space $(E\times E, \|\cdot\|)$ is a Banach space endowed with the norm $\|(x,y)\| = \|x\|+\|y\|$ for any $(x,y)\in E\times E$.

In view of Lemma 2.1, define an operator $T : E\times E \to E\times E$ as

$$T(x,y)(t) := (T_1(x,y)(t), T_2(x,y)(t)),$$

where

$$\begin{aligned} T_1(x,y)(t) = &\frac{4t}{T\Delta\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(s,x(s),y(s),x'(s),y'(s))ds \\ &-\frac{4t\eta}{T\Delta\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}g(s,x(s),y(s),x'(s),y'(s))ds \\ &+\frac{2\eta t}{\Delta\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}g(s,x(s),y(s),x'(s),y'(s))ds \\ &-\frac{2\eta\delta t}{\Delta\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}f(s,x(s),y(s),x'(s),y'(s))ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s),y(s),x'(s),y'(s))ds, \end{aligned}$$

$$\begin{aligned} T_2(x,y)(t) = &\frac{2\delta t}{\Delta\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(s,x(s),y(s),x'(s),y'(s))ds \\ &-\frac{2\delta\eta t}{\Delta\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}g(s,x(s),y(s),x'(s),y'(s))ds \\ &+\frac{4t}{\Delta T\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}g(s,x(s),y(s),x'(s),y'(s))ds \\ &-\frac{4\delta t}{\Delta T\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}f(s,x(s),y(s),x'(s),y'(s))ds \\ &+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g(s,x(s),y(s),x'(s),y'(s))ds. \end{aligned}$$

For computational convenience, we set

$$K_1 = \frac{4T^{\alpha}}{|\Delta|\alpha\Gamma(\alpha)} + \frac{T^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{2|\eta||\delta|T^{\alpha+2}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)},$$

$$K_2 = \frac{2|\eta|T^{\beta+1}}{|\Delta|\beta\Gamma(\beta)} + \frac{4|\eta|T^{\beta+1}}{|\Delta|\beta(\beta+1)\Gamma(\beta)},$$
$$K_3 = \frac{4T^{\alpha-1}}{|\Delta|\alpha\Gamma(\alpha)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{2|\eta||\delta|T^{\alpha+1}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)},$$
$$K_4 = \frac{4|\eta|T^{\beta}}{|\Delta|\beta(\beta+1)\Gamma(\beta)} + \frac{2|\eta|T^{\beta}}{|\Delta|\beta\Gamma(\beta)},$$
$$L_1 = \frac{2|\delta|T^{\alpha+1}}{|\Delta|\alpha\Gamma(\alpha)} + \frac{4|\delta|T^{\alpha+1}}{|\Delta|\alpha(\alpha+1)\Gamma(\alpha)},$$
$$L_2 = \frac{2|\eta||\delta|T^{\beta+2}}{(\beta+1)|\Delta|\beta\Gamma(\beta)} + \frac{4T^{\beta}}{|\Delta|\beta\Gamma(\beta)} + \frac{T^{\beta}}{\beta\Gamma(\beta)},$$
$$L_3 = \frac{2|\delta|T^{\alpha}}{|\Delta|\alpha\Gamma(\alpha)} + \frac{4|\delta|T^{\alpha}}{|\Delta|\alpha(\alpha+1)\Gamma(\alpha)},$$
$$L_4 = \frac{2|\eta||\delta|T^{\beta+1}}{(\beta+1)|\Delta|\beta\Gamma(\beta)} + \frac{4T^{\beta-1}}{|\Delta|\beta\Gamma(\beta)} + \frac{T^{\beta-1}}{\Gamma(\beta)}.$$

Some assumptions need to be listed to complete our results.

($H1$) $f, g : C([0,T] \times \mathbb{R}^4 \to \mathbb{R})$ are continuous functions.

($H2$) There exist real constants $m_i \geq 0 (i = 1,2,3,4)$, such that, for any $x_i \in \mathbb{R}$,

$$|f(t, x_1, x_2, x_3, x_4)| \leq m_0 + m_1|x_1| + m_2|x_2| + m_3|x_3| + m_4|x_4|.$$

($H3$) There exist real constants $n_i \geq 0 (i = 1,2,3,4)$, such that, for any $x_i \in \mathbb{R}$,

$$|g(t, x_1, x_2, x_3, x_4)| \leq n_0 + n_1|x_1| + n_2|x_2| + n_3|x_3| + n_4|x_4|.$$

($H4$) $\max_{1\leq i\leq 4}\{(K_1 + K_3 + L_1 + L_3)m_i + (K_2 + K_4 + L_2 + L_4)n_i\} < 1.$

Next we present the Leray-Schauder alternative theorem and obtain the main results on the existence of solutions to the fractional differential equations by the theorem.

Lemma 3.1 ([14]). *Let $F : E \to E$ be a completely continuous operator. Let $E(F) = \{x \in E : x = \lambda F(x)\}$ for some $0 < \lambda < 1$. Then either the set $E(F)$ is unbounded or F has at least one fixed point.*

Lemma 3.2. *Assume that* (H1) *to* (H4) *hold. Then the operator T is completely continuous.*

Proof. In view of the continuity of the function f, g, the operator T is continuous.

Let $B \subseteq E \times E$ be a bounded set. Then there exists positive constants λ_1, λ_2 such that

$$|f(s, x(s), y(s), x'(s), y'(s))| \leq \lambda_1, \quad |g(s, x(s), y(s), x'(s), y'(s))| \leq \lambda_2.$$

For $t \in [0,T], (x,y) \in B$, we have

$$\begin{aligned}|T_1(x,y)(t)| \leq& \frac{4t}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s, x(s), y(s), x'(s), y'(s))|ds \\ &+\frac{4t|\eta|}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}|g(s, x(s), y(s), x'(s), y'(s))|ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),y(s),x'(s),y'(s))|ds\\
&+\frac{2|\eta|t}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}|g(s,x(s),y(s),x'(s),y'(s))|ds\\
&+\frac{2|\eta||\delta|t}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}|f(s,x(s),y(s),x'(s),y'(s))|ds\\
\le\ &\frac{4t\lambda_1}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds+\frac{4t|\eta|\lambda_2}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}ds\\
&+\frac{\lambda_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds+\frac{2||\eta||\lambda_2 t}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}ds\\
&+\frac{2\lambda_1|\eta||\delta|t}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}ds\\
\le\ &\frac{4\lambda_1 T^{\alpha}}{|\Delta|\alpha\Gamma(\alpha)}+\frac{\lambda_1 T^{\alpha}}{\alpha\Gamma(\alpha)}+\frac{2\lambda_1|\eta||\delta|T^{\alpha+2}}{(\alpha+1)\Delta\alpha\Gamma(\alpha)}\\
&+\frac{2|\eta|\lambda_2 T^{\beta+1}}{|\Delta|\beta\Gamma(\beta)}+\frac{4|\eta|\lambda_2 T^{\beta+1}}{|\Delta|\beta(\beta+1)\Gamma(\beta)},
\end{aligned}
$$

this yields

$$
\begin{aligned}
|T_1(x,y)(t)| \le\ &\left(\frac{4T^{\alpha}}{|\Delta|\alpha\Gamma(\alpha)}+\frac{T^{\alpha}}{\alpha\Gamma(\alpha)}+\frac{2|\eta||\delta|T^{\alpha+2}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)}\right)\lambda_1\\
&+\left(\frac{2|\eta|T^{\beta+1}}{|\Delta|\beta\Gamma(\beta)}+\frac{4|\eta|T^{\beta+1}}{|\Delta|\beta(\beta+1)\Gamma(\beta)}\right)\lambda_2\\
\le\ &K_1\lambda_1+K_2\lambda_2.
\end{aligned}
$$

