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QUALITATIVE ANALYSIS OF A DIFFUSIVE COVID-19 MODEL WITH NON-MONOTONE INCIDENCE RATE*

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Abstract The paper is concerned with a diffusive susceptible-asymptomatic-infected-recovered-type COVID-19 model with non-monotone incidence rate and homogeneous zero-flux boundary conditions. First the boundedness results of the diffusive COVID-19 model are established by the technique of the comparison principle of the parabolic equations. Then, we turn our attention to the corresponding elliptic equations. A priori estimates of the solutions are given, some properties of the positive steady states and nonexistence conditions of the positive steady states are presented by energy estimates. It is found that the diffusion rate of the proposed diffusive COVID-19 model could affect the existence of the nonconstant steady states. These qualitative results will give some theoretical insights into the diffusive COVID-19 model with non-monotone incidence rate.

Keywords Diffusive COVID-19 model, non-monotone incidence rate, nonconstant steady states.

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1. Introduction

Various epidemic models have been proposed and investigated by many scholars to perform the dynamics of disease transmissions and they have obtained rich dynamical results. For instance, Liu et al. [10] showed that the stochastic nonautonomous epidemic model admits at least one nontrivial positive T-periodic solution by investigating an SEIR epidemic model with distributed delay in random environments. An and Song [1] studied a spatial susceptible-infected-susceptible (SIS) model in heterogeneous environments with vary advective rates, and their results showed that the unique disease-free equilibrium (DFE) is asymptotic stable when the basic reproduction number $\mathcal{R}_0 < 1$, and there is an endemic equilibrium as $\mathcal{R}_0 > 1$. The stability and the Hopf bifurcation were performed in [4] by employing a SIR model with the age structure of infected individuals. The asymptotic profiles of

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the endemic equilibrium for small or large mortality rate and large saturation rate are analyzed of a SIRS reaction-diffusion model by Liu and Cui in [11]. For more results about epidemic models, one can refer to Refs. [5, 14, 20, 22], and so on.

Nowadays, people are suffering from COVID-19 all over the world, and it has seriously affected people's daily life, economic and social developments. Statistics is a valid tool to collect various information about COVID-19. For example, count the number of people who are infected or have recovered so that one can provide some information to local government departments to reduce or prevent the infection of COVID-19. Nevertheless, mathematical modeling based on coupled dynamical equations can provide a more detailed analysis and prediction for the epidemic transmission, compared with the statistics method. Indeed, some dynamic models and mathematical results with respect to COVID-19 have been obtained. The stability analysis and the global bifurcation analysis of a COVID-19 transmission epidemiological model have been studied in [6]. The stochastic basic reproduction number, the dynamic properties around the disease-free equilibrium, and the endemic equilibrium of a Levy noise-driven susceptible-exposed-infected-recovered model to study the outbreak of COVID-19 are analyzed in [9]. A mathematical model for COVID-19 which incorporates multiple transmission pathways has been proposed by Yang and Wang [21], and their results indicated that the environmental reservoirs have an important influence on the transmission and spread of the coronavirus. The basic reproduction number, phase portraits, and bifurcation diagrams of a fractional-order Susceptible-Exposed-Infected-Hospitalized-Recovered (SEIHR) model for COVID-19 were presented in [18]. Early warning indicators were employed to predict the bifurcation points in the system and six cases of the subgroups interactions were studied in [16] of a COVID-19 model. For more models and results about COVID-19, one can refer to [3, 15, 17, 19] and reference therein.

Although there are a lot of mathematical models about COVID-19, it is found that most of the models are governed by coupled ordinary differential equations. As it is well known, whether they are infected, recovering, or susceptible, the random movements of individuals from different compartments in their surroundings always occur. This implies introducing the diffusion in a COVID-19 model may be more reasonable to describe the transmission of disease. Very recently, Ahmed et al. [2] proposed a diffusive COVID-19 model as follows

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S \Delta S = \mu - \dfrac{\beta S I}{1+\alpha I^2} - \mu S, & x \in \Omega,\ t > 0, \\ \dfrac{\partial A}{\partial t} - d_A \Delta A = \dfrac{\beta S I}{1+\alpha I^2} - (\sigma + \delta + \epsilon + \mu) A, & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial t} - d_I \Delta I = \sigma A - (\gamma + d + \mu) I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial R}{\partial t} - d_R \Delta R = \gamma I + \epsilon A - \mu R, & x \in \Omega,\ t > 0, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial A}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = \dfrac{\partial R}{\partial \nu} = 0, & x \in \partial\Omega,\ t \geq 0, \\ S(x,0) = S_0(x) \geq 0, \quad A(x,0) = A_0(x) \geq 0, & x \in \Omega \\ I(x,0) = I_0(x) \geq 0, \quad R(x,0) = R_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $S = S(x,t), A = A(x,t), I = I(x,t)$ and $R = R(x,t)$ are the susceptible, the asymptomatic, the infected, and the recovered or quarantine humans at spatial position x and time t, respectively. Parameter μ is the natural birth or death rate, constant d is the death rate of the infected since virus infection, the immunity rate of the asymptomatic is described by ϵ, constant α is the bilinear incidence rate constant, constant σ denotes the rate that the asymptomatic move to the infected, parameter δ describes the mortality rate of the asymptomatic induced by the virus, the rate of vaccination, quarantine or treatment is defined by γ, and β denotes the rate that the susceptible move to the asymptomatic, one can also find these notations in Ref. [2]. Moreover, Δ is the Laplacian operator in a bounded spatial domain $\Omega \subset \mathbb{R}^N (N \geq 1)$ with the smooth boundary $\partial\Omega$, ν is the outward unit normal vector along the boundary $\partial\Omega$, d_S, d_A, d_I and d_R are the diffusion rates of the susceptible, the asymptomatic, the infected, and the recovered, or quarantine humans, respectively. We assume that all parameters in model (1.1) are positive constant, and the diffusion model with the non-negative initial conditions $S_0(x) \geq 0, A_0(x) \geq 0, I_0(x) \geq 0, R_0(x) \geq 0$ for any $x \in \Omega$. The term $\frac{\beta I}{1+\alpha I^2}$ is the non-monotone incidence rate of the COVID-19 outbreak. Also, we plot a scheme diagram in Fig. 1 to better understand the scheme of establishing the model (1.1) as diffusion is disappeared.

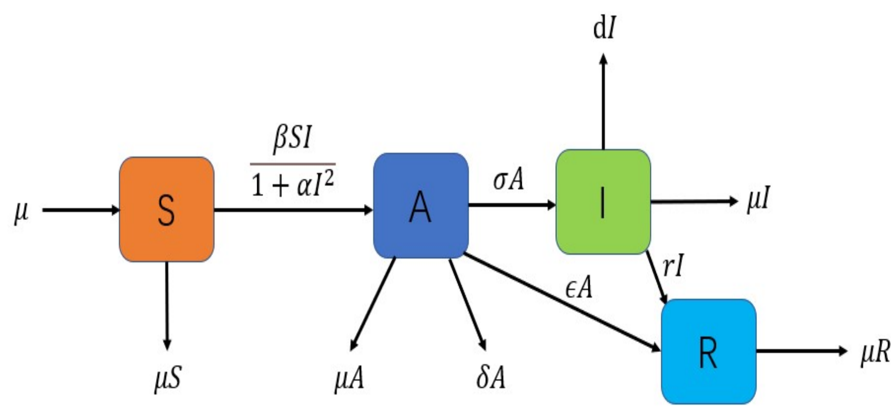

Figure 1. Scheme diagram of the COVID-19 model (1.1) as diffusion is disappeared.

For the COVID-19 model (1.1), the authors in [2] investigated the existence and the stability of the disease-free equilibrium and the endemic equilibrium when diffusion is absent. Precisely, the endemic equilibrium exists if $\mathcal{R}_0 > 1$, and its asymptotic stability conditions can be yielded by employing the Routh-Hurwitz argument. The disease-free equilibrium is stable when $\mu > 0, \sigma+\delta+\epsilon+2\mu+d+\gamma > 0$ and $\mathcal{R}_0 < 1$ hold, where $\mathcal{R}_0$ represents the basic reproduction number and takes the form

$$\mathcal{R}_0 = \frac{\sigma\beta}{(\sigma + \delta + \epsilon + \mu)(\gamma + d + \mu)}.$$

Note that only numerical experiments are performed about the diffusive COVID-19 model (1.1) in [2]. Therefore, we will give some qualitative analysis results about the diffusive COVID-19 model (1.1) with the homogeneous zero-flux boundary conditions, more precisely, the boundedness, including the uniform boundedness of the parabolic equation (1.1), and the properties, the nonexistence of the nonconstant steady states of the corresponding elliptic equations, respectively. One believes that these obtained qualitative analysis results provides the theoretical insights into the COVID-19 model (1.1) with the non-monotone incidence rate.

The outline of this paper is designed as follows. In Sect. 2, the boundedness results of solutions to diffusive COVID-19 model (1.1) are given. In Sect. 3, a priori estimates, the properties and the nonexistence of the non-constant steady states of the elliptic equation are presented. Some conclusions are drawn in Sect. 4.

