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TRAVELING WAVE SOLUTIONS, POWER SERIES SOLUTIONS AND CONSERVATION LAWS OF THE NONLINEAR DISPERSION EQUATION

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Abstract In this paper, the nonlinear dispersive equation is investigated by Lie symmetry analysis theory and bifurcation theory. The infinitesimal generators of the equation are obtained by Lie symmetry analysis. Periodic peakon solutions, single period solutions and power series solutions of the equation are acquired. And the conservation laws are obtained by the Ibragimov's method.

Keywords Lie symmetry analysis, bifurcation, traveling wave solutions, power series solutions, conservation laws.

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1. Introduction

In 1993, Rosenau and Hyman [22] investigated the effect of nonlinear dispersion on patterns formation in liquid drops by introducing the following nonlinear dispersion $K(m, n)$ equation

$$\psi_t + (\psi^m)_x + (\psi^n)_{xxx} = 0. \tag{1.1}$$

Thereafter, many scholars studied the exact solution of Eq. (1.1). In addition to studying Eq. (1.1) and its generalized equations [4, 6, 23], the solution of the following $K(2, 2)$ equation

$$\psi_t + (\psi^2)_x + (\psi^2)_{xxx} = 0 \tag{1.2}$$

as the classical solution for the $K(m, n)$ equation also deserves to be studied. In 1998, Ismail and Taha [10] obtained the numerical solution of Eq. (1.2) by the finite element and the finite difference method. Subsequently, in 2002, Wazwaz [24] acquired new soliton solutions of Eq.(1.2) by means of the Adomian decomposition method. In 2007, the Adomian method was optimized in [5, 9]. In [9], by the variational iteration method, the approximate numerical solutions of the $K(2, 2)$ equation and the compacton solutions with initial conditions were obtained. Domairry [5] investigated the exact numerical solutions of the $K(2, 2)$ equation by means of the homogeneous perturbation method. The above two methods can get the solutions of Eq. (1.2) without calculating the Adomian polynomial. Zhang and Li [28] gave the implicit loop soliton and periodic solutions of Eq. (1.2) by applying

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dynamical system theory and studied the asymptotic properties of the solutions. Li and Tang [11] used the same method to obtain the implicit analytical and loop solutions of Eq. (1.2) and discussed the convergence of the solutions. Yin and Tian [27] investigated more traveling wave solutions of the following $K(2,2)$ equation

$$\psi_t + \alpha(\psi^2)_x + (\psi^2)_{xxx} = 0 \tag{1.3}$$

by the qualitative analysis of Lenells, giving for the first time periodic compact solutions of Eq. (1.3). And they pointed out that the parameter α has an important influence on the solutions of Eq. (1.3).

Many methods are used to study exact solutions of nonlinear dispersion equation, such as Riemann-Hilbert method [13, 16, 26], bifurcation method [14, 25], Jacobi elliptic function method [21], extend tanh method [2], Lie symmetry method [1,3,12] and so on [7, 17–20]. In this paper, explicit periodic peakon solutions, single period solutions and power series solutions of Eq. (1.3) are studied by the Lie symmetry analysis method and the bifurcation method.

This article is structured as follows. Section 2, the generators are obtained by Lie symmetric analysis method. Section 3, we obtained the phase portraits and the explicit traveling wave solutions by using the dynamical system method. Section 4, the power series solutions are constructed and the convergence of the solution is proved. Section 5, it is proven that Eq. (1.3) is nonlinearly self-adjoint and conservation laws [8, 15] are constructed.

2. Lie symmetrical analysis

In this section, Lie symmetry analysis is performed to obtain the generators. Eq. (1.3) also be written in the following form

$$\psi_t + \beta\psi\psi_x + 6\psi_x\psi_{xx} + 2\psi\psi_{xxx} = 0. \tag{2.1}$$

Consider the Lie group transformation of Eq. (2.1)

$$\begin{aligned} \tilde{x} &= x + \epsilon\zeta_x(x,t,\psi) + O(\epsilon^2), \\ \tilde{t} &= t + \epsilon\zeta_t(x,t,\psi) + O(\epsilon^2), \\ \tilde{\psi} &= \psi + \varsigma(x,t,\psi) + O(\epsilon^2), \end{aligned} \tag{2.2}$$

in which ζ_x, ζ_t, ς are infinitesimal generators, ϵ is one-parameter. The corresponding vector field is

$$\mathcal{X} = \zeta_x\partial_x + \zeta_t\partial_t + \varsigma\partial_\psi. \tag{2.3}$$

The third-order prolongation of $\mathcal{X}$ is

$$pr^{(3)}\mathcal{X} = \zeta_x\frac{\partial}{\partial x} + \zeta_t\frac{\partial}{\partial t} + \varsigma\frac{\partial}{\partial\psi} + \varsigma^{xx}\frac{\partial}{\partial\psi_{xx}} + \varsigma^{xxx}\frac{\partial}{\partial\psi_{xxx}}. \tag{2.4}$$

The infinitesimal generators of (2.4) are represented as

$$\begin{aligned} \varsigma^{xx} &= D_{xx}(\varsigma - \zeta_x\psi_x - \zeta_t\psi_t) + \zeta_x\psi_{xxx} + \zeta_t\psi_{xxt}, \\ \varsigma^{xxx} &= D_{xxx}(\varsigma - \zeta_x\psi_x - \zeta_t\psi_t) + \zeta_x\psi_{xxxx} + \zeta_t\psi_{xxxt}. \end{aligned} \tag{2.5}$$

In addition, ζ_x, ζ_t, ς satisfy the invariance condition

$$pr^{(3)}(\triangle)|_{\triangle=0} = 0, \tag{2.6}$$

where $\triangle = \psi_t + \beta\psi\psi_x + 6\psi_x\psi_{xx} + 2\psi\psi_{xxx}$.

