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A ACCELERATED MODIFIED SHIFT-SPLITTING METHOD FOR NONSYMMETRIC SADDLE POINT PROBLEMS

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Abstract Recently, Huang and Su [A modified shift-splitting method for nonsymmetric saddle, *Journal of Computational and Applied Mathematics*, 2017, 317, 535–546] introduced a modified shift-splitting (denoted by MSSP) preconditioner. In this paper, based on modified shift-splitting (denoted by MSSP) iteration technique, we establish a accelerated (named after AMSSP) iterative method for nonsymmetric saddle point problems. Furthermore, we theoretically verify the AMSSP iteration method unconditionally converges to the unique solution of the saddle point problems, compute the spectral radius of the AMSSP iteration matrix. Finally, numerical examples show the spectrum of the new preconditioned matrix for the different parameters.

Keywords Saddle point problem, shift-splitting, Krylov subspace methods, convergence rate, preconditioner.

MSC(2010) 65F10, 65F15, 65F50.

1. Introduction

For solving the large sparse augmented systems of linear equations

$$Au = \begin{pmatrix} B & E \\ -E^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b, \tag{1.1}$$

where $B \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix and $E \in \mathbb{R}^{n\times m}$ is a matrix of full column rank and $n \geq m, x, f \in \mathbb{R}^n, y, g \in \mathbb{R}^m$. It appears in many different applications of scientific computing, such as constrained optimization [37],

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the finite element method for solving the Navier-Stokes equation [26–28], and constrained least squares problems and generalized least squares problems [1,34,40,41]. There have been several recent papers [2–31, 33, 37–40] for solving the augmented system (1). Santos et al. [34] studied preconditioned iterative methods for solving the singular augmented system with $A = I$. Yuan et al. [40, 41] proposed several variants of SOR method and preconditioned conjugate gradient methods for solving general augmented system (1) arising from generalized least squares problems where A can be symmetric and positive semidefinite and B can be rank deficient. The SOR-like method requires less arithmetic work per iteration step than other methods but it requires choosing an optimal iteration parameter in order to achieve a comparable rate of convergence. Golub et al. [32] presented SOR-like algorithms for solving system (1). Darvishi et al. [25] studied SSOR method for solving the augmented systems. Bai et al. [3, 4, 24] presented GSOR method, parameterized Uzawa (PU) and the inexact parameterized Uzawa (PIU) methods for solving systems (1). Zhang and Lu [42] showed the generalized symmetric SOR method for augmented systems. Peng and Li [33] studied unsymmetric block overrelaxation-type methods for saddle point. Bai and Golub [5–9, 36] presented splitting iteration methods such as Hermitian and skew-Hermitian splitting (HSS) iteration scheme and its preconditioned variants, Krylov subspace methods such as preconditioned conjugate gradient (PCG), preconditioned MINRES (PMINRES) and restrictively preconditioned conjugate gradient (RPCG) iteration schemes, and preconditioning techniques related to Krylov subspace methods such as HSS, block-diagonal, block-triangular and constraint preconditioners and so on. Bai and Wang's 2009 LAA paper [36] and Chen and Jiang's 2008 AMC paper [24] studied some general approaches about the relaxed splitting iteration methods. Wu, Huang and Zhao [38] presented modified SSOR (MSSOR) method for augmented systems. Cao, Du and Niu [18] introduced a shift-splitting preconditioner and a local shift-splitting preconditioner for saddle point problems (1). Recently, Huang and Su [29] studied a modified shift-splitting (denoted by MSSP) preconditioner to address a class of large scale sparse saddle point problems. Moreover, the authors analyze the spectral radius of the MSSP iteration matrix and estimate the sharp bounds of the eigenvalues of the corresponding iteration matrix.

For large, sparse or structure matrices, iterative methods are an attractive option. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \operatorname{span}\left\{b, \mathcal{A}b, \mathcal{A}^2 b, ..., \mathcal{A}^{n-1} b, ...\right\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are common Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices [35].

In this paper, based on modified shift-splitting (denoted by MSSP) iteration technique, we establish a accelerated (named after AMSSP) iterative method for nonsymmetric saddle point problems. Furthermore, we theoretically verify the AMSSP iteration method unconditionally converges to the unique solution of the saddle point problems, compute the spectral radius of the AMSSP iteration matrix. Finally, numerical examples show the spectrum of the new preconditioned matrix for the different parameters. However, the relaxed parameters of the modified shift-splitting methods are not optimal and only lie in the convergence region of the method.

