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SUPER-CRITICAL PROBLEMS INVOLVING THE FRACTIONAL *P*-LAPLACIAN

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Abstract This paper is concerned with the following non-local problems

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = w + h(u), \text{ in } \mathbb{R}^2,$$

where $0 < s < 1 < p < \infty$, $sp < 2$ and $0 < w \in L^\infty(\mathbb{R}^2)$. Here the nonlinearity h imposes no growth restriction. By new variational principles, a nontrivial solution to this problem is obtained. This result is new because of the super-critical nonlinearity h.

Keywords Super-critical, fractional p-Laplacian, variational principle.

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1. Introduction and main result

This paper is concerned with the following non-local problem

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = w + h(u), \text{ in } \mathbb{R}^2, \tag{1.1}$$

where $0 < s < 1 < p < \infty$, $sp < 2$ and $0 < w \in L^\infty(\mathbb{R}^2)$. The function $w = w(x)$ can be viewed as a perturbation term. On the potential function V we assume

(V_1) $V \in C(\mathbb{R}^2, \mathbb{R})$ and $0 < V_0 \leq V(x)$ for any $x \in \mathbb{R}^2$.

(V_2) $V^{-1} \in L^{\frac{1}{p-1}}(\mathbb{R}^2)$.

Moreover, on the nonlinearity h we require

(h_1) $h \in C(\mathbb{R}, \mathbb{R})$ with $h(t) \geq 0$ for $t \geq 0$ and $h(t) \leq 0$ for $t \leq 0$.

(h_2) There exists $\tau > 1$ such that $\lim_{t\to 0} \frac{h(t)}{t^\tau} = 0$.

The fractional p-Laplace operator $(-\Delta)_p^s$ is defined as follows

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^2\setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{2+ps}} dy,$$

for any $x \in \mathbb{R}^2$, where $B_\epsilon(x) = \{y \in \mathbb{R}^2 : |x-y| < \epsilon\}$.

When $p = 2$, $\mathbb{R}^2$ and $w + h(u)$ are replaced by $\mathbb{R}^N$ and $f(x,u)$ respectively, problem (1.1) can be reduced to the following classical fractional Laplacian equation

$$(-\Delta)^s u + V(x)u = f(x,u), \text{ in } \mathbb{R}^N. \tag{1.2}$$

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To expand the Feynman path integral, problem (1.2) was first studied by Laskin [14]. In the research of biology, physics and chemistry, such as combustion [3] or dislocations in crystals [10], the fractional Laplace operator is very common. Moreover, an extremely important diffusion model can be described by the fractional Laplace operator, for instance, it is applied to the limiting advection diffusion equation or the diffusion model of heterogeneous media [17]. Therefore, many results have been established on the existence, nonexistence and regularity of viscosity solutions. We can see [5,7,9,16] and the references therein. When the nonlinear term f is subcritical, Davila, del Pino and Wei [4] studied the concentration of the standing wave solution of (1.2). In [6], Fall, Mahmoudi and Valdinoci introduced the ground states and concentration phenomena to (1.2) for subcritical case. In [8], Felmer, Quaas and Tan investigated the existence of positive solutions of the problem (1.2) where $V = 1$ and f is superlinear and has subcritical growth. Moreover, they obtained some regularity, decay and symmetry results of these solutions. In [12], the existence and concentration results to (1.2) for both subcritical and critical cases were developed.

In the case of $p > 1$, the fractional p-Laplacian equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u), \text{ in } \mathbb{R}^N \tag{1.3}$$

has been widely studied by many scholars. When the nonlinearity f is critical or subcritical, many results about the existence, nonexistence and multiplicity of solutions are obtained. We can cite [1,11,15,18,20] and the references therein. As far as we know, when the nonlinearity f is supcritical, there is no relevant result.

Motivated by the above mentioned papers, this paper aims to establish the existence of a nontrivial solution of the problem (1.3) where f is supcritical, that is, without imposeing any growth restriction on the nonlinearity h for problem (1.1). We will get our conclusion by a new variational principle introduced lately in [13]. This is a new result because of the super-critical nonlinearity h.

Our main result read as follows.

Theorem 1.1. *Let (h_1)-(h_2) and (V_1)-(V_2) be satisfied. Then there is $r_0 > 0$ such that for $|w|_\infty < \frac{V_0 r_0}{2}$ problem (1.1) possesses at least one nontrivial solution.*

Remark 1.1. The conditions (V_1)-(V_2) can not deduce that V is coercive.

