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THE STUDY OF EQUILIBRIA FOR GENERALIZED GAMES IN HAUSDORFF TOPOLOGICAL VECTOR SPACES

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Abstract Recent fixed point theory of the author is used to establish equilibria to multiperson games for majorized coercive or compact type maps defined on Hausdorff topological vector spaces.

Keywords Fixed point theory, equilibrium points, majorized type maps.

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1. Introduction

Establishing the existence of equlibria arises naturally when examining various economic models [14] and in this paper we use recent fixed point theory of the author to present a unified approach to establishing very general existence results for equilibria to N–person games (indeed the theory also holds if $\{1, ..., N\}$ is replaced by an index set I if one uses the results in [9] in their full generality). The multivalued maps considered are either majorized coercive type maps or majorized compact (or condensing) type maps and our theory improves and complement results in the literature; see [1–3, 6–8, 12–17] and the references therein.

First we recall a result from [10] for majorized coercive type maps.

Theorem 1.1. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, ..., N\}$ suppose $F_i : X \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(w) \subseteq F_i(w)$ for $w \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. In addition assume there is a compact subset K of X and for each $i \in \{1, .., N\}$ a convex compact subset Y_i of X_i such that for each $x \in X \backslash K$ there exists a $j \in \{1, .., N\}$ with $S_j(x) \cap Y_j \neq \emptyset$. Also suppose for all $i \in \{1, ..., N\}$ that $z_i \notin F_i(z)$ for each $z \in X$ (here z_i denotes the projection of z on X_i). Then there exists a $x \in X$ with $S_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$.*

Remark 1.1. In Theorem 1.1 one could replace $\{X_i\}_{i=1}^N$ with $\{X_i\}_{i\in I}$ where I is an index set if we use a result in [9]. This is true for other theorems in this paper (in Section 1 and Section 2) but we will not refer to it again.

Next we consider majorized compact type maps [11].

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Theorem 1.2. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, ..., N\}$ suppose $F_i : X \to X_i$ and there exists a map $T_i : X \to X_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also suppose for each $i \in \{1, .., N\}$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Finally assume for all $i \in \{1, ..., N\}$ that $z_i \notin F_i(z)$ for each $z \in X$. Then there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$.*

Finally in this section we recall a result [10] for majorized condensing type maps.

Theorem 1.3. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space. For each $i \in \{1, ..., N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and there exists a map $S_i : X \to X_i$ with $S_i(z) \subseteq F_i(z)$ for $z \in X$, $S_i(x)$ has convex values for each $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also assume there is a compact convex subset K of X with $F(K) \subseteq K$ where $F(x) = \prod_{i=1}^N F_i(x)$ for $x \in X$. Finally suppose for each $x \in X$ there exists a $i \in \{1, ...N\}$ with $x_i \notin F_i(x)$. Then there exists a $y \in X$ with $S_{i_0}(y) = \emptyset$ for some $i_0 \in \{1, ..., N\}$.*

2. Generalized games

A generalized game (or abstract economy) is given by $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ where I is a set of players (agents), X_i is a nonempty subset of a Hausdorff topological vector space, A_i, $B_i : X \equiv \prod_{i \in I} X_i \to X_i$ are constraint correspondences and $P_i : X \to X_i$ is a preference correspondence. An equilibrium of Γ is a point $x \in X$ such that for each $i \in I$ we have $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$.

We begin with the coercive situation and we will use a maximal element result (Theorem 1.1) to establish the existence of an equilibrium point of Γ.

Theorem 2.1. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i=1}^N$ be a N-person game i.e. $\{X_i\}_{i=1}^N$ is a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i=1}^N X_i$ paracompact, and for each $i \in \{1, ..., N\}$ the constraint correspondences A_i, $B_i : X \to E_i$ and the preference correspondence $P_i : X \to E_i$. Also for each $i \in \{1, ..., N\}$ suppose $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ is upper semicontinuous (here $CK(X_i)$ denotes the family of nonempty convex compact subsets of X_i) and assume the following conditions hold for each $i \in \{1, ..., N\}$:*

$$\begin{cases} A_i : X \to X_i \ \text{ has nonempty convex values and} \\ A_i^{-1}(x) \ \text{ is open (in } X) \text{ for each } \ x \in X_i, \end{cases} \tag{2.1}$$