It follows from (2.2) that

$$
\begin{aligned}
|T_1'(x,y)(t)| \le\ &\frac{4}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,x(s),y(s),x'(s),y'(s))|ds\\
&+\frac{4|\eta|}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}|g(s,x(s),y(s),x'(s),y'(s))|ds\\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}|f(s,x(s),y(s),x'(s),y'(s))|ds\\
&+\frac{2|\eta|}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}|g(s,x(s),y(s),x'(s),y'(s))|ds\\
&+\frac{2|\eta||\delta|}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}|f(s,x(s),y(s),x'(s),y'(s))|ds\\
\le\ &\frac{4\lambda_1}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds+\frac{4|\eta|\lambda_2}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}ds\\
&+\frac{\lambda_1(\alpha-1)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}ds+\frac{2||\eta||\lambda_2}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}ds\\
&+\frac{2\lambda_1|\eta||\delta|}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}ds\\
\le\ &\frac{4\lambda_1 T^{\alpha-1}}{|\Delta|\alpha\Gamma(\alpha)}+\frac{\lambda_1 T^{\alpha-1}}{\Gamma(\alpha)}+\frac{2\lambda_1|\eta||\delta|T^{\alpha+1}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)}
\end{aligned}
$$

$$+\frac{4\lambda_2|\eta|T^\beta}{|\Delta|\beta(\beta+1)\Gamma(\beta)}+\frac{2\lambda_2|\eta|T^\beta}{|\Delta|\beta\Gamma(\beta)},$$

this yields

$$\begin{aligned}|T_1'(x,y)(t)| &\leq \left(\frac{4T^{\alpha-1}}{|\Delta|\alpha\Gamma(\alpha)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{2|\eta||\delta|T^{\alpha+1}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)}\right)\lambda_1\\ &\quad+\left(\frac{4|\eta|T^\beta}{|\Delta|\beta(\beta+1)\Gamma(\beta)}+\frac{2|\eta|T^\beta}{|\Delta|\beta\Gamma(\beta)}\right)\lambda_2\\ &\leq K_3\lambda_1+K_4\lambda_2.\end{aligned}$$

Hence

$$\begin{aligned}\|T_1(x,y)\| &= \sup_{t\in[0,T]}|T_1(x,y)(t)|+\sup_{t\in[0,T]}|T_1'(x,y)(t)|\\ &\leq (K_1+K_3)\lambda_1+(K_2+K_4)\lambda_2.\end{aligned}$$

Similarly

$$\begin{aligned}|T_2(x,y)(t)| &\leq \left(\frac{2|\delta|T^{\alpha+1}}{|\Delta|\alpha\Gamma(\alpha)}+\frac{4|\delta|T^{\alpha+1}}{|\Delta|\alpha(\alpha+1)\Gamma(\alpha)}\right)\lambda_1\\ &\quad+\left(\frac{2|\eta||\delta|T^{\beta+2}}{(\beta+1)|\Delta|\beta\Gamma(\beta)}+\frac{4T^\beta}{|\Delta|\beta\Gamma(\beta)}+\frac{T^\beta}{\beta\Gamma(\beta)}\right)\lambda_2\\ &\leq L_1\lambda_1+L_2\lambda_2,\end{aligned}$$

and

$$\begin{aligned}|T_2'(x,y)(t)| &\leq \left(\frac{2|\delta|T^{\alpha}}{|\Delta|\alpha\Gamma(\alpha)}+\frac{4|\delta|T^{\alpha}}{|\Delta|\alpha(\alpha+1)\Gamma(\alpha)}\right)\lambda_1\\ &\quad+\left(\frac{2|\eta||\delta|T^{\beta+1}}{(\beta+1)|\Delta|\beta\Gamma(\beta)}+\frac{4T^{\beta-1}}{|\Delta|\beta\Gamma(\beta)}+\frac{T^{\beta-1}}{\Gamma(\beta)}\right)\lambda_2\\ &\leq L_3\lambda_1+L_4\lambda_2.\end{aligned}$$

Hence

$$\begin{aligned}\|T_2(x,y)\| &= \sup_{t\in[0,T]}|T_2(x,y)(t)|+\sup_{t\in[0,T]}|T_2'(x,y)(t)|\\ &\leq (L_1+L_3)\lambda_1+(L_2+L_4)\lambda_2.\end{aligned}$$

In consequence, we get

$$\begin{aligned}\|T(x,y)\| &= \|T_1(x,y)\|+\|T_2(x,y)\|\\ &\leq (K_1+K_3+L_1+L_3)\lambda_1+(K_2+K_4+L_2+L_4)\lambda_2.\end{aligned}$$

Thus, from the above inequalities, it follows that the operator T is uniformly bounded.

Now it will prove that T is equicontinuous. Let $t_1,t_2\in[0,T]$ with $t_1<t_2$, we have

$$|T_1(x,y)(t_1)-T_1(x,y)(t_2)|$$

$$
\begin{aligned}
\leq & \frac{4(t_2-t_1)}{T|\Delta|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s,x(s),y(s),x'(s),y'(s))| ds \\
& + \frac{4(t_2-t_1)|\eta|}{T|\Delta|\beta\Gamma(\beta)} \int_0^T (T-s)^{\beta} |g(s,x(s),y(s),x'(s),y'(s))| ds \\
& + \frac{2|\eta|(t_2-t_1)}{|\Delta|\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} |g(s,x(s),y(s),x'(s),y'(s))| ds \\
& + \frac{2|\eta||\delta|(t_2-t_1)}{|\Delta|\alpha\Gamma(\alpha)} \int_0^T (T-s)^{\alpha} |f(s,x(s),y(s),x'(s),y'(s))| ds \\
& + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2-s)^{\alpha-1} f(s,x(s),y(s),x'(s),y'(s)) ds \right. \\
& \left. - \int_0^{t_1} (t_1-s)^{\alpha-1} f(s,x(s),y(s),x'(s),y'(s)) ds \right| \\
\leq & \frac{4(t_2-t_1)\lambda_1}{T|\Delta|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds + \frac{4(t_2-t_1)\lambda_2|\eta|}{T|\Delta|\beta\Gamma(\beta)} \int_0^T (T-s)^{\beta} ds \\
& + \frac{2|\eta|(t_2-t_1)\lambda_2}{|\Delta|\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} ds + \frac{2|\eta||\delta|(t_2-t_1)\lambda_1}{|\Delta|\alpha\Gamma(\alpha)} \int_0^T (T-s)^{\alpha} ds \\
& + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right) |f(s,x(s),y(s),x'(s),y'(s))| ds \right| \\
& + \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} |f(s,x(s),y(s),x'(s),y'(s))| ds \right| \\
\leq & \frac{4(t_2-t_1)\lambda_1 T^{\alpha-1}}{|\Delta|\alpha\Gamma(\alpha)} + \frac{4(t_2-t_1)\lambda_2|\eta|T^{\beta}}{|\Delta|\beta(\beta+1)\Gamma(\beta)} + \frac{2|\eta|(t_2-t_1)\lambda_2 T^{\beta}}{|\Delta|\beta\Gamma(\beta)} + \frac{2|\eta||\delta|(t_2-t_1)\lambda_1 T^{\alpha+1}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)} \\
& + \frac{\lambda_1}{\Gamma(\alpha)} \left| \int_0^{t_1} \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right) ds \right| + \frac{\lambda_1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right| \\
\leq & \frac{t_2-t_1}{|\Delta|} \left(\frac{4\lambda_1 T^{\alpha-1}}{\alpha\Gamma(\alpha)} + \frac{4\lambda_2|\eta|T^{\beta}}{\beta(\beta+1)\Gamma(\beta)} + \frac{2|\eta|\lambda_2 T^{\beta}}{\beta\Gamma(\beta)} + \frac{2|\eta||\delta|\lambda_1 T^{\alpha+1}}{(\alpha+1)\alpha\Gamma(\alpha)} \right) + \frac{\lambda_1}{\Gamma(\alpha+1)} (t_2^{\alpha} - t_1^{\alpha}).
\end{aligned}
$$