2. Boundedness

In this section, we want to establish the boundedness results of the solution (S, A, I, R) to the diffusive COVID-19 model (1.1).

Theorem 2.1. *Suppose that $d_S = d_A = d_I = d_R$ is valid, then for any solution (S, A, I, R) of system (1.1), it satisfies*

$$S + A + I + R \leq \max\left\{1, S_0(x) + A_0(x) + I_0(x) + R_0(x)\right\}, \tag{2.1}$$

for any $x \in \overline{\Omega}$.

Proof. Let $d_S = d_A = d_I = d_R = D$ and define $W = S + A + I + R$. In view of system (1.1), one yields

$$\begin{aligned}\frac{\partial W}{\partial t} - D\Delta W &= \mu - \mu S - (\mu + \delta)A - (d + \mu)I - \mu R \\ &\leq \mu - \mu S - \mu A - \mu I - \mu R \\ &= \mu - \mu W.\end{aligned}$$

Therefore, it is easy to check that W and

$$\max\left\{1, S_0(x) + A_0(x) + I_0(x) + R_0(x)\right\},$$

are lower and upper solutions to the following problem

$$\begin{cases}\dfrac{\partial w_1}{\partial t} - D\Delta w_1(x,t) = \mu - \mu w_1(x,t), & x \in \Omega,\ t > 0,\\ \dfrac{\partial w_1}{\partial \mathbf{n}} = 0, & x \in \partial\Omega,\ t \geq 0,\\ w_1(x,0) = S_0(x) + A_0(x) + I_0(x) + R_0(x) \geq 0, & x \in \Omega.\end{cases}$$

Thereby the comparison principle of parabolic equations shows that

$$S + A + I + R = W \leq \max\left\{1, S_0(x) + A_0(x) + I_0(x) + R_0(x)\right\},$$

for $x \in \overline{\Omega}, t \geq 0$. This finishes the proof. □

Theorem 2.2. *Suppose that $d_S, d_A, d_I, d_R > 0$, then for any solution (S, A, I, R) of system (1.1), it holds*

$$\limsup_{t\to\infty} \max_{x\in\overline{\Omega}} S(\cdot, t) \leq 1,$$

$$\limsup_{t\to\infty}\max_{x\in\overline{\Omega}} A(\cdot,t) \le \frac{\beta}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)},$$
$$\limsup_{t\to\infty}\max_{x\in\overline{\Omega}} I(\cdot,t) \le \frac{\sigma\beta}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)(\gamma+d+\mu)},$$
$$\limsup_{t\to\infty}\max_{x\in\overline{\Omega}} R(\cdot,t) \le \frac{\gamma\sigma\beta+\epsilon\beta(\gamma+d+\mu)}{2\mu\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)(\gamma+d+\mu)}.$$

Proof. From (1.1), we can find that the $S-$equation satisfies

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S\Delta S \le \mu - \mu S, & x\in\Omega,\ t>0, \\ \dfrac{\partial S}{\partial \nu} = 0, & x\in\partial\Omega,\ t\ge 0, \\ S(x,0)=S_0(x)\ge 0, & x\in\Omega, \end{cases}$$

By employing the comparison principle, we infer that there are $\varepsilon_1>0$ and $T_1>0$ such that $S(x,t)\le 1+\varepsilon_1$ for $\forall x\in\overline{\Omega}$ and $t>T_1$. Then from the $A-$equation of (1.1), one obtains

$$\begin{cases} \dfrac{\partial A}{\partial t} - d_A\Delta A \le \dfrac{\beta(1+\varepsilon_1)}{2\sqrt{\alpha}} - (\sigma+\delta+\epsilon+\mu)A, & x\in\Omega,\ t>0, \\ \dfrac{\partial A}{\partial \nu} = 0, & x\in\partial\Omega,\ t\ge T_1, \\ A(x,0)=A_0(x)\ge 0, & x\in\Omega. \end{cases}$$

We can obtain $A(x,t)\le \frac{\beta(1+\varepsilon_1)}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)}+\varepsilon_2$ for $\forall x\in\overline{\Omega}$, $\varepsilon_2>0$ and $t>T_2$, due to the comparison principle. Now using the $I-$equation of (1.1), we get

$$\begin{cases} \dfrac{\partial I}{\partial t} - d_I\Delta I \le \dfrac{\sigma\beta(1+\varepsilon_1)+2\varepsilon_2\sigma\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)} - (\gamma+d+\mu)I, & x\in\Omega,\ t>0, \\ \dfrac{\partial I}{\partial \nu} = 0, & x\in\partial\Omega,\ t\ge T_2, \\ I(x,0)=I_0(x)\ge 0, & x\in\Omega. \end{cases}$$

As such, the comparison principle again shows that

$$I(x,t)\le \frac{\sigma\beta(1+\varepsilon_1)+2\varepsilon_2\sigma\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)(\gamma+d+\mu)}+\varepsilon_3$$

for $\forall x\in\overline{\Omega}$, $\varepsilon_3>0$ and $t>T_3$. Hereafter, from $R-$equation of (1.1) and the comparison principle, we deduce

$$\begin{cases} \dfrac{\partial R}{\partial t} - d_R\Delta R \le \gamma\left(\dfrac{\sigma\beta(1+\varepsilon_1)+2\varepsilon_2\sigma\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)(\gamma+d+\mu)}+\varepsilon_3\right) \\ \qquad\qquad + \epsilon\left(\dfrac{\beta(1+\varepsilon_1)}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)}+\varepsilon_2\right) - \mu R, \\ \dfrac{\partial R}{\partial \nu} = 0, \\ R(x,0)=R_0(x)\ge 0, \end{cases}$$

for $t \geq T_3$. We can obtain

$$\begin{aligned}R(x,t) \leq & \frac{\gamma}{\mu}\left(\frac{\sigma\beta(1+\varepsilon_1)+2\varepsilon_2\sigma\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)(\gamma+d+\mu)}+\varepsilon_3\right)\\ &+\frac{\epsilon}{\mu}\left(\frac{\beta(1+\varepsilon_1)}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)}+\varepsilon_2\right)+\varepsilon_4\end{aligned}$$

for $\forall x \in \overline{\Omega}$ and $\varepsilon_4 > 0, t > T_4$. Taking $T = \max\{T_1, T_2, T_3, T_4\}$ and noting that $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are arbitrary constants, the proof is completed. $\square$

Theorem 2.3. *Suppose that $d_S, d_A, d_I, d_R > 0$, then the following inequalities are valid for any solution (S, A, I, R) of system (1.1).*

(i) There exists a positive constant c_1 depending on the initial condition such that (S, A, I, R) fulfills

$$\|S(\cdot,t)\|_{L^\infty(\Omega)}+\|A(\cdot,t)\|_{L^\infty(\Omega)}+\|I(\cdot,t)\|_{L^\infty(\Omega)}+\|R(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1,\ \forall t \geq 0. \tag{2.2}$$

(ii) There exists a positive constant c_2 independent of the initial condition such that solution (S, A, I, R) satisfies

$$\|S(\cdot,t)\|_{L^\infty(\Omega)}+\|A(\cdot,t)\|_{L^\infty(\Omega)}+\|I(\cdot,t)\|_{L^\infty(\Omega)}+\|R(\cdot,t)\|_{L^\infty(\Omega)} \leq c_2,\ \forall t \geq T, \tag{2.3}$$

for some large $T > 0$.

Proof. To check that the validity of (2.2) and (2.3), we first verify that

$$\|S(\cdot,t)\|_{L^k(\Omega)}+\|A(\cdot,t)\|_{L^k(\Omega)}+\|I(\cdot,t)\|_{L^k(\Omega)}+\|R(\cdot,t)\|_{L^k(\Omega)} \leq c_1,\ \forall t \geq 0, \tag{2.4}$$

is true. Here the basic technique to deduce (2.4) is mathematical induction. Then for $k = 1$ and system (1.1), one has

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (S+A+I+R)dx\\ =&d_S\int_\Omega \Delta S dx + d_A\int_\Omega \Delta A dx + d_I\int_\Omega \Delta I dx + d_R\int_\Omega \Delta R dx\\ &+\int_\Omega \mu dx - \int_\Omega \mu S dx - \int_\Omega (\mu+\delta)A dx - \int_\Omega (d+\mu)I dx - \int_\Omega \mu R dx\\ =&\int_\Omega \mu dx - \int_\Omega \mu S dx - \int_\Omega (\mu+\delta)A dx - \int_\Omega (d+\mu)I dx - \int_\Omega \mu R dx\\ \leq&\mu|\Omega| - \mu\int_\Omega (S+A+I+R)dx.\end{aligned}$$

