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AN ε-UNIFORMLY CONVERGENT METHOD FOR SINGULARLY PERTURBED PARABOLIC PROBLEMS EXHIBITING BOUNDARY LAYERS

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Abstract A numerical method is proposed for singularly perturbed parabolic convection-diffusion equation whose solution exhibits boundary layers near the right endpoints of the domain of consideration. The method encompasses the Crank-Nicolson scheme on a uniform mesh in temporal direction and quartic B-spline collocation method on piecewise-uniform (i.e.,Shishkin mesh) mesh in space directions, respectively. Through rigorous convergence analysis, the method has shown theoretically fourth-order convergent in space direction and second-order convergent in the time direction. We have solved two numerical examples to prove the efficiency and robustness of the method and to validate the theoretical results. Since the exact/analytical solution to the problem is not known, hence we applied the double mesh principle to compute the maximum absolute errors. Additionally, some numerical simulations are displayed to produce the conclusiveness of determining layer behaviour and their locations.

Keywords Singular perturbations, parabolic partial differential equations, collocation method, B-splines, Crank-Nicolson method, Shishkin mesh, parameter-uniform convergence.

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1. Introduction

In this paper, we consider one-dimensional singularly perturbed parabolic convection-diffusion equation (SPPCDEs) in the rectangular domain:

$$\mathbf{L}_x^{\varepsilon} Y(x,\tau) = \left(-\varepsilon \frac{\partial^2 Y}{\partial x^2} + \mathtt{a}(x,\tau)\frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial \tau} + \mathtt{b}(x,\tau) Y \right) = F(x,\tau), \tag{1.1}$$

where $0 < \varepsilon \ll 1$, $(x,\tau) \in \mathbb{D} = (0,1) \times (0,T]$, $\overline{\mathbb{D}} = [0,1] \times [0,T]$, T is finite time, $\partial\mathbb{D} = \overline{\mathbb{D}}/\mathbb{D} = \mathbb{R}_x \cup \mathbb{R}_L \cup \mathbb{R}_R$, with the conditions given as:

$$Y(x,0) = \mathtt{g}_1(x), \quad \text{on } \mathbb{R}_x = \{(x,0) : 0 \le x \le 1\},$$

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$$\begin{aligned} Y(0,\tau) &= \mathtt{g}_2(x), \quad \text{on } \mathbb{R}_L = \{(0,\tau) : 0 \le \tau \le T\}, \\ Y(1,\tau) &= \mathtt{g}_3(x), \quad \text{on } \mathbb{R}_R = \{(1,\tau) : 1 \le \tau \le T\}. \end{aligned} \tag{1.2}$$

Assuming the fact that $\mathtt{a}(x,\tau)$, $\mathtt{b}(x,\tau)$, $F(x,\tau) \in C^0(\overline{\mathbb{D}})$ such as:

$$\begin{aligned} \mathtt{a}(x,\tau) &\ge \mathtt{a}^* > 0, \quad \forall\ (x,\tau) \in \overline{\mathbb{D}}, \\ \mathtt{b}(x,\tau) &\ge \mathtt{b}^* > 0, \quad \forall\ (x,\tau) \in \overline{\mathbb{D}}. \end{aligned} \tag{1.3}$$

The problem (1.1)-(1.3) whose solution demonstrate parabolic boundary layer of width $O(\varepsilon)$ in the neighbourhood of the right side of boundary as ε approaches zero and all the characteristic curves are parallel to the right side boundary of the reduced problem. In recent years, the study of parabolic equations with convection-diffusion term has attracted the attention of many scholars due to its widespread application in various disciplines, including control theory, economics, natural science, and engineering, wherever, mass transport or energy particles flow in any physical system happen because of each convection and diffusion processes. The considered problem is used to model transport problems [13], and also model the Navier–Stokes equations having the Reynolds numbers very high [12]. Also, SPPCDEs help modelling of diffusion equation within semiconductor devices [25], financial mathematics, quantum physics, etc. In general, a singularly perturbed problem(SPP) consist a parameter ε which is very small and multiple of the highest derivative term. The conduct solution of considered problem is much ambiguous and shifts quite fast in a narrow area as ε tending to very small number. The conventional numerical methodologies, namely combination finite difference schemes and FEM with piecewise polynomial basis functions using equally spaced grid comes up short to truthfully start taking the solution, especially in the narrow zone. In reality, due to unanticipated fluctuations caused by minor perturbation parameters, the use of typical numerical techniques for analyzing SPPs on homogeneous grid can are not persistent and do not yield a correct and stable solution. Due to the sift response, it is crucial to create suitable numerical techniques that are independent of the perturbation parameter. This implies that procedures that have been designed should converge consistently to the perturbation parameter.

In recent years, several scholars have laboured to develop consistently convergent numerical techniques to solve SPPs, but much work has gone into solving SPPCDEs. Farrell et al. [8], established a numerical approach and demonstrated parameter uniform convergence using a modified mesh technique. In [22], a uniform convergent numerical scheme developed for solving parabolic equations with convection–diffusion term using upwind finite difference methods. Ramos [27] proposed a numerical algorithm and proved uniform convergence using exponentially fitted method. Gupta and Kadalbajoo [10] developed parameter uniform method using cubic B-spline and Surla and Jerkovic [35] established a computational technique using spline functions for SPP. Sakai and Usmani [32] proposed idea using trigonometric and hyperbolic B-spline. In [23], the authors established a computational technique using spline to solve SPPs. In [17], Kumar and Kadalbajoo used cubic B-spline and proved that a parameter-uniform method for SPDDEs. Kumar [16] conceived a computational technique for resolving SPP with a temporal delay. The numerical solution of SPPs have been discussed with non-polynomial quadratic splines [15], trigonometric B-spline [5], Haar wavelet approach [29]. Majumdar and Natesan proposed numerical methods using composite finite difference method and show that an ε-uniform convergence [21]. In [6], the authors have developed a method using upwind scheme on a nonuniform mesh for the spatial discretization with Euler implicit method for the time discretization and proved the uniform convergence. In [14], the authors derived method using cubic B-spline basis function on a Shishkin mesh in spatial direction with backward Euler finite difference scheme in the time direction and proved that the method has uniform convergence of $O((\Delta x)^2 + \Delta\tau)$. Moreover, a few studies based on nonuniform meshes and finite differences are available to solve SPPs (see [1–4, 18, 19, 28, 31, 33, 36]). Motivated by the preceding discussion, we created a new strategy for establishing a higher order parameter uniform numerical technique for SPPCDEs. Our primary goal is to assess the proficiencies of the collocation method employing B-spline functions during the design of a higher-order parameter-uniform method in order to offer accurate results. We first employed the Crank-Nicolson approach in the time direction using an equal-spaced mesh to obtain the semi-discrete problem. On the semidiscrete problem, we used the quartic B-spline collocation method with non-uniform mesh in the space direction so that we could achieve full discretization of the problem. We have accomplished a relentless analysis of the method and confirmed that the designed method is stable and uniformly convergent for any selection of grid size in the space and time directions, respectively. It is revealed that the method provides invariant convergence in both the parameters $\Delta\tau$ and ε, and we have demonstrated that the developed method is uniformly convergent in ε and h. Adding the results obtained at each stage we assume that our method is uniformly convergent. To our knowledge, this is the first study to analyse and develop a higher-order technique for the problem under consideration. The paper is organised as pursues: In section 2, a priori assessments for the solution of the model problem and its derivatives are provided to study the asymptotic behaviour of the exact solution. And, to get sharper estimate, we use decomposing technique of the solution into regular and singular components. In section 3, we present the temporal semi-discritization by using Crank-Nicolson method on the equidistant mesh and show the uniform convergence of the semidiscrete scheme. In section 4, we developed a piecewise uniform Shishkin mesh and propose a numerical method using quartic B-spline scheme on discretized problem. In section 5, we investigate the theoretical error assessments and convergence analysis of the method. In section 6, the accurateness and efficiency of the method has been validated by assessing two test problems through graphs and tables. In section 7 conclusion is provided.

Note: Throughout the paper, $\mathbf{C}^{k+\mu}(\mathbb{D})$ denotes the space of Hölder continuous functions with exponent μ on the domain $(\mathbb{D})$, where $\mu \in (0,1)$, and K is a positive generic constant. $\mathbf{C}^{n,m}_{\mu}(\mathbb{D} \times [0,T])$consist all functions which are n times differentiable in space direction and m times differentiable in time direction. It is always independent of ε and the mesh size. All the functions $f \in (\mathbb{D})$ are defined by $\|f\|_{\bar{\mathbb{D}}} = \sup_{x\in\bar{\mathbb{D}}} |f(x)|$.

2. Continuous problem

Here, we have analyzed the analytical behaviour of the solution of (1.1)-(1.3), as well as the bounds of derivatives of the smooth and layer components, to ensure the uniqueness and existence of the solution $Y(x,\tau)$. We suppose that the coefficient $\mathtt{a}(x,\tau)$ and $\mathtt{b}(x,\tau)$ are Hölder continuous in the both space (x) and time(τ) with exponent μ [20]. Suppose that $Y(x,\tau)$ is Hölder continuous with respect to exponent μ and $\mu \in (0,1)$, if for all (x,τ), $(x',\tau') \in \mathbb{D}$, such that

$$S_\mu = \sup\left\{ \frac{|Y(x,\tau) - Y(x',\tau')|}{(|x-x'| + |\tau-\tau'|)^{\mu/2}}, \quad (x,\tau),(x',\tau') \in \mathbb{D} \right\} < \infty. \tag{2.1}$$

For each, $k \geq 1$, $k \in \mathbf{Z}^+$ and $n, m \in \mathbf{Z}^+$, we define the subspace $\mathbf{C}_\mu^{k+\mu}(\mathbb{D})$ of $\mathbf{C}^\mu(\mathbb{D})$ which consists of functions having Hölder continuous derivative such as:

$$\mathbf{C}_\mu^{k+\mu}(\mathbb{D}) = \left\{ Y : \frac{\partial^{n+m} Y}{\partial x^n \partial \tau^m} \in \mathbf{C}^\mu(\mathbb{D}), \;\; \text{with } 0 \leq n + 2m \leq k \right\}, \tag{2.2}$$

where $\mathbf{C}^\mu(\mathbb{D})$ is set of all Hölder continuous functions with exponent μ. The Hölder norm of the solution Y is given as:

$$\|Y\|_{\mu,\mathbb{D}} \equiv \|Y\|_{\mathbb{D}} + S_\mu \equiv \sup_{(x,\tau)\in\mathbb{D}} |Y(x,\tau)| + S_\mu.$$

Now, we imposed the condition of sufficient smoothness on coefficient of $\mathtt{a}(x,\tau)$, $\mathtt{b}(x,\tau) \in \mathbf{C}_\mu^{k+\mu}(\mathbb{D})$, and also that sufficient compatibility conditions hold among them in order to $Y(x,\tau) \in \mathbf{C}^{4,2}(\mathbb{D} \times [0,T])$. Also, we suppose that the condition of smoothness on the right hand side, boundary and initial conditions are

$$F(x,\tau) \in \mathbf{C}^{2+2\mu,1+\mu}(\mathbb{D} \times [0,T]), \;\; \mathtt{g}_2(x) \in \mathbf{C}^2([0,T]),$$
$$\mathtt{g}_3(x) \in \mathbf{C}^2([0,T]), \text{ and } \mathtt{g}_1(x) \in \mathbf{C}^4(\overline{\mathbb{D}}).$$

The compatibility conditions at the corner points of the domain $(0,0)$ and $(1,0)$ are given below:

$$\begin{cases} \mathtt{g}_1(0) = \mathtt{g}_2(0), \quad \mathtt{g}_1(1) = \mathtt{g}_3(0), \\ \dfrac{d\mathtt{g}_2(0)}{d\tau} = F(0,0) + \varepsilon \dfrac{d^2 \mathtt{g}_1(0)}{dx^2} - \mathtt{a}(0,0)\dfrac{d\mathtt{g}_1(0)}{dx} - \mathtt{b}(0)\mathtt{g}_1(0), \\ \dfrac{d\mathtt{g}_3(0)}{d\tau} = F(1,0) + \varepsilon \dfrac{d^2 \mathtt{g}_1(1)}{dx^2} - \mathtt{a}(1,0)\dfrac{d\mathtt{g}_1(1)}{dx} - \mathtt{b}(1)\mathtt{g}_1(1). \end{cases} \tag{2.3}$$

From equation (2.3), we guarantee the continuity conditions for the derivative in τ up to second order. Using these compatibility conditions, we can ensure existence of unique solution $Y(x,\tau)$ in $\mathbf{C}^{k+\mu}(\overline{\mathbb{D}})$ is written by the condition in [20] of the given problem (1.1)-(1.3).

Lemma 2.1. *(Continuous Minimum Principle). Suppose* $Q(x,\tau) \in \mathbf{C}^{2,1}(\overline{\mathbb{D}})$ *and* $Q(x,\tau) \geq 0$, $\forall\ (x,\tau) \in \partial\mathbb{D}$*. Then* $\mathbf{L}_x^\varepsilon Q(x,\tau) \geq 0$, $\forall (x,\tau) \in \mathbb{D}$ *implies that* $Q(x,\tau) \geq 0, \forall\ (x,\tau) \in \overline{\mathbb{D}}$.

Proof. This lemma's proof can be simply obtained from [26] in chapter 3. □

For the stability of $\mathbf{L}_x^\varepsilon$ an ε-uniform bound for the solution of (1.1)-(1.3) can be constructed by the following lemma.