2. The AMSSP iteration method

Recently, Bai and yin [16] proposed a shift splitting precondition for solving non-Hermitian positive definite saddle point systems, where the shift splitting precondition is defined as

$$\mathcal{M} = \frac{1}{2}(\alpha I_{n+m} + A), \tag{2.1}$$

and the shift splitting of matrix A consists of the following forms

$$A = \frac{1}{2}(\alpha I_{n+m} + A) - \frac{1}{2}(\alpha I_{n+m} - A), \tag{2.2}$$

with a positive constant α and the $n+m$ identity matrix I_{n+m}.

Cao et al. [21] proposed the shift-splitting (denoted by SSP) preconditioner as follows

$$\mathcal{M}(\alpha) = \frac{1}{2}\begin{pmatrix} \alpha I_n + B & E \\ -E^T & \alpha I_m \end{pmatrix}, \tag{2.3}$$

and the shift-splitting of the matrix A is established by the following form

$$\mathcal{M}(\alpha) = \begin{pmatrix} B & E \\ -E^T & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha I_n + B & E \\ -E^T & \alpha I_m \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \alpha I_n - B & -E \\ E^T & \alpha I_m \end{pmatrix}. \tag{2.4}$$

When the 2 by 2 block parameter $\alpha = \beta$, Cao et al. [21] and Chen et al. [23] proposed a class of generalized shift-splitting (denoted by GSSP) preconditioner of form

$$\mathcal{M}(\alpha, \beta) = \frac{1}{2}\begin{pmatrix} \alpha I_n + B & E \\ -E^T & \beta I_m \end{pmatrix}, \tag{2.5}$$

and the shift-splitting of the matrix A is established by the following form

$$\mathcal{M}(\alpha, \beta) = \begin{pmatrix} B & E \\ -E^T & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha I_n + B & E \\ -E^T & \beta I_m \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \alpha I_n - B & -E \\ E^T & \beta I_m \end{pmatrix}. \tag{2.6}$$

Recently, based on the SSP preconditioner (4), Huang et al. [29] proposed a type of MSSP preconditioner as follows

$$\mathcal{P}(\alpha) = \begin{pmatrix} \alpha I_n + 2B & 2E \\ -2E^T & \alpha I_m \end{pmatrix}. \tag{2.7}$$

Then the corresponding shift-splitting of the saddle point matrix A is established by

$$A = \mathcal{P}(\alpha) - \mathcal{N}(\alpha) = \begin{pmatrix} \alpha I_n + 2B & 2E \\ -2E^T & \alpha I_m \end{pmatrix} - \begin{pmatrix} \alpha I_n + B & E \\ -E^T & \alpha I_m \end{pmatrix}. \tag{2.8}$$

In this paper, based on the MSSP preconditioner (8), we establish a accelerated (named after AMSSP) iterative method for nonsymmetric saddle point problems, which is as follows

$$\mathcal{P}(\alpha,\beta)=\begin{pmatrix}\alpha I_n+\beta B & \beta E\\ -\beta E^T & \alpha I_m\end{pmatrix}. \tag{2.9}$$

Then the corresponding shift-splitting of the saddle point matrix A is established by

$$A=\mathcal{P}(\alpha,\beta)-\mathcal{N}(\alpha,\beta)=\begin{pmatrix}\alpha I_n+\beta B & \beta E\\ -\beta E^T & \alpha I_m\end{pmatrix}-\begin{pmatrix}\alpha I_n+(\beta-1)B & (\beta-1)E\\ -(\beta-1)E^T & \alpha I_m\end{pmatrix}. \tag{2.10}$$

Without loss of generality, we define the AMSSP iteration matrix by

$$\mathcal{T}(\alpha,\beta)=\begin{pmatrix}\alpha I_n+\beta B & \beta E\\ -\beta E^T & \alpha I_m\end{pmatrix}^{-1}\begin{pmatrix}\alpha I_n+(\beta-1)B & (\beta-1)E\\ -(\beta-1)E^T & \alpha I_m\end{pmatrix}, \tag{2.11}$$

and denote the spectral radius and any eigenvalue of the iteration matrix $\mathcal{T}(\alpha,\beta)$ by $\rho(\mathcal{T}(\alpha,\beta))$ and λ, respectively. Then we can know that the AMSSP iteration method is convergent if and only if $\rho(\mathcal{T}(\alpha,\beta))<1$.