Substituting (2.5) into (2.6) yields infinitesimal generators as following form

$$\zeta_x = c_3, \qquad \zeta_t = c_1 t + c_2, \qquad \varsigma = -c_1\psi, \tag{2.7}$$

in which c_i(i=1, 2, 3) are arbitrary constants. Then vector field of Eq. (2.1) is obtained

$$\mathcal{X}_1 = t\frac{\partial}{\partial t} - \psi\frac{\partial}{\partial \psi}, \qquad \mathcal{X}_2 = \frac{\partial}{\partial t}, \qquad \mathcal{X}_3 = \frac{\partial}{\partial x}. \tag{2.8}$$

Solving the system of equation for an initial value problem with the following form

$$\begin{cases} \frac{d}{d\varepsilon}(x^*, t^*, \psi^*) = w(x^*, t^*, \psi^*), \\ (x^*, t^*, \psi^*)\,|_{\varepsilon=0} = (x, t, \psi). \end{cases} \tag{2.9}$$

Then we acquire the corresponding invariant groups $G_{[i]}$ $(i = 1, 2, 3)$.

$$\begin{cases} G_{[1]} : (x, t, \psi) \mapsto (x, te^{\varepsilon}, -\psi e^{\varepsilon}), \\ G_{[2]} : (x, t, \psi) \mapsto (x, t + \varepsilon, \psi), \\ G_{[3]} : (x, t, \psi) \mapsto (x + \varepsilon, t, \psi). \end{cases}$$

If $\psi = g(x, t)$ is a solution of Eq. (2.1), then the following functions are also the group-invariant solutions

$$\begin{cases} \psi^{(1)} = e^{\varepsilon} g(x, te^{\varepsilon}), \\ \psi^{(2)} = g(x, t - \varepsilon), \\ \psi^{(3)} = g(x - \varepsilon, t). \end{cases}$$

3. Traveling wave solutions of Eq. (2.1)

To study the traveling wave solution of the equation, consider the linear combination $\mathcal{X}_2 + c\mathcal{X}_3$. For this linear combination, we give the traveling wave transform

$$\xi = x - ct, \tag{3.1}$$

in which c is the traveling wave speed. Substituting (3.1) into Eq. (2.1), we have

$$-c\psi' + \beta\psi\psi' + 6\psi'\psi'' + 2\psi\psi''' = 0, \tag{3.2}$$

where $\beta \neq 0$, $c \neq 0$, $\psi' = \frac{d\psi}{d\xi}$. Integrating Eq. (3.2) we get

$$\frac{1}{2}\beta\psi^2 - c\psi + 2(\psi')^2 + 2\psi\psi'' = k, \tag{3.3}$$

in which k is a integral constant. Then, letting $y = \psi'$, Eq. (3.3) can be written as a singular system of the following form

$$\begin{cases} \dfrac{d\psi}{d\xi} = y, \\ \dfrac{dy}{d\xi} = \dfrac{k - \frac{1}{2}\beta\psi^2 + c\psi - 2y^2}{2\psi}. \end{cases} \tag{3.4}$$

It is obvious that there is a singular line $\psi = 0$, so perform the transformation $d\xi = 2\psi d\rho$. Eq. (3.4) can be rewritten as

$$\begin{cases} \dfrac{d\psi}{d\rho} = 2\psi y, \\ \dfrac{dy}{d\rho} = k - \dfrac{1}{2}\beta\psi^2 + c\psi - 2y^2, \end{cases} \tag{3.5}$$

with the first integral

$$H(\psi, y) = \psi^2(2y^2 + \frac{1}{4}\beta\psi^2 - \frac{2}{3}c\psi - k) = m, \tag{3.6}$$

in which m is a Hamiltonian constant.

In order to study the distribution of the equilibrium points of the system (3.5), let

$$f(\psi) = c\psi - \frac{1}{2}\beta\psi^2 + k. \tag{3.7}$$

Thus we can derive

(i) When $k = 0$, $f(\psi) = 0$ has two zero points $\bar{\psi}_1 = 0$, $\bar{\psi}_2 = \frac{c}{\beta}$;

(ii) When $k = -\frac{c^2}{2\beta}$, $f(\psi) = 0$ has two zero points $\bar{\psi}_3 = \bar{\psi}_4 = \frac{c}{\beta}$;

(iii) When $\beta > 0$, $k > -\frac{c^2}{2\beta}$ $(\beta < 0, k < -\frac{c^2}{2\beta})$, $f(\psi) = 0$ has two zero points

$$\bar{\psi}_5 = \frac{c + \sqrt{c^2 + 2\beta k}}{\beta}, \ \bar{\psi}_6 = \frac{c - \sqrt{c^2 + 2\beta k}}{\beta}.$$

Let $\mathcal{M}(\bar{\psi}_e, y_e)$ be the coefficient matrix of the system (3.5) at the equilibrium point $(\bar{\psi}_e, y_e)$,

$$\mathcal{M}(\bar{\psi}_e, y_e) = \begin{bmatrix} 2y_e & 2\bar{\psi}_e \\ -\beta\bar{\psi}_e + c & -4y_e \end{bmatrix}. \tag{3.8}$$

This results in the determinant of the coefficient at the point $(\bar{\psi}_e, 0)$ as

$$\mathcal{J}(\bar{\psi}_e, 0) = \begin{vmatrix} 0 & 2\bar{\psi}_e \\ -\beta\bar{\psi}_e + c & 0 \end{vmatrix} = 2\beta\bar{\psi}_e^2 - 2c\bar{\psi}_e. \tag{3.9}$$

From the dynamical system theory, when $\mathcal{J} > 0$, $(\bar{\psi}_e, y_e)$ is a central point; when the Poincaré index is 0 and $\mathcal{J} = 0$, $(\bar{\psi}_e, y_e)$ is a cusp; when $\mathcal{J} < 0$, $(\bar{\psi}_e, y_e)$ is a saddle point.