Then

$$
\begin{aligned}
& |T_1'(x,y)(t_1) - T_1'(x,y)(t_2)| \\
\leq & \frac{\alpha-1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2-s)^{\alpha-2} f(s,x(s),y(s),x'(s),y'(s)) ds \right. \\
& \left. - \int_0^{t_1} (t_1-s)^{\alpha-2} f(s,x(s),y(s),x'(s),y'(s)) ds \right| \\
\leq & \frac{\alpha-1}{\Gamma(\alpha)} \left| \int_0^{t_1} \left((t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}\right) |f(s,x(s),y(s),x'(s),y'(s))| ds \right| \\
& + \frac{\alpha-1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_2-s)^{\alpha-2} |f(s,x(s),y(s),x'(s),y'(s))| ds \right| \\
\leq & \frac{\lambda_1(\alpha-1)}{\Gamma(\alpha)} \left| \int_0^{t_1} \left((t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}\right) ds \right| + \frac{\lambda_1(\alpha-1)}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (t_2-s)^{\alpha-2} ds \right| \\
\leq & \frac{\lambda_1}{\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1}).
\end{aligned}
$$

Similarly, we obtain

$$\begin{aligned}
&|T_2(x,y)(t_1) - T_2(x,y)(t_2)| \\
\leq &\frac{t_2 - t_1}{|\Delta|}\left(\frac{4|\delta|\lambda_1 T^\alpha}{\alpha(\alpha+1)\Gamma(\alpha)} + \frac{2|\delta|\lambda_1 T^\alpha}{\alpha\Gamma(\alpha)} + \frac{2|\eta||\delta|\lambda_2 T^{\beta+1}}{(\beta+1)\beta\Gamma(\beta)} + \frac{4\lambda_2 T^{\beta-1}}{\beta\Gamma(\beta)}\right) \\
&+\frac{\lambda_2}{\Gamma(\beta+1)}(t_2^\beta - t_1^\beta),
\end{aligned}$$

and

$$|T_2'(x,y)(t_1) - T_2'(x,y)(t_2)| \leq \frac{\lambda_2}{\Gamma(\beta)}(t_2^{\beta-1} - t_1^{\beta-1}).$$

Therefore, when $t_2 \to t_1$, the right-hand side of the above inequality approach to zero. So the operator T is equicontinuous. It follows from the virtue of the Arzela-Ascoli theorem that the operator T is completely continuous. □

Theorem 3.1. *Assume that* (H1) *to* (H4) *hold. Then the problem* (1.1) *has at least one solution.*

Proof. We will show that the set $Z = \{(x,y) \in E \times E : (x,y) = hT(x,y),\ 0 \leq h \leq 1\}$ is bounded.

For $t \in [0,T]$, let $(x,y) \in Z$ and $(x,y) = hT(x,y)$, we get

$$x(t) = hT_1(x,y)(t), \quad y(t) = hT_2(x,y)(t).$$

Then

$$\begin{aligned}
|x(t)| \leq &|T_1(x,y)(t)| \\
\leq &\frac{4t}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,x(s),y(s),x'(s),y'(s))|ds \\
&+\frac{4t|\eta|}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}|g(s,x(s),y(s),x'(s),y'(s))|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),y(s),x'(s),y'(s))|ds \\
&+\frac{2|\eta|t}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}|g(s,x(s),y(s),x'(s),y'(s))|ds \\
&+\frac{2|\eta||\delta|t}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}|f(s,x(s),y(s),x'(s),y'(s))|ds \\
\leq &(m_0 + m_1|x(t)| + m_2|x'(t)| + m_3|y(t)| + m_4|y'(t)|) \\
&\times\left(\frac{4t}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\right. \\
&\left.+\frac{2|\eta||\delta|t}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}ds\right) \\
&+(n_0 + n_1|x(t)| + n_2|x'(t)| + n_3|y(t| + n_4|y'(t)|) \\
&\times\left(\frac{4t|\eta|}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}ds + \frac{2|\eta|t}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}ds\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{4T^{\alpha}}{|\Delta|\alpha\Gamma(\alpha)} + \frac{T^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{2|\eta||\delta|T^{\alpha+2}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)}\right) \\
&\quad (m_0 + m_1|x(t)| + m_2|x'(t)| + m_3|y(t)| + m_4|y'(t)|) \\
&\quad + \left(\frac{2|\eta|\lambda_2 T^{\beta+1}}{|\Delta|\beta\Gamma(\beta)} + \frac{4|\eta|\lambda_2 T^{\beta+1}}{|\Delta|\beta(\beta+1)\Gamma(\beta)}\right) \\
&\quad (n_0 + n_1|x(t)| + n_2|x'(t)| + n_3|y(t)| + n_4|y'(t)|) \\
&\leq K_1(m_0 + m_1|x(t)| + m_2|x'(t)| + m_3|y(t)| + m_4|y'(t)|) \\
&\quad + K_2(n_0 + n_1|x(t)| + n_2|x'(t)| + n_3|y(t)| + n_4|y'(t)|),
\end{aligned}
$$