Lemma 2.2 (Continuous stability estimate)**.** *Let $Y(x,\tau)$ be the solution of (1.1)-(1.3), then $\forall\ \varepsilon \geq 0$ the following bound holds:*

$$\|Y\|_{\overline{\mathbb{D}}} \leq \|Y\|_{\partial\mathbb{D}} + \frac{\|F\|_{\overline{\mathbb{D}}}}{\mathtt{a}^*}.$$

Proof. To illustrate the above, we describe barrier functions such as

$$H^{\pm}(x,\tau) = \|Y\|_{\partial\mathbb{D}} + \frac{\|F\|_{\overline{\mathbb{D}}}}{\mathtt{a}^*} \pm Y(x,\tau).$$

Then we have $H^{\pm}(x,\tau) \geq 0$, $\forall(x,\tau) \in \partial\mathbb{D}$. Since $\mathtt{b}(x,\tau) \geq 0$ and $\|F\| \geq F(x,\tau)$, $\forall(x,\tau) \in \partial\mathbb{D}$, and $-\mathtt{b}(x,\tau)\nu^{-1}\|F\| \pm F(x,\tau) \geq 0$. Using this inequality, we have

$$\mathbf{L}_x^{\varepsilon} H^{\pm}(x,\tau) \geq 0, \qquad \forall(x,\tau) \in \mathbb{D}.$$

Now from Lemma 2.1, we have $H^{\pm}(x,\tau) \geq 0$, $\forall(x,\tau) \in \overline{\mathbb{D}}$, which proves our required estimate. □

Lemma 2.3. *Let $\mathtt{a}(x,\tau) \in \mathbf{C}^{4+\mu}(\overline{\mathbb{D}})$, $\mathtt{b}(x,\tau), F(x,\tau) \in \mathbf{C}^{4+\mu,3+\mu/2}(\overline{\mathbb{D}})$, $\mathtt{g}_2 \in \mathbf{C}^{6+\mu}([0,T])$, $\mathtt{g}_3 \in \mathbf{C}^{6+\mu}([0,T])$, $\mu \in (0,1)$ and the compatibility conditions (2.3) are satisfied. Then the equation (1.1)-(1.3) has unique solution $Y(x,\tau)$ in $\mathbf{C}^{6+\mu,3+\mu}(\overline{\mathbb{D}})$, and*

$$\left\|\frac{\partial^{i+j}Y}{\partial x^i \partial \tau^j}\right\|_{\overline{\mathbb{D}}} \leq K\varepsilon^{-i}, \quad 1 \leq i+2j \leq 6. \tag{2.4}$$

Proof. The proof of this lemma can be done by using the approach given in [20] and [34]. □

The bounds assumed in the above lemma are not satisfactory to get $\varepsilon-$uniform error assessment. Hence, to gain more accurate bounds on the solution $Y(x,\tau)$ and its derivatives, the solution $Y(x,\tau)$ is broken down into smooth segment $R(x,\tau)$ and singular segment $S(x,\tau)$ determined as follows:

Theorem 2.1. *Let $Y(x,\tau)$ be the solution of given problem (1.1)-(1.3). The break up of the solution into smooth and singular component as:*

$$Y(x,\tau) = R(x,\tau) + S(x,\tau), \qquad \forall(x,\tau) \in \overline{\mathbb{D}}.$$

Then, $\forall\, i,j \in \mathbf{Z}^+$ such that $0 \leq i+3j \leq 4$, the smooth component $R(x,\tau)$ satisfies

$$\left\|\frac{\partial^{i+j}R}{\partial x^i \partial \tau^j}\right\|_{\overline{\mathbb{D}}} \leq K\left(1 + \varepsilon^{(3-i)} \exp\left(\frac{-\mathtt{a}^*(\mathtt{1}-\mathtt{x})}{\varepsilon}\right)\right),$$

and singular component $S(x,\tau)$ satisfies

$$\left\|\frac{\partial^{i+j}S}{\partial x^i \partial \tau^j}\right\|_{\overline{\mathbb{D}}} \leq K\left(\varepsilon^{-i} \exp\left(\frac{-\mathtt{a}^*(\mathtt{1}-\mathtt{x})}{\varepsilon}\right)\right),$$

where $\mathtt{a}(x,\tau) \geq \mathtt{a}^*$.

Proof. Suppose that the smooth segment satisfies the following non-homogeneous problem:

$$\begin{aligned} &\mathbf{L}_x^{\varepsilon} R(x,\tau) = F(x,\tau), && \forall(x,\tau) \in \mathbb{D}, \\ &R(x,\tau) = Y(x,\tau), && \forall(x,\tau) \in \mathbb{R}_x \cup \mathbb{R}_R. \end{aligned} \tag{2.5}$$

Similarly singular segment satisfies the homogeneous problem

$$\begin{aligned} &\mathbf{L}_x^{\varepsilon} S(x,\tau) = 0, && \forall(x,\tau) \in \mathbb{D}, \\ &S(x,\tau) = 0, && \forall(x,\tau) \in \mathbb{R}_x \cup \mathbb{R}_L, \end{aligned}$$

$$S(x,\tau) = Y(x,\tau) - R(x,\tau), \qquad \forall (x,\tau) \in \mathbb{R}_R. \tag{2.6}$$

Now, we decompose the smooth segment $R(x,\tau)$ into the sum of asymptotic expansion such as

$$R(x,\tau) = R_0(x,\tau) + \varepsilon R_1(x,\tau) + \varepsilon^2 R_2(x,\tau) + \varepsilon^3 R_3(x,\tau) + r(x,\tau), \quad \forall (x,\tau) \in \mathbb{D}, \tag{2.7}$$

where $R_0(x,\tau)$ is solution of following reduced non-homogeneous hyperbolic equation such that

$$\begin{cases} \left(\mathtt{a}\dfrac{\partial R_0}{\partial x} - \dfrac{\partial R_0}{\partial \tau} - \mathtt{b}R_0\right)(x,\tau) = F(x,\tau), \qquad \forall (x,\tau) \in \mathbb{D}, \\ R_0(x,\tau) = Y(x,\tau), \qquad \forall (x,\tau) \in \mathbb{R}_x \cup \mathbb{R}_R. \end{cases} \tag{2.8}$$

Also, in similar manner $R_1(x,\tau)$, $R_2(x,\tau)$ and $R_3(x,\tau)$ satisfies the nonhomogeneous hyperbolic equation such as

$$\begin{cases} \left(\mathtt{a}\dfrac{\partial R_1}{\partial x} - \dfrac{\partial R_1}{\partial \tau} - \mathtt{b}R_1\right)(x,\tau) = -\dfrac{\partial^2 R_0}{\partial x^2}, \qquad \forall (x,\tau) \in \mathbb{D}, \\ R_1(x,\tau) = 0, \qquad \forall (x,\tau) \in \mathbb{R}_x \cup \mathbb{R}_R, \end{cases} \tag{2.9}$$

$$\begin{cases} \left(\mathtt{a}\dfrac{\partial R_2}{\partial x} - \dfrac{\partial R_2}{\partial \tau} - \mathtt{b}R_2\right)(x,\tau) = -\dfrac{\partial^2 R_1}{\partial x^2}, \qquad \forall (x,\tau) \in \mathbb{D}, \\ R_2(x,\tau) = 0, \qquad \forall (x,\tau) \in \mathbb{R}_x \cup \mathbb{R}_R, \end{cases} \tag{2.10}$$

$$\begin{cases} \left(\mathtt{a}\dfrac{\partial R_3}{\partial x} - \dfrac{\partial R_3}{\partial \tau} - \mathtt{b}R_3\right)(x,\tau) = -\dfrac{\partial^2 R_2}{\partial x^2}, \qquad \forall (x,\tau) \in \mathbb{D}, \\ R_3(x,\tau) = 0, \qquad \forall (x,\tau) \in \mathbb{R}_x \cup \mathbb{R}_R, \end{cases} \tag{2.11}$$

and the residual term $r(x,\tau)$ satisfies the following problem:

$$\begin{cases} \mathbf{L}_x^{\varepsilon}\mathtt{r}(x,\tau) = -\varepsilon^4 \dfrac{\partial^2 R_3}{\partial x^2}, \qquad \forall (x,\tau) \in \mathbb{D}, \\ \mathtt{r}(x,\tau) = 0, \qquad \forall (x,\tau) \in \partial\mathbb{D}, \end{cases} \tag{2.12}$$

and smooth segment $R(x,\tau)$ satisfies the following given problem:

$$\begin{cases} \mathbf{L}_x^{\varepsilon} R(x,\tau) = F(x,\tau), \qquad \forall (x,\tau) \in \mathbb{D}, \\ R(x,\tau) = Y(x,\tau), \qquad \forall (x,\tau) \in \mathbb{R}_x \cup \mathbb{R}_R, \\ R(x,\tau) = R_0(x,\tau) + \varepsilon R_1(x,\tau) + \varepsilon^2 R_2(x,\tau) + \varepsilon^3 R_3(x,\tau), \qquad \forall (x,\tau) \in \mathbb{R}_L. \end{cases} \tag{2.13}$$

Here, $R_0(x,\tau)$, $R_1(x,\tau)$, $R_2(x,\tau)$ and $R_3(x,\tau)$ are solution of the first order hyperbolic equation (2.8) -(2.11) and independent of ε and its coefficients $\mathtt{a}(x,\tau)$ and $\mathtt{b}(x,\tau)$ are bounded. Now using [20], $\forall\ i,j \in \mathbf{Z}^+$ such that $0 \le i + 2j \le 6$, we get

$$\left\|\frac{\partial^{i+j} R_0}{\partial x^i \partial \tau^j}\right\|_{\overline{\mathbb{D}}} \le K, \quad \left\|\frac{\partial^{i+j} R_1}{\partial x^i \partial \tau^j}\right\|_{\overline{\mathbb{D}}} \le K, \quad \left\|\frac{\partial^{i+j} R_2}{\partial x^i \partial \tau^j}\right\|_{\overline{\mathbb{D}}} \le K, \ \text{ and } \ \left\|\frac{\partial^{i+j} R_3}{\partial x^i \partial \tau^j}\right\|_{\overline{\mathbb{D}}} \le K. \tag{2.14}$$

Since $\mathtt{r}(x,\tau)$ is solution of the initial boundary value problem and it is similar to the original problem (1.1), and using the Lemma 2.3 $\forall\, i,j \in \mathbf{Z}^+$ such that $0 \le i+2j \le 6$, we get the following estimate:

$$\left\| \frac{\partial^{i+j}\mathtt{r}}{\partial x^i \partial \tau^j} \right\|_{\overline{\mathbb{D}}} \le K\varepsilon^{-i}. \tag{2.15}$$

Now using above inequalities (2.8)-(2.13) into equation (2.7), after simplification we get required estimates for smooth segment as

$$\left\| \frac{\partial^{i+j}R}{\partial x^i \partial \tau^j} \right\|_{\overline{\mathbb{D}}} \le K\left(1+\varepsilon^{(3-i)}\exp\left(\frac{-\mathtt{a}^*(1-\mathtt{x})}{\varepsilon}\right)\right), \quad \text{for} \quad 0 \le i+2j \le 4. \tag{2.16}$$

Now, we calculate bounds on the singular segment $S(x,\tau)$. Since, the singular segment $S(x,\tau)$ satisfies the homogeneous initial boundary value problem (2.6). For this we introduce two barrier function such as

$$H_{\pm}(x,\tau) = K\exp(\tau)\exp\left(\frac{-\mathtt{a}^*(1-\mathtt{x})}{\varepsilon}\right) \pm S(x,\tau), \quad \forall (x,\tau) \in \overline{\mathbb{D}}. \tag{2.17}$$

We suppose that K is sufficiently large, then we get inequalities at boundaries as

$$H_{\pm}(x,0) \ge 0, \qquad \forall (x,\tau) \in \partial\mathbb{D},$$

and

$$\mathbf{L}_x^{\varepsilon} H_{\pm}(x,0) \ge 0, \qquad \forall (x,\tau) \in \mathbb{D}.$$

Then from the Lemma 2.1 we can say that $H_{\pm}(x,\tau) \ge 0$ on $\overline{\mathbb{D}}$, and we get

$$\|S(x,\tau)\| \le K\exp\left(\frac{-\mathtt{a}^*(1-\mathtt{x})}{\varepsilon}\right), \quad \forall (x,\tau) \in \mathbb{D}, \tag{2.18}$$

and from the Lemma 2.3 for $0 \le i+2j \le 6$, we get

$$\left\| \frac{\partial^{i+j}S}{\partial x^i \partial \tau^j} \right\|_{\overline{\mathbb{D}}} \le K\varepsilon^{-i}\|S\|_{\overline{\mathbb{D}}}. \tag{2.19}$$

Therefore from the equations (2.18) and (2.19), we get required derivative bound on singular component

$$\left\| \frac{\partial^{i+j}S}{\partial x^i \partial \tau^j} \right\|_{\overline{\mathbb{D}}} \le K\left(\varepsilon^{-i}\exp\left(\frac{-\mathtt{a}^*(1-\mathtt{x})}{\varepsilon}\right)\right). \tag{2.20}$$

Hence from the equations (2.16) and (2.20), we can get required result. $\square$

3. Temporal Semidiscretization

In this section, we develop numerical method using Crank-Nicolson method in time. A piecewise homogeneous grid is defined as $\overline{\mathbb{D}}^{n,m} = \Omega^n \times \Gamma^m$, where the mesh Γ^m=$\{\tau_k : \tau_k = k\Delta\tau, \quad k \le m\}$ is a homogeneous partition in time division over $[0,T]$ with homogeneous scale factor $\Delta\tau = \frac{T}{m}$. After splitting the time direction by the Crank-Nicolson technique, we obtain a system of linear ordinary differential equations such as

$$\begin{cases} -\dfrac{\varepsilon}{2}(Y_{xx}^{k+1}+Y_{xx}^{k})+\dfrac{\mathtt{a}(x,\tau_{k+1/2})}{2}(Y_{x}^{k+1}+Y_{x}^{k})+\dfrac{1}{\Delta\tau}(Y^{k+1}-Y^{k}) \\ +\dfrac{\mathtt{b}(x,\tau_{k+1/2})}{2}(Y^{k+1}+Y^{k})=\dfrac{1}{2}(F(x,\tau_{k+1})+F(x,\tau_{k})), \ \ \forall x\in\Omega_x, \\ Y^{0}(x)=Y(x,0)=\mathtt{g}_1(x), \qquad x\in\overline{\Omega}_x, \\ Y^{k+1}(0)=Y(0,\tau_{k+1})=\mathtt{g}_2(\tau), \qquad 0\le k\le m, \\ Y^{k+1}(1)=Y(1,\tau_{k+1})=\mathtt{g}_3(\tau), \qquad 0\le k\le m. \end{cases} \tag{3.1}$$

The equation (3.1) can be written in the operator form as follows:

$$\left(\frac{\Delta\tau}{2}\mathbf{L}_x^{\varepsilon}+I\right)Y^{k+1}=\left(I-\frac{\Delta\tau}{2}\mathbf{L}_x^{\varepsilon}\right)Y^{k}+\frac{\Delta\tau}{2}(F^{k+1}+F^{k}), \quad \forall x\in\Omega_x, \quad k\ge 0, \tag{3.2}$$

where $\mathbf{L}_x^{\varepsilon}=\big(-\varepsilon\frac{\partial^2}{\partial x^2}+\mathtt{a}(x,\tau_{k+1/2})\frac{\partial}{\partial x}+\mathtt{b}(x,\tau_{k+1/2})\big)I.$