In the following, part of phase portraits of the system (3.5) are given.

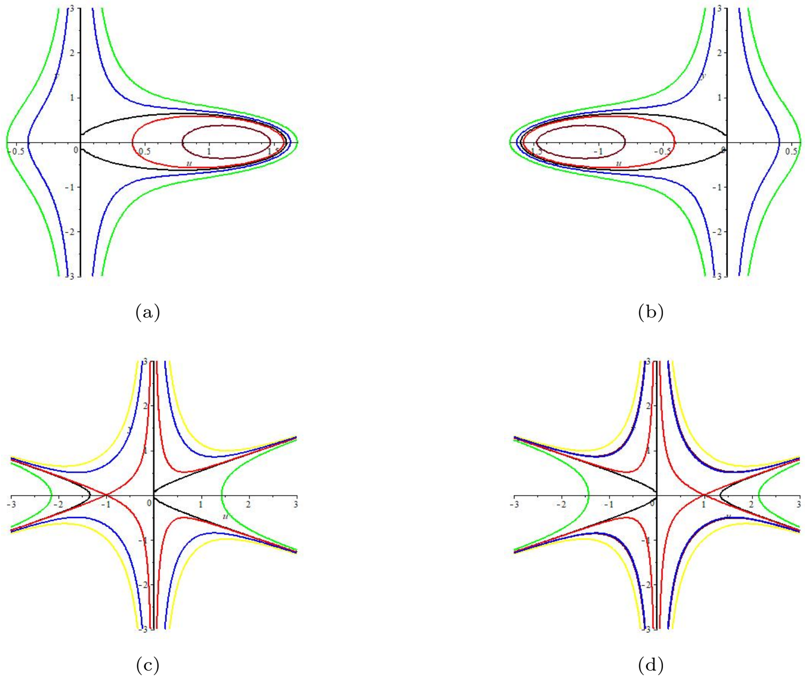


Figure 1. phase portraits of (3.5) for $k = 0$. (a) $c > 0$, $\beta > 0$; (b) $c < 0$, $\beta > 0$; (c) $c > 0$, $\beta < 0$; (d) $c < 0$, $\beta < 0$.

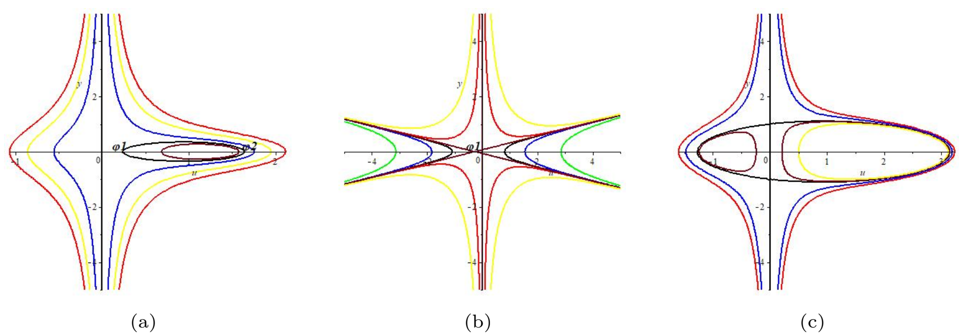


Figure 2. phase portraits of (3.5) for $k \neq 0$. (a) $c > 0$, $\beta > 0$, $-\dfrac{4c^2}{9\beta} < k < 0$; (b) $\beta < 0$, $c > 0$, $k = -\dfrac{4c^2}{9\beta}$; (c) $\beta > 0$, $c > 0$, $k > -\dfrac{4c^2}{9\beta}$.

In the following, we study the traveling wave solution corresponding to the habit of the $H(\psi, y)$=0. From(3.6), we get

$$y^2 = -\frac{1}{8}\beta\psi^2 + \frac{1}{3}c\psi + \frac{k}{2}. \tag{3.10}$$

Thus, according to (3.4) and (3.10), we have

$$\sqrt{\frac{\beta}{8}}\xi = \int_{\psi_0}^{\psi} \sqrt{\frac{-\psi^4 + \frac{8c}{3\beta}\psi^3 + \frac{4k}{\beta}\psi^2}{\psi^2}} d\psi$$
$$= \int_{\psi_0}^{\psi} \sqrt{\frac{\mathcal{K}(\psi)}{\psi^2}} d\psi. \tag{3.11}$$

Case 1 $k = 0$.

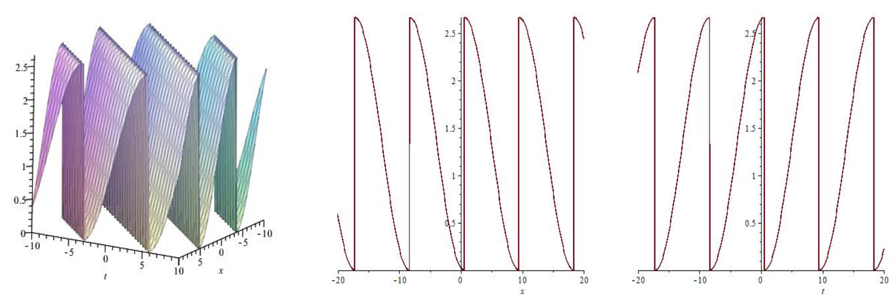
(i) When $\beta > 0$, $c > 0$ or $\beta > 0$, $c < 0$ **(Fig.1(a)–(b))**, considering the $H(\psi, y) = 0$ orbit, it follows that

$$y^2 = -\frac{1}{8}\beta\psi^2 + \frac{1}{3}c\psi$$
$$= \frac{1}{8}\beta\psi(\psi_1 - \psi). \tag{3.12}$$

From $y = \frac{d\psi}{d\xi}$, the periodic peakon solution of Eq. (2.1) is obtained

$$\psi(x,t) = \pm\frac{1}{2}\tan\left(\frac{\sqrt{2\beta}}{4}(x-ct)\right)\sqrt{\frac{{\psi_1}^2}{\tan^2\left(\frac{\sqrt{2\beta}}{4}(x-ct)\right)+1}} + \frac{1}{2}\psi_1, \tag{3.13}$$

where $\psi_1 = \frac{8c}{3\beta}$.



