Journal of Applied Analysis and Computation
Volume 13, Number 4, August 2023, 2250–2266
Website:http://www.jaac-online.com
DOI:10.11948/20220455

NON-SPURIOUS SOLUTIONS OF DISCRETE MIXED BOUNDARY VALUE PROBLEM WITH SINGULAR ϕ-LAPLACIAN*

Man Xu[1,†], Ruyun Ma[2] and Ting Wang[2]

Abstract In this paper, we consider the differential and difference problems associated with the discrete approximation of classical radial solutions of the nonlinear Dirichlet problem for the prescribed mean curvature equation in Minkowski space

$$\begin{aligned} &-\operatorname{div}\Big(\frac{\operatorname{grad} v}{\sqrt{1-|\operatorname{grad} v|^2}}\Big) = f\Big(|x|, v, \frac{dv}{dr}\Big) \quad \text{in } \mathcal{B}, \\ &v = 0 \ \text{ on } \partial\mathcal{B}, \end{aligned}$$

where $\mathcal{B}$ is the unit ball in $\mathbb{R}^N$, div denotes the divergence operator of $\mathbb{R}^N$, grad$v$ is the gradient of v, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$, $\frac{dv}{dr}$ stands for the radial derivative of v and f is a continuous function. By using lower and upper solutions, we prove the existence of solutions of the corresponding differential and difference problems, and based on the ideas of lower and upper μ-solutions show the solutions of the discrete problem can converge to the solutions of the continuous problem.

Keywords Non-spurious solution, discrete boundary value problem, singular ϕ-Laplacian, lower and upper solutions, prescribed mean curvature equation.

MSC(2010) 34A45, 34B16, 35A01, 39A27.

1. Introduction

Let $\mathbb{L}^{N+1} := \{(x,t) : x \in \mathbb{R}^N, t \in \mathbb{R}\}$ be the flat Minkowski space, endowed with the Lorentzian metric $\sum\limits_{j=1}^{N} dx_j^2 - dt^2$. In this paper we are concerned with the mixed boundary value problem

$$-(r^{N-1}\phi(u'))' = r^{N-1} f(r,u,u'), \quad r \in (0,1), \tag{1.1}$$

$$u'(0) = u(1) = 0, \tag{1.2}$$

[†]The corresponding author.
Email: xmannwnu@126.com(M. Xu), ryma@xidian.edu.cn(R. Ma), 17874158737@163.com(T. Wang)
[1]College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China
[2]College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China
*The authors were supported by the NSFC (No. 11671322) and by the grant 20JR10RA100, 21JR1RA230, 2021A-006 and NWNU-LKQN2021-17.

and its discrete approximation

$$-\frac{1}{h}\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle u_k}{h}\Big)\Big)=t_k^{N-1}f\Big(t_k,u_k,\frac{\triangle u_k}{h}\Big),\quad k=1,\cdots,n-1, \tag{1.3}$$

$$\triangle u_0=0,\qquad u_n=0, \tag{1.4}$$

where $\phi:(-a,a)\to\mathbb{R}$ $(a>0)$ is an increasing homeomorphism with $\phi(0)=0$, such an ϕ is called *singular*, f is a continuous function, $n\geq 2$ is an integer, $h=\frac{1}{n}$ is the step size, $t_k=kh$ for $k=0,1,\cdots,n$ are the grid points, $t_0=0$, $t_n=1$, $u_k:=u(t_k)$ and the differences are given by

$$\triangle u_k=\begin{cases}u_{k+1}-u_k, & k=0,1,\cdots,n-1,\\ 0, & k=n,\end{cases}$$

$$\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle u_k}{h}\Big)\Big)=\begin{cases}t_k^{N-1}\phi\Big(\frac{\triangle u_k}{h}\Big)-t_{k-1}^{N-1}\phi\Big(\frac{\triangle u_{k-1}}{h}\Big), & k=1,\cdots,n-1,\\ 0, & k=0,n.\end{cases}$$

The aim of this paper is to investigate the solvability of discrete problem (1.3)-(1.4) and consider in what sense, if any, will the solutions of discrete problem (1.3)-(1.4) converge to the solutions of the corresponding continuous problem (1.1)-(1.2).

This study mainly motivated by the numerical approximation of classical radial solutions of the nonlinear Dirichlrt problem for the prescribed mean curvature equation in $\mathbb{L}^{N+1}$:

$$-\operatorname{div}\Big(\frac{\operatorname{grad}v}{\sqrt{1-|\operatorname{grad}v|^2}}\Big)=f\Big(|x|,v,\frac{dv}{dr}\Big)\quad\text{in }\mathcal{B}, \tag{1.5}$$

$$v=0\quad\text{on }\partial\mathcal{B}, \tag{1.6}$$

where $\mathcal{B}$ is the unit ball in $\{(x,t)\in\mathbb{L}^{N+1}:t=0\}\simeq\mathbb{R}^N$, div denotes the divergence operator of $\mathbb{R}^N$, grad$v$ is the gradient of v, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$, $\frac{dv}{dr}$ stands for the radial derivative of v and f is a continuous function. Setting, as usual, $r=|x|$ and $u(r)=v(x)$, we can reduce the Dirichlet problem (1.5)-(1.6) to a problem of type (1.1)-(1.2) with $\phi(s)=\frac{s}{\sqrt{1-s^2}}$, and the solutions of (1.1)-(1.2) are just the classical radial solutions of (1.5)-(1.6).

These problems are originated in the study - in differential geometry or special relativity, of maximal or constant mean curvature hypersurfaces, i.e., spacelike submanifolds of codimension one in $\mathbb{L}^{N+1}$, having the property that their mean curvature is respectively zero or constant, for more details, see Alías and Palmer [2], Bartnik and Simon [4], Bidaut-Veron and Ratto [10], Calabi [13], Cheng and Yau [15], Treibergs [39] and the references therein. In recent years, Dirichlet problems for the prescribed mean curvature equation in $\mathbb{L}^{N+1}$ have been widely concerned by many scholars, and their attention is mainly focused on the positive solutions, we refer the reader to [5–8, 16–20, 22–26, 29, 31–34, 40, 41, 43] and the references therein. In particular, based on the detailed analysis of time map, some exact multiplicity of positive solutions have been obtained in [24, 25, 43], for the radially symmetric solutions on a ball, some existence, nonexistence and multiplicity results have been established in [6, 7], and some bifurcation results have been obtained in [20, 31] via the bifurcation method, and when $\mathcal{B}$ is a general domain in $\mathbb{R}^N$, some existence and bifurcation results have been obtained in the papers [17–19, 33]. In addition to, these concern discrete Dirichlet problems with the mean curvature operator in $\mathbb{L}^{N+1}$, we refer the reader to [9, 14, 27, 28] and the references therein.

Up to our knowledge, the study of numerical approximation of solutions of the Dirichlrt problem for the prescribed mean curvature equation in $\mathbb{L}^{N+1}$ seems lagging behind.

Recently, P. Jebelean et al. [28] considered the extremal solutions of the mixed boundary value problem

$$\begin{aligned} &-(r^{N-1}\phi(u'))' = r^{N-1}f(r,u), \quad r \in (0,1),\\ &u'(0) = u(1) = 0, \end{aligned}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\phi : (-\eta,\eta) \to \mathbb{R}$ $(\eta > 0)$ is an increasing homeomorphism with $\phi(0) = 0$ and $\phi'(y) \geq d > 0$ for all $y \in (-\eta,\eta)$. They proved the existence of minimal and maximal solutions in presence of well-ordered lower and upper solutions and developed a numerical algorithm by combining the shooting method with Euler's method for their approximation.