It follows that

$$\int_\Omega (S+A+I+R)dx \leq e^{-\mu t}\int_\Omega (S_0(x)+A_0(x)+I_0(x)+R_0(x))dx + |\Omega|(1-e^{-\mu t}),$$

for $\forall x \in \overline{\Omega},\ t \geq 0$. It implies that

$$\limsup_{t\to\infty}\int_\Omega (S+A+I+R)dx \leq |\Omega|,\quad \forall x \in \overline{\Omega}, \tag{2.5}$$

namely, (2.4) is valid as $k = 1$. In what follows, we assume that (2.4) is true for $k-1$. Multiplying the S-equation of system (1.1) by S^{k-1}, one has

$$\frac{1}{k}\frac{d}{dt}\int_\Omega S^k dx + d_S(k-1)\int_\Omega S^{k-2}|\nabla S|^2 dx = \int_\Omega \left[\mu S^{k-1} - \frac{\beta S^k I}{1+\alpha I^2} - \mu S^k\right] dx. \tag{2.6}$$

Multiplying the A-equation of system (1.1) by A^{k-1}, we have

$$\frac{1}{k}\frac{d}{dt}\int_\Omega A^k dx + d_A(k-1)\int_\Omega A^{k-2}|\nabla A|^2 dx = \int_\Omega \left[\frac{\beta S I A^{k-1}}{1+\alpha I^2} - (\sigma+\delta+\epsilon+\mu)A^k\right] dx. \tag{2.7}$$

By the same manner, multiplying the I-equation of system (1.1) by I^{k-1}, we get

$$\frac{1}{k}\frac{d}{dt}\int_\Omega I^k dx + d_I(k-1)\int_\Omega I^{k-2}|\nabla I|^2 dx = \int_\Omega \left[\sigma A I^{k-1} - (\gamma+d+\mu)I^k\right] dx. \tag{2.8}$$

Similarly, multiplying the R-equation of system (1.1) by R^{k-1}, we get

$$\frac{1}{k}\frac{d}{dt}\int_\Omega R^k dx + d_R(k-1)\int_\Omega R^{k-2}|\nabla R|^2 dx = \int_\Omega \left[\gamma I R^{k-1} + \epsilon A R^{k-1} - \mu R^k\right] dx. \tag{2.9}$$

On the basis of (2.6)-(2.9), one yields

$$\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega S^k dx + d_S(k-1)\int_\Omega S^{k-2}|\nabla S|^2 dx + \frac{1}{k}\frac{d}{dt}\int_\Omega A^k dx \\
&+ d_A(k-1)\int_\Omega A^{k-2}|\nabla A|^2 dx + \frac{1}{k}\frac{d}{dt}\int_\Omega I^k dx + d_I(k-1)\int_\Omega I^{k-2}|\nabla I|^2 dx \\
&+ \frac{1}{k}\frac{d}{dt}\int_\Omega R^k dx + d_R(k-1)\int_\Omega R^{k-2}|\nabla R|^2 dx \\
=&\mu\int_\Omega S^{k-1}dx - \mu\int_\Omega S^k dx + \int_\Omega \frac{\beta S I}{1+\alpha I^2}(A^{k-1} - S^{k-1})dx \\
&- (\sigma+\delta+\epsilon+\mu)\int_\Omega A^k dx + \sigma\int_\Omega A I^{k-1}dx - (\gamma+d+\mu)\int_\Omega I^k dx \\
&+ \gamma\int_\Omega I R^{k-1}dx + \epsilon\int_\Omega A R^{k-1}dx - \mu\int_\Omega R^k dx \\
\le&\mu\int_\Omega S^{k-1}dx - \mu\int_\Omega S^k dx + \frac{\beta}{2\sqrt{\alpha}}\int_\Omega S A^{k-1}dx - (\sigma+\delta+\epsilon+\mu)\int_\Omega A^k dx \\
&+ \sigma\int_\Omega A I^{k-1}dx - (\gamma+d+\mu)\int_\Omega I^k dx + \gamma\int_\Omega I R^{k-1}dx \\
&+ \epsilon\int_\Omega A R^{k-1}dx - \mu\int_\Omega R^k dx \\
\le&\mu\int_\Omega S^{k-1}dx - \mu\int_\Omega S^k dx + \frac{\beta}{2\sqrt{\alpha}}\int_\Omega (\varepsilon_1 S^k + C(\varepsilon_1)A^k)dx \\
&- (\sigma+\delta+\epsilon+\mu)\int_\Omega A^k dx + \sigma\int_\Omega (\varepsilon_2 A^k + C(\varepsilon_2)I^k)dx - (\gamma+d+\mu)\int_\Omega I^k dx \\
&+ \gamma\int_\Omega (\varepsilon_3 I^k + C(\varepsilon_3)R^k)dx + \epsilon\int_\Omega (\varepsilon_4 A^k + C(\varepsilon_4)R^k)dx - \mu\int_\Omega R^k dx \\
=&\mu\int_\Omega S^{k-1}dx - \left(\mu - \frac{\varepsilon_1\beta}{2\sqrt{\alpha}}\right)\int_\Omega S^k dx - \left(\sigma+\delta+\epsilon+\mu - \frac{\beta C(\varepsilon_1)}{2\sqrt{\alpha}} - \sigma\varepsilon_2 - \epsilon\varepsilon_4\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \int_\Omega A^k dx - (\gamma + d + \mu - \gamma\varepsilon_3 - \sigma C(\varepsilon_2)) \int_\Omega I^k dx \\
&- (\mu - \gamma C(\varepsilon_3) - \epsilon C(\varepsilon_4)) \int_\Omega R^k dx \\
\le& \mu \int_\Omega S^{k-1} dx - C_0 \int_\Omega (S^k + A^k + I^k + R^k) dx,
\end{aligned}
$$

where one employs the well-known ϵ-Young inequality $ab \le \varepsilon a^p + C(\varepsilon) b^q$ with $q = \frac{k}{k-1}, p = k$, and denote by

$$
\begin{aligned}
C_0 :=& k \min \Big\{ \mu - \frac{\varepsilon_1 \beta}{2\sqrt{\alpha}}, \sigma + \delta + \epsilon + \mu - \frac{\beta C(\varepsilon_1)}{2\sqrt{\alpha}} - \sigma\varepsilon_2 - \epsilon\varepsilon_4, \\
& \gamma + d + \mu - \gamma\varepsilon_3 - \sigma C(\varepsilon_2) \mu - \gamma C(\varepsilon_3) - \epsilon C(\varepsilon_4) \}
\end{aligned}
$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, C(\varepsilon_1), C(\varepsilon_2), C(\varepsilon_3)$ and $C(\varepsilon_4)$ such that $\mu - \frac{\varepsilon_1\beta}{2\sqrt{\alpha}} > 0, \sigma + \delta + \epsilon + \mu - \frac{\beta C(\varepsilon_1)}{2\sqrt{\alpha}} - \sigma\varepsilon_2 - \epsilon\varepsilon_4 > 0, \gamma + d + \mu - \gamma\varepsilon_3 - \sigma C(\varepsilon_2) > 0$ and $\mu - \gamma C(\varepsilon_3) - \epsilon C(\varepsilon_4) > 0$ are valid. It is noticed that (2.4) is true for $k-1$. This means there is a positive constant, say M_0, fulfilling $\mu \int_\Omega S^{k-1} dx \le M_0$. Therefore,

$$
\frac{d}{dt} \int_\Omega (S^k + A^k + I^k + R^k) dx \le M_0 - C_0 \int_\Omega (S^k + A^k + I^k + R^k) dx,
$$

namely

$$
\begin{aligned}
\int_\Omega (S^k + A^k + I^k + R^k) dx \le& e^{-C_0 t} \int_\Omega (S_0^k(x) + A_0^k(x) + I_0^k(x) + R_0^k(x)) dx \\
&+ \frac{M_0}{C_0}(1 - e^{-C_0 t}),
\end{aligned} \tag{2.10}
$$

for $\forall x \in \overline{\Omega},\ t \ge 0$. In addition, for some large $T > 0$, (2.10) gives

$$
\limsup_{t\to\infty} \int_\Omega (S^k + A^k + I^k + R^k) dx \le \frac{M_0}{C_0}, \quad \forall x \in \overline{\Omega}.
$$

The proof is completed. □

3. Non-constant steady states

In this section, we will give some results about the non-constant steady states to the following spatial COVID-19 model

$$
\begin{cases}
- d_S \Delta S(x) = \mu - \dfrac{\beta S(x) I(x)}{1 + \alpha I^2(x)} - \mu S(x), & x \in \Omega, \\
- d_A \Delta A(x) = \dfrac{\beta S(x) I(x)}{1 + \alpha I^2(x)} - (\sigma + \delta + \epsilon + \mu) A(x), & x \in \Omega, \\
- d_I \Delta I(x) = \sigma A(x) - (\gamma + d + \mu) I(x), & x \in \Omega, \\
- d_R \Delta R(x) = \gamma I(x) + \epsilon A(x) - \mu R(x), & x \in \Omega, \\
\dfrac{\partial S(x)}{\partial \nu} = \dfrac{\partial A(x)}{\partial \nu} = \dfrac{\partial I(x)}{\partial \nu} = \dfrac{\partial R(x)}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases} \tag{3.1}
$$

3.1. A priori estimates

Lemma 3.1 (Maximum Principle, [7, 12]). *Suppose that $F(x,\omega(x)) \in C(\overline{\Omega} \times \mathbb{R})$.*

(i) If $\omega(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta\omega(x) + F(x,\omega(x)) \geq 0, & x \in \Omega, \\ \dfrac{\partial \omega(x)}{\partial \mathbf{n}} \leq 0, & x \in \partial\Omega, \end{cases}$$

and $\omega(x_0) = \max_{x\in\overline{\Omega}} \omega(x)$, then $F(x_0,\omega(x_0)) \geq 0$.