3. Convergence of the AMSSP iteration method

To verify the convergence properties of the accelerated iterative method, we firstly need to consider the spectral properties of the AMSSP iteration method by the following form.

$$\begin{pmatrix}\alpha I_n+(\beta-1)B & (\beta-1)E\\ -(\beta-1)E^T & \alpha I_m\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}=\lambda\begin{pmatrix}\alpha I_n+\beta B & \beta E\\ -\beta E^T & \alpha I_m\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}. \tag{3.1}$$

Then the generalized eigenvalue problem $[\mathcal{P}(\alpha,\beta)-\mathcal{A}]z=\lambda\mathcal{P}(\alpha,\beta)z$ is written as the following block form

$$(\alpha I_n+B)u+(\beta-1)Ev=\lambda(\alpha I_n+\beta B)u+\lambda\beta Ev \tag{3.2}$$

and

$$-(\beta-1)E^Tu+\alpha v=-\beta\lambda E^Tu+\alpha\lambda v. \tag{3.3}$$

In order to study the convergence of AMSSP iterative method, we will give two lemmas of the following forms.

Lemma 3.1.([18]) *Let B be a symmetric positive definite matrix, E be a matrix with full column rank and $\mathcal{T}(\alpha,\beta)$ be defined as in (12). Assume λ is an eigenvalue of the AMSSP iteration matrix $\mathcal{T}(\alpha,\beta)$, and suppose $(u^T,v^T)^T$ is an eigenvector corresponding to λ. Then the eigenvalue λ of $\mathcal{T}(\alpha,\beta)$ satisfies*
(1) $\lambda\neq\pm1$.
(2) $u\neq 0$.

(3) If $v = 0$, then $\rho(\mathcal{T}(\alpha, \beta)) < 1$.

Lemma 3.2.([39]) *Both roots of the real quadratic equation $x^2 + bx + c = 0$ are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

Theorem 3.3. *Let B be a symmetric positive definite matrix, E be a matrix with full column rank and $\mathcal{T}(\alpha, \beta)$ be defined as in (12). Then for any positive iteration paramet er α, β, the spectral radius of the AMSSP iteration matrix $\mathcal{T}(\alpha, \beta)$ satisfies*

$$\rho(\mathcal{T}(\alpha, \beta)) < 1, \tag{3.4}$$

i.e., the AMSSP iteration method unconditionally converges to the unique solution of the saddle point problem (1).

Proof. From Lemma 3.1, we can find $\lambda \neq 1$. Then (15) yields the following result

$$v = \frac{\beta\lambda - \beta + 1}{\alpha(\lambda - 1)} E^T u. \tag{3.5}$$

This means by replacing the above relationship with (14)

$$\alpha^2(\lambda - 1)^2 u + \alpha(\beta\lambda - \beta + 1)(\lambda - 1)Bu + (\beta\lambda - \beta + 1)^2 EE^T u = 0. \tag{3.6}$$

Based on Lemma 3.1, we may get $x \neq 0$. Divided the sides of the equation (18) by $\frac{u^T}{u^T u}$, it holds

$$\alpha^2(\lambda - 1)^2 + \alpha(\beta\lambda - \beta + 1)(\lambda - 1)\frac{u^T Bu}{u^T u} + (\beta\lambda - \beta + 1)^2 \frac{u^T EE^T u}{u^T u} = 0. \tag{3.7}$$

For the convenience of discussing convergence, we assume that

$$\sigma = \frac{u^T Bu}{u^T u} > 0 \quad \text{and} \quad \tilde{\theta} = \frac{u^T EE^T u}{u^T u} \geq 0, \tag{3.8}$$

if $\tilde{\theta} > 0$, we use θ to denote $\tilde{\theta}$, then it holds that $\sigma \in [\sigma_{\min}, \sigma_{\max}]$ and $\theta \in [\theta_{\min}, \theta_{\max}]$ with $\theta_{\min} > 0$. After a simple calculation, we have

$$\lambda = \frac{\alpha + (\beta - 1)\sigma}{\alpha + \beta\sigma}, \quad \text{with} \quad \tilde{\theta} = 0, \tag{3.9}$$

or for $\tilde{\theta} = \theta$, it holds that

$$(\alpha^2 + \beta\sigma\alpha + 4\theta)\lambda^2 - (2\alpha^2 + 2\beta\sigma\alpha - \sigma\alpha + 4\theta)\lambda + (\alpha^2 + \beta\sigma\alpha - \sigma\alpha + \theta) = 0. \tag{3.10}$$