In 2007, I. Rachunkova et al. [37] investigated the existence of non-spurious solutions of discrete Dirichlet problems for second-order difference equation

$$\begin{aligned} &-\frac{\nabla\triangle u_k}{h^2} = f\Big(t_k, u_k, \frac{\triangle u_k}{h}\Big), \quad k = 1, \cdots, n-1,\\ &u_0 = 0, \quad u_n = 0, \end{aligned}$$

where $n \geq 2$ is an integer, h is step size, t_k are grid points, $u_k := u(t_k)$, $\triangle$ is the forward difference operator defined by $\triangle u_k = u_{k+1} - u_k$, ∇ is the backward difference operator defined by $\nabla u_k = u_k - u_{k-1}$ and f is a continuous function. Their result provides some information for the numerical approximation of solutions of the corresponding continuous problem.

Motivated by the interesting results of [6, 7, 21, 26, 28, 32], in this paper we shall show the solvability of the discrete problem (1.3)-(1.4) and the convergence of their solutions to a solution of the continuous problem (1.1)-(1.2) when the step size converges to 0.

In Section 2, we present a lower and upper solution result for continuous problem (1.1)-(1.2) that permits to consider the convergence of solutions of discrete problem (1.3)-(1.4). For the lower and upper solution method to this type of problem with periodic or Neumann boundary conditions, we refer the reader to [8, 11, 35], and the case of elliptic boundary value problem, we refer the reader to [12, 30, 42]. In Section 3, we give some notations and the fixed point reformulation of (1.3)-(1.4) and prove all possible solutions of (1.3)-(1.4) and their first differences have a prior bounds which are independent of h, based on this, we develop the well-order lower and upper solution method for (1.3)-(1.4) in two cases: for arbitrary fixed step size and for sufficiently small step size. In Section 4, the ideas of lower μ-solution and upper μ-solution from [21] are applied to show the solutions of discrete problem (1.3)-(1.4) can converge to the solutions of the continuous problem (1.1)-(1.2), and the result has an important theoretical implications for computing the numerical solutions of the prescribed mean curvature equation in Minkowski space.

For the classical results on difference equations and their comparison with differential equations, including existence, uniqueness and spurious solutions, we refer the reader to [1, 3, 36, 38] and the references therein.

2. Lower and upper solutions

In this section, we extend the lower and upper solution method in [7] to the prescribed mean curvature problem (1.1)-(1.2).

Here and hereafter, let $C[0,1]$ denote the Banach space of continuous functions on $[0,1]$ endowed with the usual norm $||\cdot||_\infty$, $C^1[0,1]$ denote the Banach space of continuously differentiable functions on $[0,1]$ endowed with the norm $||u|| = ||u||_\infty + ||u'||_\infty$. We say that a function $u \in C^1[0,1]$ is a solution of (1.1)-(1.2) if $||u'||_\infty < a$, $r^{N-1}\phi(u') \in C^1[0,1]$, and (1.1)-(1.2) is satisfied.

Definition 2.1. A lower solution of (1.1)-(1.2) is a function $\alpha \in C^1[0,1]$ such that $||\alpha'||_\infty < a$, $r^{N-1}\phi(\alpha') \in C^1[0,1]$ and

$$-(r^{N-1}\phi(\alpha'))' \le r^{N-1}f(r,\alpha,\alpha'), \quad r \in (0,1), \qquad \alpha(1) \le 0.$$

An upper solution of (1.1)-(1.2) is a function $\beta \in C^1[0,1]$ such that $||\beta'||_\infty < a$, $r^{N-1}\phi(\beta') \in C^1[0,1]$ and

$$-(r^{N-1}\phi(\beta'))' \ge r^{N-1}f(r,\beta,\beta'), \quad r \in (0,1), \qquad \beta(1) \ge 0.$$

Theorem 2.1. *If (1.1)-(1.2) has a lower solution α and an upper solution β such that $\alpha(r) \le \beta(r)$ for all $r \in [0,1]$, and if $f : [0,1] \times \mathbb{R} \times (-a,a) \to \mathbb{R}$ is continuous and satisfies*

$$f(r,u,w_2) - f(r,u,w_1) \ge 0 \text{ for } r \in [0,1], u \in [\alpha(r),\beta(r)] \text{ and} -a < w_1 \le w_2 < a, \tag{2.1}$$

then (1.1)-(1.2) has at least one solution u such that

$$\alpha(r) \le u(r) \le \beta(r)$$

for all $r \in [0,1]$.

Proof. Let $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(r,u) = \begin{cases} \beta(r), & u > \beta(r), \\ u, & \alpha(r) \le u \le \beta(r), \\ \alpha(r), & u < \alpha(r), \end{cases}$$

and define $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ by $F(r,u,w) = f(r,\gamma(r,u),w)$. We consider the auxiliary problem

$$-(r^{N-1}\phi(u'))' = r^{N-1}[F(r,u,u') - u + \gamma(r,u)], \quad r \in (0,1), \tag{2.2}$$
$$u'(0) = u(1) = 0. \tag{2.3}$$

It follows from [5] that problem (2.2)-(2.3) has at least one solution.

We show that all possible solutions u of (2.2)-(2.3) satisfy $\alpha(r) \le u(r) \le \beta(r)$ for all $r \in [0,1]$. This will conclude the proof.

Suppose that there exists some $r_0 \in [0,1]$ such that

$$\max_{r\in[0,1]}\{u(r) - \beta(r)\} = u(r_0) - \beta(r_0) > 0.$$

If $r_0 \in (0,1)$, then $u'(r_0) = \beta'(r_0)$ and there are sequences $\{r_k\}$ in $[r_0 - \varepsilon, r_0)$ and $\{r'_k\}$ in $(r_0, r_0+\varepsilon]$ converging to r_0 such that $u'(r_k)-\beta'(r_k) \geq 0$ and $u'(r'_k)-\beta'(r'_k) \leq 0$. This fact together with ϕ is an increasing homeomorphism, we have that

$$r_k^{N-1}\phi(\beta'(r_k)) - r_0^{N-1}\phi(\beta'(r_0)) \leq r_k^{N-1}\phi(u'(r_k)) - r_0^{N-1}\phi(u'(r_0)),$$

this implies

$$(r^{N-1}\phi(\beta'(r)))'_{r=r_0} \geq (r^{N-1}\phi(u'(r)))'_{r=r_0}.$$

Note that β is an upper solution of (1.1)-(1.2), it follows that

$$\begin{aligned}(r^{N-1}\phi(\beta'(r)))'_{r=r_0} \geq& (r^{N-1}\phi(u'(r)))'_{r=r_0}\\ =& r_0^{N-1}[-f(r_0,\beta(r_0),u'(r_0)) + u(r_0) - \beta(r_0)]\\ >& r_0^{N-1}[-f(r_0,\beta(r_0),\beta'(r_0))]\\ \geq& (r^{N-1}\phi(\beta'(r)))'_{r=r_0},\end{aligned}$$