(ii) If $\omega(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta\omega(x) + F(x,\omega(x)) \leq 0, & x \in \Omega, \\ \dfrac{\partial \omega(x)}{\partial \mathbf{n}} \geq 0, & x \in \partial\Omega, \end{cases}$$

and $\omega(x_0) = \min_{x\in\overline{\Omega}} \omega(x)$, then $F(x_0,\omega(x_0)) \leq 0$.

Lemma 3.2 (Harnack Inequality, [8, 13]). *Suppose that $\omega(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution of*

$$\begin{cases} \Delta\omega(x) + c(x)\omega(x) = 0, & x \in \Omega, \\ \dfrac{\partial \omega(x)}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

and satisfying no-flux boundary conditions and $c(x) \in C(\Omega) \cap L^\infty(\Omega)$. Then there is a positive constant $c_* = c_*(\|c(x)\|_\infty, \Omega)$ fulfilling $\max_{x\in\overline{\Omega}} \omega(x) \leq c_* \min_{x\in\overline{\Omega}} \omega(x)$.

Theorem 3.1 (A priori estimates). *Suppose that $d_S, d_A, d_I, d_R > 0$ are valid, then for any solution $(S(x), A(x), I(x), R(x))$ of system (3.1), one has*

$$\begin{aligned} &0 < S(x) \leq 1, \\ &0 < A(x) \leq \frac{\beta}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)}, \\ &0 < I(x) \leq \frac{\sigma\beta}{2\sqrt{\alpha}(\gamma+d+\mu)(\sigma+\delta+\epsilon+\mu)}, \\ &0 < R(x) \leq \frac{\gamma\sigma\beta + \epsilon\beta(\gamma+d+\mu)}{2\mu\sqrt{\alpha}(\gamma+d+\mu)(\sigma+\delta+\epsilon+\mu)}. \end{aligned}$$

Moreover, there is a positive constant $\widehat{C}_0$ depending on $\mu, \beta, \delta, \epsilon, \sigma, \gamma, \alpha$ and Ω such that every positive solution $(S(x), A(x), I(x), R(x))$ of system (3.1) satisfies

$$S(x) \geq \widehat{C}_0, \quad A(x) \geq \widehat{C}_0, \quad I(x) \geq \widehat{C}_0, \quad R(x) \geq \widehat{C}_0.$$

Proof. Suppose that $(S(x), A(x), I(x), R(x))$ is a positive solution of system (3.1), and denote by

$$\begin{aligned} S(x_1) = \max_{x\in\overline{\Omega}} S(x), \quad A(y_1) = \max_{x\in\overline{\Omega}} A(x), \\ I(z_1) = \max_{x\in\overline{\Omega}} I(x), \quad R(q_1) = \max_{x\in\overline{\Omega}} R(x). \end{aligned}$$

By using Lemma 3.1 and the S-equation of (3.1), we have

$$0 \le \mu - \frac{\beta S(x_1) I(x_1)}{1+\alpha I^2(x_1)} - \mu S(x_1) \le \mu - \mu S(x_1),$$

it follows that $S(x_1) \le 1$. Next due to Lemma 3.1 and the A-equation of (3.1), one has

$$0 \le \frac{\beta S(y_1) I(y_1)}{1+\alpha I^2(y_1)} - (\sigma+\delta+\epsilon+\mu)A(y_1) \le \frac{\beta}{2\sqrt{\alpha}} - (\sigma+\delta+\epsilon+\mu)A(y_1),$$

which leads to $A(y_1) \le \frac{\beta}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)}$. Hence for $x = z_1$, Lemma 3.1 and the I-equation of (3.1) give that

$$0 \le \sigma A(z_1) - (\gamma+d+\mu)I(z_1) \le \frac{\sigma\beta}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)} - (\gamma+d+\mu)I(z_1),$$

and thus we have $I(z_1) \le \frac{\sigma\beta}{2\sqrt{\alpha}(\gamma+d+\mu)(\sigma+\delta+\epsilon+\mu)}$. For the R-equation of (3.1), one obtains

$$\begin{aligned} 0 \le & \gamma I(q_1) + \epsilon A(q_1) - \mu R(q_1) \\ \le & \frac{\gamma\sigma\beta}{2\sqrt{\alpha}(\gamma+d+\mu)(\sigma+\delta+\epsilon+\mu)} + \frac{\epsilon\beta}{2\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)} - \mu R(q_1), \end{aligned}$$

then a simple computation indicates $R(q_1) \le \frac{\gamma\sigma\beta+\epsilon\beta(\gamma+d+\mu)}{2\mu\sqrt{\alpha}(\gamma+d+\mu)(\sigma+\delta+\epsilon+\mu)}$.

Now we invetigate the lower bounds of $(S(x), A(x), I(x), R(x))$. Suppose that the lower bounds of the positive solution $(S(x), A(x), I(x), R(x))$ in Theorem 3.1 is false. Then there is a sequence $\{(d_{S,j}, d_{A,j}, d_{I,j}, d_{R,j})\}_{j=1}^{\infty}$ with $d_{S,j}, d_{A,j}, d_{I,j}, d_{R,j} > \varepsilon$ and a positive solution $(S_j(x), A_j(x), I_j(x), R_j(x))$ of system (3.1) with respect to $(d_S, d_A, d_I, d_R) = (d_{S,j}, d_{A,j}, d_{I,j}, d_{R,j})$ satisfying $(S_j(x), A_j(x), I_j(x), R_j(x)) \to (S(x), A(x), I(x), R(x))$ in $[C^2(\overline{\Omega})]^4$ as $j \to \infty$ and

$$\begin{aligned} & \min_{\overline{\Omega}} S_j(x) \to 0 \text{ or } \min_{\overline{\Omega}} A_j(x) \to 0 \text{ or } \min_{\overline{\Omega}} I_j(x) \to 0 \\ & \text{or } \min_{\overline{\Omega}} R_j(x) \to 0 \text{ as } j \to \infty, \end{aligned} \tag{3.2}$$

and the positive solution $(S_j(x), A_j(x), I_j(x), R_j(x))$ fulfills

$$\begin{cases} -d_{S,j}\Delta S_j(x) = \mu - \dfrac{\beta S_j(x) I_j(x)}{1+\alpha I_j^2(x)} - \mu S_j(x), & x \in \Omega, \\ -d_{A,j}\Delta A_j(x) = \dfrac{\beta S_j(x) I_j(x)}{1+\alpha I_j^2(x)} - (\sigma+\delta+\epsilon+\mu)A_j(x), & x \in \Omega, \\ -d_{I,j}\Delta I_j(x) = \sigma A_j(x) - (\gamma+d+\mu)I_j(x), & x \in \Omega, \\ -d_{R,j}\Delta R_j(x) = \gamma I_j(x) + \epsilon A_j(x) - \mu R_j(x), & x \in \Omega, \\ \dfrac{\partial S_j(x)}{\partial \nu} = \dfrac{\partial A_j(x)}{\partial \nu} = \dfrac{\partial I_j(x)}{\partial \nu} = \dfrac{\partial R_j(x)}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

Then we obtain

$$\begin{cases} \int_\Omega \left(\mu - \frac{\beta S_j(x) I_j(x)}{1+\alpha I_j^2(x)} - \mu S_j(x) \right) dx = 0, \\ \int_\Omega \left(\frac{\beta S_j(x) I_j(x)}{1+\alpha I_j^2(x)} - (\sigma+\delta+\epsilon+\mu) A_j(x) \right) dx = 0, \\ \int_\Omega \left(\sigma A_j(x) - (\gamma+d+\mu) I_j(x) \right) dx = 0, \\ \int_\Omega \left(\gamma I_j(x) + \epsilon A_j(x) - \mu R_j(x) \right) dx = 0. \end{cases} \tag{3.4}$$

It is noticed that if (3.2) is true, then Lemma 3.2 shows

$$\begin{aligned} &\max_{\overline{\Omega}} S_j(x) \to 0 \text{ or } \max_{\overline{\Omega}} A_j(x) \to 0 \text{ or } \max_{\overline{\Omega}} I_j(x) \to 0 \\ &\text{or } \max_{\overline{\Omega}} R_j(x) \to 0 \text{ as } j \to \infty. \end{aligned} \tag{3.5}$$

Therefore, we infer that $S(x) \equiv 0$ or $A(x) \equiv 0$ or $I(x) \equiv 0$ or $R(x) \equiv 0$.