Lemma 3.1. *Consider* $\left|\frac{\partial^{i+k}Y(x,\tau)}{\partial x^i\partial\tau^k}\right|\le K$ *on* $\overline{\mathbb{D}}$, $0\le k\le 3$ *and* $0\le i+k\le 4$, *then in the temporal direction the local truncation error estimate is given by*

$$\|\overline{E}_{k+1}\|_\infty\le K(\Delta\tau)^3, \qquad 1\le k\le m, \tag{3.3}$$

where K *is constant independent of mesh point.*

Proof. The semidiscretization for local truncation error in the temporal direction is described as

$$E_{k+1}=Y(x,\tau_{k+1})-\overline{Y}(x,\tau_{k+1}), \tag{3.4}$$

where $\overline{Y}(x,\tau_{k+1})$ is the approximate solution obtained in (3.1) over one time unit, with the actual value being used as the initial condition, such as

$$\left(\frac{\Delta\tau}{2}\mathbf{L}_x^{\varepsilon}+I\right)\overline{Y}(x,\tau_{k+1})=\left(I-\frac{\Delta\tau}{2}\mathbf{L}_x^{\varepsilon}\right)Y(x,\tau_k)+\Delta\tau F^{k+1/2}, \tag{3.5}$$

and

$$Y^{k+1}(0,\tau_{k+1})=\mathtt{g}_2(\tau_{k+1}), \ \ Y^{k+1}(1,\tau_{k+1})=\mathtt{g}_3(\tau_{k+1}), \ \ k=0,1,2,...,m-1. \tag{3.6}$$

Now, we expand $Y(x,\tau_{k+1})$ making use of Taylor's series with regard to the point $(x,\tau_{k+1/2})$, we have

$$\begin{aligned} Y(x,\tau_{k+1})=Y(x,\tau_{k+1/2})&+\frac{\Delta\tau}{2}Y_\tau(x,\tau_{k+1/2})+\frac{(\Delta\tau)^2}{2^2 2}Y_{\tau\tau}(x,\tau_{k+1/2}) \\ &+\frac{(\Delta\tau)^3}{2^3 6}Y_{\tau\tau\tau}(x,\tau_{k+1/2})+O(\Delta\tau)^4. \end{aligned} \tag{3.7}$$

Similarly, we expand $Y(x,\tau_k)$ about the point $(x,\tau_{k+1/2})$, we have

$$\begin{aligned} Y(x,\tau_k)=&Y(x,\tau_{k+1/2})-\frac{\Delta\tau}{2}Y_\tau(x,\tau_{k+1/2})+\frac{(\Delta\tau)^2}{2^2 2}Y_{\tau\tau}(x,\tau_{k+1/2}) \\ &-\frac{(\Delta\tau)^3}{2^3 6}Y_{\tau\tau\tau}(x,\tau_{k+1/2})+O(\Delta\tau)^4. \end{aligned} \tag{3.8}$$

From equations (3.7) and (3.8), we have

$$\begin{aligned} Y_x(x,\tau_{k+1}) =& Y_x(x,\tau_{k+1/2}) + \frac{\Delta\tau}{2} Y_{x\tau}(x,\tau_{k+1/2}) + \frac{(\Delta\tau)^2}{2^2 2} Y_{x\tau\tau}(x,\tau_{k+1/2}) \\ &+ \frac{(\Delta\tau)^3}{2^3 6} Y_{x\tau\tau\tau}(x,\tau_{k+1/2}) + O(\Delta\tau)^4, \end{aligned} \tag{3.9}$$

$$\begin{aligned} Y_x(x,\tau_{k}) =& Y_x(x,\tau_{k+1/2}) - \frac{\Delta\tau}{2} Y_{x\tau}(x,\tau_{k+1/2}) + \frac{(\Delta\tau)^2}{2^2 2} Y_{x\tau\tau}(x,\tau_{k+1/2}) \\ &- \frac{(\Delta\tau)^3}{2^3 6} Y_{x\tau\tau\tau}(x,\tau_{k+1/2}) + O(\Delta\tau)^4, \end{aligned} \tag{3.10}$$

$$\begin{aligned} Y_{xx}(x,\tau_{k+1}) =& Y_{xx}(x,\tau_{k+1/2}) + \frac{\Delta\tau}{2} Y_{xx\tau}(x,\tau_{k+1/2}) + \frac{(\Delta\tau)^2}{2^2 2} Y_{xx\tau\tau}(x,\tau_{k+1/2}) \\ &+ \frac{(\Delta\tau)^3}{2^3 6} Y_{xx\tau\tau\tau}(x,\tau_{k+1/2}) + O(\Delta\tau)^4, \end{aligned} \tag{3.11}$$

$$\begin{aligned} Y_{xx}(x,\tau_{k}) =& Y_{xx}(x,\tau_{k+1/2}) - \frac{\Delta\tau}{2} Y_{xx\tau}(x,\tau_{k+1/2}) + \frac{(\Delta\tau)^2}{2^2 2} Y_{xx\tau\tau}(x,\tau_{k+1/2}) \\ &- \frac{(\Delta\tau)^3}{2^3 6} Y_{xx\tau\tau\tau}(x,\tau_{k+1/2}) + O(\Delta\tau)^4. \end{aligned} \tag{3.12}$$

Using equations (3.7) and (3.8) and after dividing by $\Delta\tau$, we have

$$\frac{Y(x,\tau_{k+1}) - Y(x,\tau_k)}{\Delta\tau} = Y_\tau(x,\tau_{k+1/2}) + \frac{(\Delta\tau)^2}{2^2 6} Y_{\tau\tau\tau}(x,\tau_{k+1/2}) + O(\Delta\tau)^4. \tag{3.13}$$

Similarly, from equations (3.9) and (3.12), we can find following relation:

$$\frac{Y(x,\tau_{k+1}) + Y(x,\tau_k)}{2} = Y(x,\tau_{k+1/2}) + \frac{(\Delta\tau)^2}{2^2 2} Y_{\tau\tau}(x,\tau_{k+1/2}) + O(\Delta\tau)^3, \tag{3.14}$$

$$\frac{Y_x(x,\tau_{k+1}) + Y_x(x,\tau_k)}{2} = Y_x(x,\tau_{k+1/2}) + \frac{(\Delta\tau)^2}{2^2 2} Y_{x\tau\tau}(x,\tau_{k+1/2}) + O(\Delta\tau)^3, \tag{3.15}$$

$$\frac{Y_{xx}(x,\tau_{k+1}) + Y_{xx}(x,\tau_k)}{2} = Y_{xx}(x,\tau_{k+1/2}) + \frac{(\Delta\tau)^2}{2^2 2} Y_{xx\tau\tau}(x,\tau_{k+1/2}) + O(\Delta\tau)^3. \tag{3.16}$$

Putting the values of (3.13)-(3.16) in (3.5), we get

$$\begin{aligned} \left(\frac{\Delta\tau}{2}\mathbf{L}_x^\varepsilon - I\right)\overline{Y}(x,\tau_{k+1}) =& \left(I - \frac{\Delta\tau}{2}\mathbf{L}_x^\varepsilon\right) Y(x,\tau_k) + \Delta\tau F^{k+1/2} \\ &+ \frac{(\Delta\tau)^3}{24} Y_{\tau\tau\tau}(x,\tau_{k+1/2}) + O(\Delta\tau)^4. \end{aligned} \tag{3.17}$$

Subtracting equation (3.5) from (3.17), we have

$$\left(\frac{\Delta\tau}{2}\mathbf{L}_x^\varepsilon - I\right) E^{k+1} = \frac{(\Delta\tau)^3}{24} Y_{\tau\tau\tau}(x,\tau_{k+1/2}) + O(\Delta\tau)^4. \tag{3.18}$$

Since $|Y_{\tau\tau\tau}(x,\tau_{k+1})| \le K$ and $|(\mathbf{L}_x^\varepsilon)^{-1}| \le K_1$, therefore we have

$$\|E^{k+1}\| \le K(\Delta\tau)^3. \tag{3.19}$$

$\square$

Lemma 3.2. *The estimation of the global error in the time direction taken at each different time level τ_k is given by*

$$\|E^{k+1}\| \leq K(\Delta\tau)^2, \qquad 1 \leq k \leq m, \tag{3.20}$$

where K is constant independent of mesh point.

Proof. Suppose that

$$\|E^{k+1}\| = Y(x, \tau_{k+1}) - Y^{k+1}. \tag{3.21}$$

Using equation (3.19) and Lemma 3.1, we arrive at the following global error that result at the k^{th} time step

$$\begin{aligned} \|E^{k+1}\|_\infty &= |\sum_{l=1}^{k} E_l| \\ &\leq Kk(\Delta\tau)^3, \\ &= KT(\Delta\tau)^2, \\ &= K_1(\Delta\tau)^2, \end{aligned}$$

where $K_1 = KT$. The Lemma 3.2 proof has been completed. If we consider the implications of this equation, the process (3.1) is consistent and has $2nd$ order convergent in time. □

4. The spatial Semidiscretization

To discretize the spatial domain $\overline{\Omega}_x$, first we divide the domain into a piecewise uniform mesh $\overline{\Omega}_x$ in two subintervals $\Omega_1 = [0, 1-\gamma]$ and $\Omega_2 = [1-\gamma, 1]$ such that $\overline{\Omega}_x = \Omega_1 \cup \Omega_2$. For $n \geq 2^l$, where $l \geq 2$ is an integer. Make the assumption that each subinterval has been evenly segmented into $n/2$ grid points. The transition parameter γ is defined as:

$$\gamma = \min\left\{\frac{1}{2}, \gamma_0 \varepsilon \log(n)\right\}, \qquad \gamma_0 \geq \frac{1}{a^*}.$$

The constat γ_0 is independent of the parameter ε and n. The discretization of spatial domain into coarse region Ω_1 with mesh spacing $h_i = \frac{2(1-\gamma)}{n}$ and fine region Ω_2 with mesh spacing $h_i = \frac{2\gamma}{n}$ is

$$\overline{\Omega}_x = \{0 \equiv x_0, x_1, x_2, ..., x_{n/2} = 1-\gamma, x_{n/2+1}, ..., x_n \equiv 1\},$$

where

$$x_i = \begin{cases} \dfrac{2(1-\gamma)}{n} i, & \text{if} \quad i = 0, 1, 2, 3, ..., \dfrac{n}{2}, \\ (1-\gamma) + \dfrac{2\gamma}{n}(i - n/2), & \text{if} \quad i = \dfrac{n}{2} + 1, ..., n. \end{cases}$$

Further, the spatial step size $h_i = x_i - x_{i-1}$, for $i = 1, 2, ..., n$ is defined as

$$h = \begin{cases} h_i = \dfrac{2(1-\gamma)}{n}, & \text{if} \quad i = 0, 1, 2, 3, ..., \dfrac{n}{2}, \\ h_i = (\dfrac{2\gamma}{n}), & \text{if} \quad i = \dfrac{n}{2} + 1, ..., n. \end{cases}$$

4.1. Quartic B-spline collocation method

We suppose that $L_2(\overline{\Omega}_x)$ be the space of all integrable functions on $\overline{\Omega}_x$, and let X is subspace of $L_2(\overline{\Omega}_x)$. Now we define quartic B-spline for $j=-2,-1,0,1,...,n,n+1$, [30]

$$BS_j(x)=\frac{1}{24h_j^4}\begin{cases}\chi(x-x_{j-2}), & x\in[x_{j-2},x_{j-1}]\\ \chi(x-x_{j-2})-5\chi(x-x_{j-1}), & x\in[x_{j-1},x_j]\\ \chi(x-x_{j-2})-5\chi(x-x_{j-1})+10\chi(x-x_j), & x\in[x_j,x_{j+1}]\\ \chi(x_{j+3}-x)-5\chi(x_{j+2}-x), & x\in[x_{j+1},x_{j+2}]\\ \chi(x_{j+3}-x), & x\in[x_{j+2},x_{j+3}]\\ 0, & \text{otherwise},\end{cases} \tag{4.1}$$

where $\chi=x^4$. Let $\mathbf{G}=\{BS_{-2},BS_{-1},BS_0,...,BS_n,BS_{n+1}\}$ and let $\Psi(\overline{\Omega}_x)$=Span $\mathbf{G}$. Since $BS_j(x)$ is linearly independent functions on $[0,1]$, thus $\Psi(\overline{\Omega}_x)$ is $(n+4)$-dimensional. The values of $BS_j(x)$ and its first three derivatives $BS_j'(x)$, $BS_j''(x)$,and $BS_j'''(x)$ are given in the Table 1.

Table 1. The values of basis functions $BS_j(x)$, $BS_j'(x)$, $BS_j''(x)$, and $BS_j'''(x)$ at the mesh points

	x_{j-3}	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
$BS_j(x)$	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0
$BS_j'(x)$	0	$\frac{-1}{6h}$	$\frac{-3}{6h}$	$\frac{3}{6h}$	$\frac{1}{6h}$	0
$BS_j''(x)$	0	$\frac{1}{2h^2}$	$\frac{-1}{6h^2}$	$\frac{-1}{6h^2}$	$\frac{1}{2h^2}$	0
$BS_j'''(x)$	0	$\frac{-1}{h^3}$	$\frac{3}{h^3}$	$\frac{-3}{h^3}$	$\frac{1}{h^3}$	0

Now we define the approximate solution $BS_j(x)$ of the analytic solution $Y(x,\tau_{k+1})$ in the combinations of quartic B-splines with undetermined coefficient δ_i^k at the $(k+1)th$ time level as

$$Y_h=\sum_{j=-2}^{n+1}\delta_j^k BS_j(x), \tag{4.2}$$

where δ_j^k are unknown time dependent parameters and $BS_j(x)$ are quartic B-spline functions. To solve the considered problem (1.1)-(1.3), the quartic spline function and its derivatives need to be calculated at the nodal point x_j. At all nodal points, the solution $BS_j(x)$ given in (4.2) and its derivatives yield

$$Y_h(x_j)=\frac{1}{24}(\delta_{j-2}^k+11\delta_{j-1}^k+11\delta_j^k+\delta_{j+1}^k), \tag{4.3}$$

$$Y_h'(x_j)=\frac{1}{6h}(-\delta_{j-2}^k-3\delta_{j-1}^k+3\delta_j^k+\delta_{j+1}^k), \tag{4.4}$$

$$Y_h''(x_j)=\frac{1}{2h^2}(\delta_{j-2}^k-\delta_{j-1}^k-\delta_j^k+\delta_{j+1}^k), \tag{4.5}$$

$$Y_h'''(x_j)=\frac{1}{h^3}(-\delta_{j-2}^k+3\delta_{j-1}^k-3\delta_j^k+\delta_{j+1}^k). \tag{4.6}$$