Using lemmas 3.2 and equations. (22) , we know that the eigenvalue $|\lambda| < 1$ if and only if

$$\frac{\alpha^2 + \beta\sigma\alpha - \sigma\alpha + \theta}{\alpha^2 + \beta\sigma\alpha + 4\theta} < 1 \tag{3.11}$$

and

$$\frac{2\alpha^2 + 2\beta\sigma\alpha - \sigma\alpha + 4\theta}{\alpha^2 + \beta\sigma\alpha + 4\theta} < 1 + \frac{\alpha^2 + \beta\sigma\alpha - \sigma\alpha + \theta}{\alpha^2 + \beta\sigma\alpha + 4\theta}. \tag{3.12}$$

Thus, the proof of theorem 3.3 is completed.

4. Numerical examples

To further assess the effectiveness of the block preconditioned matrix $\mathcal{P}_{MSSP}^{-1}\mathcal{A}$ combined with Krylov subspace methods, we present a sample of numerical examples which are based on two-dimensional time-harmonic Maxwell equations in mixed form in a square domain $(-1 \leq x \leq 1, -1 \leq y \leq 1)$ $(k = 0)$ with constant coefficients: find the vector field u and the multiplier p such that vector field u and the multiplier p such that

$$\begin{aligned} &\nabla \times \nabla \times u + \nabla p = f && \text{in } \Omega, \\ &\nabla \cdot u = 0 && \text{in } \Omega, \\ &u \times n = 0 && \text{on } \partial\Omega, \\ &p = 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

Here, $\Omega \in R^3$ is simply a connected polyhedron domain with a connected boundary $\partial\Omega$ and n denotes the outward unit normal on $\partial\Omega$. The datum f is a given generic source.

In all our runs we use a zero initial guess and stop the iteration when the relative residual had been reduced by at least six orders of magnitude (i.e, when $\|b-\mathcal{A}x^k\|_2 \leq 10^{-6}\|b\|_2$). For the simplicity, we take the generic source: $f = 1, P = I_n, Q = I_m$ and a finite element subdivision such as Figure 1 based on uniform grids of triangle elements. Three mesh sizes are considered: $h = \frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{18}$. The solutions of the preconditioned systems in each iteration are computed exactly. Information on the sparsity of the relevant matrices on the different meshes is given in Table 1, where $\text{nz}(A)$ denotes the number of nonzero elements of the matrix A.

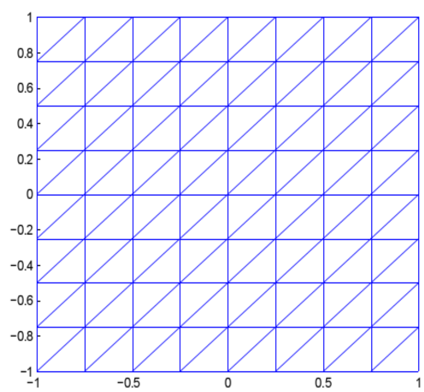

Figure 1. A uniform mesh with $h = \frac{\sqrt{2}}{4}$

Since the new preconditions have two parameters, we will test different values in numerical experiments. Numerical experiments show the spectrum of MSSP preconditioned matrix $\mathcal{P}_\alpha^{-1}A$ and AMSSP preconditioned matrix $\mathcal{P}_{\alpha,\beta}^{-1}A$ when choosing different parameters, which coincides with theoretical analysis.

In Figures 2, 4 and 6 we display the eigenvalues of the preconditioned matrix $\mathcal{P}_\alpha^{-1}A$ in the case of $h = \frac{\sqrt{2}}{8}$, $h = \frac{\sqrt{2}}{12}$ and $h = \frac{\sqrt{2}}{18}$ for different parameters. In Figures 3, 5 and 7 we display the eigenvalues of the preconditioned

Table 1. datasheet for different grids

Grid	m	n	nz(B)	nz(E)	order of A
8×8	176	49	820	462	225
16×16	736	225	3556	2190	961
32×32	3008	961	14788	9486	3969
64×64	12160	3969	60292	39438	16129