(i) If $S(x) \equiv 0$. Due to $S_j(x) \to S(x)$ as $j \to \infty$, then we have $\frac{\beta S_j(x) I_j(x)}{1+\alpha I_j^2(x)} - (\sigma + \delta + \epsilon + \mu) A_j(x) < 0$ for $\forall x \in \overline{\Omega}$ and $j \gg 1$. This implies

$$-d_{A,j} \int_\Omega \Delta A_j(x) dx = \int_\Omega \left(\frac{\beta S_j(x) I_j(x)}{1+\alpha I_j^2(x)} - (\sigma+\delta+\epsilon+\mu) A_j(x) \right) dx < 0,$$

which is a contradiction to $A_j(x)$-equation of (3.4).

(ii) If $A(x) \equiv 0$. Since $A_j(x) \to A(x)$ as $j \to \infty$, then one has $\sigma A_j(x) - (\gamma + d + \mu) I_j(x) < 0$ for $\forall x \in \overline{\Omega}$ and $j \gg 1$. This gives

$$-d_{I,j} \int_\Omega \Delta I_j(x) dx = \int_\Omega \left(\sigma A_j(x) - (\gamma+d+\mu) I_j(x) \right) dx < 0,$$

which is a contradiction to $I_j(x)$-equation of (3.4).

(iii) If $I(x) \equiv 0$. Since $I_j(x) \to I(x)$ as $j \to \infty$, then one obtains $\frac{\beta S_j(x) I_j(x)}{1+\alpha I_j^2(x)} - (\sigma + \delta + \epsilon + \mu) A_j(x) < 0$ for $\forall x \in \overline{\Omega}$ and $j \gg 1$. This implies

$$-d_{A,j} \int_\Omega \Delta A_j(x) dx = \int_\Omega \left(\frac{\beta S_j(x) I_j(x)}{1+\alpha I_j^2(x)} - (\sigma+\delta+\epsilon+\mu) A_j(x) \right) dx < 0,$$

which is a contradiction to $A_j(x)$-equation of (3.4).

(iv) If $R(x) \equiv 0$. Since $R_j(x) \to R(x)$ as $j \to \infty$, then we get $\gamma I_j(x) + \epsilon A_j(x) - \mu R_j(x) > 0$ for $\forall x \in \overline{\Omega}$ and $j \gg 1$. It follows that

$$-d_{R,j} \int_\Omega \Delta R_j(x) dx = \int_\Omega (\gamma I_j(x) + \epsilon A_j(x) - \mu R_j(x) dx > 0,$$

which is a contradiction to $R_j(x)$-equation of (3.4). Thereby, the lower bounds of the solution $S(x), A(x), I(x)$ and $R(x)$ exist. This ends the proof. □

3.2. Some properties of the positive steady states

Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots$ and $\lim_{i\to\infty} \lambda_i = \infty$, be the complete set of eigenvalues of the operator $-\Delta$ with no-flux boundary conditions in Ω, and

$$\bar{S} = \frac{1}{|\Omega|}\int_\Omega S(x)dx,\ \bar{A} = \frac{1}{|\Omega|}\int_\Omega A(x)dx,\ \bar{I} = \frac{1}{|\Omega|}\int_\Omega I(x)dx,\ \bar{R} = \frac{1}{|\Omega|}\int_\Omega R(x)dx$$

be their averages over domain Ω. Then we have

$$\int_\Omega (S-\bar{S})dx = 0,\ \int_\Omega (A-\bar{A})dx = 0,\ \int_\Omega (I-\bar{I})dx = 0,\ \int_\Omega (R-\bar{R})dx = 0.$$

Denote by $\phi = S(x)-\bar{S}, \psi = A(x)-\bar{A}, \varphi = I(x)-\bar{I}$ and $\Theta = R(x)-\bar{R}$. This gives the fact that $\int_\Omega \phi dx = 0, \int_\Omega \psi dx = 0, \int_\Omega \varphi dx = 0$ and $\int_\Omega \Theta dx = 0$ are valid.

Then we deliberate a result as follows.

Theorem 3.2. *Suppose that $d_S, d_A, d_I, d_R > 0$, then for $(S(x), A(x), I(x), R(x))$ of system (3.1), we have*

$$(i) \int_\Omega \phi^2 dx + \int_\Omega |\nabla\phi|^2 dx \leq \frac{\mu^2|\Omega|(1+\lambda_1)}{d_S^2\lambda_1^2},$$
$$(ii) \int_\Omega \psi^2 dx + \int_\Omega |\nabla\psi|^2 dx \leq \frac{\beta^2|\Omega|(1+\lambda_1)}{4\alpha d_A^2\lambda_1^2},$$
$$(iii) \int_\Omega \varphi^2 dx + \int_\Omega |\nabla\varphi|^2 dx \leq \frac{\sigma^2\beta^2|\Omega|(1+\lambda_1)}{4\alpha(\sigma+\delta+\epsilon+\mu)^2 d_I^2\lambda_1^2},$$
$$(iv) \int_\Omega \Theta^2 dx + \int_\Omega |\nabla\Theta|^2 dx \leq \frac{[\gamma\sigma\beta+\epsilon\beta(\gamma+d+\mu)]^2(1+\lambda_1)|\Omega|}{4\mu^2\alpha(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2 d_R^2\lambda_1^2},$$

where λ_1 is the first positive eigenvalue of $-\Delta$ on Ω about zero-flux boundary conditions.

Proof. Multiplying by ϕ the S-equation of (3.1) and using Cauchy-Schwarz inequality, one obtains

$$\begin{aligned} d_S\int_\Omega |\nabla\phi|^2 dx &= \int_\Omega \phi\left(\mu - \frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - \mu S(x)\right)dx \\ &\leq \mu\int_\Omega |\phi| dx \\ &\leq \mu\sqrt{|\Omega|}\left(\int_\Omega |\phi|^2 dx\right)^{\frac{1}{2}}. \end{aligned}$$

Multiplying by ψ the A-equation of (3.1) and using Cauchy-Schwarz inequality again

$$\begin{aligned} d_A\int_\Omega |\nabla\psi|^2 dx &= \int_\Omega \psi\left(\frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - (\sigma+\delta+\epsilon+\mu)A(x)\right)dx \\ &\leq \frac{\beta}{2\sqrt{\alpha}}\int_\Omega |\psi| dx \\ &\leq \frac{\beta\sqrt{|\Omega|}}{2\sqrt{\alpha}}\left(\int_\Omega |\psi|^2 dx\right)^{\frac{1}{2}}. \end{aligned}$$

In the same fashion to the I, R-equations of (3.1), we yield

$$\begin{aligned} d_I \int_\Omega |\nabla\varphi|^2 dx =& \int_\Omega \varphi\left(\sigma A(x) - (\gamma + d + \mu) I(x)\right) dx \\ \leq & \frac{\sigma\beta}{2\sqrt{\alpha}(\sigma + \delta + \epsilon + \mu)} \int_\Omega |\varphi| dx \\ \leq & \frac{\sigma\beta\sqrt{|\Omega|}}{2\sqrt{\alpha}(\sigma + \delta + \epsilon + \mu)} \left(\int_\Omega |\varphi|^2 dx\right)^{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} d_R \int_\Omega |\nabla\Theta|^2 dx =& \int_\Omega \Theta\left(\gamma I(x) + \epsilon A(x) - \mu R(x)\right) dx \\ \leq & \frac{\gamma\sigma\beta + \epsilon\beta(\gamma + d + \mu)}{2\mu\sqrt{\alpha}(\gamma + d + \mu)(\sigma + \delta + \epsilon + \mu)} \int_\Omega |\Theta| dx \\ \leq & \frac{[\gamma\sigma\beta + \epsilon\beta(\gamma + d + \mu)]\sqrt{|\Omega|}}{2\mu\sqrt{\alpha}(\gamma + d + \mu)(\sigma + \delta + \epsilon + \mu)} \left(\int_\Omega |\Theta|^2 dx\right)^{\frac{1}{2}}. \end{aligned}$$