and

$$
\begin{aligned}
|x'(t)| &\leq |T_1'(x,y)(t)| \\
&\leq \frac{4}{T|\Delta|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}|f(s,x(s),y(s),x'(s),y'(s))|ds \\
&\quad + \frac{4|\eta|}{T|\Delta|\beta\Gamma(\beta)} \int_0^T (T-s)^{\beta}|g(s,x(s),y(s),x'(s),y'(s))|ds \\
&\quad + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2}|f(s,x(s),y(s),x'(s),y'(s))|ds \\
&\quad + \frac{2|\eta|}{|\Delta|\Gamma(\beta)} \int_0^T (T-s)^{\beta-1}|g(s,x(s),y(s),x'(s),y'(s))|ds \\
&\quad + \frac{2|\eta||\delta|}{|\Delta|\alpha\Gamma(\alpha)} \int_0^T (T-s)^{\alpha}|f(s,x(s),y(s),x'(s),y'(s))|ds \\
&\leq (m_0 + m_1|x(t)| + m_2|x'(t)| + m_3|y(t)| + m_4|y'(t)|) \\
&\quad \times \left(\frac{4\lambda_1}{T|\Delta|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}ds + \frac{\lambda_1(\alpha-1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2}ds \right.\\
&\quad \left. + \frac{2\lambda_1|\eta||\delta|}{|\Delta|\alpha\Gamma(\alpha)} \int_0^T (T-s)^{\alpha}ds\right) \\
&\quad + (n_0 + n_1|x(t)| + n_2|x'(t)| + n_3|y(t| + n_4|y'(t)|) \\
&\quad \times \left(\frac{2||\eta||\lambda_2}{|\Delta|\Gamma(\beta)} \int_0^T (T-s)^{\beta-1}ds + \frac{4|\eta|\lambda_2}{T|\Delta|\beta\Gamma(\beta)} \int_0^T (T-s)^{\beta}ds\right) \\
&\leq (m_0 + m_1|x(t)| + m_2|x'(t)| + m_3|y(t)| + m_4|y'(t)|) \\
&\quad \left(\frac{4T^{\alpha-1}}{|\Delta|\alpha\Gamma(\alpha)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{2|\eta||\delta|T^{\alpha+1}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)}\right) \\
&\quad + (n_0 + n_1|x(t)| + n_2|x'(t)| + n_3|y(t)| + n_4|y'(t)|) \\
&\quad \times \left(\frac{4|\eta|T^{\beta}}{|\Delta|\beta(\beta+1)\Gamma(\beta)} + \frac{2|\eta|T^{\beta}}{|\Delta|\beta\Gamma(\beta)}\right) \\
&\leq K_3(m_0 + m_1|x(t)| + m_2|x'(t)| + m_3|y(t)| + m_4|y'(t)|) \\
&\quad + K_4(n_0 + n_1|x(t)| + n_2|x'(t)| + n_3|y(t)| + n_4|y'(t)|).
\end{aligned}
$$

Consequently, we have

$$
\|x\| = \sup_{t\in[0,T]} |x(t)| + \sup_{t\in[0,T]} |x'(t)|
$$

$$\leq (K_1+K_3)(m_0+m_1|x|+m_2|x'|+m_3|y|+m_4|y'|) \\ +(K_2+K_4)(n_0+n_1|x|+n_2|x'|+n_3|y|+n_4|y'|),$$

and

$$\begin{aligned}\|y\| &= \sup_{t\in[0,T]}|y(t)| + \sup_{t\in[0,T]}|y'(t)| \\ &\leq (L_1+L_3)(m_0+m_1|x|+m_2|x'|+m_3|y|+m_4|y'|) \\ &\quad +(L_2+L_4)(n_0+n_1|x|+n_2|x'|+n_3|y|+n_4|y'|),\end{aligned}$$

this means that

$$\begin{aligned}\|x\|+\|y\| &\leq (K_1+K_3+L_1+L_3)m_0+(K_2+K_4+L_2+L_4)n_0 \\ &\quad +((K_1+K_3+L_1+L_3)m_1+(K_2+K_4+L_2+L_4)n_1)|x| \\ &\quad +((K_1+K_3+L_1+L_3)m_2+(K_2+K_4+L_2+L_4)n_2)|x'| \\ &\quad +((K_1+K_3+L_1+L_3)m_3+(K_2+K_4+L_2+L_4)n_3)|y| \\ &\quad +((K_1+K_3+L_1+L_3)m_4+(K_2+K_4+L_2+L_4)n_4)|y'| \\ &\leq (K_1+K_3+L_1+L_3)m_0+(K_2+K_4+L_2+L_4)n_0 \\ &\quad +\mu\|x\|+\mu\|y\|,\end{aligned}$$

together with $\|(x,y)\| = \|x\|+\|y\|$, we have

$$\|(x,y)\| \leq \frac{(K_1+K_3+L_1+L_3)m_0+(K_2+K_4+L_2+L_4)n_0}{1-\mu},$$

where $\mu = \max_{1\leq i\leq 4}\{(K_1+K_3+L_1+L_3)m_i+(K_2+K_4+L_2+L_4)n_i\}$. This shows that Z is bounded. Lemma 3.1 applies and this proves that T has at least one fixed point. This implies that the boundary value problem (1.1) has at least one solution on $[0,T]$. □

4. Hyers-Ulam Stability

Definition 4.1 ([15, 23]). Let $T^*: E \to E$. E is a Banach space. The operator equation

$$T^*u = u, u \in E, \tag{4.1}$$

is Hyers-Ulam stable if for the given inequality,

$$|u(t) - T^*u(t)| \leq \varepsilon, t \in [0,T]. \tag{4.2}$$

There is $N^* > 0$ such that for any u of the equation (4.2). We can find the solution $\hat{u}$ satisfing (4.1) such that the following is the case,

$$|u(t) - \hat{u}(t)| \leq N^*\varepsilon, t \in [0,T].$$

Now, let us consider two operators $T_i : E \to E,\ i \in \{1,2\}$. Based on Definition 4.1, the coupled system

$$\begin{cases} x(t) = T_1x(t), t \in [0,T], \\ y(t) = T_2y(t), t \in [0,T], \end{cases} \tag{4.3}$$

is Hyers-Ulam stable if for the given inequality:

$$\begin{cases} |x(t) - T_1x(t)| \le \varepsilon_1, t \in [0,T], \\ |y(t) - T_2y(t)| \le \varepsilon_2, t \in [0,T]. \end{cases} \tag{4.4}$$

There are two constants $N_1, N_2 > 0$ such that for each solution (x, y) of the inequality (4.4), there exists a unique solution $(\hat{x}, \hat{y})$ exists for the system (4.3) with

$$\begin{cases} |x(t) - \hat{x}(t)| \le N_1\varepsilon_1, t \in [0,T], \\ |y(t) - \hat{y}(t)| \le N_2\varepsilon_2, t \in [0,T]. \end{cases}$$

Remark 4.1. There are two function $M_1(t),\ M_2(t)$ which depend on x and y and satisfy

$$\begin{cases} |M_1(t)| \le \varepsilon_1, \\ |M_2(t)| \le \varepsilon_2, \end{cases}$$

and

$$\begin{cases} {}^cD^\alpha \hat{x}(t) - f(t, \hat{x}(t), \hat{y}(t), \hat{x}(t), \hat{y}'(t)) = M_1(t), \quad t \in [0,T], \\ {}^cD^\beta \hat{y}(t) - g(t, \hat{x}(t), \hat{y}(t), \hat{x}'(t), \hat{y}'(t)) = M_2(t), \quad t \in [0,T]. \end{cases}$$

Theorem 4.1. *Assume $f, g : C([0,T] \times \mathbb{R}^4 \to \mathbb{R})$ are continuous functions and exist real constants $Q_1 \ge 0, Q_2 \ge 0$ such that, for all $x_i \in \mathbb{R}, i = 1,2,3,4$, the following inequalities are true*

$$\begin{aligned} H5:\quad & |f(t,x_1,x_2,x_3,x_4) - f(t,y_1,y_2,y_3,y_4)| \\ & \le Q_1(|y_1-x_1|+|y_2-x_2|+|y_3-x_3|+|y_4-x_4|); \\ H6:\quad & |g(t,x_1,x_2,x_3,x_4) - g(t,y_1,y_2,y_3,y_4)| \\ & \le Q_2(|y_1-x_1|+|y_2-x_2|+|y_3-x_3|+|y_4-x_4|); \\ H7:\quad & (K_1 + K_3 + L_1 + L_4)Q_1 + (K_2 + K_4 + L_2 + L_4)Q_2 < 1. \end{aligned}$$