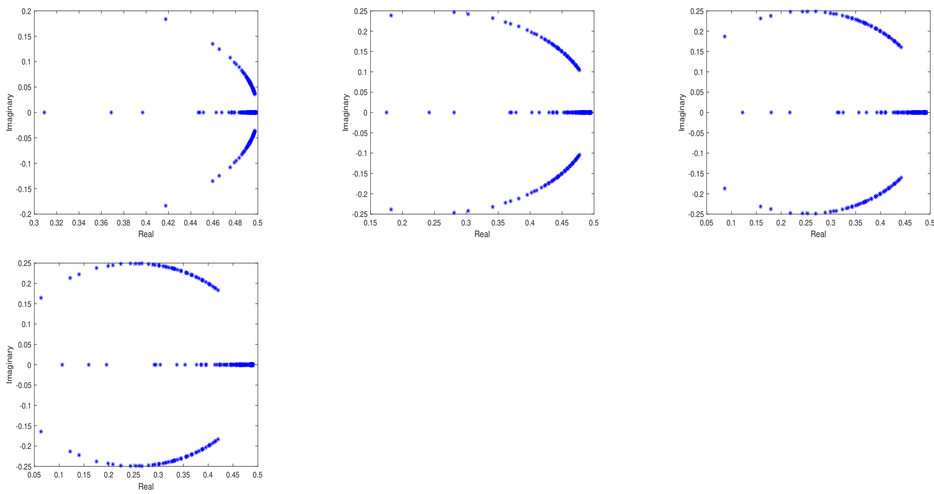

Figure 2. The eigenvalue distribution for the block preconditioned matrix $\mathcal{P}_{\alpha}^{-1}A$ when $\alpha = 0.1$(the first), $\alpha = 0.3$(the second),$\alpha = 0.5$(the third) and $\alpha = 0.6$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

Table 2. Iteration counts and time for preconditioned matrices $\mathcal{P}_{\alpha}^{-1}A$ and $\mathcal{P}_{\alpha,\beta}^{-1}A$ when choosing different parameters. Here, $h = \frac{\sqrt{2}}{8}$ denotes the size of the corresponding grid.

α	$It_{BICGSTAB(\mathcal{P}_{\alpha}^{-1}A)}$	$Res_{BICGSTAB(\mathcal{P}_{\alpha}^{-1}A)}$	α	β	$It_{BICGSTAB(\mathcal{P}_{\alpha,\beta}^{-1}A)}$	$Res_{BICGSTAB(\mathcal{P}_{\alpha,\beta}^{-1}A)}$
0.1	4.5	2.2539×10^{-7}	0.1	5	3	6.4137×10^{-7}
0.3	9	9.8896×10^{-7}	0.3	6	4.5	2.2539×10^{-7}
0.5	17.5	2.1209×10^{-7}	0.5	3	11	8.9695×10^{-7}
0.6	19.5	7.6311×10^{-7}	0.6	5	7.5	5.7425×10^{-7}
α	$It_{GMRES(\mathcal{P}_{\alpha}^{-1}A)}$	$Res_{GMRES(\mathcal{P}_{\alpha}^{-1}A)}$	α	β	$It_{GMRES(\mathcal{P}_{\alpha,\beta}^{-1}A)}$	$Res_{GMRES(\mathcal{P}_{\alpha,\beta}^{-1}A)}$
0.1	9(1)	7.7749×10^{-7}	0.1	5	7(1)	5.4037×10^{-7}
0.3	18(1)	7.6945×10^{-7}	0.3	6	9(1)	7.7749×10^{-7}
0.5	26(1)	5.5802×10^{-7}	0.5	3	19(1)	9.5444×10^{-7}
0.6	29(1)	9.2663×10^{-7}	0.6	5	15(1)	8.5127×10^{-7}

matrix $\mathcal{P}_{\alpha,\beta}^{-1}A$ in the case of $h = \frac{\sqrt{2}}{8}$, $h = \frac{\sqrt{2}}{12}$ and $h = \frac{\sqrt{2}}{18}$ for different parameters. Figures 2–7 show that the distribution of eigenvalues of the preconditioned matrix confirms our above theoretical analysis. In Tables 2–4 we show iteration counts for the preconditioned matrices $\mathcal{P}_{\alpha}^{-1}A$ and $\mathcal{P}_{\alpha,\beta}^{-1}A$, when choosing different parameters and applying to BICGSTAB and GMRES Krylov subspace iterative methods on three meshes, where $It_{BICGSTAB(\mathcal{P}_{\alpha}^{-1}A)}$ and $Res_{BICGSTAB(\mathcal{P}_{\alpha}^{-1}A)}$ are