Remark 1.2. The function $h(t) = t^{2n+1}\exp(\alpha t^2)$ for $n \in \mathbb{N}$, $2n+1 > \tau$ and $\alpha \in \mathbb{R}$ satisfying (h_1)-(h_2) is a classic example.

This paper is organized as follows. In Section 2, we state some notations and preliminaries. Section 3 is devoted to the proof of Theorem 1.1.

2. Notations and Preliminaries

Let W be a reflexive Banach space and W^* be topological dual of W. The duality pairing between W and W^* is denoted by $\langle \cdot, \cdot \rangle$ and defined by

$$\langle u, u^* \rangle = \int_{\mathbb{R}^N} u(x)u^*(x)dx, \quad \forall u \in W, u^* \in W^*.$$

Suppose that $\Psi : W \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semi-continuous map. Moreover, assume that Ψ is Gâteaux differentiable on K which is a convex and weakly closed subset of W and $D\Psi(u)$ denotes the Gâteaux derivative of Ψ at $u \in K$. Define

$$\Psi_K(u) := \begin{cases} \Psi(u), & u \in K, \\ +\infty, & u \notin K. \end{cases}$$

Consider the functional $I_K : W \to \mathbb{R} \cup \{+\infty\}$ defined by

$$I_K(u) := \Psi_K(u) - \Phi(u),$$

where $\Phi \in C^1(W, \mathbb{R})$. From Szulkin [19], we state the following definition.

Definition 2.1. $u_0 \in W$ is a critical point of I_K if

$$\langle \Phi'(u_0), u_0 - v \rangle + \Psi_K(v) - \Psi_K(u_0) \geq 0, \quad \forall v \in W. \tag{2.1}$$

Lemma 2.1. *Local minimum of I_K is a critical point of I_K.*

Proof. Let u_0 be a local minimum of I_K. By the convexity of Ψ_K, for small $t > 0$ we have

$$\begin{aligned} 0 &\leq I_K((1-t)u_0 + tv) - I_K(u_0) \\ &= \Phi(u_0 + t(v - u_0)) - \Phi(u_0) + \Psi_K((1-t)u_0 + tv) - \Psi_K(u_0) \\ &\leq \Phi(u_0 + t(v - u_0)) - \Phi(u_0) + t(\Psi_K(v) - \Psi_K(u_0)). \end{aligned}$$

Dividing by t and letting $t \to 0^+$ we can get (2.1). This completes the proof. □

Next we recall some notations. For $1 \leq q < \infty$, by $|\cdot|_q$ we denote the usual L^q-norm. The fractional Sobolev space $W^{s,p}(\mathbb{R}^2)$ is defined by

$$W^{s,p}(\mathbb{R}^2) = \left\{ u \in L^p(\mathbb{R}^2) : [u]_{s,p} < \infty \right\},$$

where $[u]_{s,p}$ denotes the Gagliardo norm, that is,

$$[u]_{s,p} = \left(\int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u(x) - u(y)|^p}{|x - y|^{2+ps}} dxdy \right)^{\frac{1}{p}},$$

and $W^{s,p}(\mathbb{R}^2)$ is equipped with the norm

$$||u||_{W^{s,p}(\mathbb{R}^2)} = \left(|u|_p^p + [u]_{s,p}^p \right)^{\frac{1}{p}}.$$

Let W denote the closing of $C_0^\infty(\mathbb{R}^2)$, with the following norm

$$||u||_W = \left(|u|_{p,V}^p + [u]_{s,p}^p \right)^{\frac{1}{p}},$$

where

$$|u|_{p,V}^p = \int_{\mathbb{R}^2} V(x)|u|^p dx.$$

Clearly the above definition is well define since any $u \in C_0^\infty(\mathbb{R}^2)$ has finite Gagliardo norm and finite norm $|u|_{p,V}$.