but this is a contradiction. If $r_0 = 1$, by using $u(1) = 0$ and $\beta(1) \geq 0$, we obtain a contradiction again. Finally, if $r_0 = 0$, then there exists $r_1 \in (0,1]$ such that $u(r) - \beta(r) > 0$ for all $r \in [0, r_1]$ and $u'(r_1) - \beta'(r_1) \leq 0$, and accordingly, we have that

$$r_1^{N-1}\phi(\beta'(r_1)) \geq r_1^{N-1}\phi(u'(r_1)).$$

Note that (2.1) and use β is an upper solution of (1.1)-(1.2) again, we can show that

$$\begin{aligned}r_1^{N-1}\phi(\beta'(r_1)) \geq& r_1^{N-1}\phi(u'(r_1))\\ =& r_1^{N-1}[-f(r_1,\beta(r_1),u'(r_1)) + u(r_1) - \beta(r_1)]\\ >& r_1^{N-1}[-f(r_1,\beta(r_1),u'(r_1))]\\ \geq& r_1^{N-1}[-f(r_1,\beta(r_1),\beta'(r_1))]\\ \geq& r_1^{N-1}\phi(\beta'(r_1)),\end{aligned}$$

clearly, this is a contradiction. Consequently, we prove that $u(r) \leq \beta(r)$ for all $r \in [0,1]$, analogously, we can prove that $\alpha(r) \leq u(r)$ for all $r \in [0,1]$. The proof is completed. □

Remark 2.1. The definitions of α and β do not impose any conditions on their derivatives at $r = 0$.

Remark 2.2. The proof of Theorem 2.1 follows the idea of the proof of Proposition 1 in [7].

3. Fixed point, a priori bound, lower and upper solutions

For $\overrightarrow{u} \in \mathbb{R}^p$ set $|\overrightarrow{u}|_\infty = \max\limits_{1\leq k\leq p} |u_k|$. For any $\overrightarrow{u} \in \mathbb{R}^p$, where $p \geq 3$ is an integer, we define

$$\Delta\overrightarrow{u} = (\Delta u_1, \cdots, \Delta u_{p-1}) \in \mathbb{R}^{p-1}$$

as follows

$$\Delta u_k = u_{k+1} - u_k, \quad (1 \le k \le p-1).$$

A solution of (1.3)-(1.4) should be a vector $\overrightarrow{u} = (u_0, \cdots, u_n) \in \mathbb{R}^{n+1}$ such that $|\frac{\Delta \overrightarrow{u}}{h}|_\infty < a$ and satisfies (1.3)-(1.4). Let us introduce the vector space

$$W^{n+1} = \{\overrightarrow{u} \in \mathbb{R}^{n+1} : \triangle u_0 = 0, u_n = 0\}$$

endowed with the orientation $(u_0, \cdots, u_n) \in \mathbb{R}^{n+1}$ and the norm $|\cdot|_\infty$.

Now, we give the fixed point reformulation of (1.3)-(1.4). Let $\tau(k) = t_k^{1-N}$, $k = 1, \cdots, n-1$. For each $h > 0$, define

$$S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}, \quad Su_k = -\tau(k) \sum_{i=1}^{k} t_i^{N-1} u_i, \quad k = 1, \cdots, n-1;$$
$$K : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}, \quad Ku_k = -\sum_{i=k}^{n-1} hu_i, \quad k = 1, \cdots, n-1.$$

It is easy to see that $K \circ \phi^{-1} \circ S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is continuous, and for a given function $\overrightarrow{g} = (g_1, \cdots, g_{n-1})$, the discrete problem

$$\begin{aligned} &-\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle u_k}{h}\Big)\Big) = t_k^{N-1} g_k, \quad k = 1, \cdots, n-1, \\ &\triangle u_0 = 0, \quad u_n = 0 \end{aligned}$$

has a unique solution $\overrightarrow{u} \in W^{n+1}$ given by

$$u_k = K \circ \phi^{-1} \circ S \circ g_k.$$

Moreover, we denote by N_f the Nemytskii operator associated to f:

$$N_f : \mathbb{R}^{n+1} \to \mathbb{R}^{n-1}, \;\; N_f(\overrightarrow{u}) = \Big(f\Big(t_1, u_1, \frac{\triangle u_1}{h}\Big), \cdots, f\Big(t_{n-1}, u_{n-1}, \frac{\triangle u_{n-1}}{h}\Big)\Big).$$

Notice that the following result.

Lemma 3.1. *A vector $\overrightarrow{u} = (u_0, \cdots, u_n) \in \mathbb{R}^{n+1}$ is a solution of (1.3)-(1.4) if and only if $\overrightarrow{u} \in W^{n+1}$ is a fixed point of the continuous operator*

$$\mathcal{A}_f : W^{n+1} \to W^{n+1}, \quad \mathcal{A}_f = K \circ \phi^{-1} \circ S \circ (hN_f).$$

Lemma 3.2. *Any fixed point $\overrightarrow{u} \in W^{n+1}$ of $\mathcal{A}_f$ satisfies*

$$\Big|\frac{\Delta \overrightarrow{u}}{h}\Big|_\infty < a \quad \textit{and} \quad |\overrightarrow{u}|_\infty < a.$$

Proof. In terms of the range of ϕ^{-1} the results are obvious, we omit the details. □

In the next result, we provide an a priori bounds to the first differences of all possible solutions of (1.3)-(1.4). This fact will play a key role later.

Lemma 3.3. *For any given $\Lambda > 0$, there exists a constant $\nu = \nu(\Lambda) \in (0, a)$, such that for any $\overrightarrow{e} = (e_1, \cdots, e_{n-1}) \in \mathbb{R}^{n-1}$ with $|\overrightarrow{e}|_\infty \leq \Lambda$, the solution $\overrightarrow{u} = (u_0, \cdots, u_n) \in \mathbb{R}^{n+1}$ of*

$$-\frac{1}{h}\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle u_k}{h}\Big)\Big) = e_k, \quad k = 1, \cdots, n-1, \tag{3.1}$$

$$\triangle u_0 = 0, \quad u_n = 0 \tag{3.2}$$

satisfies

$$\Big|\frac{\triangle \overrightarrow{u}}{h}\Big|_\infty \leq a - \nu < a. \tag{3.3}$$

Proof. Let $\overrightarrow{u}$ be a solution of (3.1)-(3.2). It is easy to see that

$$\Big| -\frac{1}{h}\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle u_k}{h}\Big)\Big)\Big| \leq \Lambda$$

for $k = 1, \cdots, n-1$. This implies

$$\Big|t_k^{N-1}\phi\Big(\frac{\triangle u_k}{h}\Big) - t_{k-1}^{N-1}\phi\Big(\frac{\triangle u_{k-1}}{h}\Big)\Big| \leq h\Lambda \tag{3.4}$$

for $k = 1, \cdots, n-1$. Using (3.4) we deduce that

$$\Big|\phi\Big(\frac{\triangle u_k}{h}\Big)\Big| \leq t_k^{1-N} \cdot kh\Lambda < +\infty$$

for $k = 1, \cdots, n-1$, and accordingly, there exists a constant $\nu = \nu(\Lambda) \in (0, a)$, such that

$$\Big|\frac{\triangle u_k}{h}\Big| \leq a - \nu < a$$

for $k = 1, \cdots, n-1$, and we complete the proof. □

The rest of this section, we develop the lower and upper solution method for (1.3)-(1.4), and we give the definition of the lower and upper solutions as follows.

