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SMALLEST EIGENVALUES AND THE EXISTENCE RESULT FOR THE BOUNDARY VALUE PROBLEM OF NONLINEAR FRACTIONAL DIFFERENTIAL SYSTEMS*

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Abstract In this paper, we first discuss the existence of smallest eigenvalues of fractional boundary value problems. Then we consider the existence of at least one positive solution for a class of nonlinear boundary value problem of fractional differential system. Compared with the existing methods, our analysis relies on the fixed point index theorem in a Cartesian product of two cones. We further construct two special operators to compute straightforwardly the fixed point index in a suitable cone. Finally, we present an illustrative example to support our main result.

Keywords Fractional differential system, eigenvalues, Green's function, positive solution, fixed point index.

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1. Introduction

Nonlinear fractional differential equations play an important role in differential equations and have wide applications in many fields, such as mathematical physics, engineering, economics, hydrology, and other fields (for instance, see [1, 2, 8, 9, 16, 18, 19]). Smallest eigenvalues and comparison of smallest eigenvalues for fractional boundary value problems are concerned by some researchers in recent years(for instance, see [6, 12, 14]). In these papers, the theory of u_0-positive operators with respect to a cone in a Banach space is generally applied to boundary value problem of several kinds of fractional linear differential equations. Meanwhile, the existence of solutions for a coupled system of nonlinear fractional differential equations is a fundamental problem, and the nature of the Green's function and fixed point theory (the cone expansion or compression fixed point theorem, Leggett-Williams fixed point theorem and Leray-Schauder fixed point theorem) are commonly em-

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ployed to deal with the problem. The key to these methods is to establish a suitable cone which is often constrained by the nature of Green's function and conditions of nonlinear terms. Nonlinear terms in a coupled system with same features have received much attention in literature, see [3–5, 7, 10, 11, 13, 17, 20–26, 28], and the references therein. For example, [24] discusses the $(n-1,1)$-type integral boundary value problem for coupled systems of nonlinear fractional differential equations when nonlinear terms in two equations are continuous and semipositone. [22] investigate the existence of positive solutions for a system of nonlinear fractional differential equations with sign-changing nonlinearities. When nonlinear terms in two equations are superlinear, [26] consider the existence of positive solutions to a singular semipositone boundary value problem of nonlinear fractional differential equations . However, only a very limited body of literature has discussed what may happen when the nonlinear term of one equation is sublinear and nonlinear term of the other equation is superlinear.

On the basis of the above work, we first get the existence of smallest eigenvalues and compare smallest eigenvalues for the fractional boundary value problems

$$D^{\alpha}_{0^+}u + \lambda_1 p(t)u = 0, \tag{1.1}$$

$$D^{\beta}_{0^+}u + \lambda_2 q(t)u = 0, \tag{1.2}$$

and satisfy

$$u(0) + u'(0) = 0, u(1) + u'(1) = 0, \tag{1.3}$$

where $0 < t < 1, 1 < \alpha, \beta \leqslant 2, \lambda_1, \lambda_2$ are real numbers, $D^{\alpha}_{0^+}$ and $D^{\beta}_{0^+}$ are the standard Caputo derivatives, $I = [0,1], \mathbb{R}^+ = [0,+\infty), p, q \in C(I, \mathbb{R}^+)$ that don't vanish identically on any nondegenerate compact subinterval of [0,1]. In this paper, the theory of u_0-positive operators with respect to a cone in a Banach space is adopted.

Secondly, we consider the existence of positive solutions to the boundary value problem of nonlinear fractional differential systems:

$$\begin{cases} D^{\alpha}_{0^+}u(t) = f_1(t,u(t)) + h_1(u(t),v(t)), \\ D^{\beta}_{0^+}v(t) = f_2(t,v(t)) + h_2(u(t),v(t)), \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, \\ v(0) + v'(0) = 0, v(1) + v'(1) = 0, \end{cases} \tag{1.4}$$

where $0 < t < 1, 1 < \alpha, \beta \leqslant 2$ are real numbers, $D^{\alpha}_{0^+}$ and $D^{\beta}_{0^+}$ are the standard Caputo derivatives, $f_i \in C(I \times \mathbb{R}^+, \mathbb{R}^+), h_i \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)(i = 1,2)$. The equations of (1.4) are derived from the furnace reheating model and various viscoelastic models. The single boundary value problem in (1.4) has also been studied, for example, in the literature [23, 27]. In fact, this kind of boundary value problem first appeared in the study of integer order boundary value problems, and the fractional order is a generalization of the integer order. To the best of our knowledge, there is very little known about the existence of solutions for the system (1.4).

Motivated by [25] and [5], we first extend the method in [5] to the nonlinear differential systems while a second-order ordinary differential system is considered. Furthermore, we construct a suitable cone and two special operators to obtain the existence results of solutions for the system (1.4) by applying the fixed point index theorem. It is noted that the nonlinear terms in the system (1.4) are different from those in [25] and the challenge is to verify that there are no fixed points on the boundary for the Cartesian product of two cones.

The paper is organized as follows. In section 2, we consider the smallest eigenvalues for the fractional boundary value problems and a simple comparison is made. In section 3, we deduce the new properties of the Green function and the existence of positive solutions of problem (1.4) is established. Finally, an example is given to illustrate the main result.