Assume that (H5) *to* (H7) *hold. Then the solution of the boundary value problem* (1.1) *is Hyers-Ulam stable.*

Proof. In view of condition and Remark 4.1, we have the following equations:

$$\begin{cases} {}^cD^\alpha \hat{x}(t) - f(t, \hat{x}(t), \hat{y}(t), \hat{x}(t), \hat{y}'(t)) = M_1(t), \quad t \in [0,T], \\ {}^cD^\beta \hat{y}(t) - g(t, \hat{x}(t), \hat{y}(t), \hat{x}'(t), \hat{y}'(t)) = M_2(t), \quad t \in [0,T], \\ \hat{x}(T) = \eta \int_0^T \hat{y}(\tau)d\tau, \quad \hat{y}(T) = \delta \int_0^T \hat{x}(\tau)d\tau, \quad \hat{x}(0) = \hat{y}(0) = 0, \end{cases} \tag{4.5}$$

so

$$\begin{aligned} & |T_1(\hat{x},\hat{y})(t) - \hat{x}(t)| \\ \le & \frac{4t}{T|\Delta|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}|M_1(t)|ds + \frac{4t|\eta|}{T|\Delta|\beta\Gamma(\beta)} \int_0^T (T-s)^\beta |M_2(t)|ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|M_1(t)|ds + \frac{2|\eta|t}{|\Delta|\Gamma(\beta)} \int_0^T (T-s)^{\beta-1}|M_2(t)|ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{2|\eta||\delta|t}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}|M_1(t)|ds\\
&\le \frac{4t\varepsilon_1}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds+\frac{4t|\eta|\varepsilon_2}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}ds\\
&+\frac{\varepsilon_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds+\frac{2||\eta||\varepsilon_2 t}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}ds+\frac{2\varepsilon_1|\eta||\delta|t}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}ds\\
&\le \frac{4t\varepsilon_1 T^{\alpha}}{T|\Delta|\alpha\Gamma(\alpha)}+\frac{4t|\eta|\varepsilon_2 T^{\beta+1}}{T|\Delta|\beta(\beta+1)\Gamma(\beta)}+\frac{\varepsilon_1 T^{\alpha}}{\alpha\Gamma(\alpha)}+\frac{2|\eta|\varepsilon_2 T^{\beta}t}{|\Delta|\beta\Gamma(\beta)}+\frac{2\varepsilon_1|\eta||\delta|T^{\alpha+1}t}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)}\\
&\le K_1\varepsilon_1+K_2\varepsilon_2.
\end{aligned}
$$

By assumption (H4) and (H5), we have

$$
\begin{aligned}
&|f(t,x(t),x'(t),y(t),y'(t))|\\
&\le |f(t,x(t),x'(t),y(t),y'(t))-f(t,0,0,0,0)|+|f(t,0,0,0,0)|\\
&\le Q_1(|x(t)|+|x'(t)|+|y(t)|+|y'(t)|)+|f(t,0,0,0,0)|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|g(t,x(t),x'(t),y(t),y'(t))|\\
&\le |g(t,x(t),x'(t),y(t),y'(t))-f(t,0,0,0,0)|+|g(t,0,0,0,0)|\\
&\le Q_2(|x(t)|+|x'(t)|+|y(t)|+|y'(t)|)+|g(t,0,0,0,0)|,
\end{aligned}
$$

which lead to

$$
\begin{aligned}
&|x(t)-\hat{x}(t)|\\
&=|T_1(x,y)-T_1(\hat{x},\hat{y})+T_1(\hat{x},\hat{y})-\hat{x}(t)|\\
&\le |T_1(x,y)-T_1(\hat{x},\hat{y})|+|T_1(\hat{x},\hat{y})-\hat{x}_2(t)|\\
&\le \frac{4t}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}dsQ_1(|x-\hat{x}|+|x'-\hat{x}'|+|y-\hat{y}|+|y'-\hat{y}'|)\\
&+\frac{4t|\eta|}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}dsQ_2(|x-\hat{x}|+|x'-\hat{x}'|+|y-\hat{y}|+|y'-\hat{y}'|)\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}dsQ_1(|x-\hat{x}|+|x'-\hat{x}'|+|y-\hat{y}|+|y'-\hat{y}'|)\\
&+\frac{2|\eta|t}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}dsQ_2(|x-\hat{x}|+|x'-\hat{x}'|+|y-\hat{y}|+|y'-\hat{y}'|)\\
&+\frac{2|\eta||\delta|t}{\Delta\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}dsQ_1(|x-\hat{x}|+|x'-\hat{x}'|+|y-\hat{y}|+|y'-\hat{y}'|)\\
&+K_1\varepsilon_1+K_2\varepsilon_2\\
&\le \left(\frac{4T^{\alpha}}{|\Delta|\alpha\Gamma(\alpha)}+\frac{T^{\alpha}}{\alpha\Gamma(\alpha)}+\frac{2|\eta||\delta|T^{\alpha+2}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)}\right)Q_1\|(x,y)-(\hat{x},\hat{y})\|\\
&+\left(\frac{2|\eta|\lambda_2 T^{\beta+1}}{|\Delta|\beta\Gamma(\beta)}+\frac{4|\eta|\lambda_2 T^{\beta+1}}{|\Delta|\beta(\beta+1)\Gamma(\beta)}\right)Q_2\|(x,y)-(\hat{x},\hat{y})\|+K_1\varepsilon_1+K_2\varepsilon_2\\
&\le K_1Q_1\|(x,y)-(\hat{x},\hat{y})\|+K_2Q_2\|(x,y)-(\hat{x},\hat{y})\|+K_1\varepsilon_1+K_2\varepsilon_2\\
&=(K_1Q_1+K_2Q_2)\|(x,y)-(\hat{x},\hat{y})\|+K_1\varepsilon_1+K_2\varepsilon_2,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&|T_1'(\hat{x},\hat{y})(t)-\hat{x}'(t)|\\
\le\ & \frac{4}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|M_1(t)|ds+\frac{4|\eta|}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}|M_2(t)|ds\\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}|M_1(t)|ds+\frac{2|\eta|}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}|M_2(t)|ds\\
&+\frac{2|\eta||\delta|}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}|M_1(t)|ds\\
\le\ & \frac{4\varepsilon_1}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds+\frac{4|\eta|\varepsilon_2}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}ds\\
&+\frac{(\alpha-1)\varepsilon_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}ds+\frac{2||\eta||\varepsilon_2}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}ds\\
&+\frac{2\varepsilon_1|\eta||\delta|}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}ds\\
\le\ & \frac{4\varepsilon_1 T^{\alpha-1}}{|\Delta|\alpha\Gamma(\alpha)}+\frac{\varepsilon_1 T^{\alpha-1}}{\Gamma(\alpha)}+\frac{2\varepsilon_1|\eta||\delta|T^{\alpha+1}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)}+\frac{4\varepsilon_2|\eta|T^{\beta}}{|\Delta|\beta(\beta+1)\Gamma(\beta)}+\frac{2\varepsilon_2|\eta|T^{\beta}}{|\Delta|\beta\Gamma(\beta)}\\
\le\ & K_3\varepsilon_1+K_4\varepsilon_2,
\end{aligned}
$$