Definition 3.1. We call $\overrightarrow{\alpha} = (\alpha_0, \cdots, \alpha_n) \in \mathbb{R}^{n+1}$ is a lower solution of (1.3)-(1.4) if $|\frac{\triangle \overrightarrow{\alpha}}{h}|_\infty < a$ and

$$-\frac{1}{h}\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle \alpha_k}{h}\Big)\Big) \leq t_k^{N-1} f\Big(t_k, \alpha_k, \frac{\triangle \alpha_k}{h}\Big), \quad k = 1, \cdots, n-1,$$
$$\triangle \alpha_0 = 0, \qquad \alpha_n \leq 0.$$

We call $\overrightarrow{\beta} = (\beta_0, \cdots, \beta_n) \in \mathbb{R}^{n+1}$ is an upper solution of (1.3)-(1.4) if $|\frac{\triangle \overrightarrow{\beta}}{h}|_\infty < a$ and

$$-\frac{1}{h}\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle \beta_k}{h}\Big)\Big) \geq t_k^{N-1} f\Big(t_k, \beta_k, \frac{\triangle \beta_k}{h}\Big), \quad k = 1, \cdots, n-1,$$
$$\triangle \beta_0 = 0, \qquad \beta_n \geq 0.$$

Theorem 3.1. *If (1.3)-(1.4) has a lower solution* $\overrightarrow{\alpha}=(\alpha_0,\cdots,\alpha_n)$ *and an upper solution* $\overrightarrow{\beta}=(\beta_0,\cdots,\beta_n)$ *such that* $\alpha_k\le\beta_k$ *for* $k=0,\cdots,n-1$*,* f *is continuous and satisfies*

$$f(r,u,w_2)-f(r,u,w_1)\ge 0 \text{ for } r\in[0,1],u\in[\alpha(r),\beta(r)] \text{ and} -a<w_1\le w_2<a, \tag{3.5}$$

where $\alpha(r)$ *and* $\beta(r)$ *are continuous functions on* $[0,1]$ *such that* $\alpha_k=\alpha(t_k)$ *and* $\beta_k=\beta(t_k)$ *for* $k=0,\cdots,n$*. Then (1.3)-(1.4) has at least one solution* $\overrightarrow{u}=(u_0,\cdots,u_n)$ *such that*

$$\alpha_k\le u_k\le\beta_k \tag{3.6}$$

for $k=0,\cdots,n$*.*

Proof. For $r\in[0,1]$, $x,z\in\mathbb{R}$, we define functions

$$\omega(r,z)=\begin{cases}\beta(r+h), & z>\beta(r+h),\\ z, & \alpha(r+h)\le z\le\beta(r+h),\\ \alpha(r+h), & z<\alpha(r+h),\end{cases}$$

$$\tilde f\Big(r,x,\frac{z-x}{h}\Big)=\begin{cases} f\Big(r,\beta(r),\dfrac{\omega(r,z)-\beta(r)}{h}\Big)-\dfrac{x-\beta(r)}{x-\beta(r)+1}, & x>\beta(r),\\ f\Big(r,x,\dfrac{\omega(r,z)-x}{h}\Big), & \alpha(r)\le x\le\beta(r),\\ f\Big(r,\alpha(r),\dfrac{\omega(r,z)-\alpha(r)}{h}\Big)+\dfrac{\alpha(r)-x}{\alpha(r)-x+1}, & x<\alpha(r),\end{cases}$$

and consider the auxiliary problem

$$-\frac{1}{h}\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle u_k}{h}\Big)\Big)=t_k^{N-1}\tilde f\Big(t_k,u_k,\frac{\triangle u_k}{h}\Big),\quad k=1,\cdots,n-1, \tag{3.7}$$

$$\triangle u_0=0,\quad u_n=0. \tag{3.8}$$

We claim that (3.7)-(3.8) has at least one solution. In fact, (3.7)-(3.8) is equivalent to the fixed point problem $\overrightarrow{u}=\mathcal{A}_{\tilde f}\overrightarrow{u}$. From $|\frac{\Delta\overrightarrow{u}}{h}|_\infty<a$, we have that $|\overrightarrow{u}|_\infty<a$, and by using the Brouwer fixed point theorem, we deduce that there exists $\overrightarrow{u}\in W^{n+1}$ such that $\overrightarrow{u}=\mathcal{A}_{\tilde f}\overrightarrow{u}$.

Let $\overrightarrow{u}=(u_0,\cdots,u_n)$ be a solution of (3.7)-(3.8). Next we prove that the solution $\overrightarrow{u}$ satisfies (3.6), which ends the proof.

Let $y_k=u_k-\beta_k$, $k=0,\cdots,n$ and suppose that

$$\max\{y_k:k=0,\cdots,n\}=y_m>0. \tag{3.9}$$

It follows from the definition of $\overrightarrow{\beta}$ and the boundary condition $u_n=0$ that $m\in\{0,1,\cdots,n-1\}$. If $m\in\{1,\cdots,n-1\}$, then we have that

$$y_{m+1}\le y_m,\qquad y_{m-1}\le y_m,$$

and

$$\triangle u_m\le\triangle\beta_m,\qquad \triangle u_{m-1}\ge\triangle\beta_{m-1},$$

this together with the monotonicity of ϕ, we have that

$$\phi\Big(\frac{\triangle u_m}{h}\Big) \le \phi\Big(\frac{\triangle \beta_m}{h}\Big), \qquad \phi\Big(\frac{\triangle u_{m-1}}{h}\Big) \ge \phi\Big(\frac{\triangle \beta_{m-1}}{h}\Big),$$

and

$$\frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle u_m}{h}\Big)\Big) \le \frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle \beta_m}{h}\Big)\Big).$$

If $m = 0$, from $\triangle\beta_0 = 0 = \triangle u_0$, we have that $\beta_0 = \beta_1$ and $u_0 = u_1$, this implies that $m = 1$ is also the maximum point, therefore, the case is concluded by the above case.

Note that (3.5) implies that f is nondecreasing on $(-a, a)$ with respect to its third variable, this together with (3.7)-(3.8), (3.9) and Definition 3.1, we have that

$$\begin{aligned}
\frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle \beta_m}{h}\Big)\Big) \ge & \frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle u_m}{h}\Big)\Big) \\
= & -t_m^{N-1}\tilde{f}\Big(t_m, u_m, \frac{\triangle u_m}{h}\Big) \\
= & -t_m^{N-1}\Big[f\Big(t_m, \beta_m, \frac{\omega(t_m, u_{m+1}) - \beta_m}{h}\Big) - \frac{u_m - \beta_m}{u_m - \beta_m + 1}\Big] \\
> & -t_m^{N-1}f\Big(t_m, \beta_m, \frac{\omega(t_m, u_{m+1}) - \beta_m}{h}\Big) \\
\ge & -t_m^{N-1}f\Big(t_m, \beta_m, \frac{\triangle \beta_m}{h}\Big) \\
\ge & \frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle \beta_m}{h}\Big)\Big),
\end{aligned}$$

but this is a contradiction. Therefore, we prove that $u_k \le \beta_k$ for $k = 0, \cdots, n$. Similarly, we can get that $u_k \ge \alpha_k$ for $k = 0, \cdots, n$. The proof is completed. □

Remark 3.1. The proof of Theorem 3.1 follows the idea of the proof of Theorem 3.1 in [37].