Equation (3.1) can be written as

$$\begin{cases} -\varepsilon \dfrac{\partial^2 \overline{Y}(x,\tau_{k+1})}{\partial x^2} + \mathtt{a}(x,\tau_{k+1/2}) \dfrac{\partial \overline{Y}(x,\tau_{k+1})}{\partial x} + \mathtt{r}(x,\tau_{k+1})\overline{Y}(x,\tau_{k+1}) = G(x,\tau_k), \forall x \in \Omega_x, \\ \overline{Y}(x,0) = \mathtt{g}_1(x), \qquad x \in \overline{\Omega}_x, \\ \overline{Y}(0,\tau_{k+1}) = \mathtt{g}_2(\tau_{k+1}), \qquad 0 \le k \le m, \\ \overline{Y}(1,\tau_{k+1}) = \mathtt{g}_3(\tau_{k+1}), \qquad 0 \le k \le m, \end{cases} \tag{4.7}$$

where $\overline{Y}(x,\tau_{k+1}) = Y^{k+1}(x)$ is solution of (4.6) , $\mathtt{r}(x,\tau_{k+1}) = \left(\frac{\mathtt{b}(x,\tau_{k+1/2})}{2} + \frac{2}{\Delta\tau}\right)$ and $G(x,\tau_k) = \varepsilon \frac{\partial^2 Y^k(x,\tau_{k+1})}{\partial x^2} - \mathtt{a}(x,\tau_{k+1/2})\frac{\partial Y(x,\tau_k)}{\partial x} - \mathtt{b}(x,\tau_{k+1/2})Y(x,\tau_k) + \frac{2}{\Delta\tau}Y(x,\tau_k) + (F(x,\tau_{k+1}) + F(x,\tau_k))$. Now at the nodal points the difference equation associated with (4.7) is given as:

$$\mathbf{L}_x^\varepsilon Y_h \equiv -\varepsilon \frac{\partial^2 Y_h(x_j,\tau_{k+1})}{\partial x^2} + \tilde{\mathtt{a}}(x_j)\frac{\partial Y_h(x_j,\tau_{k+1})}{\partial x} + \tilde{\mathtt{r}}(x_j)Y_h(x_j,\tau_{k+1}) = G(x_j,\tau_k),$$
$$j = 0,1,2,...,n, \tag{4.8}$$

and

$$Y_h(x_0,\tau_{k+1}) = \mathtt{g}_2(\tau_{k+1}), \quad Y_h(x_n,\tau_{k+1}) = \mathtt{g}_3(\tau_{k+1}), \tag{4.9}$$

where $\mathtt{a}(x_j,\tau_{k+1/2}) = \tilde{\mathtt{a}}(x_j)$ and $\mathtt{r}(x_j,\tau_{k+1/2}) = \tilde{\mathtt{r}}(x_j)$. Using equation (4.2) into equations (4.8)-(4.9), we get

$$-\varepsilon \sum_{i=-2}^{n+1} \delta_i^k BS_i''(x_j) + \tilde{\mathtt{a}}(x_j)\sum_{i=-2}^{n+1} \delta_i^k BS_i'(x_j) + \tilde{\mathtt{r}}(x_j)\sum_{i=-2}^{n+1} \delta_i^k BS_i(x_j) = G(x_j,\tau_k),$$
$$j = 0,1,2,...,n, \tag{4.10}$$

and

$$\sum_{i=-2}^{n+1} \delta_i^k BS_i(x_0) = \mathtt{g}_2(\tau_{k+1}), \quad \sum_{i=-2}^{n+1} \delta_i^k BS_i(x_n) = \mathtt{g}_3(\tau_{k+1}). \tag{4.11}$$

Putting the values of $Y_h(x_j,\tau_{k+1})$, $Y_h'(x_j,\tau_{k+1})$ and $Y_h''(x_j,\tau_{k+1})$ from equations (4.3)- (4.5) into equations (4.10)-(4.11) we get linear system of equation of size $(n+3)\times(n+4)$ as follows:

$$\left(\frac{-\varepsilon}{4h_j^2} - \frac{\tilde{\mathtt{a}}(x_j)}{12h_j} + \frac{\tilde{\mathtt{r}}(x_j)}{48}\right)\delta_{j-2}^{k+1} + \left(\frac{\varepsilon}{4h_j^2} - \frac{3\tilde{\mathtt{a}}(x_j)}{12h_j} + \frac{11\tilde{\mathtt{r}}(x_j)}{48}\right)\delta_{j-1}^{k+1} + \left(\frac{\varepsilon}{4h_j^2} + \frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}\right.$$
$$\left. + \frac{11\tilde{\mathtt{r}}(x_j)}{48}\right)\delta_j^{k+1} + \left(\frac{-\varepsilon}{4h_j^2} + \frac{\tilde{\mathtt{a}}(x_j)}{12h_j} + \frac{\tilde{\mathtt{r}}(x_j)}{48}\right)\delta_{j+1}^{k+1} = G(x_j,\tau_k),$$
$$0 \le j \le n, \ \ 0 \le k \le m, \tag{4.12}$$

and boundary conditions

$$\delta_{-2}^{k+1} + 11\delta_{-1}^{k+1} + 11\delta_0^{k+1} + \delta_1^{k+1} = 24\mathtt{g}_2(\tau_{k+1}), \tag{4.13}$$
$$\delta_{n-2}^{k+1} + 11\delta_{n-1}^{k+1} + 11\delta_n^{k+1} + \delta_{n+1}^{k+1} = 24\mathtt{g}_3(\tau_{k+1}), \tag{4.14}$$

where

$$\begin{aligned} G(x_j,\tau_k) =&\left(\frac{\varepsilon}{4h_j^2}+\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{s}}(x_j)}{48}\right)\delta_{j-2}^k+\left(\frac{-\varepsilon}{4h_j^2}+\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{11\tilde{\mathtt{s}}(x_j)}{48}\right)\delta_{j-1}^k \\ &+\left(\frac{-\varepsilon}{4h_j^2}-\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{11\tilde{\mathtt{s}}(x_j)}{48}\right)\delta_j^k+\left(\frac{\varepsilon}{4h_j^2}-\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{s}}(x_j)}{48}\right)\delta_{j+1}^k \\ &+(F(x,\tau_{k+1})+F(x,\tau_k)). \end{aligned} \tag{4.15}$$

The above system of equation (4.12) can be written as

$$X_1(j)\delta_{j-2}^{k+1}+X_2(j)\delta_{j-1}^{k+1}+X_3(j)\delta_j^{k+1}+X_4(j)\delta_{j+1}^{k+1}=G(x_j,\tau_k), \qquad 0\le j\le n, \tag{4.16}$$

where

$$\begin{aligned} X_1(j)&=\left(\frac{-\varepsilon}{4h_j^2}-\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{r}}(x_j)}{48}\right), \qquad X_2(j)=\left(\frac{\varepsilon}{4h_j^2}-\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{11\tilde{\mathtt{r}}(x_j)}{48}\right), \\ X_3(j)&=\left(\frac{\varepsilon}{4h_j^2}+\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{11\tilde{\mathtt{r}}(x_j)}{48}\right), \qquad X_4(j)=\left(\frac{-\varepsilon}{4h_j^2}+\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{r}}(x_j)}{48}\right). \end{aligned}$$

From equations (4.12), (4.13), and (4.16), we get system of equations with $n+4$ unknowns and $n+3$ equations. We still need one more equation to get the one and only solution for each time level. To accomplish this, we differentiate equation (4.6) with respect to x and use equations (4.3)-(4.6) to approximation the solution at $x=0$.

$$T_1(0)\delta_{-2}^{k+1}+T_2(0)\delta_{-1}^{k+1}+T_3(0)\delta_0^{k+1}+T_4(0)\delta_1^{k+1}=G'(x_0,\tau_k), \tag{4.17}$$

where

$$\begin{aligned} T_1(0)&=(\mathtt{e}_0h^3-4d_0h^2+12\mathtt{a}_0h-24\epsilon), \quad T_2(0)=(11\mathtt{e}_0h^3-12d_0h^2-12\mathtt{a}_0h-72\epsilon), \\ T_3(0)&=(11\mathtt{e}_0h^3+124d_0h^2-12\mathtt{a}_0h-72\epsilon), \quad T_4(0)=(\mathtt{e}_0h^3+4d_0h^2+12\mathtt{a}_0h+24\epsilon), \end{aligned}$$

and

$$\begin{aligned} G'(x_0,\tau_k) =&(-\tilde{\mathtt{e}}_0h^3+4\tilde{d}_0h^2-12\mathtt{a}_0h+24\epsilon)+(-11\tilde{\mathtt{e}}_0h^3+12\tilde{d}_0h^2+12\mathtt{a}_0h+72\epsilon) \\ &+(-11\tilde{\mathtt{e}}_0h^3-12\tilde{d}_0h^2-12\mathtt{a}_0h-72\epsilon)+(\tilde{\mathtt{e}}_0h^3+4\tilde{d}_0h^2-12\mathtt{a}_0h \\ &+24\epsilon)+(F'(x,\tau_{k+1})+F'(x,\tau_k)), \end{aligned} \tag{4.18}$$

where $d_0=\mathtt{a}_x(0)+\mathtt{r}(0)$, $e_0=\mathtt{r}_x(0)$, $\tilde{d}_0=\mathtt{a}_x(0)+\mathtt{s}_x(0)$, $\mathtt{s}(x)=\left(\frac{2}{\Delta\tau}-\mathtt{b}(x)\right)$. The equations (4.16)-(4.17), constitutes a linear system of equations with $n+4$ equations in $n+4$ unknowns $\delta_{-2}^{k+1},\delta_{-1}^{k+1},...,\delta_{n-1}^{k+1},\delta_n^{k+1},\delta_{n+2}^{k+1}$.

After eliminating we get $(n+1)$ linear equations in $(n+1)$ unknowns, at the kth time level, which can be written in the matrix form as:

$$PV^{k+1}=\overline{G}, \qquad k=0,1,2,3,...,m-1, \tag{4.19}$$

where the matrix P is given by

$$M = \begin{pmatrix} A_1(0) & A_2(0) & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_1(1) & A_2(1) & X_2(1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & X_1(2) & X_2(2) & X_3(2) & X_4(2) & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & X_1(n-1) & X_2(n-1) & X_3(n-1) & X_4(n-1) \\ 0 & \cdots & 0 & 0 & 0 & \overline{T}_1 & \overline{T}_2 & \overline{T}_3 \end{pmatrix},$$

where

$$\begin{aligned}
&\overline{T}_1 = X_1(n+1) - X_4(n+1), \;\; \overline{T}_2 = X_2(n+1) - 11X_4(n+1), \\
&\overline{T}_3 = X_3(n+1) - 11X_4(n+1), \\
&A_1(0) = \left(\frac{11X_1(0) - X_3(0)T_2(0)}{X_2(0) - 11X_1(0)} - 11T_1(0)\right), \\
&A_2(0) = \left(\frac{X_1(0) - X_4(0)T_2(0)}{X_2(0) - 11X_1(0)} - T_1(0)\right), \\
&A_1(1) = \left(\frac{11X_1(0) - X_3(0)X_1(1)}{X_2(0) - 11X_1(0)} - X_2(1)\right), \\
&A_2(1) = \left(\frac{X_1(0) - X_4(0)X_1(1)}{X_2(0) - 11X_1(0)} - X_3(1)\right), \\
&X_1(j) = \left(\frac{\varepsilon}{4h_j^2} - \frac{\mathtt{\tilde{a}}(x_j)}{12h_j} + \frac{\mathtt{\tilde{r}}(x_j)}{48}\right), \qquad X_2(j) = \left(-\frac{\varepsilon}{4h_j^2} - \frac{3\mathtt{\tilde{a}}(x_j)}{12h_j} + \frac{11\mathtt{\tilde{r}}(x_j)}{48}\right), \\
&X_3(j) = \left(-\frac{\varepsilon}{4h_j^2} + \frac{3\mathtt{\tilde{a}}(x_j)}{12h_j} + \frac{1\tilde{1}\mathtt{r}(x_j)}{48}\right), \qquad X_4(j) = \left(\frac{\varepsilon}{4h_j^2} + \frac{\mathtt{\tilde{a}}(x_j)}{12h_j} + \frac{\mathtt{\tilde{r}}(x_j)}{48}\right), \\
&j = 0, 1, 2, ..., n-1.
\end{aligned}$$

The vectors V^{k+1} and $\overline{G}$ are given as:

$$V = \begin{pmatrix} \delta_0^{k+1} \\ \delta_1^{k+1} \\ \delta_2^{k+1} \\ \vdots \\ \vdots \\ \delta_{n-1}^{k+1} \\ \delta_n^{k+1} \\ \delta_{n+1}^{k+1} \end{pmatrix}, \qquad \overline{G} = \begin{pmatrix} \tilde{g}(0, \tau_k) \\ \tilde{g}(x_1, \tau_k) \\ \tilde{g}(x_2, \tau_k) \\ \vdots \\ \vdots \\ \tilde{g}(x_{n-1}, \tau_k) \\ \tilde{g}(x_n, \tau_k) \\ \tilde{g}(x_{n+1}, \tau_k) \end{pmatrix}, \qquad k = 0, 1, 2, ..., m,$$

$$\tilde{g}(0, \tau_k) = \hat{g}(\mathtt{0}, \tau_k) - 24X_1(0)\mathtt{g}_2(\tau_{k+1}) - \frac{24X_1(0)(T_1(0)\mathtt{g}_2(\tau_{k+1}) - \hat{g}_x(x, \tau_k))}{T_2(0) - 11T_1(0)}$$

$$-\frac{X_2(0)(\hat{g}_x(0,\tau_k)-24T_1(0)\mathrm{g}_2(\tau_{k+1}))}{T_2(0)-11T_1(0)},$$
$$\tilde{g}(x_1,\tau_k)=\hat{g}(x_1,\tau_k)-\frac{\hat{g}_x(0,\tau_k)-24T_1(0)\mathrm{g}_2(\tau_{k+1})}{T_2(0)-11T_1(0)},$$
$$\tilde{g}(x_j,\tau_k)=\hat{g}(x_1,\tau_k),\qquad j=2,3,4,...,n-1,$$
$$\tilde{g}(x_n,\tau_k)=\hat{g}(x_n,\tau_k)-24X_4(n)\mathrm{g}_3(\tau_{k+1}).$$

5. Error Analysis

Here, we discuss the error bounds for the quartic B-spline and its derivatives up to order four. Let us consider the quartic B-spline interpolationg function $Y_h(x)$ satisfies the following condition:

$$Y_h(x_j)=Y(x_j),\quad j=0,1,2,...,n. \tag{5.1}$$
$$Y_h^{(3)}(x_j)=Y^{(3)}(x_j)-\frac{1}{12}h^2Y^{(5)}(x_j)+\frac{1}{240}h^4Y^{(7)}(x_j)+O(h^7),\quad j=0,1,2,...,n. \tag{5.2}$$