(a) (b) (c)

Figure 3. (a) The 3D plot of ψ via (3.13) for $\beta > 0$, $c > 0$; (b) Wave propagation along the x-axis; (c) Wave propagation along the t-axis.

(ii) When $\beta < 0$, $c > 0$ or $\beta < 0$, $c < 0$ **(Fig.1(c)–(d))**, the orbit determined by $H(\psi, y) = 0$ yields $\mathcal{K}(\psi) = \psi^3(\psi - \psi_1)$, $\psi_1 = \frac{8c}{3\beta}$. Therefore, we get from (3.11) that

$$\sqrt{\frac{\beta}{8}}\xi = \int \sqrt{\frac{\mathcal{K}(\psi)}{\psi^2}} d\psi. \tag{3.14}$$

Thus, we get the solution of Eq. (2.1)

$$\psi(x,t) = \frac{4e^{\delta^2} + 4e^{\delta}\psi_1 + (\psi_1)^2}{8e^{\delta}}, \tag{3.15}$$

in which $\delta = \frac{\sqrt{2\beta(x-t)}}{4}$.

Case 2 $\quad k \neq 0$

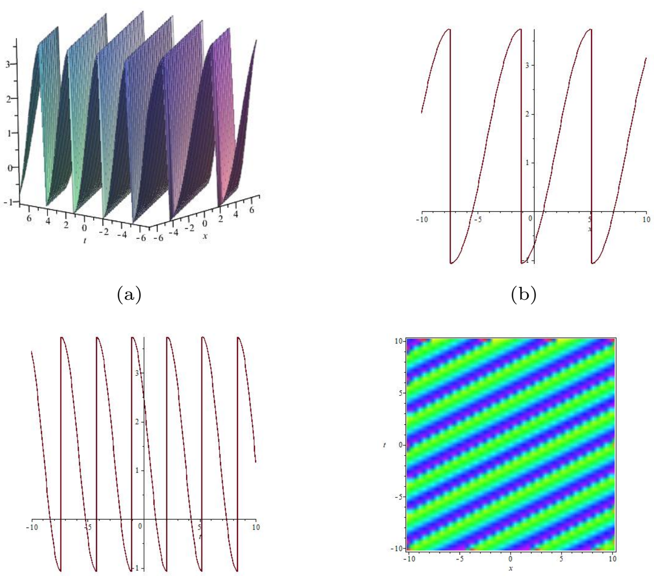
(i) When $c > 0$, $\beta > 0$, $-\frac{4c^2}{9\beta} < k < 0$ **(Fig.2(a))**, from the orbit defined by $H(\psi, y) = 0$, we obtain

$$\frac{\psi d\psi}{\sqrt{\psi^2\left(\psi-\psi_1\right)\left(\psi_2-\psi\right)}} = \sqrt{\frac{\beta}{8}}d\xi. \tag{3.16}$$

Integrating the above equation, we have single period wave solution

$$\psi(x,t) = \frac{1}{2}\delta\sqrt{\frac{\left(\psi_1-\psi_2\right)^2}{\delta^2+1}} + \frac{1}{2}\left(\psi_1+\psi_2\right), \quad 0 < \psi_1 < \psi < \psi_2, \tag{3.17}$$

where $\delta = \tan(\frac{\sqrt{2a}}{4}(x-ct))$, $\psi_1 = \frac{-2\left(-2c+\sqrt{9\beta k+4c^2}\right)}{3\beta}$, $\psi_2 = \frac{2\left(2c+\sqrt{9\beta k+4c^2}\right)}{3\beta}$.



(a) (b) (c) (d)

Figure 4. (a) The 3D plot of ψ via (3.17) for $\beta > 0$, $c > 0$; (b)Wave propagation along the x-axis; (c) Wave propagation along the t-axis; (d) Density plot.

(ii) When $c > 0$, $\beta < 0$, $k = -\frac{4c^2}{9\beta}$**(Fig.2(b))**, considering the $H(\psi, y) = 0$ orbit, it follows that

$$\sqrt{\frac{\beta}{8}}\xi = \int_{\psi}^{+\infty}\sqrt{\frac{\psi^2\left(\psi-\psi_1\right)^2}{\psi^2}}d\psi, \quad \psi_1 < 0 < \psi. \tag{3.18}$$

Solving Eq. (3.18), we acquire the solution of Eq.(2.1):

$$\psi(x,t) = e^{\delta} + \psi_1, \tag{3.19}$$

where $\delta = \frac{\sqrt{-2\beta}(x-ct)}{4}$, $\psi_1 = \frac{4c}{3\beta}$.

(iii) When $c > 0$, $\beta > 0$, $k > -\frac{4c^2}{9\beta}$ **(Fig.2(c))**, from the orbit defined by $H(\psi, y) = 0$, we get

$$\sqrt{\frac{\beta}{8}}d\xi = \frac{\psi d\psi}{\sqrt{\psi^2\left(\psi-\psi_2\right)\left(\psi_1-\psi\right)}}, \quad \psi_2 < \psi < 0 < \psi_1. \tag{3.20}$$

Integrating Eq. (3.20), we obtain the periodic solution of Eq. (2.1)

$$\begin{aligned}\psi(x,t) = &-\frac{1}{2}\tan\left(\delta_1 + \arctan\left(\delta_2\right)\sqrt{\frac{\left(\psi_1-\psi_2\right)^2}{\tan^2\left(\delta_1+\arctan\left(\delta_2\right)\right)+1}}\right)\\ &+\frac{1}{2}\left(\psi_1+\psi_2\right),\end{aligned} \tag{3.21}$$

where $\delta_1 = \frac{\sqrt{2\beta}(x-ct)}{4}$, $\delta_2 = \frac{\psi_1+\psi_2}{2\sqrt{-\psi_1\psi_2}}$.