From the Poincaré's inequality

$$\begin{aligned} \int_\Omega \phi^2 dx \leq \frac{1}{\lambda_1}\int_\Omega |\nabla\phi|^2 dx, \quad \int_\Omega \psi^2 dx \leq \frac{1}{\lambda_1}\int_\Omega |\nabla\psi|^2 dx, \\ \int_\Omega \varphi^2 dx \leq \frac{1}{\lambda_1}\int_\Omega |\nabla\varphi|^2 dx, \quad \int_\Omega \Theta^2 dx \leq \frac{1}{\lambda_1}\int_\Omega |\nabla\Theta|^2 dx, \end{aligned}$$

one yields

$$\begin{aligned} d_S \int_\Omega |\nabla\phi|^2 dx \leq & \mu\sqrt{|\Omega|}\left(\int_\Omega |\phi|^2 dx\right)^{\frac{1}{2}} \leq \mu\sqrt{\frac{|\Omega|}{\lambda_1}}\left(\int_\Omega |\nabla\phi|^2 dx\right)^{\frac{1}{2}}, \\ d_A \int_\Omega |\nabla\psi|^2 dx \leq & \frac{\beta\sqrt{|\Omega|}}{2\sqrt{\alpha}}\left(\int_\Omega |\psi|^2 dx\right)^{\frac{1}{2}} \leq \frac{\beta}{2\sqrt{\alpha}}\sqrt{\frac{|\Omega|}{\lambda_1}}\left(\int_\Omega |\nabla\psi|^2 dx\right)^{\frac{1}{2}}, \\ d_I \int_\Omega |\nabla\varphi|^2 dx \leq & \frac{\sigma\beta\sqrt{|\Omega|}}{2\sqrt{\alpha}(\sigma + \delta + \epsilon + \mu)}\left(\int_\Omega |\varphi|^2 dx\right)^{\frac{1}{2}} \\ \leq & \frac{\sigma\beta}{2\sqrt{\alpha}(\sigma + \delta + \epsilon + \mu)}\sqrt{\frac{|\Omega|}{\lambda_1}}\left(\int_\Omega |\nabla\varphi|^2 dx\right)^{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} d_R \int_\Omega |\nabla\Theta|^2 dx \leq & \frac{[\gamma\sigma\beta + \epsilon\beta(\gamma + d + \mu)]\sqrt{|\Omega|}}{2\mu\sqrt{\alpha}(\gamma + d + \mu)(\sigma + \delta + \epsilon + \mu)}\left(\int_\Omega |\Theta|^2 dx\right)^{\frac{1}{2}} \\ \leq & \frac{\gamma\sigma\beta + \epsilon\beta(\gamma + d + \mu)}{2\mu\sqrt{\alpha}(\gamma + d + \mu)(\sigma + \delta + \epsilon + \mu)}\sqrt{\frac{|\Omega|}{\lambda_1}}\left(\int_\Omega |\nabla\Theta|^2 dx\right)^{\frac{1}{2}}. \end{aligned}$$

Some direct calculations give that

$$\int_\Omega |\nabla\phi|^2 dx \leq \frac{\mu^2|\Omega|}{d_S^2\lambda_1}, \quad \int_\Omega |\nabla\psi|^2 dx \leq \frac{\beta^2|\Omega|}{4\alpha d_A^2\lambda_1},$$

$$\int_\Omega |\nabla\varphi|^2 dx \leq \frac{\sigma^2\beta^2|\Omega|}{4\alpha(\sigma+\delta+\epsilon+\mu)^2 d_I^2\lambda_1},$$
$$\int_\Omega |\nabla\Theta|^2 dx \leq \frac{[\gamma\sigma\beta+\epsilon\beta(\gamma+d+\mu)]^2|\Omega|}{4\mu^2\alpha(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2 d_R^2\lambda_1}.$$

By employing the Poincaré's inequality, one deduces

$$\int_\Omega \phi^2 dx + \int_\Omega |\nabla\phi|^2 dx \leq \frac{\mu^2|\Omega|(1+\lambda_1)}{d_S^2\lambda_1^2},$$
$$\int_\Omega \psi^2 dx + \int_\Omega |\nabla\psi|^2 dx \leq \frac{\beta^2|\Omega|(1+\lambda_1)}{4\alpha d_A^2\lambda_1^2},$$
$$\int_\Omega \varphi^2 dx + \int_\Omega |\nabla\varphi|^2 dx \leq \frac{\sigma^2\beta^2|\Omega|(1+\lambda_1)}{4\alpha(\sigma+\delta+\epsilon+\mu)^2 d_I^2\lambda_1^2},$$
$$\int_\Omega \Theta^2 dx + \int_\Omega |\nabla\Theta|^2 dx \leq \frac{[\gamma\sigma\beta+\epsilon\beta(\gamma+d+\mu)]^2(1+\lambda_1)|\Omega|}{4\mu^2\alpha(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2 d_R^2\lambda_1^2}.$$

The proof is completed. □

Theorem 3.3. *Suppose that* $d_S, d_A, d_I, d_R > 0, 0 < 4\beta < 3\lambda_1 d_A$ *and the assumption* $\frac{\sigma^2\beta^2}{4(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2} < 1$*, one has*

$$\frac{16\alpha(3d_A\lambda_1-4\beta)(\gamma+d+\mu)^6(\sigma+\delta+\epsilon+\mu)^6 d_A\lambda_1}{[\sigma^3\beta^4+4\sigma\beta^2(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2]^2} \leq \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega |\nabla\psi|^2 dx} \leq \frac{4\beta^2}{3d_S^2\lambda_1^2}$$

and

$$\frac{16\alpha(3d_A\lambda_1-4\beta)(\gamma+d+\mu)^6(\sigma+\delta+\epsilon+\mu)^6 d_A\lambda_1^2}{(1+\lambda_1)[\sigma^3\beta^4+4\sigma\beta^2(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2]^2}$$
$$\leq \frac{\int_\Omega (|\nabla\phi|^2+\phi^2)dx}{\int_\Omega (|\nabla\psi|^2+\psi^2)dx} \leq \frac{4\beta^2(1+\lambda_1)}{3d_S^2\lambda_1^3},$$

where λ_1 *is the first positive eigenvalue of* $-\Delta$ *on* Ω *about zero-flux boundary conditions.*

Proof. Multiplying the S-equation of (3.1) by ϕ, we have

$$\begin{aligned}
0 &= \int_\Omega \left(d_S\Delta S + \mu - \frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - \mu S(x)\right)\phi dx \\
&= -d_S\int_\Omega |\nabla\phi|^2 dx + \mu\int_\Omega \phi dx - \beta\int_\Omega \frac{S(x)I(x)}{1+\alpha I^2(x)}\phi dx - \mu\int_\Omega S(x)\phi dx \\
&= -d_S\int_\Omega |\nabla\phi|^2 dx - \beta\int_\Omega \frac{S(x)I(x)}{1+\alpha I^2(x)}\phi dx - \mu\int_\Omega \phi^2 dx \\
&= -d_S\int_\Omega |\nabla\phi|^2 dx - \beta\int_\Omega \frac{I(x)(1+\alpha\bar{I}^2)\phi^2}{(1+\alpha I^2(x))(1+\alpha\bar{I}^2(x))}dx \\
&\quad -\beta\int_\Omega \frac{\bar{S}(1-\alpha\bar{I}I(x))\phi\psi}{(1+\alpha I^2(x))(1+\alpha\bar{I}^2(x))}dx - \mu\int_\Omega \phi^2 dx,
\end{aligned}$$

it follows that

$$\begin{aligned}&\beta\int_\Omega \frac{\bar{S}(1-\alpha\bar{I}I(x))\phi\psi}{(1+\alpha I^2(x))(1+\alpha\bar{I}^2(x))}dx\\ =&-d_S\int_\Omega|\nabla\phi|^2dx-\beta\int_\Omega\frac{I(x)(1+\alpha\bar{I}^2)\phi^2}{(1+\alpha I^2(x))(1+\alpha\bar{I}^2(x))}dx-\mu\int_\Omega\phi^2dx.\end{aligned}$$

Obviously, $\int_\Omega \phi\psi dx<0$ is true since $\frac{\sigma^2\beta^2}{4(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}<1$. Thereby,

$$\begin{aligned}d_S\int_\Omega|\nabla\phi|^2dx=&-\beta\int_\Omega\frac{I(x)(1+\alpha\bar{I}^2)\phi^2}{(1+\alpha I^2(x))(1+\alpha\bar{I}^2(x))}dx-\mu\int_\Omega\phi^2dx\\ &-\beta\int_\Omega\frac{\bar{S}(1-\alpha\bar{I}I(x))\phi\psi}{(1+\alpha I^2(x))(1+\alpha\bar{I}^2(x))}dx\\ \leq&\beta\int_\Omega|\phi\psi|dx,\end{aligned}$$

namely

$$\begin{aligned}d_S\int_\Omega|\nabla\phi|^2dx\leq&\beta\int_\Omega|\phi\psi|dx\\ \leq&\frac{d_S\lambda_1}{4}\int_\Omega\phi^2dx+\frac{\beta^2}{d_S\lambda_1}\int_\Omega\psi^2dx\\ \leq&\frac{d_S}{4}\int_\Omega|\nabla\phi|^2dx+\frac{\beta^2}{d_S\lambda_1^2}\int_\Omega|\nabla\psi|^2dx.\end{aligned}$$

As a result, one has

$$\frac{3d_S}{4}\int_\Omega|\nabla\phi|^2dx\leq\frac{\beta^2}{d_S\lambda_1^2}\int_\Omega|\nabla\psi|^2dx. \tag{3.6}$$

On the other hand, multiplying the A-equation of (3.1) by ψ, one yields

$$\begin{aligned}0=&\int_\Omega\left(d_A\Delta A(x)+\frac{\beta S(x)I(x)}{1+\alpha I^2(x)}-(\sigma+\delta+\epsilon+\mu)A(x)\right)\psi dx\\ =&-d_A\int_\Omega|\nabla\psi|^2dx+\beta\int_\Omega\left(\frac{S(x)I(x)}{1+\alpha I^2(x)}-\frac{\bar{S}\bar{I}}{1+\alpha\bar{I}^2}\right)\psi dx\\ &-(\sigma+\delta+\epsilon+\mu)\int_\Omega\psi^2dx\\ \leq&-d_A\int_\Omega|\nabla\psi|^2dx+\beta\int_\Omega\frac{I(x)(1+\alpha\bar{I}^2)\phi\psi}{(1+\alpha I^2(x))(1+\alpha\bar{I}^2(x))}dx\\ &+\beta\int_\Omega\frac{\bar{S}(1-\alpha\bar{I}I(x))\psi^2}{(1+\alpha I^2(x))(1+\alpha\bar{I}^2(x))}dx.\end{aligned}$$