which lead to

$$
\begin{aligned}
&|x'(t)-\hat{x}'(t)|\\
=\ & |T_1'(x,y)-T_1'(\hat{x},\hat{y})+T_1'(\hat{x},\hat{y})-\hat{x}'(t)|\\
\le\ & |T_1'(x,y)-T_1'(\hat{x},\hat{y})|+|T_1'(\hat{x},\hat{y})-\hat{x}'(t)|\\
\le\ & \frac{4}{T|\Delta|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}dsQ_1(|x-\hat{x}|+|x'-\hat{x}'|+|y-\hat{y}|+|y'-\hat{y}'|)\\
&+\frac{4|\eta|}{T|\Delta|\beta\Gamma(\beta)}\int_0^T (T-s)^{\beta}dsQ_2(|x-\hat{x}|+|x'-\hat{x}'|+|y-\hat{y}|+|y'-\hat{y}'|)\\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}dsQ_1(|x-\hat{x}|+|x'-\hat{x}'|+|y-\hat{y}|+|y'-\hat{y}'|)\\
&+\frac{2|\eta|}{|\Delta|\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}dsQ_2(|x-\hat{x}|+|x'-\hat{x}'|+|y-\hat{y}|+|y'-\hat{y}'|)\\
&+\frac{2|\eta||\delta|}{|\Delta|\alpha\Gamma(\alpha)}\int_0^T (T-s)^{\alpha}dsQ_1(|x-\hat{x}|+|x'-\hat{x}'|+|y-\hat{y}|+|y'-\hat{y}'|)\\
&+K_3\varepsilon_1+K_4\varepsilon_2\\
\le\ & \left(\frac{4T^{\alpha-1}}{|\Delta|\alpha\Gamma(\alpha)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{2|\eta||\delta|T^{\alpha+1}}{(\alpha+1)|\Delta|\alpha\Gamma(\alpha)}\right)Q_1\|(x_1,y_1)-(x_2,y_2)\|\\
&+\left(\frac{2|\eta|\lambda_2T^{\beta}}{|\Delta|\beta\Gamma(\beta)}+\frac{4|\eta|\lambda_2T^{\beta}}{|\Delta|\beta(\beta+1)\Gamma(\beta)}\right)Q_2\|(x_1,y_1)-(x_2,y_2)\|+K_3\varepsilon_1+K_4\varepsilon_2\\
\le\ & K_3Q_1\|(x,y)-(\hat{x},\hat{y})\|+K_4Q_2\|(x,y)-(\hat{x},\hat{y})\|+K_3\varepsilon_1+K_4\varepsilon_2\\
=\ & (K_3Q_1+K_4Q_2)\|(x,y)-(\hat{x},\hat{y})\|+K_3\varepsilon_1+K_4\varepsilon_2.
\end{aligned}
$$

Similarly,

$$|y(t)-\hat{y}(t)|\le(L_1Q_1+L_2Q_2)\|(x,y)-(\hat{x},\hat{y})\|+(L_1\varepsilon_1+L_2\varepsilon_2),$$

$$|y'(t) - \hat{y}'(t)| \le (L_3Q_1 + L_4Q_2)\|(x, y) - (\hat{x}, \hat{y})\| + (L_3\varepsilon_1 + L_4\varepsilon_2).$$

From the above certificate it follows that

$$\begin{aligned}\|(x, y) - (\hat{x}, \hat{y})\| \le{}& (K_1 + K_3 + L_1 + L_3)Q_1\|(x, y) - (\hat{x}, \hat{y})\| \\ &+(K_2 + K_4 + L_2 + L_4)Q_2\|(x, y) - (\hat{x}, \hat{y})\| \\ &+(K_1 + K_3 + L_1 + L_3)\varepsilon_1 + (K_2 + K_4 + L_2 + L_4)\varepsilon_2.\end{aligned}$$

According to conditions $Q_0 = (K_1+K_3+L_1+L_3)Q_1+(K_2+K_4+L_2+L_4)Q_2 < 1$, we get

$$\begin{aligned}\|(x, y) - (\hat{x}, \hat{y})\| &\le \frac{(K_1 + K_3 + L_1 + L_3)\varepsilon_1 + (K_2 + K_4 + L_2 + L_4)\varepsilon_2}{1 - ((K_1 + K_3 + L_1 + L_3)Q_1 + (K_2 + K_4 + L_2 + L_4)Q_2)} \\ &\le N_1\varepsilon_1 + N_2\varepsilon_2,\end{aligned}$$

where

$$N_1 = \frac{K_1 + K_3 + L_1 + L_3}{1 - ((K_1 + K_3 + L_1 + L_3)Q_1 + (K_2 + K_4 + L_2 + L_4)Q_2)},$$

$$N_2 = \frac{K_2 + K_4 + L_2 + L_4}{1 - ((K_1 + K_3 + L_1 + L_3)Q_1 + (K_2 + K_4 + L_2 + L_4)Q_2)}.$$

Therefore, the boundary value problem (1.1) is Hyers-Ulam stable. □

5. Numerical simulation

In this section, the two examples are provided to show the flexibility of these criteria. In addition, the approximate graphs of solutions are presented by using the iterative methods, and the Hyers-Ulam stability of the coupling system is verified by numerical simulation.

Example 5.1. Consider the following system of coupled fractional differential equations

$$\begin{cases} {}^cD^{\frac{4}{3}}x(t) = \dfrac{1}{30 + t^2} + \dfrac{|x(t)| + |y(t)| + |x'(t)| + |y'(t)|}{30 + 30t^2}, & t \in [0, 1], \\ {}^cD^{\frac{5}{3}}y(t) = \dfrac{1}{60 + t^2} + \dfrac{|x(t)| + |y(t)| + |x'(t)| + |y'(t)|}{60 + 60t^2}, & t \in [0, 1], \\ x(1) = 3\displaystyle\int_0^T y(\tau)d\tau,\ y(1) = 2\displaystyle\int_0^T x(\tau)d\tau,\ x(0) = y(0) = 0, \end{cases} \tag{5.1}$$

where $\alpha = \frac{4}{3},\ \beta = \frac{5}{3},\ T = 1,\ \eta = 3,\ \delta = 2.$

From the inequalities

$$\begin{aligned} f(t, x(t), y(t), x'(t), y'(t)) &= \frac{1}{30 + t^2} + \frac{|x(t)| + |y(t)| + |x'(t)| + |y'(t)|}{30 + 30t^2} \\ &\le \frac{1}{30} + \frac{1}{30}|x(t)| + \frac{1}{30}|(t)| + \frac{1}{30}|x'(t)| + \frac{1}{30}|y'(t)|, \end{aligned}$$

and

$$g(t, x(t), y(t), x'(t), y'(t)) = \frac{1}{60 + t^2} + \frac{|x(t)| + |y(t)| + |x'(t)| + |y'(t)|}{60 + 60t^2}$$

$$\leq \frac{1}{60} + \frac{1}{60}|x(t)| + \frac{1}{60}|(t)| + \frac{1}{60}|x'(t)| + \frac{1}{60}|y'(t)|,$$

we have $m_0 = m_1 = m_2 = m_3 = m_4 = \frac{1}{30}$, $n_0 = n_1 = n_2 = n_3 = n_4 = \frac{1}{60}$, $\mu = 0.8345$. Thus, all conditions of Theorem 3.1 are satisfied. Therefore, there exists at least one solution of problem (5.1) on $[0, 1]$.