$$A_i(x) \subseteq \overline{B_i}(x) \ \text{ for } \ x \in X, \tag{2.2}$$

$$\begin{cases} \text{there exists a map } \ S_i : X \to X_i \ \text{ with } \ S_i(z) \subseteq (A_i \cap P_i)(z) \ \text{ for } \ z \in X, \\ S_i(x) \ \text{ is convex valued for each } \ x \in X, \ S_i^{-1}(z) \ \text{ is open (in } X) \\ \text{for each } \ z \in X_i \ \text{ and } \ x_i \notin (A_i \cap P_i)(x) \ \text{ for } \ x \in X \end{cases} \tag{2.3}$$

and

$$\begin{cases} \text{if } \ x \in X \ \text{ with } \ x_i \in \overline{B_i}(x) \ \text{ and } \ S_i(x) = \emptyset \ \text{ for} \\ \text{a } \ i \in \{1, ..., N\}, \ \text{ then } \ A_i(x) \cap P_i(x) = \emptyset. \end{cases} \tag{2.4}$$

Finally suppose there exists a compact subset K of X and for each $i \in \{1,...,N\}$ a convex compact set Y_i of X_i such that for each $x \in X \backslash K$ there exists a $j \in \{1,....,N\}$ with $S_j(x) \cap Y_j \neq \emptyset$. Then there exists an equilibrium point of Γ i.e. there exists a $x \in X$ with $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in \{1,...,N\}$.

Proof. For each $i \in \{1,...,N\}$ let

$$M_i = \{x \in X : \ x_i \notin \overline{B_i}(x)\}$$

and note M_i is open in X since $\overline{B_i} : X \to CK(X_i)$ is upper semicontinuous. Let $F_i : X \to X_i$ and $T_i : X \to X_i$ be given by

$$F_i(x) = \begin{cases} A_i(x) \cap P_i(x), & x \notin M_i \\ A_i(x), & x \in M_i \end{cases}$$

and

$$T_i(x) = \begin{cases} S_i(x), & x \notin M_i \\ A_i(x), & x \in M_i. \end{cases}$$

For each $i \in \{1,...,N\}$ note $T_i(w) \subseteq F_i(w)$ for $w \in X$, $T_i(x)$ has convex values for each $x \in X$ and for each $y \in X_i$ note

$$\begin{aligned} T_i^{-1}(y) &= \{z \in X : \, y \in T_i(z)\} \\ &= \{z \in M_i : \, y \in A_i(z)\} \cup \{z \in X \backslash M_i : \, y \in S_i(z)\} \\ &= [M_i \cap \{z \in X : \, y \in A_i(z)\}] \cup [(X \backslash M_i) \cap \{z \in X : \, y \in S_i(z)\}] \\ &= \left[M_i \cap A_i^{-1}(y)\right] \cup \left[(X \backslash M_i) \cap S_i^{-1}(y)\right] \\ &= \left[M_i \cup S_i^{-1}(y)\right] \cap A_i^{-1}(y) \end{aligned}$$

(note $S_i^{-1}(y) \subseteq A_i^{-1}(y)$ since $S_i(z) \subseteq (A_i \cap P_i)(z)$ for $z \in X$) which is open in X. Next we claim for $i \in \{1,...,N\}$ that $x_i \notin F_i(x)$ for $x \in X$. To see this fix $i \in \{1,..,N\}$ and $x \in X$. First consider $x \in M_i$. Then $x_i \notin \overline{B_i}(x)$ so $x_i \notin A_i(x)$ from (2.2) i.e. $x_i \notin F_i(x)$ if $x \in M_i$. Next suppose $x \notin M_i$. Then $x_i \notin (A_i \cap P_i)(x) = F_i(x)$ from (2.3). Consequently $x_i \notin F_i(x)$ for $x \in X$ and $i \in \{1,....,N\}$.

Now let K and Y_i be as in the statement of Theorem 2.1. Note if $x \in X \backslash K$ then there exists a $j \in \{1,...,N\}$ with $S_j(x) \cap Y_j \neq \emptyset$, so if $x \in X \backslash K$ and $x \notin M_j$ then $T_j(x) \cap Y_j = S_j(x) \cap Y_j \neq \emptyset$ whereas if $x \in X \backslash K$ and $x \in M_j$ then $\emptyset \neq S_j(x) \cap Y_j \subseteq (A_j \cap P_j)(x) \cap Y_j \subseteq A_j(x) \cap Y_j = T_j(x) \cap Y_j$.