Let $Y(x)\in\mathbb{C}^5(\overline{\Omega})$, then we have the following consistency relationship for any quartic spline [9]

$$\Phi Y_h'(x_j)=\frac{4}{h}\big(-Y_{j-2}-3Y_{j-1}+3Y_j+Y_{j+1}\big),\quad j=2,3,4,...,n-1, \tag{5.3}$$
$$\Phi Y_h''(x_j)=\frac{12}{h^2}\big(Y_{j-2}-Y_{j-1}-Y_j+Y_{j+1}\big),\quad j=2,3,4,...,n-1, \tag{5.4}$$
$$\Phi Y_h^{(3)}(x_j)=\frac{1}{h^2}\big(-Y_{j-2}+3Y_{j-1}-3Y_j+Y_{j+1}\big),\quad j=2,3,4,...n-1, \tag{5.5}$$

where Φ is the discrete operator defined as:

$$\Phi Y_j=\varrho_{j-2}+11\varrho_{j-1}+11\varrho_j+\varrho_{j+1},\quad j=2,3,4,...,n-1. \tag{5.6}$$

Theorem 5.1. *Let $Y_h(x_j)$ be the quartic spline interpolant of $Y(x_j)$ and $Y(x)\in\mathbb{C}^8(\overline{\Omega})$, then we have*

$$\Phi Y_h'(x_j)=24Y_j'-12hY_j''+8h^2Y_j'''-3h^3Y_j^{(4)}+\frac{6}{5}h^4Y_j^{(5)}-\frac{11}{30}h^5Y_j^{(6)}+O(h^6), \tag{5.7}$$
$$\Phi Y_h''(x_j)=24Y_j''-12hY_j'''+8h^2Y_j^{(4)}-3h^3Y_j^{(5)}+\frac{16}{15}h^4Y_j^{(6)}-\frac{3}{10}h^5Y_j^{(8)}+O(h^6), \tag{5.8}$$
$$\Phi Y_h'''(x_j)=24Y_j'''-12hY_j^{(4)}+6h^2Y_j^{(5)}-2h^3Y_j^{(6)}+\frac{3}{5}h^4Y_j^{(7)}-\frac{3}{20}h^5Y_j^{(8)}+O(h^6). \tag{5.9}$$

Proof. From equations (5.3) and (5.7) and using interpolating condition (5.1), and expanding $Y(x)$ in Taylor's series, in the right side of equation (5.7) we get,

$$\Phi Y_h'(x_j)=24Y_j'-12hY_j''+8h^2Y_j'''-3h^3Y_j^{(4)}+\frac{6}{5}h^4Y_j^{(5)}-\frac{11}{30}h^5Y_j^{(6)}+O(h^6). \tag{5.10}$$

In similar way we can prove the remaining relations (5.8) and (5.9). □

Theorem 5.2. *Let $Y_h(x_j)$ be the quartic spline of $Y(x) \in \mathbb{C}^8(\overline{\Omega})$ and satisfies the interpolating conditions (5.1) and (5.2). Then we have following relation for $i = 0, 1, 2, 3, ..., n$,*

$$Y_h'(x_j) = Y'(x_j) + \frac{1}{720}h^4Y^{(5)}(x_j) - \frac{1}{2016}h^6Y^{(7)}(x_j) + O(h^8), \tag{5.11}$$

$$Y_h''(x_j) = Y''(x_j) - \frac{1}{240}h^4Y^{(6)}(x_j) + \frac{1}{6048}h^6Y^{(8)}(x_j) + O(h^8), \tag{5.12}$$

$$Y_h'''(x_j) = Y'''(x_j) - \frac{1}{12}h^4Y^{(5)}(x_j) + \frac{1}{240}h^4Y^{(7)}(x_j) + O(h^6). \tag{5.13}$$

Proof. Let any function $S(x) \in \mathbb{C}^8(\overline{\Omega})$, we have following relation,

$$\Phi S_j = S_{j-2} + 11S_{j-1} + 11S_j + S_{j+1}, \qquad j = 2, 3, 4, ..., n-1. \tag{5.14}$$

Using Taylor's series expansion for S_j, each term on the right side of equation (5.14), we get

$$\Phi S_j = 24S_j - 12hS_j' + 8h^2S_j'' - 3h^3S_j''' + \frac{7}{6}h^4S_j^{(4)} - \frac{7}{20}h^5S_j^{(5)} + O(h^6). \tag{5.15}$$

Since $S_j = Y^{(3)}(x_j) - \frac{1}{12}h^2Y^{(5)}(x_j) + \frac{1}{240}h^4Y^{(7)}(x_j)$. From equation (5.15), we get

$$\begin{aligned}&\Phi(Y^{(3)}(x_j) - \frac{1}{12}h^2Y^{(5)}(x_j) + \frac{1}{240}h^4Y^{(7)}(x_j))\\ =&24Y_j''' - 12hY_j^{(4)} + 6h^2Y_j^{(5)} - 2h^3Y_j^{(6)} + \frac{3}{5}h^4Y_j^{(7)} - \frac{3}{20}h^5Y_j^{(8)}.\end{aligned} \tag{5.16}$$

From equations (5.16) and (5.10), we get

$$\Phi(Y_{hj}''' - Y_j''' + \frac{h^2}{12}Y_j^{(5)} - \frac{h^4}{240}Y_j^{(5)}) = O(h^6). \tag{5.17}$$

Now, we set $q_j = Y_{hj}''' - Y_j''' + \frac{h^2}{12}Y_j^{(5)} - \frac{h^4}{240}Y_j^{(5)}$ and interpoloting condition (5.2), we have

$$\Phi q_j = O(h^6), \qquad j = 2, 3, 4, ..., n-1, \tag{5.18}$$

and $q_0 = q_1 = q_n = 0$. Therefore from equation (5.17) and (5.18), we can get

$$Y_{hj}''' = Y_j''' + \frac{h^2}{12}Y_j^{(5)} - \frac{h^4}{240}Y_j^{(5)} + O(h^6). \tag{5.19}$$

Now, we consider the consistency relation of quartic B-spline for $j = 1, 2, 3, ..., n-1$, we have

$$Y_{hj}''' = \frac{1}{h^2}(Y_{hj-1} - 2Y_{hj} + Y_{hj+1}) - \frac{h}{24}(Y_{hj+1}^{(3)} - Y_{hj-1}^{(3)}). \tag{5.20}$$

Now, using equations (5.1) and (5.13) in equation (5.20), we get

$$\begin{aligned}Y_{hj}''' =&\frac{1}{h^2}(Y_{j-1} - 2Y_j + Y_{j+1}) - \frac{1}{h^2}[(Y_{j+1}''' - \frac{1}{12}h^2Y_{j+1}^{(5)} + \frac{1}{240}h^4Y_{j+1}^{(7)})\\ &- (Y_{j-1}''' - \frac{1}{12}h^2Y_{j-1}^{(5)} + \frac{1}{240}h^4Y_{j-1}^{(7)})], \qquad j = 2, 3, 4, ..., n-1.\end{aligned} \tag{5.21}$$

Now, expanding each $Y_{j\pm1}$ by Taylor's series about x_j, we get

$$Y''_{hj} = Y''(x_j) - \frac{1}{240}h^4Y^{(6)}(x_j) + \frac{1}{6048}h^6Y^{(8)}(x_j) + O(h^8), \quad j = 1, 2, 3, ..., n-1. \tag{5.22}$$

Now, we prove the consistency relation for $j = 0, n$, we have

$$Y''_{hj} = Y''_{hj+1} - \frac{h}{2}(Y'''_{hj} - Y'''_{hj+1}), \quad j = 0, \tag{5.23}$$

$$Y''_{hj} = Y''_{hj-1} - \frac{h}{2}(Y'''_{hj} + Y'''_{hj-1}), \quad j = n. \tag{5.24}$$

Putting equations (5.13) and (5.22) in (5.23) and (5.24) and expanding each term by Taylor's series about x_j, we get complete proof of equation (5.23). For the last relation, we use following consistency relation of quartic spline for $j = 1, 2, 3, ..., n$.

$$Y'_{hj} = \frac{h^2}{24}(Y'''_{hj-2}+5Y'''_{hj-1}+2Y'''_{hj})+\frac{h}{12}(Y''_{hj-2}+Y''_{hj-1}+Y''_{hj})+\frac{1}{h}(Y'_{hj}-Y'_{hj-1}), \tag{5.25}$$

for $j = 0, n.$ we get

$$Y'_{hj} = Y'_{hj+2} - \frac{h^3}{3}(Y'''_{hj} + 4Y'''_{hj+1} + 2Y'''_{hj+2}). \tag{5.26}$$

Now using equation (5.1) in equation (5.25) and (5.26) ,we get

$$Y'_{hj} = \frac{h^2}{24}(Y'''_{j-2} + 5Y'''_{j-1} + 2Y'''_j) + \frac{h}{12}(Y''_{j-2} + Y''_{j-1} + Y''_j) + \frac{1}{h}(Y'_j - Y'_{j-1}),$$
$$j = 1, 2, 3, \ldots, n, \tag{5.27}$$

$$Y'_j = Y'_{j+2} - \frac{h^3}{3}(Y'''_j + 4Y'''_{j+1} + 2Y'''_{j+2}), \quad j = 0, n. \tag{5.28}$$

Expanding each term in the right side using Taylor's series about x_j, we get

$$Y'_h(x_j) = Y'(x_j) + \frac{1}{720}h^4Y^{(5)}(x_j) - \frac{1}{2016}h^6Y^{(7)}(x_j) + O(h^8). \tag{5.29}$$

In similar fashion, we prove the equation (5.13). $\square$

Now the truncation error is defined as $\mathtt{e}(x) = Y(x) - Y_h(x)$ can be obtained by using Taylor series expansion for $\mathtt{e}(x_j + \theta h)$ as

$$\begin{aligned}\mathtt{e}(x_j + \theta h) =& \frac{-(10\theta^2 - \theta)}{720}h^5Y^{(5)}(x_j) - \frac{(5\theta^4 - \theta^2)}{720}h^6Y^{(6)}(x_j)\\ &+ \frac{(7\theta^2 - 5\theta)}{720}h^7Y^{(7)}(x_j) + O(h^8), \ \ 0 \le \theta \le 1.\end{aligned} \tag{5.30}$$

5.1. Stability Analysis

Here, we discuss the stability of the proposed numerical method (4.19) by means of Von-Neumann technique.

Theorem 5.3. *The proposed numerical method (4.19) is unconditionally stable.*

Proof. To demonstrate the stability of the proposed numerical method, we suppose that $F(x,\tau)=0$. We suppose that

$$\omega_j^k=\xi^k e^{ij\sigma h}, \tag{5.31}$$

where $i=\sqrt{-1}$, ξ is the amplitude and σ wave number of the error. Putting equation (5.31) in the equation (4.12), we have

$$\begin{aligned}
&\xi^{k+1}e^{ij\sigma h}\Bigg(e^{-2i\sigma h}\left(\frac{\varepsilon}{4h_j^2}-\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{r}}(x_j)}{48}\right)+e^{-i\sigma h}\left(-\frac{\varepsilon}{4h_j^2}-\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{11\tilde{\mathtt{r}}(x_j)}{48}\right)\\
&+\left(-\frac{\varepsilon}{4h_j^2}+\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{1\tilde{1}\mathtt{r}(x_j)}{48}\right)+e^{i\sigma h}\left(\frac{\varepsilon}{4h_j^2}+\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{r}}(x_j)}{48}\right)\Bigg)\\
=&\xi^{k}e^{ij\sigma h}\Bigg(e^{-2i\sigma h}\left(\frac{-\varepsilon}{4h_j^2}+\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{s}}(x_j)}{48}\right)+e^{-i\sigma h}\left(\frac{\varepsilon}{4h_j^2}+\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{1\tilde{1}\mathtt{s}(x_j)}{48}\right)\\
&+\left(\frac{\varepsilon}{4h_j^2}-\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{11\tilde{\mathtt{s}}(x_j)}{48}\right)+e^{i\sigma h}\left(\frac{-\varepsilon}{4h_j^2}-\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{s}}(x_j)}{48}\right)\Bigg),\quad 0\le k\le m-1.
\end{aligned} \tag{5.32}$$

After simplifying the above equation, we get

$$\begin{aligned}
&\xi^{k+1}\Bigg((\cos(2\sigma h)-i\sin(2\sigma h))\left(\frac{\varepsilon}{4h_j^2}-\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{r}}(x_j)}{48}\right)+\cos(\sigma h)\left(\frac{-\tilde{\mathtt{a}}(x_j)}{4h_j}+\frac{\tilde{\mathtt{r}}(x_j)}{4}\right)\\
&+i\sin(\sigma h)\left(\frac{-\varepsilon}{2h_j^2}+\frac{\tilde{\mathtt{a}}(x_j)}{3h_j}-\frac{5\tilde{\mathtt{r}}(x_j)}{24}\right)+\left(\frac{\varepsilon}{4h_j^2}+\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{11\tilde{\mathtt{r}}(x_j)}{48}\right)\Bigg)\\
=&\xi^{k}\Bigg((\cos(2\sigma h)-i\sin(2\sigma h))\left(\frac{-\varepsilon}{4h_j^2}+\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{s}}(x_j)}{48}\right)+\cos(\sigma h)\left(\frac{\tilde{\mathtt{a}}(x_j)}{4h_j}+\frac{\tilde{\mathtt{s}}(x_j)}{4}\right)\\
&+i\sin(\sigma h)\left(\frac{\varepsilon}{2h_j^2}-\frac{\tilde{\mathtt{a}}(x_j)}{3h_j}-\frac{5\tilde{\mathtt{s}}(x_j)}{24}\right)+\left(\frac{-\varepsilon}{4h_j^2}-\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{11\tilde{\mathtt{s}}(x_j)}{48}\right)\Bigg),\\
&0\le k\le m-1.
\end{aligned} \tag{5.33}$$

For our convenience, we define the following notations:

$$\begin{aligned}
L_1&=\left(\frac{\varepsilon}{4h_j^2}-\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{r}}(x_j)}{48}\right),\quad L_2=\left(\frac{-\tilde{\mathtt{a}}(x_j)}{4h_j}+\frac{\tilde{\mathtt{r}}(x_j)}{4}\right),\\
L_3&=\left(\frac{-\varepsilon}{2h_j^2}+\frac{\tilde{\mathtt{a}}(x_j)}{3h_j}-\frac{5\tilde{\mathtt{r}}(x_j)}{24}\right),\quad L_4=\left(\frac{\varepsilon}{4h_j^2}+\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{11\tilde{\mathtt{r}}(x_j)}{48}\right),\\
L_5&=\left(\frac{-\varepsilon}{4h_j^2}+\frac{\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{\tilde{\mathtt{s}}(x_j)}{48}\right),\quad L_6=\left(\frac{\tilde{\mathtt{a}}(x_j)}{4h_j}+\frac{\tilde{\mathtt{s}}(x_j)}{4}\right),\\
L_7&=\left(\frac{\varepsilon}{2h_j^2}-\frac{\tilde{\mathtt{a}}(x_j)}{3h_j}-\frac{5\tilde{\mathtt{s}}(x_j)}{24}\right),\quad L_8=\left(\frac{-\varepsilon}{4h_j^2}-\frac{3\tilde{\mathtt{a}}(x_j)}{12h_j}+\frac{11\tilde{\mathtt{s}}(x_j)}{48}\right).
\end{aligned}$$