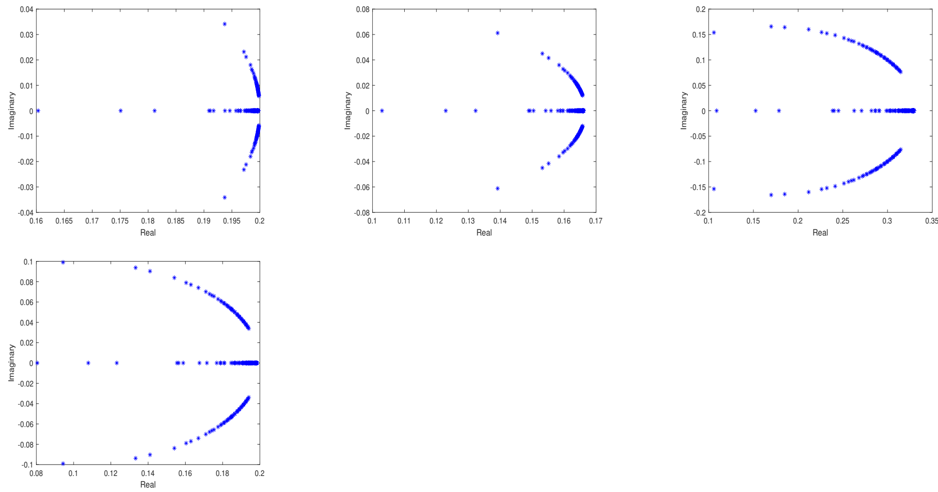

Figure 3. The eigenvalue distribution for the block preconditioned matrix $\mathcal{P}_{\alpha,\beta}^{-1}A$ when $\alpha = 0.1, \beta = 5$(the first), $\alpha = 0.3, \beta = 6$(the second),$\alpha = 0.5, \beta = 3$(the third) and $\alpha = 0.6, \beta = 5$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

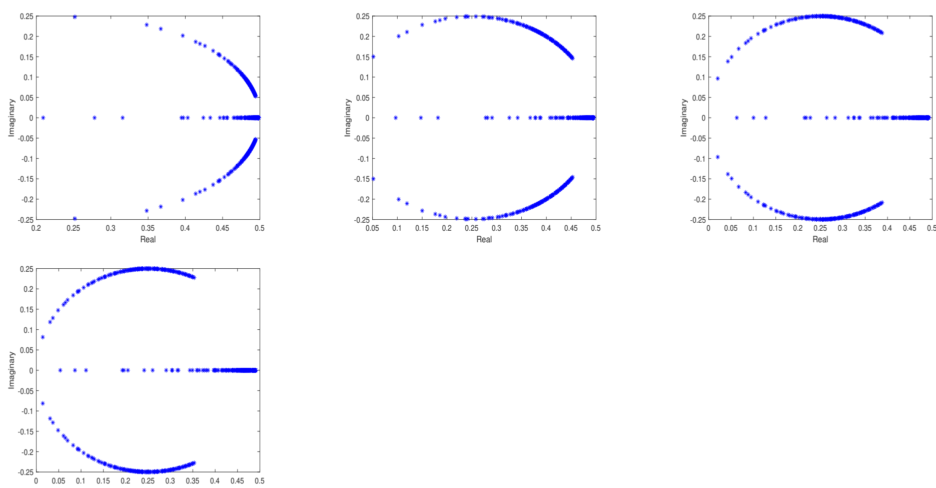

Figure 4. The eigenvalue distribution for the block preconditioned matrix $\mathcal{P}_{\alpha}^{-1}A$ when $\alpha = 0.1$(the first), $\alpha = 0.3$(the second),$\alpha = 0.5$(the third) and $\alpha = 0.6$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

the iteration numbers and relative residual of the preconditioned matrices $\mathcal{P}_{\alpha,\beta}^{-1}A$ when applying to BICGSTAB Krylov subspace iterative methods, respectively. $It_{GMRES(\mathcal{P}_{\alpha}^{-1}A)}$ and $Res_{GMRES(\mathcal{P}_{\alpha,\beta}^{-1}A)}$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{P}_{\alpha}^{-1}A$ when applying to GMRES Krylov subspace iterative methods, respectively. $It_{BICGSTAB(\mathcal{P}_{\alpha,\beta}^{-1}A)}$, $Res_{BICGSTAB(\mathcal{P}_{\alpha,\beta}^{-1}A)}$, $It_{GMRES(\mathcal{P}_{\alpha,\beta}^{-1}A)}$, $Res_{GMRES(\mathcal{P}_{\alpha,\beta}^{-1}A)}$ have similar definitions.