4. Power series solutions

In this section, we consider linear combinations of generators and reduce Eq. (2.1) to ordinary differential equations by symmetric reduction, thereby constructing power series solutions of Eq. (2.1).

4.1. Symmetric reductions

4.1.1 Considering the linear combination $\mathcal{X}_1 + \mathcal{X}_2 = (1+t)\frac{\partial}{\partial t} - \psi\frac{\partial}{\partial \psi}$, the corresponding characteristic equation is

$$\frac{dx}{0} = \frac{dt}{1+t} = \frac{d\psi}{-\psi}. \tag{4.1}$$

Solving Eq. (4.1), we get

$$\psi(x,t) = \frac{\phi(\upsilon)}{t+1}, \tag{4.2}$$

where $\upsilon = x$. Substituting (4.2) into Eq. (2.1), we obtain

$$\beta\phi\phi' + 2\phi\phi''' + 6\phi'\phi'' - \phi = 0. \tag{4.3}$$

4.1.2 For the linear combination $\mathcal{X}_1 + \mathcal{X}_3 = \frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \psi\frac{\partial}{\partial \psi}$, the corresponding characteristic equation is

$$\frac{dx}{1} = \frac{dt}{t} = \frac{d\psi}{-\psi}. \tag{4.4}$$

Then we get

$$\psi(x,t) = \frac{\phi(\upsilon)}{t}, \tag{4.5}$$

in which $\upsilon = \frac{t}{e^x}$. Substituting (4.5) into Eq. (2.1), we have

$$6\phi'\phi'' + 2\phi\phi''' + 6(\phi')^2 + 6\phi\phi'' + (2+\beta)\phi\phi' - \phi' + \phi = 0. \tag{4.6}$$

4.2. Power series solutions

In the following we first consider Eq. (4.3). Suppose that the power series solution of Eq. (4.3) takes the form

$$\phi(\upsilon) = \sum_{n=0}^{\infty} q_n \upsilon^n, \tag{4.7}$$

in which q_n are the coefficients to be determined. Substituting (4.7) into Eq. (4.3), we get

$$\begin{aligned} &\beta \sum_{n=0}^{\infty} \sum_{k=0}^{n} (n+1-k)\, q_k q_{n+1-k} \upsilon^n + 6 \sum_{n=0}^{\infty} \sum_{k=0}^{n} (k+1)(n-k+1) \\ &\times (n-k+2)\, \upsilon^n q_{k+1} q_{n-k+2} + 2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} (n+1-k)(n+2-k) \\ &\times (n+3-k)\, q_k q_{n+3-k} \upsilon^n - \sum_{n=0}^{\infty} q_n \upsilon^n = 0. \end{aligned} \tag{4.8}$$

Comparing coefficients for Eq. (4.8), q_{n+3} $(n \geq 0)$ is obtained as follows:

$$\begin{aligned} q_{n+3} = &\frac{-1}{2(n+1)(n+2)(n+3)\, q_0} \Bigg\{ \beta (n+1)\, q_0 q_{n+1} + 6(n+1) \\ &\times (n+2)\, q_1 q_{n+2} - q_n + \sum_{k=1}^{n} \big[\beta (n+1-k)\, q_k q_{n+1-k} \\ &+ 6(k+1)(n+1-k)(n+2-k)\, q_k q_{n+2-k} \\ &+ 2 \sum_{k=0}^{\infty} (n+1-k)(n+2-k)(n+3-k)\, q_k q_{n+3-k} \big] \Bigg\}. \end{aligned} \tag{4.9}$$

According to (4.8), we get

$$q_3 = \frac{q_0 - \beta q_0 q_1}{12 q_0}. \tag{4.10}$$

If $q_0 \neq 0$ and q_1, q_2 are arbitrary constants, q_{n+3} are determined by (4.9). In this way, we obtain all the coefficients of (4.7). Next, the convergence of the power series solution (4.7) is proved for Eq. (4.3). According to (4.9), we have following inequality

$$\begin{aligned} |q_{n+3}| \leq M \Bigg\{ &\sum_{k=1}^{n} \big[|q_k|\, |q_{n+1-k}| + (k+1)\, |q_{k+1}|\, |q_{n+2-k}| + |q_k|\, |q_{n+3-k}| \big] \\ &+ |q_n| + |q_{n+1}| + |q_{n+2}| \Bigg\}, \end{aligned} \tag{4.11}$$

where $M = \max \left\{ \left|\frac{\beta}{2}\right|, \left|\frac{3q_1}{q_0}\right|, \left|\frac{1}{2q_0}\right| \right\}$.