Lemma 2.2. *Let (V_1)-(V_2) hold. Then the embedding $W \hookrightarrow L^\nu(\mathbb{R}^2)$ is compact for $\nu \in [1, p_s^*)$ where $p_s^* = 2p/(2 - sp)$.*

Proof. It follows from (V_1) that the embedding $W \hookrightarrow W^{s,p}(\mathbb{R}^2)$ is continuous. Hence, $W \hookrightarrow L^\nu(\mathbb{R}^2)$ is continuous for $\nu \in [p, p_s^*)$. Note that

$$\int_{\mathbb{R}^2} |u|dx \leq \left(\int_{\mathbb{R}^2} V^{-\frac{1}{p-1}}(x)dx\right)^{\frac{p-1}{p}} ||u||_W \text{ for } u \in W.$$

Hence, by interpolation $W \hookrightarrow L^\nu(\mathbb{R}^2)$ is continuous for $\nu \in [1, p_s^*)$. Next, let $\{u_j\}$ be a bounded sequence in W. Going if necessary to a subsequence, $u_j \rightharpoonup u_0$ in W. Given $\epsilon > 0$, from (V_2) and the boundedness of $\{u_j\}$, one has

$$\int_{|x|>R} |u_j - u_0|dx \leq \left(\int_{|x|>R} V^{-\frac{1}{p-1}}(x)dx\right)^{\frac{p-1}{p}} ||u_j - u_0||_W \leq \frac{\epsilon}{2}$$

for some $R > 0$. Moreover, since $W \hookrightarrow L^1(B_R)$ is compact, there holds

$$\int_{B_R} |u_j - u_0|dx \leq \frac{\epsilon}{2}$$

for j large enough. Hence, $u_j \to u_0$ in $L^1(\mathbb{R}^2)$. By interpolation inequality, we have

$$|u_j - u_0|_\nu \leq |u_j - u_0|_1^\beta |u_j - u_0|_{\nu_0}^{1-\beta} \to 0,$$

for some $1 \leq \nu < \nu_0 < p_s^*$ and $0 < \beta \leq 1$. This completes the proof. □

Define convex subset

$$K = K(r) := \{u \in W \cap L^\infty(\mathbb{R}^2) : |u|_\infty \leq r\},$$

for some $r > 0$ to be determined later and $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} h(t), & |t| \leq r, \\ \frac{h(r)}{r}t, & |t| \geq r. \end{cases}$$

Then $f(u) = h(u)$ for $u \in K(r)$.

Therefore, our purpose is transformed into seeking a solution to the truncation equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = w + f(u), \text{ in } \mathbb{R}^2 \tag{2.2}$$

in $K(r)$ for some appropriate $r > 0$. Associated with (2.2), we consider the energy functional $I(u)$, for each $u \in W$,

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|u(x)-u(y)|^p}{|x-y|^{2+ps}}dxdy + \frac{1}{p}\int_{\mathbb{R}^2} V(x)|u|^p dx - \int_{\mathbb{R}^2} wudx - \int_{\mathbb{R}^2} F(u)dx,$$

where $F(t) = \int_0^t f(z)dz$. Define $\Phi : W \to \mathbb{R}$ by

$$\Phi(u) = \int_{\mathbb{R}^2} wudx + \int_{\mathbb{R}^2} F(u)dx.$$

Clearly $\Phi \in C^1(W, \mathbb{R})$. Define $\Psi : W \to \mathbb{R}$ by

$$\Psi(u) = \frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|u(x)-u(y)|^p}{|x-y|^{2+ps}}dxdy + \frac{1}{p}\int_{\mathbb{R}^2} V(x)|u|^p dx.$$

Furthermore, define $I_K : W \to \mathbb{R} \cup \{+\infty\}$ by

$$I_K(u) := \Psi_K(u) - \Phi(u), \tag{2.3}$$

where

$$\Psi_K(u) := \begin{cases} \Psi(u), & u \in K, \\ +\infty, & u \notin K. \end{cases}$$

Finally we recall a regularity result (see [2]).

Lemma 2.3. *Assume $g \in L^\infty(\mathbb{R}^2)$ and (V_1)-(V_2) hold. If $u \in W$ is a weak solution of the equation $(-\Delta)_p^s u + V(x)|u|^{p-2}u = g(x)$, then $V_0|u|_\infty \le |g|_\infty$.*

3. Proof of Theorem 1.1

First we state the following variational principle for problem (1.1).