It then follows that

$$\begin{aligned}d_A\int_\Omega|\nabla\psi|^2dx\leq&\beta\int_\Omega\frac{I(x)(1+\alpha\bar{I}^2)\phi\psi}{(1+\alpha I^2(x))(1+\alpha\bar{I}^2(x))}dx\\ &+\beta\int_\Omega\frac{\bar{S}(1-\alpha\bar{I}I(x))\psi^2}{(1+\alpha I^2(x))(1+\alpha\bar{I}^2(x))}dx\end{aligned}$$

$$\leq \frac{\sigma^3\beta^4 + 4\sigma\beta^2(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}{8\sqrt{\alpha}(\gamma+d+\mu)^3(\sigma+\delta+\epsilon+\mu)^3}\int_\Omega |\phi\psi|dx + \beta\int_\Omega \psi^2 dx$$
$$\leq \frac{\sigma^3\beta^4 + 4\sigma\beta^2(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}{8\sqrt{\alpha}(\gamma+d+\mu)^3(\sigma+\delta+\epsilon+\mu)^3}\int_\Omega |\phi\psi|dx + \frac{\beta}{\lambda_1}\int_\Omega |\nabla\psi|^2 dx.$$

Consequently, if $0 < 4\beta < 3\lambda_1 d_A$ is valid, we have

$$\begin{aligned}
&\frac{d_A\lambda_1 - \beta}{\lambda_1}\int_\Omega |\nabla\psi|^2 dx \\
\leq& \frac{\sigma^3\beta^4 + 4\sigma\beta^2(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}{8\sqrt{\alpha}(\gamma+d+\mu)^3(\sigma+\delta+\epsilon+\mu)^3}\int_\Omega |\phi\psi|dx \\
\leq& \frac{d_A\lambda_1}{4}\int_\Omega \psi^2 dx + \frac{[\sigma^3\beta^4 + 4\sigma\beta^2(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2]^2}{64\alpha(\gamma+d+\mu)^6(\sigma+\delta+\epsilon+\mu)^6 d_A\lambda_1}\int_\Omega |\phi|^2 dx \\
\leq& \frac{d_A}{4}\int_\Omega |\nabla\psi|^2 dx + \frac{[\sigma^3\beta^4 + 4\sigma\beta^2(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2]^2}{64\alpha(\gamma+d+\mu)^6(\sigma+\delta+\epsilon+\mu)^6 d_A\lambda_1^2}\int_\Omega |\nabla\phi|^2 dx.
\end{aligned}$$

Henceforth

$$\frac{3d_A\lambda_1 - 4\beta}{4\lambda_1}\int_\Omega |\nabla\psi|^2 dx \leq \frac{[\sigma^3\beta^4+4\sigma\beta^2(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2]^2}{64\alpha(\gamma+d+\mu)^6(\sigma+\delta+\epsilon+\mu)^6 d_A\lambda_1^2}\int_\Omega |\nabla\phi|^2 dx. \tag{3.7}$$

From (3.6) and (3.7), one obtains

$$\frac{16\alpha(3d_A\lambda_1 - 4\beta)(\gamma+d+\mu)^6(\sigma+\delta+\epsilon+\mu)^6 d_A\lambda_1}{[\sigma^3\beta^4 + 4\sigma\beta^2(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2]^2} \leq \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega |\nabla\psi|^2 dx} \leq \frac{4\beta^2}{3d_S^2\lambda_1^2}.$$

Now due to the Poincaré's inequality, we have

$$\int_\Omega (|\nabla\phi|^2 + \phi^2)dx \leq \frac{1+\lambda_1}{\lambda_1}\int_\Omega \nabla|\phi|^2 x, \quad \int_\Omega (|\nabla\psi|^2 + \psi^2)dx \leq \frac{1+\lambda_1}{\lambda_1}\int_\Omega \nabla|\psi|^2 x.$$

We thus obtain

$$\frac{\int_\Omega (|\nabla\phi|^2 + \phi^2)dx}{\int_\Omega (|\nabla\psi|^2 + \psi^2)dx} \leq \frac{(1+\lambda_1)\int_\Omega |\nabla\phi|^2 dx}{\lambda_1\int_\Omega (|\nabla\psi|^2 + \psi^2)dx} \leq \frac{(1+\lambda_1)\int_\Omega |\nabla\phi|^2 dx}{\lambda_1\int_\Omega |\nabla\psi|^2 dx} \leq \frac{4\beta^2(1+\lambda_1)}{3d_S^2\lambda_1^3},$$

and

$$\begin{aligned}
\frac{\int_\Omega (|\nabla\phi|^2 + \phi^2)dx}{\int_\Omega (|\nabla\psi|^2 + \psi^2)dx} \geq& \frac{\lambda_1\int_\Omega (|\nabla\phi|^2 + \phi^2)dx}{(1+\lambda_1)\int_\Omega |\nabla\psi|^2 dx} \geq \frac{\lambda_1\int_\Omega |\nabla\phi|^2 dx}{(1+\lambda_1)\int_\Omega |\nabla\psi|^2 dx} \\
\geq& \frac{16\alpha(3d_A\lambda_1 - 4\beta)(\gamma+d+\mu)^6(\sigma+\delta+\epsilon+\mu)^6 d_A\lambda_1^2}{(1+\lambda_1)[\sigma^3\beta^4 + 4\sigma\beta^2(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2]^2}.
\end{aligned}$$

The proof is finished. □

3.3. Nonexistence of the positive steady states

Theorem 3.4. *Suppose that $\sigma > 2(d+\mu)+\gamma, \epsilon+\gamma > 2\mu$ and $\beta > \mu+\delta$ are valid, then system (3.1) has no nonconstant steady state as $d_S > d_S^*, d_A > d_A^*, d_I > d_I^*$ and $d_R > d_R^*$, where*

$$\begin{aligned}
d_S^* &= \frac{1}{\lambda_1}\left(\frac{\sigma^2\beta^3 - 8\mu(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2} + \frac{\beta}{4\sqrt{\alpha}}\right),\\
d_A^* &= \frac{1}{\lambda_1}\left(\frac{\sigma^2\beta^3}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2} + \frac{\beta(1+4\sqrt{\alpha})-4\sqrt{\alpha}(\delta+\mu)}{4\sqrt{\alpha}} + \frac{\epsilon+\sigma}{2}\right),\\
d_I^* &= \frac{\sigma-2(d+\mu)-\gamma}{2\lambda_1},\\
d_R^* &= \frac{\epsilon+\gamma-2\mu}{2\lambda_1},
\end{aligned}$$

and λ_1 is the first positive eigenvalue of $-\Delta$ on Ω with respect to no-flux boundary conditions.

Proof. Multiplying the S-equation of (3.1) by ϕ, we have

$$\begin{aligned}
d_S\int_\Omega |\nabla\phi|^2 dx =& \int_\Omega \left(\mu - \frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - \mu S(x)\right)\phi dx\\
=& \int_\Omega \left(\mu - \frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - \mu S(x)\right)\phi dx\\
&- \int_\Omega \left(\mu - \frac{\beta\bar{S}\bar{I}}{1+\alpha\bar{I}^2} - \mu\bar{S}\right)\phi dx\\
=& \int_\Omega (\mu - \mu S(x))\,\phi dx - \int_\Omega \frac{\beta S(x)I(x)}{1+\alpha I^2(x)}\phi dx\\
=& S_1 + S_2,
\end{aligned}$$

where

$$S_1 = \int_\Omega (\mu - \mu S(x))\,\phi dx, \quad S_2 = -\int_\Omega \frac{\beta S(x)I(x)}{1+\alpha I^2(x)}\phi dx.$$

Then we can obtain

$$S_1 - \int_\Omega \left(\mu - \mu\bar{S}\right)\phi dx = -\mu\int_\Omega \phi^2 dx,$$

and

$$\begin{aligned}
S_2 + \int_\Omega \frac{\beta\bar{S}\bar{I}}{1+\alpha\bar{I}^2}\phi dx =& \beta\int_\Omega\left(\frac{\bar{S}\bar{I}}{1+\alpha\bar{I}^2} - \frac{S(x)I(x)}{1+\alpha I^2(x)}\right)\phi dx\\
=& \int_\Omega \frac{\beta(\alpha I\bar{I}\bar{S} - S(x))\phi\psi}{(1+\alpha\bar{I}^2)(1+\alpha I^2)}dx - \int_\Omega \frac{\beta\bar{I}(1+\alpha I\bar{I})}{(1+\alpha\bar{I}^2)(1+\alpha I^2)}\phi^2 dx\\
\leq& \frac{\sigma^2\beta^3}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}\int_\Omega \phi^2 dx\\
&+ \frac{\sigma^2\beta^3}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}\int_\Omega \psi^2 dx.
\end{aligned}$$