Then all the conditions in Theorem 1.1 are satisfied so there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1,...,N\}$. Note for each $i \in \{1,...,N\}$ that A_i has nonempty values (see (2.1)) so as a result we have $x \notin M_i$. Thus $x \notin M_i$ with $T_i(x) = \emptyset$ for all $i \in \{1,...,N\}$ i.e. $x_i \in \overline{B_i}(x)$ and $S_i(x) = \emptyset$ for $i \in \{1,...,N\}$. Now this together with (2.4) will yield $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in \{1,...,N\}$. □

Remark 2.1. (i). If $\{x \in X : \ S_i(x) \neq \emptyset\} = \{x \in X : \ (A_i \cap P_i)(x) \neq \emptyset\}$ for a $i \in \{1,...,N\}$ then (2.4) holds. Of course in Theorem 2.1 one could have (2.4) as

$$\begin{cases} \text{if } x \in X \ \text{ with } \ x_i \in \overline{B_i}(x) \ \text{ and } \ S_i(x) = \emptyset \ \text{ for all } \ i \in \{1,...,N\}, \\ \text{then } \ A_j(x) \cap P_j(x) = \emptyset \ \text{ for all } \ j \in \{1,...,N\}. \end{cases}$$

(ii). Note in the proof of Theorem 2.1 the assumption "$x_i \notin (A_i \cap P_i)(x)$ for $x \in X$" in (2.3) can be replaced by "$x_i \notin (A_i \cap P_i)(x)$ for $x \notin M_i$".

(iii). In (2.1) we assumed A_i has convex values but this can easily be removed if we replace A_i (in the appropriate places in the statement and proof) with $co\, A_i$.

This Remark is also true in the other results in this paper but we will not refer to it again. Next we consider the compact situation and we will use Theorem 1.2 to establish the existence of an equilibrium point of Γ.

Theorem 2.2. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i=1}^N$ be a N-person game i.e. $\{X_i\}_{i=1}^N$ is a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i=1}^N X_i$ paracompact, and for each $i \in \{1, ..., N\}$ the constraint correspondences A_i, $B_i : X \to E_i$, the preference correspondence $P_i : X \to E_i$ and $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ is upper semicontinuous. Also for each $i \in \{1, ..., N\}$ suppose* (2.1), (2.2), (2.3) *and* (2.4) *hold and in addition there is a compact convex subset K_i with $A_i(X) \subseteq K_i \subseteq X_i$. Then there exists an equilibrium point of Γ i.e. there exists a $x \in X$ with $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$.*

Proof. For $i \in \{1, ..., N\}$ let M_i, F_i and T_i be as in Theorem 2.1 and note $T_i(w) \subseteq F_i(w)$ for $w \in X$, $T_i(x)$ has convex values for each $x \in X$, $T_i^{-1}(y)$ is open (in X) for each $y \in X_i$ and $x_i \notin F_i(x)$ for $x \in X$. Let $i \in \{1, ..., N\}$ and let K_i be as in Theorem 2.2 and note $F_i(X) \subseteq A_i(X) \subseteq K_i \subseteq X_i$ (note if $x \notin M_i$ then $F_i(x) = A_i(x) \cap P_i(x) \subseteq A_i(x)$ whereas if $x \in M_i$ then $F_i(x) = A_i(x)$). Thus all the conditions in Theorem 1.2 are satisfied so there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$ and as in Theorem 2.1 (A_i has nonempty values) note $x \notin M_i$. Thus $x \notin M_i$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$ i.e. $x_i \in \overline{B_i}(x)$ and $S_i(x) = \emptyset$ for $i \in \{1, ..., N\}$ so from (2.4) we have $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$. □

Next we consider the condensing map situation where we will use Theorem 1.3.

Theorem 2.3. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i=1}^N$ be a N-person game i.e. $\{X_i\}_{i=1}^N$ is a family of convex sets each in a Hausdorff topological vector space E_i and for each $i \in \{1, ..., N\}$ the constraint correspondences A_i, $B_i : X \to E_i$, the preference correspondence $P_i : X \to E_i$ and $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ is upper semicontinuous. Also for each $i \in \{1, ..., N\}$ suppose* (2.1), (2.2), (2.3) *and* (2.4) *hold and in addition there is a compact convex subset K of X with $A(K) \subseteq K$ where $A(x) = \prod_{i=1}^N A_i(x)$ for $x \in X$. Then there exists a $x \in X$ with $x_{i_0} \in \overline{B_{i_0}}(x)$ and $A_{i_0}(x) \cap P_{i_0}(x) = \emptyset$ for some $i_0 \in \{1, ..., N\}$.*