2. Smallest eigenvalues of the fractional boundary value problem

Definition 2.1 ([16]). For a function $f(t)$ given in the interval $[0,+\infty)$ the expression

$$D^{\alpha}_{0+}f(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n\int_0^t\frac{f(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouvilles fractional derivative of order α.

Let B be a Banach space on $\mathbb{R}$. A closed nonemty subset P of B is a cone. P is said to be solid if the interior P^o of P is nonempty. A cone P is reproducing if $B=P-P$, i.e., given $w\in B$, there exist $u,v\in P$ such that $w=u-v$. [15] proves that every solid cone is reproducing.

Cones generate a natural partial ordering on a real Banach space. Let P be a cone in a Banach space B. If $u,v\in B, u\leq v$ with respect to P if $v-u\in P$. If both $M,N:B\to B$ are bounded linear operators, $M\leq N$ with respect to P if $Mu\leq Nu$ for all $u\in P$. A bounded linear operator $M:B\to B$ is u_0-positive with respect to P if there exists $u_0\in P\setminus\{0\}$, there exist $k_1(u)>0$ and $k_2(u)>0$ such that $k_1u_0\leq Mu\leq k_2u_0$ with respect to P.

For (1.1), (1.2) and (1.3), the solution of (1.1) and (1.3) is equivalent to

$$u(t)=\lambda_1\int_0^1 G_1(t,s)p(s)u(s)ds,$$

where

$$G_1(t,s)=\begin{cases}\dfrac{(1-s)^{\alpha-1}(1-t)+(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & 0\leqslant s\leqslant t\leqslant 1,\\ \dfrac{(1-s)^{\alpha-1}(1-t)}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

Similarly, the solution of (1.2) and (1.3) is equivalent to

$$u(t)=\lambda_2\int_0^1 G_2(t,s)q(s)u(s)ds,$$

where

$$G_2(t,s)=\begin{cases}\dfrac{(1-s)^{\beta-1}(1-t)+(t-s)^{\beta-1}}{\Gamma(\beta)}+\dfrac{(1-s)^{\beta-2}(1-t)}{\Gamma(\beta-1)}, & 0\leqslant s\leqslant t\leqslant 1,\\ \dfrac{(1-s)^{\beta-1}(1-t)}{\Gamma(\beta)}+\dfrac{(1-s)^{\beta-2}(1-t)}{\Gamma(\beta-1)}, & 0\leqslant t\leqslant s\leqslant 1.\end{cases}$$

Define the Banach space

$$B=\{u\in C^1[0,1]:u(0)+u'(0)=0,u(1)+u'(1)=0\},$$

with the norm $\|u\|=\|u\|_0+\|u'\|_0$, where

$$\|u\|_0=\max_{t\in[0,1]}|u(t)|,\quad \|u'\|_0=\max_{t\in[0,1]}|u'(t)|.$$

Next, we define the linear operators

$$Mu(t)=\int_0^1 G_1(t,s)p(s)u(s)ds,$$

and

$$Nu(t)=\int_0^1 G_2(t,s)q(s)u(s)ds.$$

Define the cone of B as

$$P=\{u\in B:u(t)\geq 0,t\in[0,1]\}.$$

Lemma 2.1. *The cone P is solid in B and reproducing.*

The proof method is similar to lemma 3.2 of [6]. The proof is omitted here.

Lemma 2.2. *The bounded linear operators M and N are u_0-positive with respect to P.*

Proof. Define

$$\Omega=\{u\in B:u(t)>0,t\in[0,1),u'(1)<0\}.$$

Firstly, we prove $\Omega\subset P^o$. Let $u\in\Omega$, then $u(0)>0$. Meanwhile, there exists $\varepsilon_1>0$ such that $u(0)>\varepsilon_1$. Since $u\in C[0,1]$, there exists $\delta_1>0$ such that $u(t)>\varepsilon_1,t\in[0,\delta_1)$. And $u'(1)<0$, then there exists $\varepsilon_2>0$ such that $u'(1)+\varepsilon_2<0$. Since $u'\in C[0,1]$, there exists $\delta_2>0$ such that $u'(t)+\varepsilon_2<0,t\in(\delta_2,1]$. Therefore

$$u(t)=u(1)-\int_t^1 u'(s)ds\geq u(1)+\varepsilon_2(1-t),\quad t\in(\delta_2,1].$$

There also exists $\varepsilon_3>0$ such that $u(t)>\varepsilon_3,\quad t\in[\delta_1,\delta_2]$.