Theorem 3.1. *Suppose $K \subset W$ is convex and weakly closed. If the following two propositions are true:*
(i) The functional $I_K : W \to \mathbb{R} \cup \{+\infty\}$ defined in (2.3) has a critical point $\bar{u} \in W$ as in Definition 2.1, and;
(ii) there exists $\bar{v} \in K$ such that $(-\Delta)_p^s \bar{v} + V(x)|\bar{v}|^{p-2}\bar{v} = D\Phi(\bar{u}) = w + f(\bar{u})$ in the weak sense.
Then $\bar{u} \in K$ is a weak solution of the equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = w + f(u).$$

Proof. Since $\bar{u}$ is a critical point of $I_K(u) := \Psi_K(u) - \Phi(u)$, by the Definition 2.1 one has

$$\Psi_K(v) - \Psi_K(\bar{u}) \ge \langle D\Phi(\bar{u}), v - \bar{u}\rangle, \quad \forall v \in W, \tag{3.1}$$

which yields

$$\begin{aligned} &\frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|v(x)-v(y)|^p}{|x-y|^{2+ps}}dxdy - \frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{2+ps}}dxdy \\ &+ \frac{1}{p}\int_{\mathbb{R}^2} V(x)|v|^p dx - \frac{1}{p}\int_{\mathbb{R}^2} V(x)|\bar{u}|^p dx \ge \langle D\Phi(\bar{u}), v - \bar{u}\rangle, \quad \forall v \in W. \end{aligned} \tag{3.2}$$

The assumption (ii) implies that there exists $\bar{v} \in K$ such that

$$\begin{aligned} &\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|\bar{v}(x)-\bar{v}(y)|^{p-2}(\bar{v}(x)-\bar{v}(y))(h(x)-h(y))}{|x-y|^{2+ps}}dxdy \\ &+ \int_{\mathbb{R}^2} V(x)|\bar{v}|^{p-2}\bar{v}h dx = \int_{\mathbb{R}^2} D\Phi(\bar{u})h dx, \quad \forall h \in W. \end{aligned} \tag{3.3}$$

Now putting $h = \bar{v} - \bar{u}$ in (3.3), one has

$$\begin{aligned} &\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|\bar{v}(x)-\bar{v}(y)|^{p-2}(\bar{v}(x)-\bar{v}(y))((\bar{v}-\bar{u})(x)-(\bar{v}-\bar{u})(y))}{|x-y|^{2+ps}}dxdy \\ &+ \int_{\mathbb{R}^2} V(x)|\bar{v}|^{p-2}\bar{v}(\bar{v}-\bar{u})dx = \int_{\mathbb{R}^2} D\Phi(\bar{u})(\bar{v}-\bar{u})dx, \quad \forall h \in W. \end{aligned} \tag{3.4}$$

Next by puting $v = \bar{v}$ in (3.2) and combining (3.4), there holds

$$\begin{aligned}
&\frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\bar{v}(x)-\bar{v}(y)|^p}{|x-y|^{2+ps}}dxdy-\frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{2+ps}}dxdy\\
&+\frac{1}{p}\int_{\mathbb{R}^2}V(x)(|\bar{v}|^p-|\bar{u}|^p)dx\\
\geq&\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\bar{v}(x)-\bar{v}(y)|^{p-2}(\bar{v}(x)-\bar{v}(y))((\bar{v}-\bar{u})(x)-(\bar{v}-\bar{u})(y))}{|x-y|^{2+ps}}dxdy\\
&+\int_{\mathbb{R}^2}V(x)|\bar{v}|^{p-2}\bar{v}(\bar{v}-\bar{u})dx.
\end{aligned}\tag{3.5}$$