Proof. For $i \in \{1, ..., N\}$ let M_i, F_i and T_i be as in Theorem 2.1 and note $T_i(w) \subseteq F_i(w)$ for $w \in X$, $T_i(x)$ has convex values for each $x \in X$, $T_i^{-1}(y)$ is open (in X) for each $y \in X_i$ and $x_i \notin F_i(x)$ for $x \in X$. Let $F(x) = \prod_{i=1}^N F_i(x)$ for $x \in X$ and since $F_i(x) \subseteq A_i(x)$ for $x \in X$ then $F(K) \subseteq A(K) \subseteq K$. Thus all the conditions in Theorem 1.3 are satisfied so there exists a $x \in X$ with $T_{i_0}(x) = \emptyset$ for some $i_0 \in \{1, ..., N\}$. Note $x \notin M_{i_0}$ (A_{i_0} has nonempty values) and $T_{i_0}(x) = \emptyset$ i.e. $x_{i_0} \in \overline{B_{i_0}}(x)$ and (see (2.4)) $A_{i_0}(x) \cap P_{i_0}(x) = \emptyset$. □

For Theorem 2.3 note in (2.3) we could replace "for each $i \in \{1, ..., N\}$ and for each $x \in X$ we have $x_i \notin (A_i \cap P_i)(x)$" with "for each $x \in X$ there exists a $i \in \{1, ..., N\}$ with $x_i \notin (A_i \cap P_i)(x)$" and this would guarantee that for each $x \in X$ there exists a $i \in \{1, ..., N\}$ with $x_i \notin F_i(x)$, which is all that is needed to apply

Theorem 1.3.

Next we generalize Theorem 2.1 by introducing another map. Before we present our result we first discuss a generalization of majorized type maps [3,6,8,14–17]. Let Z and W be convex sets in a Hausdorff topological vector space with Z paracompact. Suppose for each $y \in Z$ there exist maps $A_y : Z \to W$, $B_y : Z \to W$ and an open set U_y containing y with $B_y(z) \subseteq A_y(z)$ for every $z \in U_y$, B_y is convex valued and $(B_y)^{-1}(x)$ is open (in Z) for each $x \in W$. We claim that there exist maps $S : Z \to W$, $\phi : Z \to W$ with $S(z) \subseteq \phi(z)$ for $z \in Z$, S is convex valued and $S^{-1}(x)$ is open (in Z) for each $x \in W$. To see this note $\{U_y\}_{y\in Z}$ is an open covering of Z and since Z is paracompact there exists [4,5] a locally finite open covering $\{V_y\}_{y\in Z}$ of Z with $y \in V_y$ and $\Omega_y = \overline{V_y} \subseteq U_y$ for each $y \in Z$. Now for each $y \in Z$ let

$$Q_y(z) = \begin{cases} A_y(z), & z \in \Omega_y \\ W, & z \in Z\backslash\Omega_y \end{cases} \quad \text{and} \quad R_y(z) = \begin{cases} B_y(z), & z \in \Omega_y \\ W, & z \in Z\backslash\Omega_y. \end{cases}$$

Now R_y is convex valued and for any $x \in W$ note

$$\begin{aligned} &(R_y)^{-1}(x) \\ &= \{z \in Z : x \in R_y(z)\} \\ &= \{z \in Z\backslash\Omega_y : x \in R_y(z) = W\} \cup \{z \in \Omega_y : x \in R_y(z) = B_y(z)\} \\ &= (Z\backslash\Omega_y) \cup \{z \in \Omega_y : x \in B_y(z)\} = (Z\backslash\Omega_y) \cup [\Omega_y \cap \{z \in Z : x \in B_y(z)\}] \\ &= (Z\backslash\Omega_y) \cup \left[\Omega_y \cap B_y^{-1}(x)\right] = Z \cap \left[(Z\backslash\Omega_y) \cup B_y^{-1}(x)\right] = (Z\backslash\Omega_y) \cup B_y^{-1}(x) \end{aligned}$$

which is open in Z. Also for each $z \in Z$ note $R_y(z) \subseteq Q_y(z)$ (to see this note if $z \in \Omega_y$ then it is immediate since $\Omega_y \subseteq U_y$ whereas if $z \in Z\backslash\Omega_y$ then it is also immediate since $Q_y(z) = W = R_y(z)$). Let $S : Z \to W$ and $\phi : Z \to W$ be given by

$$\phi(z) = \bigcap_{y\in Z} Q_y(z) \quad \text{and} \quad S(z) = \bigcap_{y\in Z} R_y(z) \quad \text{for} \quad z \in Z.$$