Related iterative mimulation methods reference to [8, 26]. The iterative method indeed is the successive iterative method for finding the fixed point of operator T. we know the operator T is contraction and the iterative method converges with the rate of geometric progression. In the iteration process, the x_{n-1} obtained is substituted into the x_n and x_n gradually converges to the final approximate solution x^*.

The iterative sequences of solutions of fractional differential equations are defined by formula (2.2) and (2.3). The simulated iterative process curve and approximate solution of fractional differential equations (5.1) are given by simple iterative method and numerical simulation. For the convenience of calculation, let $x(0) = y(0) = 0$. The iteration sequences are as follows,

$$\begin{aligned}
x_{n+1}(t) = & \frac{4t}{\Gamma(\frac{4}{3})}\int_0^1 (1-s)^{\frac{1}{3}} \left(\frac{1}{30+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{30+30s^2}\right) ds \\
& - \frac{12t}{5\Gamma(\frac{5}{3})}\int_0^1 (1-s)^{\frac{5}{3}} \left(\frac{1}{60+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{60+60s^2}\right) ds \\
& + \frac{2t}{\Gamma(\frac{5}{3})}\int_0^1 (1-s)^{\frac{2}{3}} \left(\frac{1}{60+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{60+60s^2}\right) ds \\
& - \frac{9t}{2\Gamma(\frac{4}{3})}\int_0^1 (1-s)^{\frac{4}{3}} \left(\frac{1}{30+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{30+30s^2}\right) ds \\
& + \frac{1}{\Gamma(\frac{4}{3})}\int_0^t (t-s)^{\frac{1}{3}} \left(\frac{1}{30+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{30+30s^2}\right) ds,
\end{aligned}$$

$$\begin{aligned}
y_{n+1}(t) = & \frac{2t}{\Gamma(\frac{4}{3})}\int_0^1 (1-s)^{\frac{1}{3}} \left(\frac{1}{30+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{30+30s^2}\right) ds \\
& - \frac{18t}{5\Gamma(\frac{5}{3})}\int_0^1 (1-s)^{\frac{5}{3}} \left(\frac{1}{60+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{60+60s^2}\right) ds \\
& + \frac{2t}{\Gamma(\frac{5}{3})}\int_0^1 (1-s)^{\frac{2}{3}} \left(\frac{1}{60+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{60+60s^2}\right) ds \\
& - \frac{3t}{\Gamma(\frac{4}{3})}\int_0^1 (1-s)^{\frac{4}{3}} \left(\frac{1}{30+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{30+30s^2}\right) ds \\
& + \frac{1}{\Gamma(\frac{5}{3})}\int_0^t (t-s)^{\frac{2}{3}} \left(\frac{1}{60+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{60+60s^2}\right) ds,
\end{aligned}$$

$$\begin{aligned}
x'_{n+1}(t) = & \frac{2}{\Gamma(\frac{4}{3})}\int_0^1 (1-s)^{\frac{1}{3}} \left(\frac{1}{30+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{30+30s^2}\right) ds \\
& - \frac{18}{5\Gamma(\frac{5}{3})}\int_0^1 (1-s)^{\frac{30}{3}} \left(\frac{1}{60+s^2} + \frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{60+60s^2}\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{\Gamma(\frac{5}{3})}\int_0^1(1-s)^{\frac{2}{3}}\left(\frac{1}{60+s^2}+\frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{60+60s^2}\right)ds\\
&-\frac{9}{2\Gamma(\frac{4}{3})}\int_0^1(1-s)^{\frac{4}{3}}\left(\frac{1}{30+s^2}+\frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{30+30s^2}\right)ds\\
&+\frac{1}{3\Gamma(\frac{4}{3})}\int_0^t(t-s)^{-\frac{2}{3}}\left(\frac{1}{30+s^2}+\frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{30+30s^2}\right)ds,
\end{aligned}
$$

$$
\begin{aligned}
y'_{n+1}(t)=&\frac{2}{\Gamma(\frac{4}{3})}\int_0^1(1-s)^{\frac{1}{3}}\left(\frac{1}{30+s^2}+\frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{30+30s^2}\right)ds\\
&-\frac{18}{5\Gamma(\frac{5}{3})}\int_0^1(1-s)^{\frac{30}{3}}\left(\frac{1}{60+s^2}+\frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{60+60s^2}\right)ds\\
&+\frac{2}{\Gamma(\frac{5}{3})}\int_0^1(1-s)^{\frac{2}{3}}\left(\frac{1}{60+s^2}+\frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{60+60s^2}\right)ds\\
&-\frac{3}{\Gamma(\frac{4}{3})}\int_0^1(1-s)^{\frac{4}{3}}\left(\frac{1}{30+s^2}+\frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{30+30s^2}\right)ds\\
&+\frac{2}{3\Gamma(\frac{5}{3})}\int_0^t(t-s)^{-\frac{1}{3}}\left(\frac{1}{60+s^2}+\frac{|x_n(s)|+|y_n(s)|+|x'_n(s)|+|y'_n(s)|}{60+60s^2}\right)ds.
\end{aligned}
$$

After several iterations, the approximate solution of fractional differential equations (5.1) can be obtained by using the numerical simulation. The absolute errors for the iterative method to problem (5.1) are shown in Table 1, which demonstrates the applicability of the iterative method. $E(n)$ is the error between each iteration and the previous iteration. Figure 1 and Figure 3 show the 10-iteration process of the solution $x(t), y(t)$ of the equation. Figure 2 and Figure 4 show the 20-iteration process of the solution $x(t), y(t)$ of the equation.

Table 1. The absolute errors in Example 5.1

n	$E(n)$ for $x_n(t)$	$E(n)$ for $y_n(t)$	$E(n)$ for $x'_n(t)$	$E(n)$ for $y'_n(t)$
1	3.2888139608e-02	9.1047973005e-03	8.5757601317e-02	4.0383372232e-02
2	1.2888350941e-02	3.9088210835e-03	1.5689159005e-02	1.5875059068e-02
3	5.0305867537e-03	1.5762105991e-03	6.2301029147e-03	6.2133360329e-03
4	1.9557818768e-03	6.1635524997e-04	2.4385422622e-03	2.4199576778e-03
5	7.5909844990e-04	2.3912247866e-04	9.4825485923e-04	9.3992363819e-04
6	2.9449158144e-04	9.2684321853e-05	3.6797753973e-04	3.6471049156e-04
7	1.1424018376e-04	3.5938546041e-05	1.4273857438e-04	1.4148281585e-04
8	4.4317165208e-05	1.3939666044e-05	5.5368969528e-05	5.4884891015e-05
9	1.7192221440e-05	5.4075457600e-06	2.1479001296e-05	2.1291665969e-05
...	...	...	...	...
20	3.7564895465e-08	7.4539699555e-09	4.5751245646e-08	5.7316489753e-08