On the flip side, from the convexity of Ψ, one has

$$\begin{aligned}
&\frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{2+ps}}dxdy-\frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\bar{v}(x)-\bar{v}(y)|^p}{|x-y|^{2+ps}}dxdy\\
&+\frac{1}{p}\int_{\mathbb{R}^2}V(x)(|\bar{u}|^p-|\bar{v}|^p)dx\\
\geq&\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\bar{v}(x)-\bar{v}(y)|^{p-2}(\bar{v}(x)-\bar{v}(y))((\bar{u}-\bar{v})(x)-(\bar{u}-\bar{v})(y))}{|x-y|^{2+ps}}dxdy\\
&+\int_{\mathbb{R}^2}V(x)|\bar{v}|^{p-2}\bar{v}(\bar{u}-\bar{v})dx.
\end{aligned}\tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}
&\frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\bar{v}(x)-\bar{v}(y)|^p}{|x-y|^{2+ps}}dxdy-\frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{2+ps}}dxdy\\
&+\frac{1}{p}\int_{\mathbb{R}^2}V(x)(|\bar{v}|^p-|\bar{u}|^p)dx\\
=&\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\bar{v}(x)-\bar{v}(y)|^{p-2}(\bar{v}(x)-\bar{v}(y))((\bar{v}-\bar{u})(x)-(\bar{v}-\bar{u})(y))}{|x-y|^{2+ps}}dxdy\\
&+\int_{\mathbb{R}^2}V(x)|\bar{v}|^{p-2}\bar{v}(\bar{v}-\bar{u})dx.
\end{aligned}\tag{3.7}$$

Note that $|t_1|^p - |t_2|^p < p|t_1|^{p-2}t_1(t_1 - t_2)$ for $t_1 \neq t_2 \in \mathbb{R}$ and $p > 1$. Then (3.7) yields $\bar{v} = \bar{u}$ a.e. in $\mathbb{R}^2$. Hence, to make use of (3.3), we get the required result. This completes the proof. □

The following lemma shows that the set K is weakly closed.

Lemma 3.1. *Assume that $r > 0$ is fixed then $K(r)$ is weakly closed in W.*

Proof. Let $\{u_n\}$ be a sequence in $K(r)$ such that $u_n \rightharpoonup u_0$ in W. Clearly $u_0 \in W$ because of the reflexivity of W. Going if necessary to a subsequence, $u_n(x) \to u_0(x)$ a.e. $x \in \mathbb{R}^2$. This yields that $|u_0(x)| = \lim_{n\to\infty} |u_n(x)| \leq r$ for a.e. $x \in \mathbb{R}^2$. Therefore, $|u_0|_\infty \leq r$. □

Notice that the condition (h_2) implies that there exists $\delta > 0$ such that

$$|h(t)| \leq |t|^\tau, \text{ whenever } |t| < \delta. \tag{3.8}$$

Then we have the following lemmas.

Lemma 3.2. *For all $u \in K(r)$ with $0 < r < \delta$, one has*

$$|D\Phi(u)|_\infty \le r^\tau + |w|_\infty.$$

Proof. Using (3.8) and the definition of $D\Phi(u)$, we get the desired result. □

Lemma 3.3. *There exists $0 < r_0 < \delta$ such that $r^\tau \le \frac{V_0}{2} r$, $\forall r \in (0, r_0]$. Furthermore, if $|w|_\infty \le \frac{V_0 r_0}{2}$, one has $r^\tau + |w|_\infty \le V_0 r_0$.*

We need the following lemma to address condition (ii) in Theorem 3.1.

Lemma 3.4. *If $|w|_\infty \le \frac{V_0 r_0}{2}$, for each $\bar{u} \in K(r_0)$ there exists $v \in K(r_0)$ such that*

$$(-\Delta)_p^s v + V(x)|v|^{p-2} v = w + f(\bar{u}).$$

Proof. Let $D\Phi(\bar{u}) = w + f(\bar{u})$. Clearly $D\Phi(\bar{u}) \in L^\infty(\mathbb{R}^2)$ because of $\bar{u} \in K(r_0)$. From the compactness of $W \hookrightarrow L^1(\mathbb{R}^2)$, the functional

$$T(u) = \frac{1}{p} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u(x) - u(y)|^p}{|x - y|^{2+ps}} dxdy + \frac{1}{p} \int_{\mathbb{R}^2} V(x)|u|^p dx - \int_{\mathbb{R}^2} D\Phi(\bar{u}) u dx$$

is well defined on W. Moreover, $T(u)$ has its minimum at some $v \in W$ which satisfies

$$(-\Delta)_p^s v + V(x)|v|^{p-2} v = D\Phi(\bar{u}) = w + f(\bar{u}).$$

It follows from Lemma 2.3, 3.2 and 3.3 that

$$V_0 |v|_\infty \le |D\Phi(\bar{u})|_\infty \le r_0^\tau + |w|_\infty \le V_0 r_0,$$

which implies $|v|_\infty \le r_0$. Therefore, $v \in K(r_0)$. □

Proof of Theorem 1.1. Let r_0 be as in Lemma 3.3 and $|w|_\infty \le \frac{V_0 r_0}{2}$. Also, let $K = K(r_0)$. We divide the proof into the following two steps.