Now S is convex valued and $S(z) \subseteq \phi(z)$ for every $z \in Z$. It remains to show $S^{-1}(x)$ is open for each $x \in W$. Fix $x \in W$ and let $u \in S^{-1}(x)$. We now claim there exists an open set W_u containing u with $u \in W_u \subseteq S^{-1}(x)$, so then as a result $S^{-1}(x)$ is open. To prove our claim note since $\{V_y\}_{y\in Z}$ is locally finite there exists an open neighborhood N_u of u (in Z) such that $\{y \in Z : N_u \cap V_y \neq \emptyset\} = \{y_1, ..., y_m\}$ (a finite set). Now if $y \notin \{y_1, ..., y_m\}$ then $\emptyset = V_y \cap N_u = \overline{V_y} \cap N_u = \Omega_y \cap N_u$ so $R_y(z) = W$ for all $z \in N_u$, and as a result

$$S(z) = \bigcap_{y\in Z} R_y(z) = \bigcap_{i=1}^{m} R_{y_i}(z) \quad \text{for all} \quad z \in N_u.$$

Now $S^{-1}(x) = \{z \in Z : x \in S(z)\}$ whereas

$$\{z \in N_u : x \in S(z)\} = \left\{z \in N_u : x \in \bigcap_{i=1}^{m} R_{y_i}(z)\right\} = N_u \cap \left[\cap_{i=1}^{m} (R_{y_i})^{-1}(x)\right]$$

so

$$u \in N_u \cap \left[\cap_{i=1}^{m} (R_{y_i})^{-1}(x)\right] \subseteq S^{-1}(x)$$

and our claim is true (note $N_u \cap \left[\cap_{i=1}^{m} (R_{y_i})^{-1}(x)\right]$ is an open neighborhood of u).

Theorem 2.4. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i=1}^N$ be a N-person game i.e. $\{X_i\}_{i=1}^N$ is a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i=1}^N X_i$ paracompact, and for each $i \in \{1, ..., N\}$ the constraint correspondences A_i, $B_i : X \to E_i$ and the preference correspondence $P_i : X \to E_i$. For each $i \in \{1, ..., N\}$ suppose $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ is upper semicontinuous,* (2.1), (2.2) *hold and also suppose the following:*

$$\begin{cases} \text{there exist maps } S_i,\, \phi_i : X \to X_i \text{ with } S_i(z) \subseteq \phi_i(z) \text{ for } z \in X, \\ S_i(x) \text{ is convex valued for each } x \in X,\ S_i^{-1}(z) \text{ is open (in } X) \\ \text{for each } z \in X_i \text{ and } x_i \notin \phi_i(x) \text{ for } x \in X, \end{cases} \tag{2.5}$$

$$S_i(x) \subseteq A_i(x) \text{ for } x \in X \tag{2.6}$$

and

$$\begin{cases} \text{if } x \in X \text{ with } x_i \in \overline{B_i}(x) \text{ and } S_i(x) = \emptyset \text{ for} \\ \text{a } i \in \{1, ..., N\}, \text{ then } A_i(x) \cap P_i(x) = \emptyset. \end{cases} \tag{2.7}$$

Finally suppose there exists a compact subset K of X and for each $i \in \{1, ..., N\}$ a convex compact set Y_i of X_i such that for each $x \in X \backslash K$ there exists a $j \in \{1, ..., N\}$ with $S_j(x) \cap Y_j \neq \emptyset$. Then there exists an equilibrium point of Γ i.e. there exists a $x \in X$ with $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$.