Set $\varepsilon=\frac{1}{2}\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$ and $U=\{v\in B:\|u-v\|<\varepsilon\}$, then $|v(t)-u(t)|<\varepsilon,|v'(t)-u'(t)|<\varepsilon,\ v\in U$. And $u(t)>\varepsilon_1\geq 2\varepsilon,t\in[0,\delta_1)$ and $u(t)>\varepsilon_3\geq 2\varepsilon,t\in[\delta_1,\delta_2]$, then

$$v(t)>u(t)-\varepsilon\geq\varepsilon,\ t\in[0,\delta_2].$$

For $t \in (\delta_2, 1]$, we get

$$v(t) - u(t) = v(1) - u(1) - \int_t^1 (v'(s) - u'(s))ds \geq v(1) - u(1) - \varepsilon(1-t).$$

Then

$$\begin{aligned} v(t) &\geq u(t) + v(1) - u(1) - \varepsilon(1-t) \\ &\geq u(1) + \varepsilon_2(1-t) + v(1) - u(1) - \varepsilon(1-t) \\ &\geq \varepsilon(1-t) + v(1) > 0. \end{aligned}$$

So $v(t) > 0, t \in [0, 1)$. And

$$v'(1) \leq u'(1) + \varepsilon < -\varepsilon_2 + \varepsilon < -\varepsilon < 0.$$

Therefore $v \in \Omega \subset P$. Then we have $\Omega \subset P^o$.

Secondly, we prove $M : P \setminus \{0\} \to \Omega$. Let $u \in P \setminus \{0\}$. So there exists a compact interval $[a, b] \subset [0, 1]$ such that $u(t) > 0, p(t) > 0$ for $t \in [a, b]$. For $G_1(t, s) > 0, t, s \in (0, 1]$, we have

$$\begin{aligned} Mu(t) &= \int_0^1 G_1(t,s)p(s)u(s)ds \\ &\geq \int_a^b G_1(t,s)p(s)u(s)ds \\ &> 0. \end{aligned}$$

Then from

$$\begin{aligned} (Mu)'(t) = &-\int_0^1 [\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}]p(s)u(s)ds \\ &+ \int_0^t \frac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds, \end{aligned}$$

we can get

$$(Mu)'(1) = -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds \leq -\int_a^b \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds < 0.$$

So, $M : P \setminus \{0\} \to \Omega \subset P^o$.

Finally, we choose $u_0 \in P \setminus \{0\}$ and let $u \in P \setminus \{0\}$. So $Mu \in \Omega \subset P^o$. Hence we can choose sufficiently small $k_1 > 0$ and sufficiently large $k_2 > 0$ so that $Mu - k_1u_0 \in P^o, u_0 - \frac{1}{k_2}Mu \in P^o$. So we get $k_1u_0 \leq Mu \leq k_2u_0$ with respect to P. Thus the bounded linear operators M is u_0-positive with respect to P. Meanwhile, we can prove the bounded linear operators N is u_0-positive with respect to P at the same way. □

Theorem 2.1. *The operators $M, N : P \to P$ are compact.*

Proof. Clearly $M : P \to P$.

Next, we prove that M is a compact operator. Let $L > 0$, $K = \{u \in P : \|u\| \leq L\}$, Then

$$|Mu(t)| = |\int_0^1 [\frac{(1-s)^{\alpha-1}(1-t)}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}]p(s)u(s)ds$$

$$
\begin{aligned}
&+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds| \\
&\le \|p\|_0 L|\int_0^1 [\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}]ds| + \|p\|_0 L|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds| \\
&= \|p\|_0 L(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}).
\end{aligned}
$$

$$
\begin{aligned}
|(Mu)'(t)| &= |-\int_0^1 [\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}]p(s)u(s)ds \\
&\quad +\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}p(s)u(s)ds| \\
&\le \|p\|_0 L(\frac{1}{\Gamma(\alpha+1)}+\frac{2}{\alpha\Gamma(\alpha)}).
\end{aligned}
$$

So M is uniformly bounded.

Let $\|u\| \le L$, $\forall\varepsilon > 0$, there exists $\delta < \frac{\varepsilon}{\|p\|_0 L\frac{3\alpha+3}{\Gamma(\alpha+1)}}(0 < \delta < 1)$ and $\delta < (\frac{\varepsilon}{\|p\|_0 L\frac{3\alpha+3}{\Gamma(\alpha+1)}})^{\frac{1}{\alpha}}(\delta > 1)$ for $0 \le t_1 < t_2 \le 1, |t_1 - t_2| < \delta$ so that

$$
\begin{aligned}
&|Mu(t_1) - Mu(t_2)| \\
&= |\int_0^1 [\frac{(1-s)^{\alpha-1}(1-t_1)}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}(1-t_1)}{\Gamma(\alpha-1)}]p(s)u(s)ds \\
&\quad +\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - \int_0^1 [\frac{(1-s)^{\alpha-1}(1-t_2)}{\Gamma(\alpha)} \\
&\quad +\frac{(1-s)^{\alpha-2}(1-t_2)}{\Gamma(\alpha-1)}]p(s)u(s)ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds| \\
&\le \|p\|_0 L|(t_2-t_1)\int_0^1 [\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}]ds| \\
&\quad +\|p\|_0 L|\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds - \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds| \\
&\le \|p\|_0 L\delta(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}) \\
&\quad +\|p\|_0 L|\frac{t_1^\alpha}{\Gamma(\alpha+1)} - \frac{t_2^\alpha-(t_2-t_1)^\alpha}{\Gamma(\alpha+1)} + \frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}| \\
&\le \|p\|_0 L\delta\frac{\alpha+1}{\Gamma(\alpha+1)} + \|p\|_0 L\frac{2\delta^\alpha}{\Gamma(\alpha+1)} < \varepsilon.
\end{aligned}
$$

$$
\begin{aligned}
&|(Mu)'(t_1) - (Mu)'(t_2)| \\
&\le \|p\|_0 L|\int_0^{t_1} \frac{(t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds - \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds| \\
&= \|p\|_0 L|\frac{t_1^{\alpha-1}}{\Gamma(\alpha)} - \frac{t_2^{\alpha-1}-(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}| \\
&\le \|p\|_0 L\frac{2\delta^\alpha}{\Gamma(\alpha)} < \varepsilon.
\end{aligned}
$$

So M is equicontinuous. In the same way we can show that N is also compact. $\square$

The following two results can be proved by using the method of [28].