Defining a power series

$$\mathcal{R} = \mathcal{R}(\upsilon) = \sum_{n=0}^{\infty} r_n \upsilon^n, r_i = |q_i|\,, i = 0, 1, 2, \cdots, \tag{4.12}$$

and

$$\begin{aligned} r_{n+3} = M \Bigg\{ &\sum_{k=1}^{n} \big[(k+1)\, r_{k+1} r_{n+2-k} + r_k r_{n+1-k} + r_k r_{n+3-k} \big] \\ &r_n + r_{n+1} + r_{n+2} \Bigg\}, \end{aligned} \tag{4.13}$$

in which $n = 0, 1, 2, \cdots$, $|q_n| \leq r_n$. Therefore, $\mathcal{R} = \mathcal{R}(\upsilon) = \sum\limits_{n=0}^{\infty} r_n \upsilon^n$ is a majorant series of (4.7). Then, we prove $\mathcal{R} = \mathcal{R}(\upsilon)$ has a positive radius of convergence.

$$\begin{aligned}
\mathcal{R}(\upsilon) &= r_0 + r_1\upsilon + r_2\upsilon^2 + \sum_{n=0}^{\infty} r_{n+3}\upsilon^{n+3} \\
&= r_0 + r_1\upsilon + r_2\upsilon^2 \\
&\quad + M\Bigg\{\sum_{n=0}^{\infty} r_n\upsilon^{n+3} + \sum_{n=0}^{\infty} r_{n+1}\upsilon^{n+3} + \sum_{n=0}^{\infty} r_{n+2}\upsilon^{n+3} \\
&\quad + \sum_{n=0}^{\infty}\sum_{k=1}^{n}\left[(k+1)\, r_{k+1}r_{n+2-k} + r_k r_{n+1-k} + r_k r_{n+2-k}\right]\upsilon^{n+3}\Bigg\} \\
&= r_0 + r_1\upsilon + r_2\upsilon^2 + M\left\{\upsilon^3\mathcal{R} + (\mathcal{R} - r)\,\upsilon^2 + (\mathcal{R} - r_0 - r_1\upsilon)\,\upsilon\right. \\
&\quad + (\mathcal{R} - r_0 - r_1\upsilon)'\,(\mathcal{R} - r_0 - r_1\upsilon)\,\upsilon + \upsilon(R - r_0)^2 \\
&\quad \left. + (\mathcal{R} - r_0)\left(\mathcal{R} - r_0 - r_1\upsilon - r_2\upsilon^2\right)\right\}.
\end{aligned} \tag{4.14}$$

Considering the implicit functional equation about υ,

$$\begin{aligned}
F(\upsilon, R) = &R - r_0 - r_1\upsilon - r_2\upsilon^2 - M\Big[\upsilon^3 R + (R - r_0)\,\upsilon^2 \\
&+ (R - r_0 - r_1\upsilon)\,\upsilon + (R - r_0 - r_1\upsilon)'\,(R - r_0 - r_1\upsilon)\,\upsilon \\
&+ \upsilon(R - r_0)^2 + (R - r_0)\left(R - r_0 - r_1\upsilon - r_2\upsilon^2\right)\Big].
\end{aligned} \tag{4.15}$$

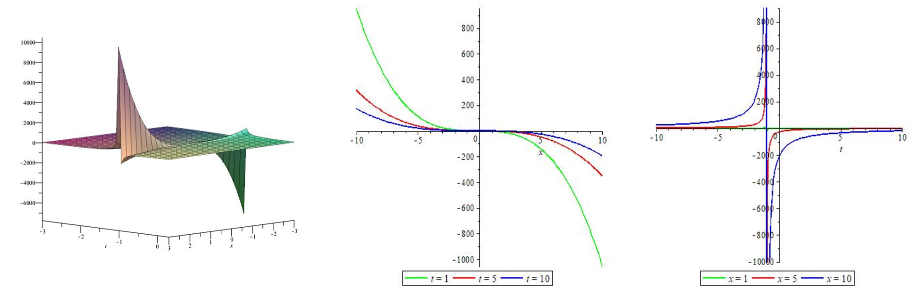
Since $F(0, r_0) = 0$, $F_{\mathcal{R}}^{'}(0, r_0) = 1 \neq 0$, by the implicit function theorem, the series (4.7) is convergent.

Hence, the explicit power series solution of Eq. (2.1) is

$$\begin{aligned}
\psi(x,t) = \frac{1}{t+1}\Bigg\{&q_0 + q_1 x + q_2 x^2 + \sum_{n=0}^{\infty}\frac{-(x)^{n+3}}{2(n+1)(n+2)(n+3)q_0} \\
&\Big\{\beta(n+1)q_0 q_{n+1} - q_n + 6(n+1)(n+2)q_1 q_{n+2} \\
&+ \sum_{k=1}^{n}\Big[6\,(k+1)\,(n-1+k)\,(n+2-k)\,q_k q_{n+2-k} \\
&+\beta\,(n+1-k)\,q_k q_{n+1-k} + 2\sum_{k=0}^{\infty}(n+1-k) \\
&(n+2-k)\,(n+3-k)\,q_k q_{n+3-k}\Big]\Big\}\Bigg\}.
\end{aligned} \tag{4.16}$$

Using the same method we acquire the power series solution of Eq. (4.6) as follows

$$\begin{aligned}
\phi(\upsilon) = &q_0 + q_1\upsilon + q_2\upsilon^2 + \sum_{n=0}^{\infty}\frac{\upsilon^{n+3}}{2(n+1)(n+2)(n+3)q_0}\Big\{(n+1)\,q_{n+1} \\
&- 6\,(n+1)\,(n+2)\,q_1 q_{n+2} - 6\,(n+1)\,q_1 q_{n+1} - q_n - 6\,(n+1)\,(n+2) \\
&\times q_0 q_{n+2} - 2\,(2+\beta)\,(n+1)\,q_0 q_{n+1} - \sum_{k=1}^{n}\Big[6\,(k+1)\,(n-k+1)
\end{aligned}$$



(a) (b) (c)

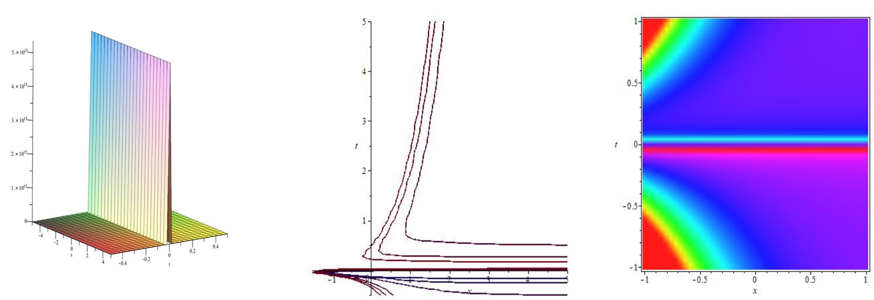
Figure 5. (a) The 3D plot of ψ via (4.16); (b)The plot of ψ via (4.16) for $t = 1$, $t = 5$, $t = 10$; (c) The plot of ψ via (4.16) for $x = 1$, $x = 5$, $x = 10$;.