Theorem 3.1 is valid for an arbitrary fixed step size h. We also want to consider the convergence of solutions of (1.3)-(1.4) and our consideration there can be restricted to a sufficiently small step, to this end it will be necessary to develop a lower and upper solution method for each sufficiently small step size h .

Theorem 3.2. *If (1.3)-(1.4) has a lower solution $\overrightarrow{\alpha} = (\alpha_0, \cdots, \alpha_n)$ and an upper solution $\overrightarrow{\beta} = (\beta_0, \cdots, \beta_n)$ such that $\alpha_k \le \beta_k$ for $k = 0, \cdots, n-1$, and assume that there exists $M > 0$ such that*

$$|f(r, u, w)| \le M \qquad \text{for } r \in [0, 1], u \in [\alpha(r), \beta(r)] \text{ and } w \in (-a, a), \tag{3.10}$$

where $\alpha(r)$ and $\beta(r)$ are continuous functions on $[0, 1]$ such that $\alpha_k = \alpha(t_k)$ and $\beta_k = \beta(t_k)$ for $k = 0, \cdots, n$. Then there exists $n^* \ge 2$, such that for each $n : n \ge n^*$, (1.3)-(1.4) has at least one solution $\overrightarrow{u} = (u_0, \cdots, u_n)$ and it satisfies (3.6).

Proof. Argue as in the proof of Theorem 3.1, we can get that (3.7)-(3.8), (3.9) and

$$\frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle u_m}{h}\Big)\Big) \le \frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle \beta_m}{h}\Big)\Big) \tag{3.11}$$

for $m = 0, \cdots, n-1$. From (3.9), we have that $u_m > \beta_m$ for some $m \in \{0, 1, \cdots, n-1\}$, this yields that there exists $c > 0$, such that $\beta_m + c = u_m$. In fact, we can show that for sufficiently large n,

$$u_m > \beta_m \Rightarrow u_{m+1} \geq \beta_{m+1}.$$

On one hand, from (3.10) and Lemma 3.3, for any solution $\overrightarrow{u}$ of (3.7)-(3.8), we have that there exists a constant $\nu = \nu(M) \in (0, a)$, such that

$$\Big|\frac{\triangle u_k}{h}\Big| \leq a - \nu < a \quad \text{ for } k = 0, \cdots, n-1. \tag{3.12}$$

On the other hand, from Definition 3.1, we have that there exists a $\rho > 0$ such that

$$\Big|\frac{\triangle \alpha_k}{h}\Big| \leq \rho, \quad \Big|\frac{\triangle \beta_k}{h}\Big| \leq \rho \tag{3.13}$$

for each $n \geq 2$, $h = \frac{1}{n}$ and $k = 0, \cdots, n-1$. From (3.12) and (3.13), we have that

$$\begin{aligned}
u_{m+1} =& u_m + \triangle u_m \\
=& \beta_m + c + \triangle u_m \\
=& \beta_{m+1} - \triangle \beta_m + c + \triangle u_m \\
\geq& \beta_{m+1} + c - |\triangle \beta_m| - |\triangle u_m| \\
\geq& \beta_{m+1} + c - \rho h - (a-\nu)h \\
\geq& \beta_{m+1}
\end{aligned}$$

if $n \geq n^*$ and $n^* = \frac{\rho + a - \nu}{c}$.

Therefore, from the definition of $\omega(r, z)$, we have that $\omega(t_m, u_{m+1}) = \beta_{m+1}$.

And accordingly, by (3.7)-(3.8), (3.9) and Definition 3.1, we get that

$$\begin{aligned}
&\frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle u_m}{h}\Big)\Big) - \frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle \beta_m}{h}\Big)\Big) \\
=& -t_m^{N-1}\tilde{f}\Big(t_m, u_m, \frac{\triangle u_m}{h}\Big) - \frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle \beta_m}{h}\Big)\Big) \\
=& -t_m^{N-1} f\Big(t_m, \beta_m, \frac{\omega(t_m, u_{m+1}) - \beta_m}{h}\Big) + \frac{t_m^{N-1} y_m}{y_m + 1} - \frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle \beta_m}{h}\Big)\Big) \\
=& -t_m^{N-1} f\Big(t_m, \beta_m, \frac{\triangle \beta_m}{h}\Big) + \frac{t_m^{N-1} y_m}{y_m + 1} - \frac{1}{h}\nabla\Big(t_m^{N-1}\phi\Big(\frac{\triangle \beta_m}{h}\Big)\Big) \\
\geq& \frac{t_m^{N-1} y_m}{y_m + 1} \\
>& 0,
\end{aligned}$$

but this contradicts with (3.11). Therefore, we prove that $u_k \leq \beta_k$ for $k = 0, \cdots, n$. Similarly, we can prove that $u_k \geq \alpha_k$ for $k = 0, \cdots, n$. The proof is completed. □

Example 3.1. Assume that there exist $s_1, s_2 \in (0, \infty)$ such that

$$f(r, -s_1, 0) \geq 0, \quad f(r, s_2, 0) \leq 0 \quad \text{ for all } r \in [0, 1],$$

and there exists $M > 0$ such that

$$|f(r, u, w)| \leq M \quad \text{ for } r \in [0, 1], u \in [-s_1, s_2] \text{ and } w \in (-a, a).$$

Then by Theorem 3.2, there exists $n^* \geq 2$ such that for each $n \geq n^*$, problem (1.3)-(1.4) has a solution $\overrightarrow{u} = (u_0, \cdots, u_n)$, and it satisfies

$$-s_1 \leq u_k \leq s_2$$

for $k = 0, \cdots, n$.

4. Non-spurious solutions

In this section, we consider the convergence of solutions of (1.3)-(1.4). The ideas of this section from the work of R. Gaines [21].

We need the following definition.

Definition 4.1. Let $\mu > 0$ be a constant. We call function $\alpha \in C^1[0,1]$ is a lower μ-solution of (1.1)-(1.2) if $||\alpha'||_\infty < a$, $r^{N-1}\phi(\alpha') \in C^1[0,1]$ and

$$\begin{aligned}&(r^{N-1}\phi(\alpha'))' + r^{N-1}f(r, \alpha, \alpha') \geq \mu, \qquad r \in (0,1),\\ &\alpha'(0) \leq -\mu, \qquad \alpha(1) \leq -\mu.\end{aligned}$$

We call function $\beta \in C^1[0,1]$ is an upper μ-solution of (1.1)-(1.2) if $||\beta'||_\infty < a$, $r^{N-1}\phi(\beta') \in C^1[0,1]$ and

$$\begin{aligned}&(r^{N-1}\phi(\beta'))' + r^{N-1}f(r, \beta, \beta') \leq -\mu, \qquad r \in (0,1),\\ &\beta'(0) \geq \mu, \qquad \beta(1) \geq \mu.\end{aligned}$$

Lemma 4.1. *If (1.1)-(1.2) has a lower μ-solution α and an upper μ-solution β, then there exists a constant $\delta(\mu) > 0$ such that*

$$\begin{aligned}&\frac{1}{h}\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle\alpha_k}{h}\Big)\Big) + t_k^{N-1}f\Big(t_k, \alpha_k, \frac{\triangle\alpha_k}{h}\Big) \geq \frac{\mu}{2}, \quad k = 1, \cdots, n-1,\\ &\triangle\alpha_0 \leq -\frac{\mu}{2}, \qquad \alpha_n \leq -\frac{\mu}{2}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{h}\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle\beta_k}{h}\Big)\Big) + t_k^{N-1}f\Big(t_k, \beta_k, \frac{\triangle\beta_k}{h}\Big) \leq -\frac{\mu}{2}, \quad k = 1, \cdots, n-1,\\ &\triangle\beta_0 \geq \frac{\mu}{2}, \qquad \beta_n \geq \frac{\mu}{2}\end{aligned}$$

for each $h < \delta(\mu)$.