Theorem 2.2. *B, P, M and N are defined above. Then M (and N) has an eigenvalue that is simple, positive and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $P^o \backslash \{0\}$.*

Theorem 2.3. *B, P, M and N are defined above. Let $p(t) \leq q(t)$ on $[0,1]$. Λ_1, Λ_2 are the eigenvalues of M and N, corresponding to an essentially unique eigenvector $u_1, u_2 \in P^o$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0,1]$.*

The following theorem can be proved by Theorem 2.2 and Theorem 2.3.

Theorem 2.4. *Assume the hypothesis of Theorem 2.3. Then there exist smallest positive eigenvalue λ_1 and λ_2 of (1.1),(1.3) and (1.2),(1.3), respectively, each of which is simple and less than the absolute value of any other eigenvalue for corresponding problem, and eigenvectors corresponding to λ_1 and λ_2 may be chosen to belong to P^o.*

3. The existence of nonlinear fractional differential systems

In this section, we construct a cone which is the Cartesian product of two cones and change the problem into the fixed point problem in the constructed cone. We first give some necessary definitions and preliminary results as following.

It is well known that $C[0,1]$ is a Banach space with the norm given by $\|u\| = \max_{t\in[0,1]} |u(t)|$. Let $C^+[0,1] = \{u \in C[0,1] : u(t) \geq 0, t \in [0,1]\}$, $K = \{u \in C^+[0,1] : u(t) \geq \frac{1}{8}\|u\|, t \in [\frac{1}{4}, \frac{3}{4}]\}$.

Lemma 3.1 ([23]). *Let $f(t) \in C^+[0,1]$ be a given function. Then the boundary value problem*

$$\begin{cases} D^{\alpha}_{0^+} u(t) = f(t), \quad 0 < t < 1, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0 \end{cases} \tag{3.1}$$

has a unique solution

$$u(t) = \int_0^1 G_1(t,s) f(s) ds,$$

here $G_1(t,s)$ is the Green function of boundary value problem (3.1). (G_1, G_2) is called the Green's function of the boundary value problem (1.4).

Lemma 3.2 ([23]). *Let $f(t) \in C[0,1]$ be a given function. Then function $G_1(t,s)$ has the following properties:*

(i) $G_1(t,s) \in C([0,1] \times [0,1])$, and $G_1(t,s) > 0$ for $t, s \in (0,1)$;

(ii) There exists a positive function $\gamma \in C(0,1)$ such that

$$\min_{1/4 \leq t \leq 3/4} G_1(t,s) \geq \gamma(s) H(s)$$

$$\max_{0 \leq t \leq 1} G_1(t,s) \leqslant H(s), \quad s \in (0,1),$$

where

$$H(s) = \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)},$$
$$\gamma(s) = \frac{1}{4}\frac{(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2}}{2(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2}}.$$

Remark 3.1 ([23]). Clearly, From the expression of $\gamma(s)$, we see that $\gamma(s) \geq \frac{1}{8}$.

Define operator $A : C[0,1] \to C[0,1]$ by

$$(Au)(t) = \int_0^1 G_1(t,s)u(s)ds.$$

Clearly, A is a completely continuous linear operator and $A(K) \subset K$. By Theorem 2.4, we know that A has a positive first eigenvalue λ_1.

For $\lambda \in [0,1], u, v \in C^+[0,1]$, we define the mappings $A_v(\lambda,\cdot), B_u(\lambda,\cdot) : C^+[0,1] \to C^+[0,1]$ and $T_\lambda(\cdot,\cdot) : C^+[0,1] \times C^+[0,1] \to C^+[0,1] \times C^+[0,1]$ by

$$A_v(\lambda,u)(t) = \int_0^1 G_1(t,s)[(1-\lambda)u^2(s) + \lambda(f_1(t,u(t)) + h_1(u(t),v(t)))]ds,$$
$$B_u(\lambda,v)(t) = \int_0^1 G_2(t,s)[(1-\lambda)\sqrt{v(s)} + \lambda(f_2(t,v(t)) + h_2(u(t),v(t)))]ds,$$

and

$$T_\lambda(u,v)(t) = (A_v(\lambda,u)(t), B_u(\lambda,v)(t)).$$

It is clear that the existence of a positive solution of system (1.4) is equivalent to the existence of a nontrivial fixed point of T_1 in $K \times K$.