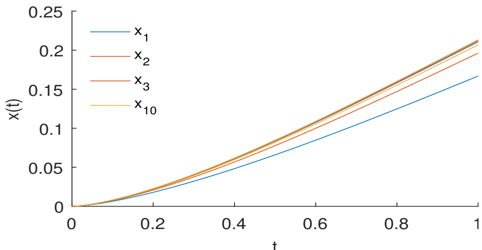

Figure 1. 10-iteration process of $x(t)$

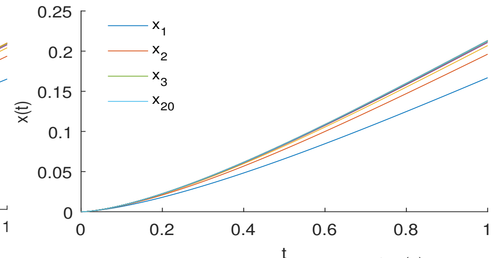

Figure 2. 20-iteration process of $x(t)$

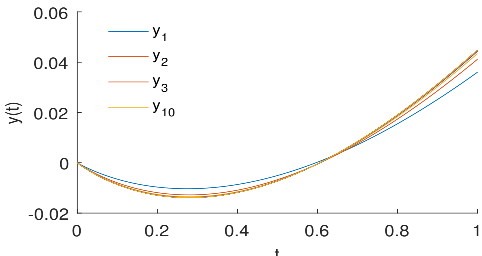

Figure 3. 10-iteration process of $y(t)$

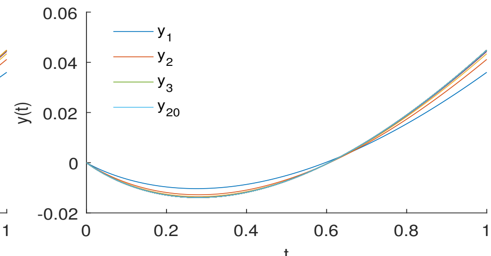

Figure 4. 20-iteration process of $y(t)$

Example 5.2. Consider the following system of coupled fractional differential equations

$$\begin{cases} {}^cD^{\frac{4}{3}}x(t)=\dfrac{1}{25+t^2}\left(\dfrac{|x(t)|}{1+|x(t)|}+\dfrac{|y(t)|}{1+|y(t)|}+\dfrac{|x'(t)|}{1+|x'(t)|}+\dfrac{|y'(t)|}{1+|y'(t)|}\right),\ t\in[0,1],\\ {}^cD^{\frac{5}{3}}y(t)=\dfrac{1}{30+t^2}\left(\dfrac{|x(t)|}{1+|x(t)|}+\dfrac{|y(t)|}{1+|y(t)|}+\dfrac{|x'(t)|}{1+|x'(t)|}+\dfrac{|y'(t)|}{1+|y'(t)|}\right),\ t\in[0,1],\\ x(1)=10\displaystyle\int_0^T y(\tau)d\tau,\ y(1)=6\int_0^T x(\tau)d\tau,\ x(0)=y(0)=0, \end{cases} \tag{5.2}$$

where $\alpha=\frac{4}{3},\ \beta=\frac{5}{3},\ T=1,\ \eta=10,\ \delta=6.$

From the inequalities

$$\begin{aligned} &|f(t,x_1(t),y_1(t),x_1'(t),y_1'(t))-f(t,x_2(t),y_2(t),x_2'(t),y_2'(t))|\\ =&\left|\frac{1}{25+t^2}\left(\frac{|x_1(t)|}{1+|x_1(t)|}-\frac{|x_2(t)|}{1+|x_2(t)|}+\frac{|y_1(t)|}{1+|y_1(t)|}-\frac{|y_2(t)|}{1+|y_2(t)|}\right.\right.\\ &\left.\left.+\frac{|x_1'(t)|}{1+|x_1'(t)|}-\frac{|x_2'(t)|}{1+|x_2'(t)|}+\frac{|y_1'(t)|}{1+|y_1'(t)|}-\frac{|y_2'(t)|}{1+|y_2'(t)|}\right)\right|\\ \le&\frac{1}{25}\left(|x_1-x_2|+|x_1'-x_2'|+|y_1-y_2|+|y_1'-y_2'|\right) \end{aligned}$$

and

$$\begin{aligned} &|g(t,x_1(t),y_1(t),x_1'(t),y_1'(t))-g(t,x_2(t),y_2(t),x_2'(t),y_2'(t))|\\ =&\left|\frac{1}{30+t^2}\left(\frac{|x_1(t)|}{1+|x_1(t)|}-\frac{|x_2(t)|}{1+|x_2(t)|}+\frac{|y_1(t)|}{1+|y_1(t)|}-\frac{|y_2(t)|}{1+|y_2(t)|}\right.\right.\\ &\left.\left.+\frac{|x_1'(t)|}{1+|x_1'(t)|}-\frac{|x_2'(t)|}{1+|x_2'(t)|}+\frac{|y_1'(t)|}{1+|y_1'(t)|}-\frac{|y_2'(t)|}{1+|y_2'(t)|}\right)\right| \end{aligned}$$

$$\leq \frac{1}{30}\left(|x_1 - x_2| + |x_1' - x_2'| + |y_1 - y_2| + |y_1' - y_2'|\right),$$

we have $Q_1 = \frac{1}{25}$, $Q_2 = \frac{1}{30}$, $Q_0 = 0.3401$. Thus, all conditions of Theorem 4.1 are satisfied. Therefore, the problem (5.2) is Hyers-Ulam stable on $[0, 1]$.

Next, the stability of the coupled system is proved by graphs, that is, the following inequality is verified,

$$E(\varepsilon_1, \varepsilon_2) = N_1\varepsilon_1 + N_2\varepsilon_2 - \|(x_1, y_1) - (x_2, y_2)\| \geq 0.$$

Let $\varepsilon_1, \varepsilon_2 \in [0, 1]$, $t \in [0, 1]$. The approximation to the left of the inequality is represented by graphs. Figure 5, Figure 6 and Figure 7 are sectional views of the equation $E(\varepsilon_1, \varepsilon_2)$ with respect to $\varepsilon_1, \varepsilon_2, t$. It can be seen from the image that the inequality is true. Therefore, the problem (5.2) is Hyers-Ulam stable.

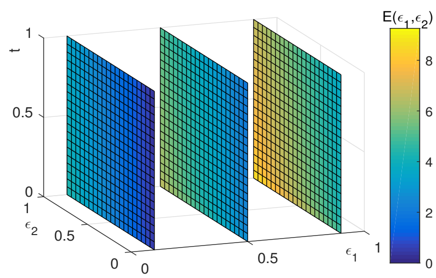

Figure 5. Approximate value of $E(\varepsilon_1, \varepsilon_2)$

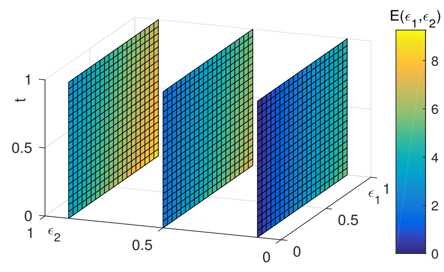

Figure 6. Approximate value of $E(\varepsilon_1, \varepsilon_2)$

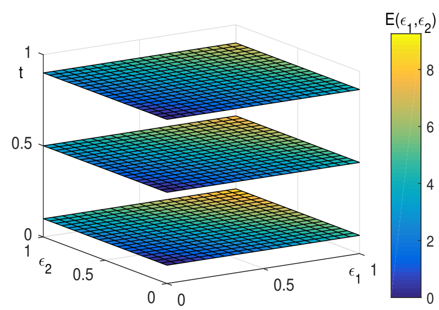

Figure 7. Approximate value of $E(\varepsilon_1, \varepsilon_2)$

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