Thus, equation (5.33) reduces to

$$\begin{aligned}
&\xi^{k+1}\{(L_1\cos(2\sigma h)+L_2\cos(\sigma h)+L_4)-i(L_1\sin(2\sigma h)+L_3\sin(\sigma h))\}\\
=&\xi^{k}\{(L_5\cos(2\sigma h)+L_6\cos(\sigma h)+L_8)+i(L_7\sin(2\sigma h)+L_8\sin(\sigma h))\},
\end{aligned}$$

$$0 \leq k \leq m-1. \tag{5.34}$$

Suppose $\xi^{k+1} = \frac{\xi^k}{\kappa}$, with $\kappa = \kappa(\sigma h)$ is amplification factor of error and it is free from the time. Then from the above equation, we have

$$\begin{aligned} &\kappa\{(L_5\cos(2\sigma h) + L_6\cos(\sigma h) + L_8) + i(L_7\sin(2\sigma h) + L_8\sin(\sigma h))\} \\ =&\{(L_1\cos(2\sigma h) + L_2\cos(\sigma h) + L_4) - i(L_1\sin(2\sigma h) + L_3\sin(\sigma h))\}, \\ &0 \leq k \leq m-1, \end{aligned} \tag{5.35}$$

and

$$\kappa = \frac{[(L_1\cos(2\sigma h) + L_2\cos(\sigma h) + L_4) - i(L_1\sin(2\sigma h) + L_3\sin(\sigma h))]}{[(L_5\cos(2\sigma h) + L_6\cos(\sigma h) + L_8) + i(L_7\sin(2\sigma h) + L_8\sin(\sigma h))]}. \tag{5.36}$$

This implies that

$$|\kappa|^2 \leq \frac{[(L_1\cos(2\sigma h) + L_2\cos(\sigma h) + L_4) - (L_1\sin(2\sigma h) + L_3\sin(\sigma h))]^2}{[(L_5\cos(2\sigma h) + L_6\cos(\sigma h) + L_8) + (L_7\sin(2\sigma h) + L_8\sin(\sigma h))]^2}. \tag{5.37}$$

Since

$$[(L_5\cos(2\sigma h) + L_6\cos(\sigma h) + L_8) + (L_7\sin(2\sigma h) + L_8\sin(\sigma h))]^2 > 0, \tag{5.38}$$

and

$$\begin{aligned} &[(L_1\cos(2\sigma h) + L_2\cos(\sigma h) + L_4) - (L_1\sin(2\sigma h) + L_3\sin(\sigma h))] \\ \leq&[(L_5\cos(2\sigma h) + L_6\cos(\sigma h) + L_8) + (L_7\sin(2\sigma h) + L_8\sin(\sigma h))]^2. \end{aligned} \tag{5.39}$$

Using the equations (5.38) and (5.39) in equation (5.37), we get

$$|\kappa| \leq 1. \tag{5.40}$$

From equation (5.40), we have seen that the suggested numerical technique is unconditionally stable. □

5.2. Convergence Analysis

In this subsection, we discuss the convergence of the proposed algorithm. We shall used previous results and the lemma to establish the parameter uniform convergence.

Theorem 5.4. *If $Y_h(x)$ be the quartic B-spline collocation approximation for the space $\Phi_4(\overline{\Omega})$, then bound on solution $Y_h(x)$ of the considered problem (1.1)-(1.3) is given by*

$$|Y(x)| \leq K, \qquad x \in \overline{\Omega}_x, \tag{5.41}$$

where K is positive constant.

Proof. From equation (4.19), we observed that the matrix P is strictly diagonally dominant with $\frac{2}{\mathtt{r}_j}(-\mathtt{a}_j + \sqrt{(\mathtt{a}_j^2 - 12\mathtt{r}_j\varepsilon)}) < h < \frac{2}{\mathtt{r}_j}(\mathtt{a}_j + \sqrt{(\mathtt{a}_j^2 j - 12\mathtt{r}_j\varepsilon)})$, $\forall\ \ j = 0, 1, 2, ...n$. Thus, we have

$$\|P^{-1}\|_\infty \leq \frac{1}{\min_{0\leq i\leq n[|\mathtt{a}_{ii}| - \sum_{i\neq j}|a_{ij}|]}} = \max\left(\frac{1}{\Delta_i(P)}\right), \tag{5.42}$$

where $\Delta_i(P) = |a_{ii}| - \Sigma_{i\neq j}|a_{ij}| > 0$, for $i = 0, 1, 2, ..., n$. Therefore, we can say that

$$\|P^{-1}\|_\infty \leq K. \tag{5.43}$$

Thus $\|Y^{k+1}\| \leq \|P^{-1}\|\|G\| \leq K$, which show that $\delta_0^{k+1}, \delta_1^{k+1}, ..., \delta_{n-1}^{k+1}, \delta_n^{k+1}$ are bounded. Now using the boundary conditions, the coefficients $\delta_{-2}^{k+1}, \delta_{-1}^{k+1}$ and δ_{n+1}^{k+1} are also bounded. Therefore

$$|Y(x)| = |\sum_{i=-2}^{n+1} \delta_i^{k+1} Y_h(x)| \leq \max |\delta_i^{k+1}| \sum_{i=-2}^{n+1} |Y_h(x)|, \quad \forall x \in \overline{\Omega}. \tag{5.44}$$

Since we know that the quartic B-spline basis function $\{Y_{hj}\}_{j=-2}^{n+1}$, satisfying following inequality

$$\sum_{i=-2}^{n+1} |Y_{hi}(x_j)| \leq 47, \quad j = 0, 1, 2, 3, ..., n-1. \tag{5.45}$$

Therefore, from equations (5.44) and (5.45), we have

$$|Y(x)| \leq K, \qquad x \in \overline{\Omega}_x. \tag{5.46}$$

□

Theorem 5.5. *Let $Y(x)$ be the sufficiently smooth solution of the given problem (1.1)-(1.3) and $Y_h(x)$ be quartic B-spline collocation approximation on the nonuniform mesh. The error term must satisfy the following ε-uniform error estimation for sufficiently large n as:*

$$\|Y(x) - Y_h(x)\|_\infty = \begin{cases} Kn^{-4}(\log(n))^{10}, & \forall x \in \overline{\Omega}_1 \\ Kn^{-4}, & \forall x \in \overline{\Omega}_2. \end{cases} \tag{5.47}$$

Proof. To calculate the error $\|Y(x) - Y_h(x)\|_\infty$, we suppose $T(x)$ be the unique quartic B-spline interpolation from space $\Phi_4(\overline{\Omega})$ to the solution $Y(x)$ of problem (1.1)-(1.3). If $G(x) \in \mathbf{C}^4[0,1]$, and $Y(x) \in \mathbf{C}^5[0,1]$, then it follows from [7,11] that

$$\|D^j(Y(x) - T(x))\|_\infty \leq \varsigma_j |Y^{(6)}(x)| h_\lambda^{6-j}, \qquad j = 0, 1, 2, 3, 4 \tag{5.48}$$

where $h_\lambda = \max\{h_{\lambda_1}, h_{\lambda_2}\}$ and ς_j are constant independent of h_λ and n. Let

$$T(x) = \sum_{i=-2}^{n+1} \tilde{\delta}_i BS_i(x). \tag{5.49}$$

Now from equation (5.47), we have

$$\|\mathbf{L}Y(x_i) - \mathbf{L}Y_h(x_i)\|_\infty \leq Kh_\lambda^4, \tag{5.50}$$

where $K = (\varepsilon\varsigma_2 + \varsigma_1\|\mathtt{a(z)}\|h_\lambda + \varsigma_0\|\mathtt{r(z)}\|h_\lambda^2)\|Y^{(6)}(x)\|h_\lambda^6$. Now we suppose that $\mathbf{L}Y(x_i) = \tilde{G}(x_i), \forall i$ where $\tilde{G}(x_i) = [\tilde{G}(x_0), \tilde{G}(x_1), \tilde{G}(x_2), ..., \tilde{G}(x_{n-1}), \tilde{G}(x_n)]^T$ which leads to system of equation

$$MW = \tilde{G}(x), \tag{5.51}$$

where $W = [w_0, w_1, ..., w_{n-1}, w_n]^T$. From equations (4.19) and (5.51), it is clear that the i^{th} coordinate of vector $M(Y - W)$ satisfies the following inequality:

$$\|[M(Y-W)]_i\| = \|[G-\tilde{G}]_i\| \le Kh_\lambda^6. \tag{5.52}$$

Therefore, the i^{th} coordinate of $[M(Y - W)]$ is the i^{th} equation

$$\begin{aligned}&(-6\varepsilon - 4\mathtt{a}_i + \mathtt{r}_i h_i^2)\alpha_{i-2} + (6\varepsilon - 12\mathtt{a}_i + 11\mathtt{r}_i h_i^2)\alpha_{i-1} + (6\varepsilon + 12\mathtt{a}_i + 11\mathtt{r}_i h_i^2)\alpha_i\\ &+ (-6\varepsilon + 4\mathtt{a}_i + \mathtt{r}_i h_i^2)\alpha_{i+1} = \chi_i, \qquad 0 \le i \le n,\end{aligned} \tag{5.53}$$

where $\alpha_i = \delta_i - w_i$ and $\chi_i = 48h_i^2[G(x_i) - \tilde{G}(x_i)]$ for $2 \le i \leqq n-1$, and from equation (5.53), we can easily prove that $|\chi_i| \le \tau h_\lambda^6$. Let $\chi_i = \max_{2\le i\le n-1} |\chi_i|$. Also we suppose that $\alpha_i = [\alpha_0, \alpha_1, \alpha_2, ..., \alpha_{n-1}, \alpha_n]^T$. Now we have to introduce error term $\hat{E}_i = |\alpha_i|$ and $E = \max_{2\le i\le n-1} |\hat{E}_i|$. From equation (5.53), we have

$$\begin{aligned}&(6\varepsilon + 12\mathtt{a}_i + 11\mathtt{r}_i h_i^2)\alpha_i = (6\varepsilon + 4\mathtt{a}_i - \mathtt{r}_i h_i^2)\alpha_{i-2} - (6\varepsilon - 12\mathtt{a}_i + 11\mathtt{r}_i h_i^2)\alpha_{i-1}\\ &- (-6\varepsilon + 4\mathtt{a}_i + \mathtt{r}_i h_i^2)\alpha_{i+1} + \chi_i, \qquad 2 \le i \le n-1.\end{aligned} \tag{5.54}$$

Now using the condition from differential equation $0 \le \mathtt{a}^* \le \mathtt{a}(x)$, $0 \le \mathtt{r}^* \le \mathtt{r}(x)$ and taking the absolute values with sufficiently small h_λ, we obtain

$$(2\mathtt{r}^* h^2 + 8\mathtt{a}^* h + 12\varepsilon)\tilde{e} \le \chi_i \le Kh_\lambda^6. \tag{5.55}$$

Therefore we have

$$\tilde{E} \le \frac{Kh_\lambda^4}{(2\mathtt{r}^* h^2 + 8\mathtt{a}^* h + 12\varepsilon)}, \qquad i = 2, 3, ..., n-1. \tag{5.56}$$

Now we estimate the values $\tilde{E}_{-2}$, $\tilde{E}_{-1}$, $\tilde{E}_0$, $\tilde{E}_n$, and $\tilde{E}_{n+1}$. Using the first and last equation of the system $M(Y - W) = (G - \tilde{G})$, with boundary conditions and auxiliary equation, we get

$$\begin{aligned}\tilde{E}_0 &\le \frac{8\mathtt{r}^* Kh_\lambda^4}{(24\varepsilon + 18\mathtt{a}^* h)(2\mathtt{r}^* h^2 + 8\mathtt{a}^* h + 12\varepsilon)},\\ \tilde{E}_n &\le \frac{8\mathtt{r}^* Kh_\lambda^4}{(24\varepsilon - 18\mathtt{a}^* h)(2\mathtt{r}^* h^2 + 8\mathtt{a}^* h + 12\varepsilon)},\end{aligned} \tag{5.57}$$

and

$$\begin{aligned}\tilde{E}_{-1} &\le \frac{(18\varepsilon + 4\mathtt{a}^* h + 12\mathtt{r}^* h^2)Kh_\lambda^4}{(64\varepsilon + 48\mathtt{a}^* h)(2\mathtt{r}^* h^2 + 8\mathtt{a}^* h + 12\varepsilon)},\\ \tilde{E}_{n+1} &\le \frac{(18\varepsilon - 4\mathtt{a}^* h + 12\mathtt{r}^* h^2)Kh_\lambda^4}{(64\varepsilon - 48\mathtt{a}^* h)(2\mathtt{r}^* h^2 + 8\mathtt{a}^* h + 12\varepsilon)}.\end{aligned} \tag{5.58}$$

Also

$$\tilde{E}_{-2} \le \frac{(64\varepsilon - 24\mathtt{a}^* h + 8\mathtt{r}^* h^2)Kh_\lambda^4}{(36\varepsilon + 8\mathtt{a}^* h + 16\mathtt{r}^* h^2)(2\mathtt{r}^* h^2 + 8\mathtt{a}^* h + 12\varepsilon)}. \tag{5.59}$$

Therefore from equations (5.56)- (5.59), and putting the value of K, we get

$$\hat{E} = \max_{-2\le i\le n+1} \{\hat{E}\} \le K_1 \|Y^{(6)}(x)\| h_\lambda^4, \tag{5.60}$$

where K_1 is constant independent of ε. The above inequality enables us to estimate $\|Y_h(x) - T(x)\|_\infty$ and hence $\|Y_h(x) - Y(x)\|_\infty$. Thus, we have

$$Y_h(x) - T(x) = \sum_{i=-2}^{n+1} (\delta_i - w_i) BS_i(x) \tag{5.61}$$

$$|Y_h(x) - T(x)| \leq \max |\delta_i - w_i| \sum_{i=-2}^{n+1} |BS_i(x)|$$

$$|Y_h(x) - T(x)| \leq \max |\alpha_i| \sum_{i=-2}^{n+1} |BS_i(x)|. \tag{5.62}$$

From equations (5.60) and (5.61) and $\sum_{i=-2}^{n+1} |BS_i(x)| \leq 47$, we have

$$|Y_h(x) - T(x)| \leq K h_\lambda^4. \tag{5.63}$$

Also

$$\|Y(x) - Y_h(x)\|_\infty \leq \|Y(x) - T(x)\|_\infty + \|T(x) - Y_h(x)\|_\infty.$$