Lemma 3.3. *$T_\lambda : K \times K \to K \times K$ is completely continuous.*

Proof. For $(u,v) \in K \times K$, we show that $T_\lambda(u,v) \in K \times K$,i.e., $A_v(\lambda,u) \in K \times K$ and $B_u(\lambda,v) \in K \times K$. By Lemma 3.2 and Remark 3.1, we have

$$\begin{aligned}
A_v(\lambda,u)(t) &= \int_0^1 G_1(t,s)[(1-\lambda)u^2(s) + \lambda(f(v(s)))]ds \\
&\geq \int_0^1 \gamma(s)H(s)[(1-\lambda)u^2(s) + \lambda(f(v(s)))]ds \\
&\geq \frac{1}{8}\int_0^1 H(s)[(1-\lambda)u^2(s) + \lambda(f(v(s)))]ds \\
&\geq \frac{1}{8}\|A_v(\lambda,u)\|, \quad t \in [\frac{1}{4},\frac{3}{4}].
\end{aligned}$$

Similarly,

$$B_u(\lambda,v) \geq \frac{1}{8}\|B_u(\lambda,v)\|, \quad t \in [\frac{1}{4},\frac{3}{4}].$$

Consequently $A_v(\lambda,u) \in K$ and $B_u(\lambda,v) \in K$, thus $T_\lambda(K \times K) \in K \times K$. Obviously, $T_\lambda : K \times K \to K \times K$ is completely continuous. □

Remark 3.2 ([5]). In fact, denoting $T(\lambda, u, v)(t) \in K$, thus $\overline{T([0,1] \times K \times K)}$ is a compact set by the Arzelà-Ascoli theorem.

Lemma 3.4. *Let X is a retract of real Banach space E. For every bounded open set $U \subset X$, suppose $A : \overline{U} \to X$ is completely continuous and has no fixed point on ∂U, then there exists an integer $i(A, U, X)$(called the fixed point index of A with respect to X in U) that satisfies the following conditions*

(i) *Normalization: if $A : \overline{U} \to U$ is a constant operator, then $i(A, U, X) = 1$;*

(ii) *Decomposition: if U_1, U_2 are disjoint subsets and open (with respect to X), and A has no fixed point on $\partial U \setminus (U_1 \cup U_2)$, then*

$$i(A, U, X) = i(A, U_1, X) + i(A, U_2, X);$$

(iii) *Homotopy invariance: if $H : [0,1] \times \overline{U} \to X$ is completely continuous and there has $H(t, x) \neq x$, for $(t, x) \in [0,1] \times \partial U$, then $i(H(t, \cdot), U, X)$ has nothing to do with t;*

(iv) *Retention: if Y is a retract of X, $A(\overline{U}) \subset Y$, then*

$$i(A, U, X) = i(A, U \cap Y, Y);$$

(v) *Excision property: let V(with respect to X) be an open set, $V \subset U$ and A has no fixed point on $\partial U \setminus V$, then*

$$i(A, U, X) = i(A, V, X);$$

(vi) *Solution property: if $i(A, U, X) \neq 0$, then A has at least a fixed point on U.*

From the Dugundji theorem, we know that every non-empty convex closed set of real Banach space E is a retract of E. Therefore, any cones of E are retracts of E.

Let $P \subset E$ be a closed convex cone in Banach space E. For $r > 0$, let $P_r = \{u \in P : \|u\| < r\}, \partial P_r = \{u \in P : \|u\| = r\}$; then ∂P_r is the boundary of P_r in P.

Lemma 3.5 ([8,28]). *Let $A : P \to P$ be completely continuous. We have:*

(i) if $\|Au\| > \|u\|, \quad \forall u \in \partial P_r$, then $i(A, P_r, P) = 0$;

(ii) if $\|Au\| < \|u\|, \quad \forall u \in \partial P_r$, then $i(A, P_r, P) = 1$.

Lemma 3.6 ([5]). *Let E be a Banach space and let $K_i \subset E(i = 1, 2)$ be a closed convex cone in E. For $r_i > 0(i = 1, 2)$, denote $K_{r_i} = \{u \in K_i : \|u\| < r_i\}, \partial K_{r_i} = \{u \in K_i : \|u\| = r_i\}$. Suppose $A_i : K_i \to K_i$ is completely continuous. If $u_i \neq A_i u_i$, for $u_i \in \partial K_{r_i}$, then*

$$i(A, K_{r_1} \times K_{r_2}, K_1 \times K_2) = i(A, K_{r_1}, K_1) \cdot i(A, K_{r_2}, K_2),$$

where $A(u, v) = (A_1 u, A_2 v), \forall (u, v) \in K_1 \times K_2$.

3.1. The proof of main result

Theorem 3.1. *Suppose $f_i \in C(I \times \mathbb{R}^+, \mathbb{R}^+), h_i \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)(i = 1, 2)$ satisfy the conditions:*

(H_1) $\limsup\limits_{u \to 0^+} \max\limits_{t \in I} \dfrac{f_1(t, u)}{u} < \lambda_1 < \liminf\limits_{u \to +\infty} \min\limits_{t \in I} \dfrac{f_1(t, u)}{u}$;

(H_2) $\limsup\limits_{v \to \infty+} \max\limits_{t \in I} \dfrac{f_2(t, v)}{v} < \lambda_1 < \liminf\limits_{v \to 0^+} \min\limits_{t \in I} \dfrac{f_2(t, v)}{v}$;