$$\begin{aligned}&\times (n-k+2)\, q_{k+1}q_{n-k+2} + 2\,(n-k+1)\,(n-k+2)\,(n-k+3)\\&\times q_k q_{n+3-k} + 6\,(k+1)\,(n-k+1)\, q_{k+1}q_{n-k+1} + 6\,(n-k+1)\\&\times (n-k+2) \times q_k q_{n-k+2} + (2+\beta)\,(n+1-k)\, q_k q_{n+1-k}\Big]\Big\}.\end{aligned} \tag{4.17}$$

Substituting (4.17) into (4.5), we have

$$\psi(x,t) = \frac{\phi(\upsilon)}{t}, \tag{4.18}$$

in which $\upsilon = \frac{t}{e^x}$.



(a) (b) (c)

Figure 6. (a) The 3D plot of ψ via (4.18); (b)Contour plot ; (c) Density plot.

5. Conservation laws

In this section, we give the conservation laws of the $K(2,2)$ equation by Ibragimov's method.

Firstly, considering the $K(2,2)$ equation

$$\mathcal{F} = \psi_t + \beta\psi\psi_x + 4\psi_x\psi_{xx} + (2\psi\psi_{xx})_x = 0. \tag{5.1}$$

Defining the Lagrangian function of Eq. (5.1)

$$\mathcal{L} = \theta\mathcal{F} = \theta[\psi_t + \beta\psi\psi_x + 4\psi_x\psi_{xx} + (2\psi\psi_{xx})_x], \tag{5.2}$$

where $\theta(x, t, \psi)$ is dependent variable.

The adjoint equation of Eq. (5.1) has the following form

$$\mathcal{F}^* = \frac{\delta\mathcal{L}}{\delta\psi} = 0, \tag{5.3}$$

in which $\frac{\delta}{\delta\psi} = \frac{\partial}{\partial\psi} - D_t\left(\frac{\partial}{\partial\psi_t}\right) - D_x\left(\frac{\partial}{\partial\psi_x}\right) + {D_x}^2\left(\frac{\partial}{\partial\psi_{xx}}\right) - {D_x}^3\left(\frac{\partial}{\partial\psi_{xxx}}\right)$.

Because $\theta(2\psi\psi_{xx})_x = (2\theta\psi\psi_{xx})_x - \theta_x(2\psi\psi_{xx})$, Eq. (5.2) is equivalent to the second-order form

$$\mathcal{L} = \theta\psi_t + \theta\beta\psi\psi_x + 4\theta\psi_x\psi_{xx} - \theta_x(2\psi\psi_{xx}). \tag{5.4}$$

Therefore, the adjoint equation to Eq. (2.1) is

$$\mathcal{F}^* = -\theta_t - \beta\psi\theta_x - 2\psi\theta_{xxx}. \tag{5.5}$$

If the adjoint equation satisfies

$$\mathcal{F}^* = \lambda\mathcal{F}, \tag{5.6}$$

in which λ is an undetermined coefficient, solving Eq. (5.6) yields

$$\theta(x, t, \psi) = c_1 + c_2\sin\left(\frac{\sqrt{2\beta}}{2}x\right) + c_3\cos\left(\frac{\sqrt{2\beta}}{2}x\right), \tag{5.7}$$

where c_1, c_2, c_3 are arbitrary constants. Therefore, according to [8], Eq. (2.1) is nonlinearly self-adjoint.

Next, we give the definition and theorem related to the conservation laws.

Definition 5.1. A vector filed $\mathcal{C}(x, t, \psi, \psi_x, \psi_t, \cdots)$ has two components,

$$\mathcal{C} = \mathcal{C}(\mathcal{C}^1, \mathcal{C}^2). \tag{5.8}$$

If each solution $\psi = \psi(x, t)$ of Eq. (2.1) satisfies

$$D_i(\mathcal{C}^i) = D_t\mathcal{C}^t + D_x\mathcal{C}^x = 0, \tag{5.9}$$

then the vector field $\mathcal{C}(x, t, \psi, \psi_x, \psi_t, \cdots)$ is a conserved vector. Eq. (5.9) is said to be a conservation law of Eq. (2.1).

Theorem 5.1. *For the generator*

$$\mathcal{X} = \zeta_x\frac{\partial}{\partial_x} + \zeta_t\frac{\partial}{\partial_t} + \varsigma\frac{\partial}{\partial_\psi}, \tag{5.10}$$

the conservation law of Eq. (2.1) is expressed as

$$D_x\mathcal{C}^x + D_t\mathcal{C}^t = 0, \tag{5.11}$$

the conserved vector $\mathcal{C} = \mathcal{C}(\mathcal{C}^x, \mathcal{C}^x)$ *is given by*

$$\begin{aligned}\mathcal{C}^x &= \zeta_x \mathcal{L} + \mathcal{W}\left(\frac{\partial \mathcal{L}}{\partial \psi_x} - D_x \frac{\partial \mathcal{L}}{\partial \psi_{xx}} + D_{xx} \frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\right) + (D_x \mathcal{W}) \left[\frac{\partial \mathcal{L}}{\partial \psi_{xx}}\right.\\ &\quad \left. - D_x \left(\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\right)\right] + (D_{xx} \mathcal{W}) \left(\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\right),\\ \mathcal{C}^t &= \zeta_t \mathcal{L} + \mathcal{W}\left(\frac{\partial \mathcal{L}}{\partial \psi_t} - D_t \frac{\partial \mathcal{L}}{\partial \psi_{tt}}\right) + (D_t \mathcal{W}) \frac{\partial \mathcal{L}}{\partial \psi_{tt}},\end{aligned} \tag{5.12}$$