Now from equation (5.63) and (5.48), we have

$$|Y(x) - Y_h(x)| \leq K \|Y^{(6)}(x)\| h_\lambda^4. \tag{5.64}$$

Now, we discuss the convergence of the developed method on each subinterval $\overline{\Omega}_i = (x_{i-1}, x_i)$, $\forall i = 1, 2, 3, ..., n$ separately. As we have dealt with continuous problem, we first decomposed the solution of discrete problem into smooth component $V(x)$ and singular component $W(x)$ respectively. Thus $Y_h(x) = V(x) + W(x)$, where $V(x)$ is the solution of non-homogeneous problem given by

$$\begin{aligned} &\mathbf{L}(V(x)) = G(x), \qquad &&\forall x \in \overline{\Omega}_x, \\ &V(0) = \mathtt{v}(0), \qquad &&V(1) = \mathtt{v}(1), \end{aligned}$$

and $W(x)$ is the solution of the homogeneous problem

$$\begin{aligned} &\mathbf{L}(W(x)) = 0, \qquad &&\forall x \in \overline{\Omega}_x, \\ &W(0) = \mathtt{w}(0), \qquad &&W(1) = \mathtt{w}(1). \end{aligned}$$

From the above relation, we can define error term in the spatial direction such as

$$Y_h(x) - Y(x) = (V(x) - \mathtt{v}(x)) + (W(x) - \mathtt{w}(x)), \qquad \forall x \in \overline{\Omega}_x. \tag{5.65}$$

Since each quartic B-spline basis function covered by five elements at each finite subinterval $\overline{\Omega}_x$, therefore the B-spline collocation approximation $Y_h(x)$ of $Y(x)$, on $x \in \overline{\Omega}_z$ is given by

$$\begin{aligned} Y_h(x) = &\delta_{i-2}^{k+1} BS_{i-2}(x) + \delta_{i-1}^{k+1} BS_{i-1}(x) + \delta_i^{k+1} BS_i(x) \\ &+ \delta_{i+1}^{k+1} BS_{i+1}(x) + \delta_{i+2}^{k+1} BS_{i+2}(x), \end{aligned} \tag{5.66}$$

and it is obvious that on $\overline{\Omega}_i$

$$|Y_h(x)| \leq \max_{\Omega_i} |Y(x)|. \tag{5.67}$$

Therefore, from the equation (5.64), we can easily write as

$$|Y(x) - Y_h(x)| \leq K|Y^{(6)}(x)|h_i^{(4)}, \qquad x \in \overline{\Omega}_x. \tag{5.68}$$

Now, we obtain the estimates on the solution $Y(x)$ and its derivatives of semidiscretized problem (4.8):

$$|Y^{(m)}(x)| \leq K\left(1 + \varepsilon^{-m} \exp\left(\frac{-\mathtt{a}^*(1-x)}{\varepsilon}\right)\right), \qquad x \in \overline{\Omega}_x. \tag{5.69}$$

Therefore, from equation (5.69) and (5.68), we have

$$|Y(x) - Y_h(x)| \leq Kh_i^{(4)}\left(1 + \varepsilon^{-m} \exp\left(\frac{-\mathtt{a}^*(1-x)}{\varepsilon}\right)\right), \qquad x \in \overline{\Omega}_x. \tag{5.70}$$

Now, we calculate the error part that varies with the quantity of γ i.e, $\gamma_0\varepsilon\log(n) \geq \frac{1}{2}$ and $\gamma_0\varepsilon\log(n) \leq \frac{1}{2}$.
Case I: Consider $\gamma = \frac{1}{2}$, then the mesh is uniform with space size $h_i = \frac{1}{n}$ and $\gamma_0\varepsilon\log(n) \geq \frac{1}{2}$ and it gives $\varepsilon^{-1} \leq K(\log(n))$. Now we suppose that $\varepsilon \to 0$ and $\exp\left(\frac{-\mathtt{a}^*(1-x)}{\varepsilon}\right) \to 0$ in the equation (5.70). Then we easily get the error estimate as:

$$\|Y(x) - Y_h(x)\|_\infty \leq Kn^{-4}(\log(n))^{10}, \qquad 0 \leq i \leq n. \tag{5.71}$$

Case II: In this case we consider $K\varepsilon\log(n) \leq \frac{1}{2}$. Then the mesh is uniform in fine mesh region Ω_1 with space size $h_i = \frac{2\gamma}{n}$ for i satisfies $1 \leq i \leq \frac{n}{2}$, while the mesh is uniform in the coarse region Ω_2 with space size $h_i = \frac{2(1-\gamma)}{n}$ for i satisfies $\frac{n}{2} + 1 \leq i \leq n$. Therefore, from the equation (5.68) in the fine mesh Ω_1, the error estimate is given by

$$\begin{aligned}
|Y(x) - Y_h(x)| \leq& |V(x) - \mathtt{v}(x)| + |W(x) - \mathtt{w}(x)| \\
\leq& Kh_i^{(4)} \max(|\mathtt{v}^6(x)| + |\mathtt{w}^6(x)|) \\
\leq& Kh_i^{(4)}\left(\left(1 + \varepsilon^{-3}\exp\left(\frac{-\mathtt{a}^*(1-x)}{\varepsilon}\right)\right) + \left(\varepsilon^{-6}\exp\left(\frac{-\mathtt{a}^*(1-x)}{\varepsilon}\right)\right)\right).
\end{aligned} \tag{5.72}$$

Here the fine mesh region $\Omega_1 = [0, 1-\gamma]$, we have $\frac{h_i}{\varepsilon} = \frac{2\tau}{n\varepsilon} = Kn^{-1}\log(n)$, for $1 \leq i \leq \frac{n}{2}$. Now we again suppose that $\varepsilon \to 0$ and $\exp\left(\frac{-\mathtt{a}^*(1-x)}{\varepsilon}\right) \to 0$, $\forall x \in \Omega_i$, and $k \in \mathbb{Z}$. Then the inequality (5.72), become:

$$|Y(x) - Y_h(x)| \leq Kn^{-4}(\log(n))^{10}, \qquad \forall x \in \Omega_i, \quad 1 \leq i \leq \frac{n}{4}. \tag{5.73}$$

Now we find the error estimate in the coarse mesh region $\Omega_2 = [1-\gamma, 1]$. Since both singular component $W(x)$ and $\mathtt{w}(x)$ are small in $\Omega_2 = [1-\gamma, 1]$. Therefore from the equation (5.67) and (5.68), on Ω_i for $n/4 \leq i \leq n/2$, we obtain

$$\begin{aligned}
|Y(x) - Y_h(x)| =& |V(x) - \mathtt{v}(x) + W(x) - \mathtt{w}(x)| \\
\leq& |V(x) - \mathtt{v}(x)| + |W(x) - \mathtt{w}(x)| \\
\leq& Kh_i^{(4)} \max_{\Omega_i}(|\mathtt{v}^6(x)| + 2\max|\mathtt{w}(x)|)
\end{aligned}$$

$$
\begin{aligned}
&\leq K\left(h_i^{(4)} + \exp\big(\frac{-\mathtt{a}^*(1-x)}{\varepsilon}\big)\right)\\
&\leq K\left(\left(\frac{4(1-\gamma)}{n}\right)^4 + \exp^{(\frac{-\mathtt{a}^*\gamma_0}{\varepsilon})}\right)\\
&\leq K\left(\left(\frac{256(1-\gamma)^4}{n^4}\right) + \exp^{(-\mathtt{a}^*\gamma_0\log(n))}\right)\\
&\leq K(n^{-4} + n^{-\mathtt{a}^*\gamma_0})\\
&\leq Kn^{-4}, \qquad \forall x \in [1-\gamma, \gamma]. \qquad (5.74)
\end{aligned}
$$

From equation (5.73) and (5.74), we conclude that

$$
\|Y(x) - Y_h(x)\|_\infty = \begin{cases} Kn^{-4}(\log(n))^{10}, & \forall x \in \overline{\Omega}_1 \\ Kn^{-4}, & \forall x \in \overline{\Omega}_2. \end{cases} \qquad (5.75)
$$

□

Theorem 5.6. *Let $Y_h(x,\tau)$ be the approximation to the solution $Y(x,\tau)$ of the problem (1.1)-(1.3) at $(k+1)$th time level of the fully discretized scheme after the temporal discretization. Then,*

$$
\sup_{0\leq\varepsilon\leq 1}\max_{0\leq n\leq 1}|Y(x) - Y_h(x)| = \begin{cases} K((\Delta\tau)^2 + n^{-4}(\log(n))^{10}), & \forall x \in \overline{\Omega}_1, \\ K((\Delta\tau)^2 + n^{-4}), & \forall x \in \overline{\Omega}_2. \end{cases} \qquad (5.76)
$$

Proof. The proof is the consequence of Theorem 5.5 and Lemma 3.2. □

6. Numerical Experiments

In order to demonstrate improved accuracy for various values of ϵ and step size n, we addressed two problems to demonstrate the efficacy and efficiency of the proposed technique. We contrasted our findings with those reported in the literature. There is no analytical solution provided for problems 6.1 and 6.2. So, we apply the double mesh approach as follows to determine the maximum pointwise errors and order of convergence

$$
|E_\varepsilon^{n,m}| = |Y_{hj}^{n,m} - Y_{hj}^{2n,2m}|,
$$

where $Y_{hj}^{n,m}$ and $Y_{hj}^{2n,2m}$ are numerical solutions obtained by the use of the quartic B-spline method on a shishkin mesh in the spatial direction and a uniform mesh in the time direction. To calculate ε-uniform maximum pointwise error as:

$$
E_{n,m} = \max E_\varepsilon^{n,m}.
$$

By using the formula, one can determine the order of convergence of the approach by

$$
R_\varepsilon^{n,m} = \log_2\left(\frac{E_\varepsilon^{n,m}}{E_\varepsilon^{2n,2m}}\right),
$$

and the ε-uniform order of convergence is calculated by

$$
R_{n,m} = \log_2\left(\frac{E_{n,m}}{E_{2n,2m}}\right).
$$

Problem 6.1. *Consider the SPPCDE with* $a(x) = (2 + x^2)$, $b(x) = x$, $F(x, \tau) = 10\tau^2 \exp(-\tau)x(1-x)$, *and initial and boundary conditions are* $Y(x, 0) = 0$, $Y(0, \tau) = 0$, $Y(1, \tau) = 0$.

Problem 6.2. *Consider the SPPCDE with* $a(x) = (2-x^2)$, $b(x) = x^2+1+\cos(\pi x)$, $F(x, \tau) = \sin(\pi x)$, *and initial and boundary conditions are* $Y(x, 0) = 0$, $Y(0, \tau) = 0$, $Y(1, \tau) = 0$.

We apply proposed method (4.19) to solve the problems 6.1 and 6.2 for distinct values of perturbation parameter ε and spatial size n. We have computed maximum pointwise error (MPE) and their order of convergence. From the Tables 2 and 5, we can observe that the constructed method with Shishkin mesh is ε-uniformly convergent for various values of ε and n. Because of this, we are able to say that the order of convergence attained computationally is higher than one that was predicted in the section before it. It has been shown that theoretical rate of convergence of the developed method is fourth order in the spatial direction and second order in the time direction. Moreover in Tables 3 and 4, we compare our numerical results with those obtained by Clavero et al. [6] and Kadalbajoo et al. [14]. The comparison between these tables makes it very clear that the proposed strategy is superior to [6, 14]. We have plotted the zoomed figure of numerical solutions of the problem 6.1 and 6.2 for $\varepsilon = 2^{-20}$ and $n = 64, 128$ respectively, in figures $5, 6, 11$ and 12. From, the figures $5, 6, 11$, and 12 we observed that the large amount of the mesh point gather in the boundary layer, when the very small values of ε, and hence we also observed that the width of boundary layer decreases and becomes sharper at the ends points. Therefore, we have observed that the significance of Shishkin mesh over the uniform mesh. The surface plot of numerical solution for various values of ε, *i.e.* ($\varepsilon = 2^{-06}, 2^{-12}, 2^{-18}, 2^{-24}$) and $m = n = 128$ are presented in the Figures $1, 2, 7$ and 8. The Log log plot of maximum pointwise errors for the problem 6.1 and 6.2 are plotted in Figures in $3, 4, 9$ and 10.

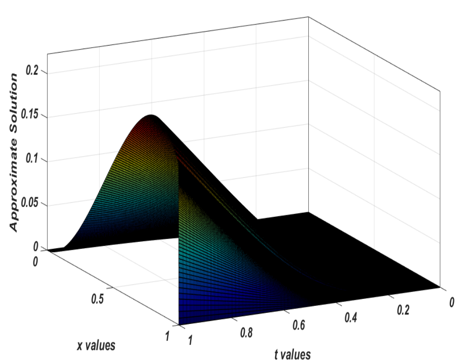

Figure 1. Approximate solutions of the Problem 6.1 for $n = 128$ and $\varepsilon = 2^{-10}$

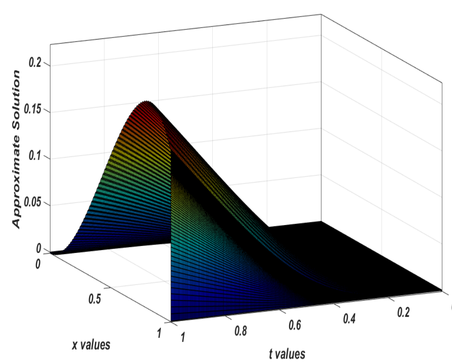

Figure 2. Approximate solutions of the Problem 6.1 for $n = 128$ and $\varepsilon = 2^{-20}$

Table 2. Maximum Pointwise errors $E^{n,m}$ of Problem 6.1 using proposed method (4.19).