Proof. For $i \in \{1, ..., N\}$ let M_i be as in Theorem 2.1 and let $\Phi_i : X \to X_i$ and $T_i : X \to X_i$ be given by

$$\Phi_i(x) = \begin{cases} \phi_i(x), & x \notin M_i \\ A_i(x), & x \in M_i \end{cases} \quad \text{and} \quad T_i(x) = \begin{cases} S_i(x), & x \notin M_i \\ A_i(x), & x \in M_i. \end{cases}$$

For each $i \in \{1, ..., N\}$ note $T_i(w) \subseteq \Phi_i(w)$ for $w \in X$ (see (2.5)), $T_i(x)$ has convex values for each $x \in X$ and for each $y \in X_i$ note

$$\begin{aligned} T_i^{-1}(y) &= \{z \in M_i : y \in A_i(z)\} \cup \{z \in X \backslash M_i : y \in S_i(z)\} \\ &= [M_i \cap \{z \in X : y \in A_i(z)\}] \cup [(X \backslash M_i) \cap \{z \in X : y \in S_i(z)\}] \\ &= \left[M_i \cap A_i^{-1}(y)\right] \cup \left[(X \backslash M_i) \cap S_i^{-1}(y)\right] \\ &= \left[M_i \cup S_i^{-1}(y)\right] \cap A_i^{-1}(y) \end{aligned}$$

(note, see (2.6), $S_i^{-1}(y) \subseteq A_i^{-1}(y)$) which is open in X. Next we claim for $i \in \{1, ..., N\}$ that $x_i \notin \Phi_i(x)$ for $x \in X$. To see this fix $i \in \{1, .., N\}$ and $x \in X$. First consider $x \in M_i$ and then $x_i \notin \overline{B_i}(x)$ so $x_i \notin A_i(x)$ from (2.2) i.e. $x_i \notin \Phi_i(x)$ if $x \in M_i$. Next suppose $x \notin M_i$ and then $x_i \notin \phi_i(x) = \Phi_i(x)$ from (2.5). Consequently $x_i \notin \Phi_i(x)$ for $x \in X$ and $i \in \{1, ..., N\}$. Now let K and Y_i be as in the statement of Theorem 2.4. Note if $x \in X \backslash K$ then there exists a $j \in \{1, ..., N\}$ with $S_j(x) \cap Y_j \neq \emptyset$, so if $x \in X \backslash K$ and $x \notin M_j$ then $T_j(x) \cap Y_j = S_j(x) \cap Y_j \neq \emptyset$ whereas if $x \in X \backslash K$ and $x \in M_j$ then $\emptyset \neq S_j(x) \cap Y_j \subseteq A_j(x) \cap Y_j = T_j(x) \cap Y_j$ from (2.6).

Then all the conditions in Theorem 1.1 are satisfied so there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$. Note as in Theorem 2.1 (A_i has nonempty values) for each $i \in \{1, ..., N\}$ we have $x \notin M_i$ so $x \notin M_i$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$ i.e. $x_i \in \overline{B_i}(x)$ and $S_i(x) = \emptyset$ for $i \in \{1, ..., N\}$. Now from (2.7) we have $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$. □

Remark 2.2. Suppose for each $i \in \{1, ..., N\}$ and for each $x \in X$ there exist maps $A_{i,x}, B_{i,x} : X \to X_i$ and an open set $U_{i,x}$ containing x with $B_{i,x}(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $B_{i,x}$ is convex valued, $(B_{i,x})^{-1}(z)$ is open (in X) for each $z \in X_i$ and $w_i \notin A_{i,x}(w)$ for each $w \in U_{i,x}$. From the discussion before Theorem 2.4 (with $Z = X$, $W = X_i$) there exist maps $S_i : X \to X_i$ and $\phi_i : X \to X_i$ with $S_i(w) \subseteq \phi_i(w)$ for $w \in X$, S_i is convex valued and $(S_i)^{-1}(z)$ is open for each $z \in X_i$; here

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in \Omega_{i,x} \\ X_i, & z \in X \backslash \Omega_{i,x} \end{cases} \quad \text{and} \quad R_{i,x}(z) = \begin{cases} B_{i,x}(z), & z \in \Omega_{i,x} \\ X_i, & z \in X \backslash \Omega_{i,x} \end{cases}$$

and

$$\phi_i(z) = \bigcap_{x \in X} Q_{i,x}(z) \quad \text{and} \quad S_i(z) = \bigcap_{x \in X} R_{i,x}(z) \quad \text{for} \quad z \in X$$

where $\{V_{i,x}\}_{x \in X}$ is a locally finite open covering of X with $x \in V_{i,x}$ and $\Omega_{i,x} = \overline{V_{i,x}} \subseteq U_{i,x}$ for each $x \in X$. Now let $i \in \{1, ..., N\}$ and $w \in X$. Note there exists a $y \in X$ with $w \in \Omega_{i,y}$ (recall $\{V_{i,x}\}_{x \in X}$ is a locally finite open covering of X) so $\phi_i(w) = \bigcap_{x \in X} Q_{i,x}(w) \subseteq Q_{i,y}(w) = A_{i,y}(w)$ and since $w_i \notin A_{i,y}(w)$ for each $w \in U_{i,y}$ we have $w_i \notin \phi_i(w)$ for $w \in X$. Thus (2.5) holds in this situation.