(H_3) $\lim\limits_{u\to 0^+}\dfrac{h_1(u,v)}{u}=0$, *uniformly with respect to* $v\in\mathbb{R}^+$*;*

(H_4) $\lim\limits_{v\to +\infty}\dfrac{h_2(u,v)}{v}=0$, *uniformly with respect to* $u\in\mathbb{R}^+$, *and* $\lim\limits_{u\to+\infty}h_2(u,v)=0$, *uniformly with respect to* $v\in[0,Q]$, $\forall Q>0$. *Then system* (1.4) *has at least one positive solution.*

Proof. From the definition of $A_v(\lambda,u)(t)$, we have

$$A_v(0,u)(t)=\int_0^1 G_1(t,s)u^2(s)ds.$$

Then for every $u\in K$, we have that

$$\|A_v(0,u)\|\leq\int_0^1 M(s)u^2(s)ds\leq\int_0^1 M(s)ds\|u\|^2.$$

Set $r_0=(\int_0^1 M(s)ds)^{-1}$, then for $r\in(0,r_0)$ and $u\in\partial K_r$, it follows $\|A_v(0,u)|<\|u\|$. By Lemma 3.5, we have

$$i(A_v(0,\cdot),K_r,K)=1,\quad \forall r\in(0,r_0). \tag{3.2}$$

On the other hand, for every $u\in K$, we have

$$\begin{aligned}\|A_v(0,u)\|\geq A_v(0,u)(\frac{1}{2})&=\int_0^1 G_1(\frac{1}{2},s)u^2(s)ds\\&\geq\frac{1}{8}\int_{\frac{1}{4}}^{\frac{3}{4}}M(s)u^2(s)ds\\&\geq\frac{1}{512}\int_{\frac{1}{4}}^{\frac{3}{4}}M(s)ds\|u\|^2.\end{aligned}$$

Set $R_0=(\frac{1}{512}\int_{\frac{1}{4}}^{\frac{3}{4}}M(s)ds)^{-1}(>r_0)$, if $R>R_0$, then

$$\|A_v(0,u)\|\geq\frac{1}{R_0}\|u\|^2=\frac{\|u\|}{R_0}\|u\|=\frac{R}{R_0}\|u\|>\|u\|,\quad\forall u\in\partial K_R.$$

By Lemma 3.5, we have

$$i(A_v(0,\cdot),K_R,K)=0,\quad\forall R>R_0. \tag{3.3}$$

Similarly, we have

$$\|B_u(0,v)\|\leq\int_0^1 M(s)ds\sqrt{\|v\|},\quad \|B_u(0,v)\|\geq\frac{1}{16\sqrt{2}}\int_{\frac{1}{4}}^{\frac{3}{4}}M(s)ds\sqrt{\|v\|}.$$

Set $\overline{R}_0=(\int_0^1 M(s)ds)^2$, $\overline{r}_0=\frac{1}{16\sqrt{2}}(\int_{\frac{1}{4}}^{\frac{3}{4}}M(s)ds)^2$, then for $0<\overline{r}_0<\overline{R}_0<+\infty$. For $r_1\in(0,\overline{r}_0)$, $R_1>\overline{R}_0$, it is clear that

$$\|B_u(0,v)\|>\|v\|,\quad\forall v\in\partial K_{r_1},\quad \|B_u(0,v)\|<\|v\|,\ \forall\ v\in\partial K_{R_1}.$$

Thus by Lemma 3.5, we have

$$i(B_u(0,\cdot),K_{r_1},K)=0,\quad \forall r_1\in(0,\overline{r}_0), \tag{3.4}$$

$$i(B_u(0,\cdot),K_{R_1},K)=1,\quad \forall R_1>\overline{R}_0. \tag{3.5}$$

By Lemma 3.4, Lemma 3.6 and (3.2)-(3.5), we have

$$\begin{aligned} &i(T_0,(K_R\setminus\overline{K}_r)\times(K_{R_1}\setminus\overline{K}_{r_1}),K\times K)\\ =&i(A_v(0,\cdot),K_R\setminus\overline{K}_r,K)\cdot i(B_u(0,\cdot),K_{R_1}\setminus\overline{K}_{r_1},K)\\ =&-1. \end{aligned} \tag{3.6}$$

Finally, we show that

$$\begin{aligned} &i(T_\lambda,(K_{R_2}\setminus\overline{K}_{r_2})\times(K_{R_3}\setminus\overline{K}_{r_3}),K\times K)\\ =&\,i(T_0,(K_{R_2}\setminus\overline{K}_{r_2})\times(K_{R_3}\setminus\overline{K}_{r_3}),K\times K), \end{aligned}$$

where $r_2\in(0,r_0),R_2>R_0,r_3\in(0,\overline{r}_0)$ and $R_3>\overline{R}_0$ will be determined later.

By Lemma 3.2, Lemma 3.4 and Remark 3.1, we will verify that

$$(u,v)\neq T_\lambda(u,v),\quad (u,v)\in\partial[(K_{R_2}\setminus\overline{K}_{r_2})\times(K_{R_3}\setminus\overline{K}_{r_3})]. \tag{3.7}$$

Now we show that it is valid.