$\varepsilon \downarrow$	$n = 2^5$ $\Delta\tau = 1/2^5$	$n = 2^6$ $\Delta\tau = 1/2^6$	$n = 2^7$ $\Delta\tau = 1/2^7$	$n = 2^8$ $\Delta\tau = 1/2^8$	$n = 2^9$ $\Delta\tau = 1/2^9$
2^{-08}	$3.2417e-03$	$5.3041e-04$	$7.4123e-05$	$9.5666e-06$	$1.6511e-06$
	2.6116	2.8391	2.9538	2.5345	
2^{-10}	$3.5450e-03$	$5.7044e-04$	$7.6956e-05$	$9.8069e-06$	$1.1489e-06$
	2.6580	2.8900	2.9721	3.0936	
2^{-12}	$3.6228e-03$	$5.6951e-04$	$7.7660e-05$	$9.8714e-06$	$1.2108e-06$
	2.6693	2.8744	2.9758	3.0274	
2^{-14}	$3.6424e-03$	$5.7146e-04$	$7.7886e-05$	$9.8860e-06$	$1.2246e-06$
	2.6722	2.8763	2.9737	3.0131	
2^{-16}	$3.6473e-03$	$5.7195e-04$	$7.7885e-05$	$9.8895e-06$	$1.2277e-06$
	2.6729	2.8767	2.9774	3.0099	
2^{-18}	$3.6485e-03$	$5.7207e-04$	$7.7881e-05$	$9.8904e-06$	$1.2284e-06$
	2.6730	2.8770	2.9771	3.0093	
2^{-20}	$3.6488e-03$	$5.7210e-04$	$7.7884e-05$	$9.8906e-06$	$1.2286e-06$
	2.6730	2.8769	2.9773	3.0090	
2^{-22}	$3.6489e-03$	$5.7210e-04$	$7.7884e-05$	$9.8906e-06$	$1.2286e-06$
	2.6730	2.8769	2.9771	3.0090	
2^{-24}	$3.6489e-03$	$5.7211e-04$	$7.7884e-05$	$9.8906e-06$	$1.2286e-06$
	2.6730	2.8769	2.9771	3.0090	
$E^{n,m}$	$3.6489e-03$	$5.7211e-04$	$7.7884e-05$	$9.8906e-06$	$1.2286e-06$
$R_{n,m}$	2.6730	2.8769	2.9771	3.009	

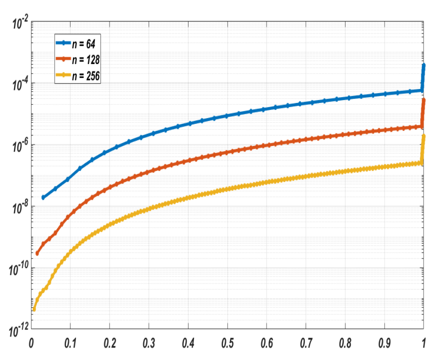

Figure 3. Loglog plot of errors the Problem 6.1 for $n = 128$ and $\varepsilon = 2^{-10}$

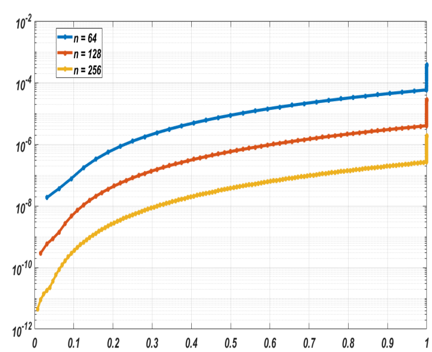

Figure 4. Loglog plot of errors of the Problem 6.1 for $n = 128$ and $\varepsilon = 2^{-20}$

7. Conclusion

In this paper, we constructed a high order numerical method for solving a class of singularly perturbed parabolic convection-diffusion problem with the boundary layer. The method comprised Crank-Nicolson scheme in the temporal direction on uniform mesh and quartic B-spline basis function in the spatial direction on non-

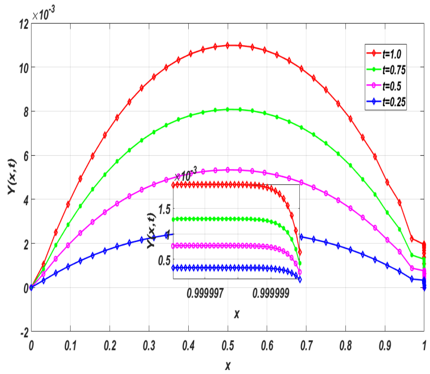

Figure 5. Approximate solutions of the Problem 6.1 at various time level with $n = 64$ and $\varepsilon = 2^{-20}$.

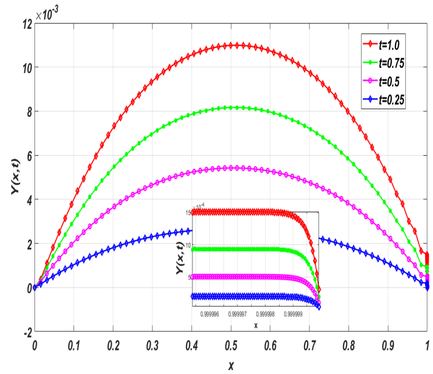

Figure 6. Approximate solutions of the Problem 6.1 at various time level with $n = 128$ and $\varepsilon = 2^{-20}$.

Table 3. Comparison of $E^{n,m}$ for Problem 6.1 between proposed method(PM) and [6,14].

	Methods	ε			
		2^{-08}	2^{-12}	2^{-16}	2^{-20}
$n = 64, m = 40$	PM	$7.3143 - 04$	$7.7052e - 04$	$7.7296e - 04$	$7.7311e - 04$
	[6]	$9.9801e - 03$	$1.1062e - 03$	$1.1123e - 02$	$1.1127e - 02$
	[14]	$1.3036e - 03$	$2.4394e - 03$	$2.7221e - 03$	$2.7486e - 03$
$n = 128, m = 80$	PM	$1.6094e - 04$	$9.1412e - 04$	$1.8520e - 04$	$1.8528e - 04$
	[6]	$5.8721e - 03$	$5.8677e - 03$	$5.8286e - 03$	$5.0641e - 03$
	[14]	$1.2822e - 03$	$1.3043e - 03$	$8.1807e - 03$	$6.4156e - 03$
$n = 256, m = 160$	PM	$5.1430e - 05$	$5.3509e - 05$	$5.2178e - 05$	$5.2178e - 05$
	[6]	$3.0833e - 03$	$3.0818e - 03$	$3.0536e - 03$	$2.5678e - 03$
	[14]	$6.3983e - 04$	$6.1306e - 04$	$4.3726e - 03$	$3.2554e - 04$

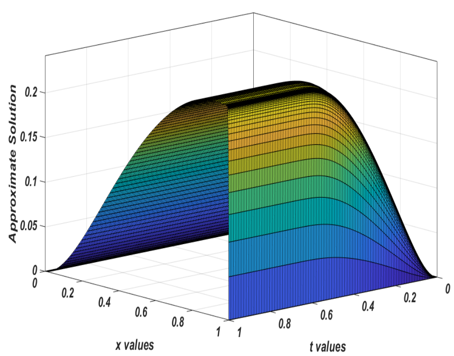

Figure 7. Approximate solutions of the Problem 6.2 for $n = 128$ and for $\varepsilon = 2^{-10}$.

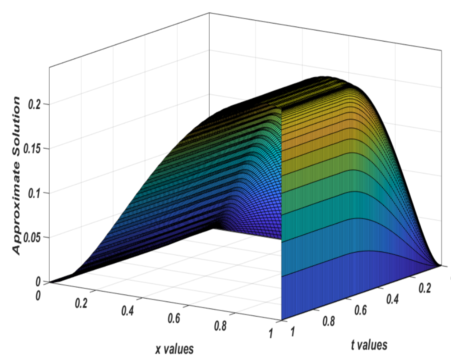

Figure 8. Approximate solutions of the Problem 6.2 for $n = 128$ and $\varepsilon = 2^{-20}$

uniform mesh. We have done comprehensive analysis and got parameter-uniform error estimates which show high order accuracy with regard to space and time. It can be seen from the graphs that the boundary layer width continuously relies on ε

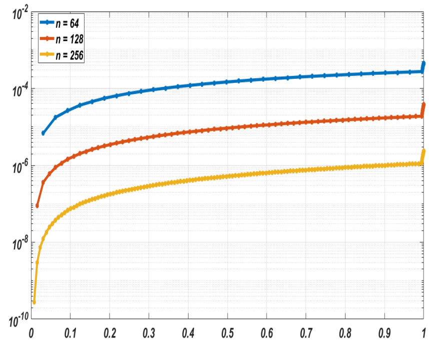

Figure 9. Loglog plot of errors of the Problem 6.2 for $n = 64, 128, 256$ and $\varepsilon = 2^{-10}$.

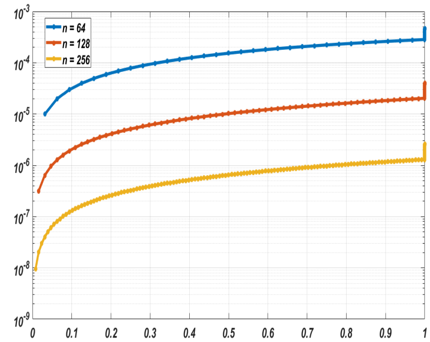

Figure 10. Loglog plot of errors of the Problem 6.2 for $n = 64, 128, 256$ and $\varepsilon = 2^{-20}$.

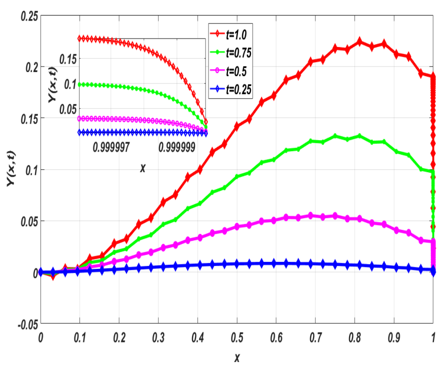

Figure 11. Approximate solutions of the Problem 6.2 at various time level with $n = 64$ and $\varepsilon = 2^{-20}$.

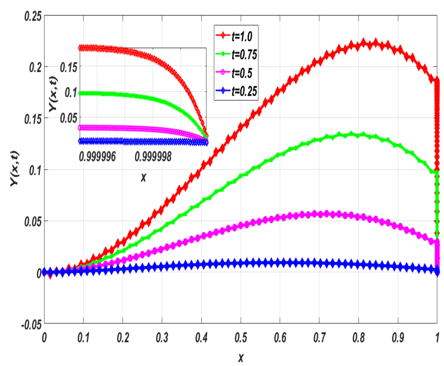

Figure 12. Approximate solutions of the Problem 6.2 at various time level with $n = 128$ and $\varepsilon = 2^{-20}$.

Table 4. Comparison of $E^{n,m}$ for Problem 6.2 between proposed method(PM) and [14], [6].

	Methods	ε			
		2^{-08}	2^{-12}	2^{-16}	2^{-20}
$n = 64, m = 40$	PM	$7.3143e-04$	$5.2957e-04$	$5.3187e-04$	$5.3199e-04$
	[6]	$1.5419e-02$	$1.6499e-02$	$1.6558e-02$	$1.6561e-02$
	[14]	$1.6693e-03$	$2.6772e-03$	$3.1358e-03$	$3.1573e-03$
$n = 128, m = 80$	PM	$2.3584e-04$	$2.6036e-04$	$2.6188e-04$	$2.6197e-04$
	[6]	$9.0213e-03$	$9.9728e-03$	$1.0026e-02$	$1.0029e-02$
	[14]	$7.6966e-04$	$1.0383e-03$	$1.4896e-03$	$1.5190e-03$
$n = 256, m = 160$	PM	$7.8162e-05$	$8.2432e-05$	$8.2752e-05$	$8.2772e-05$
	[6]	$4.9681e-03$	$5.6225e-03$	$5.6598e-03$	$5.6619e-03$
	[14]	$3.5999e-04$	$5.3072e-04$	$6.6507e-04$	$7.5426e-04$

and it decreases as ε decreases. Also, numerical results presented in the tables verify the theoretical estimate. The execution of the developed method for the considered problems was studied by calculating the maximum pointwise error presented in

Table 5. Maximum Pointwise errors $E^{n,m}$ of Problem 6.2, using proposed method (4.19).

$\varepsilon\downarrow$	$n=2^5$ $\Delta\tau=1/2^5$	$n=2^6$ $\Delta\tau=1/2^6$	$n=2^7$ $\Delta\tau=1/2^7$	$n=2^8$ $\Delta\tau=1/2^8$	$n=2^9$ $\Delta\tau=1/2^9$
2^{-08}	$1.3494e-03$	$2.1931e-04$	$6.2680e-05$	$8.8802e-06$	$1.2078e-06$
	2.6212	2.8828	2.8193	2.8782	
2^{-10}	$3.3079e-03$	$4.5175e-04$	$6.7584e-05$	$9.3250e-06$	$1.3271e-06$
	2.8723	2.7408	2.8575	2.8128	
2^{-12}	$3.3506e-03$	$4.5970e-04$	$6.9204e-05$	$9.4918e-06$	$1.5004e-06$
	2.8656	2.7317	2.8661	2.6613	
2^{-14}	$3.3704e-03$	$4.6169e-04$	$6.9613e-05$	$9.5669e-06$	$1.3763e-06$
	2.8679	2.7294	2.8632	2.7972	
2^{-16}	$3.3774e-03$	$4.6218e-04$	$6.9716e-05$	$9.5858e-06$	$1.3211e-06$
	2.8694	2.7289	2.8625	2.8591	
2^{-18}	$3.3791e-03$	$4.6230e-04$	$6.9742e-05$	$9.5906e-06$	$1.3331e-06$
	2.8698	2.7287	2.8623	2.8469	
2^{-20}	$3.3796e-03$	$4.6232e-04$	$6.9748e-05$	$9.8906e-06$	$1.3361e-06$
	2.8700	2.7287	2.8623	2.8880	
2^{-22}	$3.3797e-03$	$4.6233e-04$	$6.9750e-05$	$9.5921e-06$	$1.3368e-06$
	2.8700	2.7287	2.8623	2.8431	
2^{-24}	$3.3797e-03$	$4.6233e-04$	$6.9750e-05$	$9.5921e-06$	$1.3370e-06$
	2.8700	2.7287	2.8623	2.8428	
$E^{n,m}$	$3.3797e-03$	$4.6233e-04$	$6.9750e-05$	$9.8906e-06$	$1.5004e-06$
$R_{n,m}$	2.8700	2.7287	2.8623	2.8880	

Tables 2 and 5. From these tables, we can see that the suggested method provides more precise outcomes with high order convergence than those of the methods considered in [6,14]. Also, we have connived numerical results in Figures 1, 2, 7 and 8 and observed the physical phenomenon of the considered problems. Our future research would comprise the analysis of such properties. Also, we extend the result to solve the high-dimensional problem of the form:

$$Y_\tau(\mathbf{z},\tau)-\varepsilon.\Delta Y(\mathbf{z},\tau)+\mathtt{a}(\mathbf{z},\tau).\nabla Y(\mathbf{z},\tau)+\mathtt{b}(\mathbf{z},\tau)Y(\mathbf{z},\tau)=F(\mathbf{z},\tau),$$

where $\mathbf{z}=(x_1,x_2)$, which is given in [24].

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