in which

$$\begin{aligned}&\mathcal{W} = \varsigma - \zeta_x \psi_x - \zeta_t \psi_t,\\ &\mathcal{L} = \left[c_1 + c_2 \sin\left(\frac{\sqrt{2\beta}}{2} x\right) + c_3 \cos\left(\frac{\sqrt{2\beta}}{2} x\right)\right] (\psi_t + \beta \psi \psi_x + 6 \psi_x \psi_{xx} + 2 \psi \psi_{xxx}).\end{aligned}$$

Then, by applying **Theorem 5.1** to the conserved vector $\mathcal{C} = \mathcal{C}(\mathcal{C}^x, \mathcal{C}^t)$, the conservation laws are obtained.

(I) For the generator $\chi = t\frac{\partial}{\partial t} - \psi \frac{\partial}{\partial \psi}$, we have $\mathcal{W} = -\psi - t\psi_t$. According to (5.11), we obtain

$$\begin{aligned}\mathcal{C}^x =& \sqrt{2\beta}\,(t\psi_x\psi_t + \psi\psi_{xt} + 2\psi\psi_x) \left[c_2 \cos\left(\frac{\sqrt{2\beta}}{2} x\right) - c_3 \sin\left(\frac{\sqrt{2\beta}}{2} x\right)\right]\\ & - 2t\,(\psi_t\psi_{xx} + 2\psi_x\psi_{xt} + \psi\psi_{xxt}) \left[c_2 \sin\left(\frac{\sqrt{2\beta}}{2} x\right) + c_3 \cos\left(\frac{\sqrt{2\beta}}{2} x\right)\right]\\ & - 4\left({\psi_x}^2 + \psi\psi_{xx}\right) \left[c_1 + c_2 \sin\left(\frac{\sqrt{2\beta}}{2} x\right) + c_3 \cos\left(\frac{\sqrt{2\beta}}{2} x\right)\right]\\ & - c_1 t\,(\beta\psi\psi_t + 2\psi_t\psi_{xx} + 4\psi_x\psi_{xt} + 2\psi\psi_{xxt}) - c_1\beta\psi^2,\\ \mathcal{C}^t =& (2t\psi\psi_{xxx} + 6t\psi_x\psi_{xx} + \beta t\psi\psi_x - \psi) \left[c_1 + c_2 \sin\left(\frac{\sqrt{2\beta}}{2} x\right)\right.\\ & \left. + c_3 \cos\left(\frac{\sqrt{2\beta}}{2} x\right)\right].\end{aligned} \tag{5.13}$$

(II) For the generator $\chi = \dfrac{\partial}{\partial t}$. From (5.11) we get

$$\begin{aligned}\mathcal{C}^x =& -2\,(\psi_t\psi_{xx} + 2\psi_x\psi_{xt} + \psi\psi_{xxt}) \left[c_1 + c_2 \sin\left(\frac{\sqrt{2\beta}}{2} x\right)\right.\\ & \left. + c_3 \cos\left(\frac{\sqrt{2\beta}}{2} x\right)\right] + \sqrt{2\beta}\,(\psi_x\psi_t + \psi\psi_{xt}) \left[c_2 \cos\left(\frac{\sqrt{2\beta}}{2} x\right)\right.\\ & \left. - c_3 \sin\left(\frac{\sqrt{2\beta}}{2} x\right)\right] - c_1\beta\psi\psi_t,\\ \mathcal{C}^t =& (6t\psi_x\psi_{xx} + 2t\psi\psi_{xxx} + \beta t\psi\psi_x + t\psi_t - \psi_t) \left[c_1 + c_2 \sin\left(\frac{\sqrt{2\beta}}{2} x\right)\right.\\ & \left. + c_3 \cos\left(\frac{\sqrt{2\beta}}{2} x\right)\right].\end{aligned} \tag{5.14}$$

(III) For the generator $\chi = \dfrac{\partial}{\partial x}$. In the same way as above, we obtain

$$\begin{aligned}
\mathcal{C}^x &= -2\left(3\psi_x\psi_{xx} + \psi\psi_{xxx}\right)\left[c_1 + c_2\sin\left(\frac{\sqrt{2\beta}}{2}x\right) + c_3\cos\left(\frac{\sqrt{2\beta}}{2}x\right)\right] \\
&\quad + \sqrt{2\beta}\left({\psi_x}^2 + \psi\psi_{xx}\right)\left[c_2\cos\left(\frac{\sqrt{2\beta}}{2}x\right) - c_3\sin\left(\frac{\sqrt{2\beta}}{2}x\right)\right] - c_1\beta\psi\psi_x, \\
\mathcal{C}^t &= \left(t\psi_t + \beta t\psi\psi_x + 6t\psi_x\psi_{xx} + 2t\psi\psi_{xxx} - \psi_x\right)\left[c_1 + c_2\sin\left(\frac{\sqrt{2\beta}}{2}x\right)\right. \\
&\quad \left. + c_3\cos\left(\frac{\sqrt{2\beta}}{2}x\right)\right].
\end{aligned} \tag{5.15}$$

6. Conclusions

In this paper, the nonlinear dispersion equation is considered. We obtain the generators using the Lie symmetry analysis method. Then we give partial phase portraits to obtain exact periodic peakon solutions, single period solutions and other forms of traveling wave solutions. In addition, the analytic power series solutions of the equation are given. Finally, we prove that the equation is nonlinearly self-adjoint and construct the conservation laws.

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