(i) We can prove that $(u,v)\neq T_\lambda(u,v)$ for all $\lambda\in[0,1]$ and $(u,v)\in\partial K_{r_2}\times K$. In fact, if there exist $\lambda_0\in[0,1]$ and $(u_0,v_0)\in\partial K_{r_2}\times K$, such that $(u_0,v_0)=T_{\lambda_0}(u_0,v_0)$, then (u_0,v_0) satisfies the following equation

$$\begin{cases} D^\alpha_{0^+}u_0(t)=(1-\lambda_0)u_0^2(t)+\lambda_0f_1(t,u_0(t))+\lambda_0h_1(u_0(t),v_0(t)),\\ u_0(0)+u_0'(0)=0,u_0(1)+u_0'(1)=0. \end{cases} \tag{3.8}$$

By (H_1) and (H_3), we choose $\varepsilon\in(0,\frac{\lambda_1}{2})$ and $0<r_2<\min\{r_0,\lambda_1-\varepsilon\}$ such that

$$f_1(t,u)\le(\lambda_1-2\varepsilon)u,\quad \text{for } 0<t<1\ 0<u\le r_2, \tag{3.9}$$

$$h_1(u,v)\le\varepsilon u,\quad \text{for}\ \ 0<u\le r_2,v\in R^+. \tag{3.10}$$

Hence, by (3.8)-(3.10), we get

$$\begin{aligned} u_0(t)=&\int_0^1G_1(t,s)[(1-\lambda_0)u_0^2(s)+\lambda_0f_1(s,u_0(s))+\lambda_0h_1(u_0(s),v_0(s))]ds\\ \leqslant&\int_0^1G_1(t,s)[(1-\lambda_0)u_0^2(s)+\lambda_0(\lambda_1-2\varepsilon)u_0(s)+\lambda_0\varepsilon u_0(s)]ds\\ \leqslant&\int_0^1G_1(t,s)[(1-\lambda_0)r_2u_0(s)+\lambda_0(\lambda_1-\varepsilon)u_0(s)]ds\\ \leqslant&\int_0^1G_1(t,s)(\lambda_1-\varepsilon)u_0(s)ds\\ =&(\lambda_1-\varepsilon)Au_0(t),\quad t\in[0,1]. \end{aligned}$$

Moreover, $u_0\leqslant(\lambda_1-\varepsilon)^nA^nu_0\leqslant(\lambda_1-\varepsilon)^n\|A^n\|\|u_0\|$. Thus $r(A)=\lim\limits_{n\to\infty}\sqrt[n]{\|A^n\|}\ge\dfrac{1}{\lambda_1-\varepsilon}>\dfrac{1}{\lambda_1}$, which is a contradiction.

(ii) By conditions (H_1) and (H_3), we have $\varepsilon > \frac{1-\frac{\lambda_1}{64}\int_{\frac{1}{4}}^{\frac{3}{4}} M(s)ds}{\frac{1}{64}\int_{\frac{1}{4}}^{\frac{3}{4}} M(s)ds}$, when $0 < \lambda_1 < \frac{64}{\int_{\frac{1}{4}}^{\frac{3}{4}} M(s)ds}$ (or else $\varepsilon > 0$, when $\lambda_1 \geqslant \frac{64}{\int_{\frac{1}{4}}^{\frac{3}{4}} M(s)ds}$) and $N > 0$ such that

$$f_1(t,u) \geqslant (\lambda_1+\varepsilon)u, \quad t \in I, \quad u \geqslant N.$$

Let $C = (\lambda_1+\varepsilon)^2 + \max_{t\in I, 0\leqslant u\leqslant N} f_1(t,u)$, it is obvious that

$$\begin{aligned} &f_1(t,u) \geqslant (\lambda_1+\varepsilon)u - C, \quad \forall t \in I, \quad u \geqslant 0,\\ &u^2 \geqslant (\lambda_1+\varepsilon)u - (\lambda_1+\varepsilon)^2 \geqslant (\lambda_1+\varepsilon)u - C, \quad \forall u \geqslant 0. \end{aligned}$$

If there exist $\lambda_0 \in I$ and $(u_0, v_0) \in \partial K \times K$ such that $(u_0, v_0) = T_{\lambda_0}(u_0, v_0)$, then (u_0, v_0) satisfies (3.8). Then we can show that

$$\begin{aligned} u_0(t) &= \int_0^1 G_1(t,s)[(1-\lambda_0)u_0^2(s) + \lambda_0 f_1(s,u_0(s)) + \lambda_0 h_1(u_0(s),v_0(s))]ds\\ &\geqslant \frac{1}{8}\int_{\frac{1}{4}}^{\frac{3}{4}} M(s)[(\lambda_1+\varepsilon)u_0(s) - C]ds\\ &\geqslant \frac{1}{8}\int_{\frac{1}{4}}^{\frac{3}{4}} M(s)[(\lambda_1+\varepsilon)\frac{1}{8}\|u_0\| - C]ds, \quad t \in I. \end{aligned}$$

Hence, we have

$$\|u_0\| \leqslant \frac{\frac{c}{8}\int_{\frac{1}{4}}^{\frac{3}{4}} M(s)ds}{\frac{1}{64}\int_{\frac{1}{4}}^{\frac{3}{4}} M(s)(\lambda_1+\varepsilon)ds - 1} \triangleq \overline{R}.$$

Taking $R_2 > \max\{R_0, \overline{R}\}$, $(u,v) \neq T_\lambda(u,v)$ for any $(u,v) \in \partial K_{R_2} \times K$, $\lambda \in I$.