Step 1. There is $\bar{u} \in K$ such that $I_K(\bar{u}) = \inf_{u \in W} I_K(u)$. Therefore, it follows from Lemma 2.2 that $\bar{u}$ is a critical point of I_K.

Let $\kappa = \inf_{u \in W} I_K(u)$. From the definition of Ψ_K, we have $\kappa = \inf_{u \in K} I_K(u)$. By Lemma 2.2, for $u \in K$ one has

$$\begin{aligned}\Phi(u) &= \int_{\mathbb{R}^2} wudx + \int_{\mathbb{R}^2} F(u)dx = \int_{\mathbb{R}^2} wudx + \int_{|u| \le r_0} H(u)dx \\ &\le |w|_\infty \int_{\mathbb{R}^2} |u|dx + C \int_{|u| \le r_0} |u|^{\tau+1} dx \le |w|_\infty \int_{\mathbb{R}^2} |u|dx + C|u|_\infty^\tau \int_{|u| \le r_0} |u|dx \\ &\le C||u||_W.\end{aligned}$$

Therefore, for $u \in K$ there holds

$$I_K(u) = \Psi_K(u) - \Phi(u) \ge \frac{1}{p}||u||_W^p - C||u||_W, \tag{3.9}$$

which yields $\kappa > -\infty$. Now, suppose that $\{u_n\} \subset K$ is a minimizing sequence, i.e. $I_K(u_n) \to \kappa$. The definition of K and (3.9) mean that $\{u_n\}$ is bounded. Passing to a subsequence if necessary, there is $\bar{u} \in W$ such that

$$\begin{aligned}&u_n \rightharpoonup \bar{u} \quad \text{in } W,\\ &u_n(x) \to \bar{u}(x) \quad \text{a.e. } x \in \mathbb{R}^2,\end{aligned}$$

which yields that $|\bar{u}|_\infty \le r_0$. It follows that $\bar{u} \in K$. We now assert that $\Phi(u_n) \to \Phi(\bar{u})$ as $n \to \infty$. In fact, it follows from

$$|wu_n| + F(\bar{u}) \le |w|_\infty |u_n| + r_0^\tau |u_n|,$$

the compactness of $W \hookrightarrow L^1(\mathbb{R}^2)$ and the dominated convergence theorem that $\Phi(u_n) \to \Phi(\bar{u})$. On the flip side, by weak lower semi-continuity we have $\Psi_K(\bar{u}) \le \liminf_{n\to\infty} \Psi_K(u_n)$. Therefore,

$$I_K(\bar{u}) \le \lim \inf_{n\to\infty} I_K(u_n) = \kappa \le I_K(\bar{u}),$$

it comes to the *step 1.*

Step 2. There is $v \in K$ such that $(-\Delta)_p^s v + V(x)|v|^{p-2}v = w + f(\bar{u})$.

As a consequence of Lemma 3.4, using the fact that $\bar{u} \in K$, we get the desired result of *Step 2.*

Using Theorem 3.1, *Step 1* and *Step 2* mean that $\bar{u}$ is a solution of (2.2). Next we show that $\bar{u}$ is non-trivial. Consider $0 < e \in K$. It follows from the definition of K that $te \in K$ for $0 < t < 1$. Furthermore,

$$\begin{aligned} &I(\bar{u}) \le I(te) \\ =&\frac{1}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|te(x)-te(y)|^p}{|x-y|^{2+ps}}dxdy + \frac{1}{p}\int_{\mathbb{R}^2} V(x)|te|^p dx - \int_{\mathbb{R}^2} wtedx - \int_{\mathbb{R}^2} F(te)dx \\ =&t\left(\frac{t^{p-1}}{p}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|e(x)-e(y)|^p}{|x-y|^{2+ps}}dxdy + \frac{t^{p-1}}{p}\int_{\mathbb{R}^2} V(x)|e|^p dx - \int_{\mathbb{R}^2} wedx\right) < 0 \end{aligned}$$

for t small enough. This implies that $\bar{u}$ is non-trivial. Finally, using the fact that $\bar{u} \in K$, we have $\bar{u}$ is a non-trivial solution of (1.1). This completes the proof. □

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