Proof. Since β is an upper μ-solution of (1.1)-(1.2) and ϕ is continuous, we have that there exists $\xi_k^1 \in (t_k, t_{k+1})$ such that

$$\frac{\Delta\beta_k}{h} = \frac{\beta(t_{k+1}) - \beta(t_k)}{h} = \beta'(\xi_k^1) = \beta'(t_k) + \big[\,\beta'(\xi_k^1) - \beta'(t_k)\,\big] \tag{4.1}$$

for $k = 0, \cdots, n-1$, and there exists $\xi_k^2 \in (t_{k-1}, t_{k+1})$ such that

$$\frac{\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle\beta_k}{h}\Big)\Big)}{h}$$

$$
\begin{aligned}
&= \frac{t_k^{N-1}\phi\Big(\frac{\triangle\beta_k}{h}\Big) - t_{k-1}^{N-1}\phi\Big(\frac{\triangle\beta_{k-1}}{h}\Big)}{h} \\
&= \Big((\xi_k^2)^{N-1}\phi\big(\beta'(\xi_k^2)\big)\Big)' \\
&= \Big(t_k^{N-1}\phi\big(\beta'(t_k)\big)\Big)' + \Big[\Big((\xi_k^2)^{N-1}\phi\big(\beta'(\xi_k^2)\big)\Big)' - \Big(t_k^{N-1}\phi(\beta'(t_k))\Big)'\Big] \qquad (4.2)
\end{aligned}
$$

for $k = 1, \cdots, n-1$.

Assume that there exists $\delta_1(\mu) > 0$ such that if $|r_1 - r_2| < \delta_1(\mu)$ then

$$\Big|\Big(r_1^{N-1}\phi(\beta'(r_1))\Big)' - \Big(r_2^{N-1}\phi(\beta'(r_2))\Big)'\Big| \le \frac{\mu}{4}. \qquad (4.3)$$

Let

$$S = \{(r, y, z) : y = \beta(r), |z - \beta'(r)| < 2a\}.$$

Assume that there exists $\sigma(\mu) > 0$ such that if $|z_1 - z_2| < \sigma(\mu)$ then

$$|r^{N-1}f(r, y, z_1) - r^{N-1}f(r, y, z_2)| \le \frac{\mu}{4} \qquad (4.4)$$

for $(r, y, z_1), (r, y, z_2) \in S$.

Assume that there exists $\delta_2(\mu) > 0$ such that if $|r_1 - r_2| < \delta_2(\mu)$ then

$$|\beta'(r_1) - \beta'(r_2)| < \min\{\sigma(\mu), 2a\}. \qquad (4.5)$$

By (4.1)-(4.5) and Definition 4.1, if $h < \min\{\delta_1(\mu), \delta_2(\mu)\}$, then we have that

$$
\begin{aligned}
&\frac{1}{h}\nabla\Big(t_k^{N-1}\phi\Big(\frac{\triangle\beta_k}{h}\Big)\Big) + t_k^{N-1}f\Big(t_k, \beta_k, \frac{\triangle\beta_k}{h}\Big) \\
=&\Big(t_k^{N-1}\phi\big(\beta'(t_k)\big)\Big)' + \Big[\Big((\xi_k^2)^{N-1}\phi\big(\beta'(\xi_k^2)\big)\Big)' - \Big(t_k^{N-1}\phi(\beta'(t_k))\Big)'\Big] \\
&+ t_k^{N-1}f\Big(t_k, \beta_k, \beta'(t_k) + \big(\beta'(\xi_k^1) - \beta'(t_k)\big)\Big) \\
\le&\Big(t_k^{N-1}\phi\big(\beta'(t_k)\big)\Big)' + \frac{\mu}{4} + t_k^{N-1}f\Big(t_k, \beta_k, \beta'(t_k)\Big) + \frac{\mu}{4} \\
\le& -\mu + \frac{\mu}{4} + \frac{\mu}{4} \\
=& -\frac{\mu}{2}.
\end{aligned}
$$

From (4.1), we have that

$$\frac{\Delta\beta_0}{h} = \beta'(t_0) + \big[\,\beta'(\xi_0^1) - \beta'(t_0)\,\big].$$

Similarly, we assume that there exists $\delta_3(\mu) > 0$ such that if $|r_1 - r_2| < \delta_3(\mu)$ then

$$|\beta'(r_1) - \beta'(r_2)| \le \frac{\mu}{2}.$$

From this fact and $\beta'(0) \ge \mu$, we can conclude that if $h < \delta_3(\mu)$, then $\triangle\beta_0 \ge \frac{\mu}{2}$. $\beta_n = \beta(t_n) = \beta(1) \ge \frac{\mu}{2}$ is obvious. In a similar argument, we can get the conclusion of α. The proof is completed. □

Next, we give a result which describes the sense in which the solutions we shall obtain for the discrete problem will converge to the solutions of the continuous problem. We first let

$$n_m \to +\infty \text{ as } m \to +\infty, \quad h_m = \frac{1}{n_m},$$
$$t_k^m = kh_m \text{ for } k = 0, \cdots, n \text{ and } t_0^m = 0, t_n^m = 1.$$

Assume that (1.3)-(1.4) has a solution $\overrightarrow{u}^m = (u_0^m, \cdots, u_n^m)$ for $h = h_m$ and $m \geq m_0$. Define the continuous function $u^m(r)$ by linear interpolation such that $u^m(t_k^m) = u_k^m$, i.e.

$$u^m(r) = u_k^m + \frac{u_{k+1}^m - u_k^m}{h_m}(r - t_k^m), \quad t_k^m \leq r \leq t_{k+1}^m.$$

Define $y_k^m = \frac{u_k^m - u_{k-1}^m}{h_m}$, $\overrightarrow{y}^m = (y_1^m, \cdots, y_n^m)$ and $y^m(r)$ on $[0,1]$ by

$$y^m(r) = \begin{cases} y_k^m + \dfrac{y_{k+1}^m - y_k^m}{h_m}(r - t_k^m), & t_k^m \leq r \leq t_{k+1}^m, \\ y_1^m, & 0 \leq r \leq t_1^m. \end{cases}$$

We have the following result.