Hence

$$d_S \int_\Omega |\nabla\phi|^2 dx \leq \frac{\sigma^2\beta^3 - 8\mu(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}\int_\Omega \phi^2 dx + \frac{\sigma^2\beta^3}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}\int_\Omega \psi^2 dx. \tag{3.8}$$

Multiplying the A-equation of (3.1) by ψ, we have

$$\begin{aligned}
d_A \int_\Omega |\nabla\psi|^2 dx =& \int_\Omega \left(\frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - (\sigma+\delta+\epsilon+\mu)A(x)\right)\psi dx \\
=& \int_\Omega \left(\frac{\beta S(x)I(x)}{1+\alpha I^2(x)} - (\sigma+\delta+\epsilon+\mu)A(x)\right)\psi dx \\
&- \int_\Omega \left(\frac{\beta\bar{S}\bar{I}}{1+\alpha\bar{I}^2} - (\sigma+\delta+\epsilon+\mu)\bar{A}\right)\psi dx \\
=& \int_\Omega \frac{\beta S(x)I(x)}{1+\alpha I^2(x)}\psi dx - \int_\Omega (\sigma+\delta+\epsilon+\mu)A(x)\psi dx \\
=& A_1 + A_2,
\end{aligned}$$

where

$$A_1 = \int_\Omega \frac{\beta S(x)I(x)}{1+\alpha I^2(x)}\psi dx, \quad A_2 = -\int_\Omega (\sigma+\delta+\epsilon+\mu)A(x)\psi dx.$$

Henceforth, one yields

$$\begin{aligned}
A_1 - \int_\Omega \frac{\beta\bar{S}\bar{I}}{1+\alpha\bar{I}^2}\psi dx =& \beta\int_\Omega \left(\frac{S(x)I(x)}{1+\alpha I^2(x)} - \frac{\bar{S}\bar{I}}{1+\alpha\bar{I}^2}\right)\psi dx \\
=& \int_\Omega \frac{\beta I(x)\phi\psi}{1+\alpha I^2(x)}dx + \int_\Omega \frac{\beta\bar{S}(1-\alpha I(x)\bar{I})}{(1+\alpha\bar{I}^2)(1+\alpha I^2)}\psi^2 dx \\
\leq& \frac{\beta}{2\sqrt{\alpha}}\int_\Omega |\phi\psi| dx + \beta\int_\Omega \psi^2 dx \\
\leq& \frac{\beta}{4\sqrt{\alpha}}\int_\Omega \phi^2 dx + \frac{\beta(1+4\sqrt{\alpha})}{4\sqrt{\alpha}}\int_\Omega \psi^2 dx,
\end{aligned}$$

and

$$A_2 + \int_\Omega (\sigma+\delta+\epsilon+\mu)\bar{A}\psi dx = -(\sigma+\delta+\epsilon+\mu)\int_\Omega \psi^2 dx.$$

Therefore, one can obtain

$$d_A \int_\Omega |\nabla\psi|^2 dx \leq \frac{\beta}{4\sqrt{\alpha}}\int_\Omega \phi^2 dx + \frac{\beta(1+4\sqrt{\alpha}) - 4\sqrt{\alpha}(\sigma+\delta+\epsilon+\mu)}{4\sqrt{\alpha}}\int_\Omega \psi^2 dx. \tag{3.9}$$

In what follows, multiplying the I-equation of (3.1) by φ, we get

$$d_I \int_\Omega |\nabla\varphi|^2 dx = \int_\Omega [\sigma A(x) - (\gamma+d+\mu)I(x)]\varphi dx$$

$$
\begin{aligned}
&=\sigma \int_\Omega \varphi\psi dx - \int_\Omega (\gamma+d+\mu)\varphi^2 dx \\
&\leq \frac{\sigma}{2}\int_\Omega \psi^2 dx + \frac{\sigma-2(\gamma+d+\mu)}{2}\int_\Omega \varphi^2 dx. \qquad (3.10)
\end{aligned}
$$

Also, multiplying the R-equation of (3.1) by Θ gives

$$
\begin{aligned}
d_R \int_\Omega |\nabla\Theta|^2 dx &= \int_\Omega [\gamma I(x) + \epsilon A(x) - \mu R(x)]\Theta dx \\
&= \int_\Omega \gamma\Theta\varphi dx + \int_\Omega \epsilon\psi\Theta dx - \int_\Omega \mu\Theta^2 dx \\
&\leq \frac{\epsilon}{2}\int_\Omega \psi^2 dx + \frac{\gamma}{2}\int_\Omega \varphi^2 dx + \frac{\epsilon+\gamma-2\mu}{2}\int_\Omega \Theta^2 dx. \qquad (3.11)
\end{aligned}
$$

On the basis of (3.8)-(3.11), one yields

$$
\begin{aligned}
& d_S \int_\Omega |\nabla\phi|^2 dx + d_A \int_\Omega |\nabla\psi|^2 dx + d_I \int_\Omega |\nabla\varphi|^2 dx + d_R \int_\Omega |\nabla\Theta|^2 dx \\
\leq & \left(\frac{\sigma^2\beta^3 - 8\mu(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2} + \frac{\beta}{4\sqrt{\alpha}}\right)\int_\Omega \phi^2 dx \\
& + \left(\frac{\sigma^2\beta^3}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2} + \frac{\beta(1+4\sqrt{\alpha})-4\sqrt{\alpha}(\delta+\mu)}{4\sqrt{\alpha}} + \frac{\epsilon+\sigma}{2}\right) \\
& \times \int_\Omega \psi^2 dx + \frac{\sigma-2(d+\mu)-\gamma}{2}\int_\Omega \varphi^2 dx + \frac{\epsilon+\gamma-2\mu}{2}\int_\Omega \Theta^2 dx \\
\leq & \frac{1}{\lambda_1}\left(\frac{\sigma^2\beta^3 - 8\mu(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2} + \frac{\beta}{4\sqrt{\alpha}}\right)\int_\Omega |\nabla\phi|^2 dx \\
& + \frac{1}{\lambda_1}\left(\frac{\sigma^2\beta^3}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2} + \frac{\beta(1+4\sqrt{\alpha})-4\sqrt{\alpha}(\delta+\mu)}{4\sqrt{\alpha}} + \frac{\epsilon+\sigma}{2}\right) \\
& \times \int_\Omega |\nabla\psi|^2 dx + \frac{\sigma-2(d+\mu)-\gamma}{2\lambda_1}\int_\Omega |\nabla\varphi|^2 dx + \frac{\epsilon+\gamma-2\mu}{2\lambda_1}\int_\Omega |\nabla\Theta|^2 dx.
\end{aligned}
$$

Let

$$
\begin{aligned}
& d_S^* = \frac{1}{\lambda_1}\left(\frac{\sigma^2\beta^3 - 8\mu(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2} + \frac{\beta}{4\sqrt{\alpha}}\right), \\
& d_A^* = \frac{1}{\lambda_1}\left(\frac{\sigma^2\beta^3}{8(\gamma+d+\mu)^2(\sigma+\delta+\epsilon+\mu)^2} + \frac{\beta(1+4\sqrt{\alpha})-4\sqrt{\alpha}(\delta+\mu)}{4\sqrt{\alpha}} + \frac{\epsilon+\sigma}{2}\right), \\
& d_I^* = \frac{\sigma-2(d+\mu)-\gamma}{2\lambda_1}, \quad d_R^* = \frac{\epsilon+\gamma-2\mu}{2\lambda_1}.
\end{aligned}
$$

Consequently, it is easy to check that if $d_S > d_S^*, d_A > d_A^*, d_I > d_I^*$ and $d_R > d_R^*$ are valid, one obtains $\nabla\phi = \nabla\psi = \nabla\varphi = \nabla\Theta = 0$. This implies the solution $(S(x), A(x), I(x), R(x))$ must be a constant steady state of system (3.1). This ends the proof. □

4. Conclusions

In this present paper, we deal with a diffusive COVID-19 model with the non-monotone incidence rate and the homogeneous zero-flux boundary conditions. By using the technique of the comparison principle of the parabolic equations, the boundedness of the diffusive COVID-19 model is first established. Especially, the conclusions show that the positive solution of the diffusive COVID-19 model is uniformly bounded in $L^\infty(\Omega)$. Next, a priori estimates, the properties of the positive steady states, and nonexistence of the positive steady states to the corresponding elliptic equations are presented by the maximum principle and some energy estimates, respectively. An interesting finding is that the diffusion rates d_S, d_A, d_I and d_R of the susceptible, the asymptomatic, the infected, and the recovered or quarantine humans can lead to the nonexistence of the non-constant steady states. These qualitative results enhance the theoretical research of the diffusive COVID-19 model. More dynamical results, for example, bifurcations, about such a diffusive COVID-19 model with the non-monotone incidence rate will be considered.

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