In the above we additionally assume for each $i \in \{1, ..., N\}$ that $B_{i,x}(z) \subseteq A_i(z)$ for every $z \in U_{i,x}$. Now let $i \in \{1, ..., N\}$ and $w \in X$. Note there exists a $y \in X$ with $w \in \Omega_{i,y}$ so $S_i(w) = \bigcap_{x \in X} R_{i,x}(w) \subseteq R_{i,y}(w) = B_{i,y}(w) \subseteq A_i(w)$. Thus (2.6) holds in this situation.

Theorem 2.5. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i=1}^N$ be a N-person game i.e. $\{X_i\}_{i=1}^N$ is a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i=1}^N X_i$ paracompact, and for each $i \in \{1, ..., N\}$ the constraint correspondences $A_i, B_i : X \to E_i$ and the preference correspondence $P_i : X \to E_i$. For each $i \in \{1, ..., N\}$ suppose $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ is upper semicontinuous,* (2.1), (2.2), (2.5), (2.6), (2.7) *hold and also assume there is a compact convex subset K_i with $A_i(X) \subseteq K_i \subseteq X_i$ and $\phi_i(X) \subseteq K_i$. Then there exists an equilibrium point of Γ i.e. there exists a $x \in X$ with $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$.*

Proof. For $i \in \{1, ..., N\}$ let M_i, Φ_i and T_i be as in Theorem 2.4 and note $T_i(w) \subseteq \Phi_i(w)$ for $w \in X$, $T_i(x)$ has convex values for each $x \in X$, $T_i^{-1}(y)$ is open (in X) for each $y \in X_i$ and $x_i \notin \Phi_i(x)$ for $x \in X$. Let $i \in \{1, ..., N\}$ and $\Phi_i(X) \subseteq K_i$ (note if $x \notin M_i$ then $\Phi_i(x) = \phi_i(x)$ whereas if $x \in M_i$ then $\Phi_i(x) = A_i(x)$). Thus all the conditions in Theorem 1.2 are satisfied so there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$ and as in Theorem 2.1 (A_i has nonempty values) note $x \notin M_i$ i.e. $x \notin M_i$ with $T_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$ i.e. $x_i \in \overline{B_i}(x)$ and $S_i(x) = \emptyset$ for $i \in \{1, ..., N\}$ so from (2.7) we have $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in \{1, ..., N\}$. □

Theorem 2.6. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i=1}^N$ be a N-person game i.e. $\{X_i\}_{i=1}^N$ is a family of convex sets each in a Hausdorff topological vector space E_i with $X =$*

$\prod_{i=1}^{N} X_i$ *paracompact, and for each* $i \in \{1, ..., N\}$ *the constraint correspondences* A_i, $B_i : X \to E_i$ *and the preference correspondence* $P_i : X \to E_i$. *For each* $i \in \{1, ..., N\}$ *suppose* $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ *is upper semicontinuous,* (2.1), (2.2), (2.5), (2.6), (2.7) *hold and also assume there is a compact convex subset* K *of* X *with* $A(K) \subseteq K$ *and* $\phi(K) \subseteq K$ *where* $A(x) = \prod_{i=1}^{N} A_i(x)$ *and* $\phi(x) = \prod_{i=1}^{N} \phi_i(x)$ *for* $x \in X$. *Then there exists a* $x \in X$ *with* $x_{i_0} \in \overline{B_{i_0}}(x)$ *and* $A_{i_0}(x) \cap P_{i_0}(x) = \emptyset$ *for some* $i_0 \in \{1, ..., N\}$.