Lemma 4.2. *If $\overrightarrow{u}^m = (u_0^m, \cdots, u_n^m)$ is a solution of (1.3)-(1.4) for $h = h_m$ and $m \geq m_0$, and there exist constant $R_1 \geq 0$ and $R_2 \geq 0$ such that*

$$|\overrightarrow{u}^m|_\infty \leq R_1, \quad |\overrightarrow{y}^m|_\infty \leq R_2.$$

Then there exists a sequence $\{u^{k(m)}(r)\}$ and a solution $u(r)$ of (1.1)-(1.2) such that

$$\max_{r\in[0,1]} \left|u^{k(m)}(r) - u(r)\right| \to 0 \quad \text{and} \quad \max_{r\in[0,1]} \left|y^{k(m)}(r) - u'(r)\right| \to 0$$

as $m \to +\infty$.

Proof. By Arzela-Ascoli theorem it can be shown that there exists a subsequence $\{u^{k(m)}(r)\}$ such that $\{u^{k(m)}(r)\}$ converges uniformly to a continuous function $u(r)$, and $\{y^{k(m)}(r)\}$ converges uniformly to a continuous function $y(r)$. From Lemma 3.1 and the continuity of f, we can in a direct way prove the limit $u(r)$ is a solution of continuous problem (1.1)-(1.2). The conclusion $y(r) = u'(r)$ follows immediately from the fact

$$|u(\tau) - u(r) - y(r)(\tau - r)| \leq H|\tau - r|^2.$$

The proof is completed. □

We have the following convergence result.

Theorem 4.1. *If (1.1)-(1.2) has a lower μ-solution α and an upper μ-solution β such that $\alpha(r) \leq \beta(r)$ for all $r \in [0,1]$, and if $f : [0,1] \times \mathbb{R} \times (-a,a) \to \mathbb{R}$ is continuous and satisfies*

$$f(r,u,w_2) - f(r,u,w_1) \geq 0 \text{ for } r \in [0,1], u \in [\alpha(r), \beta(r)] \text{ and} -a < w_1 \leq w_2 < a, \tag{4.6}$$

and there exists $M > 0$ *such that*

$$|f(r,u,w)| \le M \quad \text{for } r \in [0,1], u \in [\alpha(r), \beta(r)] \text{ and } w \in (-a,a). \tag{4.7}$$

Then

(i) problem (1.1)-(1.2) has at least one solution u *such that* $\alpha(r) \le u(r) \le \beta(r)$ *for all* $r \in [0,1]$;

(ii) there exists a constant $\delta(\mu) > 0$, *such that for* $h < \delta(\mu)$, *problem (1.3)-(1.4) has at least one solution* $\overrightarrow{u} = (u_0, \cdots, u_n)$ *satisfying* $\alpha(t_k) \le u_k \le \beta(t_k)$ *for* $k = 0, \cdots, n$;

(iii) the solutions $\overrightarrow{u} = (u_0, \cdots, u_n)$ *of (1.3)-(1.4) with* $\alpha(t_k) \le u_k \le \beta(t_k)$ *for* $k = 0, \cdots, n$, *converge to the solutions of (1.1)-(1.2) in the following sense:*

Let u *be a solution of (1.1)-(1.2). For any* $\varepsilon > 0$, *there exists a* $h(\varepsilon) > 0$, *such that if* $h \le h(\varepsilon)$, *then there is a solution* $\overrightarrow{u} = (u_0, \cdots, u_n)$ *of (1.3)-(1.4) such that*

$$\max_{r\in[0,1]} |u(r, \overrightarrow{u}) - u(r)| \le \varepsilon,$$
$$\max_{r\in[0,1]} |y(r, \overrightarrow{u}) - u'(r)| \le \varepsilon,$$

where

$$u(r, \overrightarrow{u}) := u_k + \frac{u_{k+1} - u_k}{h}(r - t_k), \quad t_k \le r \le t_{k+1},$$
$$y(r, \overrightarrow{u}) := \begin{cases} \dfrac{u_k - u_{k-1}}{h} + \dfrac{u_{k+1} - 2u_k + u_{k-1}}{h^2}(r - t_k), & t_k \le r \le t_{k+1}, \\ \dfrac{u_1 - u_0}{h}, & 0 \le r \le t_1. \end{cases}$$

Proof. (i) Clearly, α can be a lower solution and β can be an upper solution of (1.1)-(1.2). Therefore, from (4.6) and Theorem 2.1, we have the conclusion (i).

(ii) From Lemma 4.1, we have that there exists a $\delta(\mu) > 0$ such that

$$\overrightarrow{\alpha} = (\alpha_0, \cdots, \alpha_n) := (\alpha(t_0), \cdots, \alpha(t_n))$$

and

$$\overrightarrow{\beta} = (\beta_0, \cdots, \beta_n) := (\beta(t_0), \cdots, \beta(t_n))$$

being, respectively, a lower and an upper solution of (1.3)-(1.4) for each $h < \delta(\mu)$, and $\alpha_k \le \beta_k$ for $k = 0, \cdots, n-1$. This together with (4.7), all of the assumptions of Theorem 3.2 are satisfied, and the conclusion (ii) follows from there.

(iii) Assume the conclusion is false, then there exist a $\varepsilon > 0$ and a sequence $\{h_m\}$ such that $h_m \to 0$, and for $h = h_m = \frac{1}{n_m}$, the problem (1.3)-(1.4) has a solution $\overrightarrow{u}^m = (u_0^m, \cdots, u_n^m)$ such that for every solution $u(r)$ of (1.1)-(1.2) one of the inequalities

$$\max_{r\in[0,1]} |u(r, \overrightarrow{u}) - u(r)| > \varepsilon, \tag{4.8}$$
$$\max_{r\in[0,1]} |y(r, \overrightarrow{u}) - u'(r)| > \varepsilon \tag{4.9}$$

holds.

By hypothesis, for sufficiently large m, there exist constant $R_1 \ge 0$ and $R_2 \ge 0$ such that

$$|\overrightarrow{u}^m|_\infty \le R_1 \quad \text{and} \quad |\overrightarrow{y}^m|_\infty \le R_2.$$

Therefore, from Lemma 4.2, we have that there exist a subsequence $\{u^{k(m)}(r)\}$ and a solution $u(r)$ of (1.1)-(1.2) such that

$$\max_{r\in[0,1]}\left|u^{k(m)}(r)-u(r)\right|\to 0 \quad \text{and} \quad \max_{r\in[0,1]}\left|y^{k(m)}(r)-u'(r)\right|\to 0$$

as $m\to+\infty$. However, this contract with (4.8) and (4.9). □

Acknowledgements

We are very grateful to the anonymous referees for their valuable suggestions.

References

[1] R. P. Agarwal, *Difference equations and inequalities, in: Theory, Methods, and Applications, second ed.,* in: Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2000.

[2] L. J. Alías and B. Palmer, *On the Gaussian curvature of maximal surfaces and the Calabi-Bernstein theorem*, Bull. London Math. Soc., 2001, 33, 454–458.

[3] U. M. Ascher, R. M. M. R. Mattheij and D. Russell, *Numerical solutions of boundary value problems for ordinary differential equations*, SIAM Philadelphia PA, 1995.

[4] R. Bartnik and L. Simon, *Spacelike hypersurfaces with prescribed boundary values and mean curvature*, Comm. Math. Phys., 1982–1983, 87, 131–152.

[5] C. Bereanu, P. Jebelean and J. Mawhin, *Radial solutions for some nonlinear problems involving mean curvature operators in Euclidean and Minkowski spaces*, Proc. Amer. Math. Soc., 2009, 137, 171–178.