(iii) By conditions (H_2) and (H_4), there exist $\varepsilon > 0$ and $0 < \eta < \frac{1}{(\lambda_1+\varepsilon)^2}$ such that

$$f_2(t,v) \geqslant (\lambda_1+\varepsilon)v, \quad 0 \leqslant v \leqslant \eta. \tag{3.11}$$

By $0 < \eta < \frac{1}{(\lambda_1+\varepsilon)^2}$, it is not difficult to show that

$$\sqrt{v} \geqslant (\lambda_1+\varepsilon)v, \quad 0 \leqslant v \leqslant \eta. \tag{3.12}$$

Taking $0 < r_3 < \min(r_0, \eta)$, by (3.11)-(3.12) and the proof in the same way as (i), we have $(u,v) \neq T_\lambda(u,v)$ for any $(u,v) \in \partial K \times K_{r_3}$, $\lambda \in I$.

(iv) By conditions (H_2) and (H_4), there exist $\varepsilon \in (\frac{\lambda_1\int_0^1 M(s)ds - 1}{\int_0^1 M(s)ds}, \frac{\lambda_1}{2})$, when $\frac{1}{\int_0^1 M(s)ds} < \lambda_1 < \frac{2}{\int_0^1 M(s)ds}$ (or else $\varepsilon \in (0, \frac{\lambda_1}{2})$) and $N > 0$ such that

$$f_2(t,v) \leqslant (\lambda_1 - 2\varepsilon)v, \quad h_2(u,v) \leqslant \varepsilon v, \quad \forall t \in I, \ u \in \mathbb{R}^+, v \geqslant N.$$

Let $C = \frac{1}{\lambda_1-\varepsilon} + \max_{t\in I, u\in\mathbb{R}^+, 0\leqslant v\leqslant N}(f_2(t,v) + h_2(u,v))$, then it is not difficult to get that

$$f_2(t,v) + h_2(u,v) \leqslant (\lambda_1-\varepsilon)v + C, \quad \forall t \in I, \ u \in \mathbb{R}^+, v \geqslant 0,$$

and

$$\sqrt{v} \leqslant (\lambda_1 - \varepsilon)v + C, \qquad v \geqslant 0.$$

If there have $\lambda_0 \in I$ and $(u_0, v_0) \in K \times \partial K$ such that $(u_0, v_0) = T_{\lambda_0}(u_0, v_0)$, then

$$\begin{aligned} v_0(t) &= \int_0^1 G_2(t,s)[(1-\lambda_0)\sqrt{v_0(s)} + \lambda_0(f_2(s,u_0(s)) + h_2(u_0(s), v_0(s)))]ds \\ &\leqslant \int_0^1 M(s)[(\lambda_1 - \varepsilon)v_0(s) + C]ds \\ &\leqslant \int_0^1 M(s)[(\lambda_1 - \varepsilon)\|v_0\| + C]ds, \quad t \in I. \end{aligned}$$

Hence, we have

$$\|v_0\| \leqslant \frac{C\int_0^1 M(s)ds}{1 - (\lambda_1 - \varepsilon)\int_0^1 M(s)ds} \triangleq \overline{R'}.$$

Taking $R_3 > \max\{\overline{R_0}, \overline{R'}\}$, $(u,v) \neq T_\lambda(u,v)$ for any $(u,v) \in K \times \partial K_{R_3}$, $\lambda \in I$. In this way, through the above four steps it is not difficult to get (3.7). By Remark 3.2 and the homotopy invariance of fixed point index and (3.6), we obtain

$$i(T_1, (K_{R_2} \setminus \overline{K}_{r_2}) \times (K_{R_3} \setminus \overline{K}_{r_3}), K \times K) = -1.$$

Hence, T_1 has a fixed point in $(K_{R_2} \setminus \overline{K}_{r_2}) \times (K_{R_3} \setminus \overline{K}_{r_3})$, and system (1.4) has at least one positive solution namely. □

3.2. An example

Example 3.1. To illustrate the usefulness of the above result, we discuss the following system

$$\begin{cases} D_{0^+}^{\frac{3}{2}} u(t) = (2+t)u(t)\sin u(t) + u^3(t)\cos v(t), \quad 0 < t < 1, \\ D_{0^+}^{\frac{3}{2}} v(t) = (1+t)v^{\frac{1}{2}}(t) + \frac{v^{\frac{1}{3}}(t)}{\exp(u(t))}, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, \\ v(0) + v'(0) = 0, v(1) + v'(1) = 0, \end{cases} \tag{3.13}$$

Let $f_1(t,u) = (2+t)u\sin u$, $f_2(t,v) = (1+t)v^{\frac{1}{2}}$, $h_1(u,v) = u^3\cos v$, $h_2(u,v) = \frac{v^{\frac{1}{3}}}{\exp(u)}$. Obviously, $f_i, h_i (i = 1,2)$ satisfy the conditions of Theorem 3.1. Therefore, system (3.13) has at least one positive solution.

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