Proof. For $i \in \{1, ..., N\}$ let M_i, Φ_i and T_i be as in Theorem 2.4 and note $T_i(w) \subseteq \Phi_i(w)$ for $w \in X$, $T_i(x)$ has convex values for each $x \in X$, $T_i^{-1}(y)$ is open (in X) for each $y \in X_i$ and $x_i \notin \Phi_i(x)$ for $x \in X$. Also note $\Phi(K) \subseteq K$ where $\Phi(x) = \prod_{i=1}^{N} \Phi_i(x)$ for $x \in X$ (note if $x \notin M_i$ then $\Phi_i(x) = \phi_i(x)$ whereas if $x \in M_i$ then $\Phi_i(x) = A_i(x)$). Thus all the conditions in Theorem 1.3 are satisfied so there exists a $x \in X$ with $T_{i_0}(x) = \emptyset$ for some $i_0 \in \{1, ..., N\}$. Note $x \notin M_{i_0}$ (A_{i_0} has nonempty values) and $T_{i_0}(x) = \emptyset$ i.e. $x_{i_0} \in \overline{B_{i_0}}(x)$ and (see (2.7)) $A_{i_0}(x) \cap P_{i_0}(x) = \emptyset$. □

For Theorem 2.6 note in (2.5) we could replace "for each $i \in \{1, ..., N\}$ and for each $x \in X$ we have $x_i \notin \phi_i(x)$" with "for each $x \in X$ there exists a $i \in \{1, ..., N\}$ with $x_i \notin \phi_i(x)$" and this would guarantee that for each $x \in X$ there exists a $i \in \{1, ..., N\}$ with $x_i \notin \Phi_i(x)$, which is all that is needed to apply Theorem 1.3.

Remark 2.3. It is also possible to first prove Theorem 2.3 and Theorem 2.6 when $N = 1$ (we call these Corollary 2.1 and Corollary 2.2) and then the case of general N can be established by considering

$$A(x) = \prod_{i=1}^{N} A_i(x), \;\; B(x) = \prod_{i=1}^{N} B_i(x) \;\; \text{and} \;\; P(x) = \prod_{i=1}^{N} P_i(x) \;\; \text{for} \;\; x \in X$$

and using Corollary 2.1 and Corollary 2.2 (we refer the reader to [11] to illuminate this strategy).

References

[1] M. U. Ali and A. Pitea, *Existence theorem for integral inclusions by a fixed point theorem for multivalued implicit–type contractive maps*, Nonlinear Analysis: Modelling and Control, 2021, 26, 334–348.

[2] M. Balaj, *Existence results for quasi–equilibrium problems under a weaker equilibrium condition*, Operation Research Letters, 2021, 49, 333–337.

[3] X. Ding and K. Tan, *A minimax inequality with applications to existence of equilibrium point and fixed point theorems*, Colloquium Math., 1992, 63, 233–247.

[4] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[5] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[6] C. D. Horvath, *Contractibility and generalized convexity*, Jour. Math. Anal. Appl., 1991, 156, 341–357.

[7] A. Latif, R. F. Al Subaie and M. O. Alansari, *Fixed points of generalized multivalued contractive mappings in metric type spaces*, Journal of Nonlinear and Variational Analysis, 2022, 6, 123–138.

[8] L. J. Lim, S. Park and Z. Yu, *Remarks on fixed points, maximal elements and equilibria of generalized games*, Jour. Math. Anal. Appl., 1999, 233, 581–596.

[9] D. O'Regan, *Collectively fixed point theory in the compact and coercive cases*, Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Mathematica, 2022, 30(2), 193–207.

[10] D. O'Regan, *Maximal elements and equilibria for generalized majorized maps*, Nonlinear Analysis: Modelling and Control, 2023, 28, 116–132.

[11] D. O'Regan, *Existence of equilibria for N–person games*, Fixed Point Theory, to appear.

[12] A. Petrusel and G. Petrusel, *Common fixed point theory for a pair of multivalued operators*, Carpathian Jour. Math., 2022, 38, 707–714.

[13] I. A. Rus, *Sets with structure, mappings and fixed point property: Fixed point structures*, Fixed Point Theory, 2022, 23, 689–706.

[14] N. C. Yanelis and N. D. Prabhakar, *Existence of maximal elements and equlibria in linear topological spaces*, J. Math. Econom., 1983, 12, 233–245.

[15] X. Yuan and E. Tarafdar, *Maximal elements and equilibria of generalized games for condensing correspondences*, Jour. Math. Anal. Appl., 1996, 203, 13–30.

[16] X. Yuan and E. Tarafdar, *Existence of equilibria of generalized games without compactness and paracompactness*, Nonlinear Analysis, 1996, 26, 893–902.

[17] X. Yuan, *The study of equilibria for abstract economies in topological vector spaces-a unified approach*, Nonlinear Analysis, 1999, 37, 409–430.