[6] C. Bereanu, P. Jebelean and P. J. Torres, *Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space*, J. Funct. Anal., 2013, 264, 270–287.

[7] C. Bereanu, P. Jebelean and P. J. Torres, *Multiple positive radial solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space*, J. Funct. Anal., 2013, 265, 644–659.

[8] C. Bereanu and J. Mawhin, *Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian*, J. Differential Equations, 2007, 243, 536–557.

[9] C. Bereanu and J. Mawhin, *Boundary value problems for second-order nonlinear difference equations with discrete ϕ-Laplacian and singular ϕ*, J. Difference Equ. Appl., 2008, 14(10–11), 1099–1118.

[10] M. F. Bidaut-Veron and A. Ratto, *Spacelike graphs with prescribed mean curvature*, Differential Integral Equations, 1997, 10(5), 1003–1017.

[11] A. Cabada, P. Habets and R. L. Pouso, *Lower and upper solutions for the periodic problem associated with a ϕ-Laplacian equation. In: International Conference on Differential Equations*, World Scientific Publishing, River Edge, 2000.

[12] A. Cabada and R. L. Pouso, *Extremal solutions of strongly nonlinear discontinuous second-order equations with nonlinear functional boundary conditions*, Nonlinear Anal., 2000, 42(8), 1377–1396.

[13] E. Calabi, *Examples of Bernstein problems for some nonlinear equations*, Proc. Sympos. Pure Math., 1970, 15, 223–230.

[14] T. Chen, R. Ma and Y. Liang, *Multiple positive solutions of second-order nonlinear difference equations with discrete singular ϕ-Laplacian*, J. Difference Equ. Appl., 2019, 25(1), 38–55.

[15] S. Cheng and S. T. Yau, *Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces*, Ann. of Math., 1976, 104, 407–419.

[16] I. Coelho, C. Corsato, F. Obersnel and P. Omari, *Positive solutions of the Dirichlet problem for the one-dimensional Minkowski-curvature equation*, Adv. Nonlinear Stud., 2012, 12(3), 621–638.

[17] C. Corsato, F. Obersnel, P. Omari and S. Rivetti, *Positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space*, J. Math. Anal. Appl., 2013, 405, 227–239.

[18] G. Dai, *Bifurcation and nonnegative solutions for problems with mean curvature operator on general domain*, Indiana Univ. Math. J., 2018, 67(6), 2103–2121.

[19] G. Dai, *Global structure of one-sign solutions for problem with mean curvature operator*, Nonlinearity, 2018, 31(11), 5309–5328.

[20] G. Dai, *Bifurcation and positive solutions for problem with mean curvature operator in Minkowski space*, Calc. Var. Partial Differential Equations, 2016, 55, 1–17.

[21] R. Gaines, *Difference equations associated with boundary value problems for second order nonlinear ordinary differential equations*, SIAM J. Numer. Anal., 1974, 11, 411–434.

[22] D. Gurban and P. Jebelean, *Positive radial solutions for multiparameter Dirichlet systems with mean curvature operator in Minkowski space and Lane-Emden type nonlinearities*, J. Differential Equations, 2019, 266(9), 5377–5396.

[23] D. Gurban, P. Jebelean and C. Serban, *Non-potential and non-radial Dirichlet systems with mean curvature operator in Minkowski space*, Discrete Contin. Dyn. Syst., 2020, 40, 133–151.

[24] S. Huang, *Classification and evolution of bifurcation curves for the one dimensional Minkowski curvature problem and its applications*, J. Differential Equations, 2018, 264, 5977–6011.

[25] S. Huang, *Bifurcation diagrams of positive solutions for one-dimensional Minkowski-curvature problem and its applications*, Discrete Contin. Dyn. Syst., 2019, 39(6), 3443–3462.

[26] S. Huang and M. Huang, *Bifurcation curves of positive solutions for the Minkowski-curvature problem with cubic nonlinearity*, Electron, J. Qual. Theory Differ. Equ., 2021, 41, 29.

[27] P. Jebelean and C. Popa, *Numerical solutions to singular ϕ-Laplacian with Dirichlet boundary conditions*, Numer. Algorithms, 2014, 67, 305–318.

[28] P. Jebelean, C. Popa and C. Serban, *Numerical extremal solutions for a mixed problem with singular ϕ-Laplacian*, NoDEA Nonlinear Differential Equations Appl., 2014, 21, 289–304.

[29] Y. H. Lee, I. Sim and R. Yang, *Bifurcation and Calabi-Bernstein type asymptotic property of solutions for the one-dimensional Minkowski-curvature equation*, J. Math. Anal. Appl., 2022, 507(1), 16.

[30] Y. Li, *Positive radial solutions for elliptic equations with nonlinear gradient terms on the unit ball*, Mediterr. J. Math., 2020, 17, 1–13.

[31] R. Ma, H. Gao and Y. Lu, *Global structure of radial positive solutions for a prescribed mean curvature problem in a ball*, J. Funct. Anal., 2016, 270, 2430–2455.

[32] R. Ma, L. Wei and Z. Chen, *Evolution of bifurcation curves for one-dimensional Minkowski-curvature problem*, Appl. Math. Lett., 2020, 103, 1–8.

[33] R. Ma and M. Xu, *Multiple positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space*, Discrete Contin. Dyn. Syst. Ser. B, 2019, 24, 2701–2718.

[34] R. Ma and M. Xu, *Existence of infinitely many radial nodal solutions for a Dirichlet problem involving mean curvature operator in Minkowski space*, Electron. J. Qual. Theory Differ. Equ., 2020, 27, 14.

[35] R. Ma, M. Xu and Z. He, *Nonconstant positive radial solutions for Neumann problem involving the mean extrinsic curvature operator*, J. Math. Anal. Appl., 2020, 484(2), 13.

[36] J. Myjak, *Boundary value problems for nonlinear differential and difference equations of the second order*, Zeszyty Nauk. Uniw. Jagiello. Prace Mat., 1971, 15, 113–123.

[37] I. Rachunkova and C. C. Tisdell, *Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions*, Nonlinear Anal., 2007, 67, 1236–1245.

[38] H. B. Thompson and C. Tisdell, *Boundary value problems for systems of difference equations associated with systems of second-order ordinary differential equations*, Appl. Math. Lett., 2002, 15, 761–766.

[39] A. E. Treibergs, *Entire spacelike hypersurfaces of constant mean curvature in Minkowski space*, Invent. Math., 1982, 66, 39–56.

[40] R. Yang, Y. H. Lee and I. Sim, *Bifurcation of nodal radial solutions for a prescribed mean curvature problem on an exterior domain*, J. Differential Equations, 2020, 268, 4464–4490.

[41] R. Yang, I. Sim and Y. H. Lee, *$\frac{\pi}{4}$-tangentiality of solutions for one-dimensional Minkowski-curvature problems*, Adv. Nonlinear Anal., 2020, 9(1), 1463–1479.

[42] X. Zhang, *Lower and upper solutions for delay evolution equations with nonlocal and impulsive conditions*, Electron. J. Differential Equations, 2022, 31, 14.

[43] X. Zhang and M. Feng, *Bifurcation diagrams and exact multiplicity of positive solutions of one-dimensional prescribed mean curvature equation in Minkowski space*, Commun. Contemp. Math., 2019, 21, 1–17.