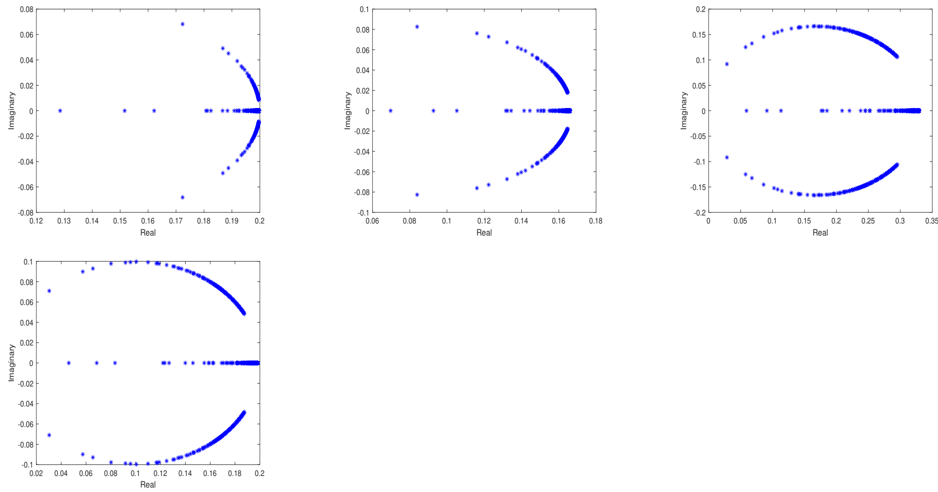

Figure 5. The eigenvalue distribution for the block preconditioned matrix $\mathcal{P}_{\alpha,\beta}^{-1}A$ when $\alpha = 0.1, \beta = 5$(the first), $\alpha = 0.3, \beta = 6$(the second),$\alpha = 0.5, \beta = 3$(the third) and $\alpha = 0.6, \beta = 5$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

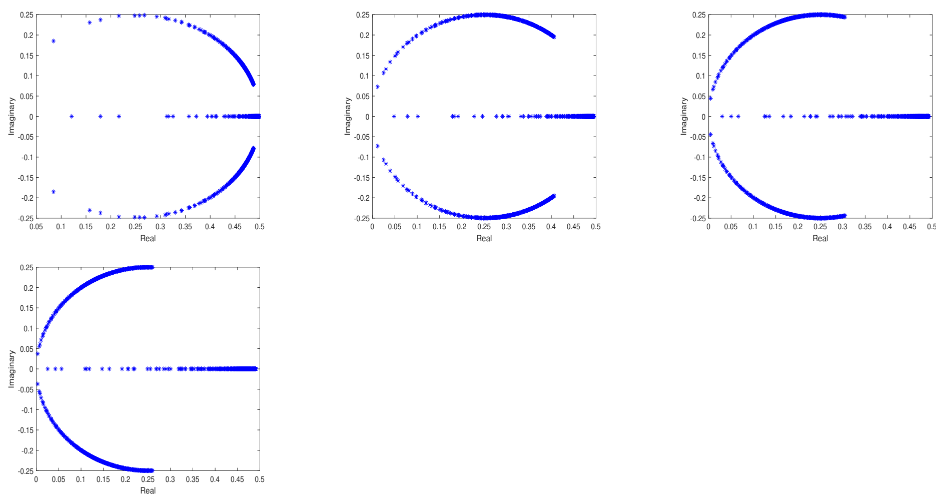

Figure 6. The eigenvalue distribution for the block preconditioned matrix $\mathcal{P}_{\alpha}^{-1}A$ when $\alpha = 0.1$(the first), $\alpha = 0.3$(the second),$\alpha = 0.5$(the third) and $\alpha = 0.6$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

Remark 4.1. From the above figures and tables, we know that the modified block preconditioner $\mathcal{P}_{\alpha,\beta}^{-1}$ has the same spectral clustering as the preconditioner $\mathcal{P}_{\alpha}^{-1}A$ when choosing the optimal parameters.

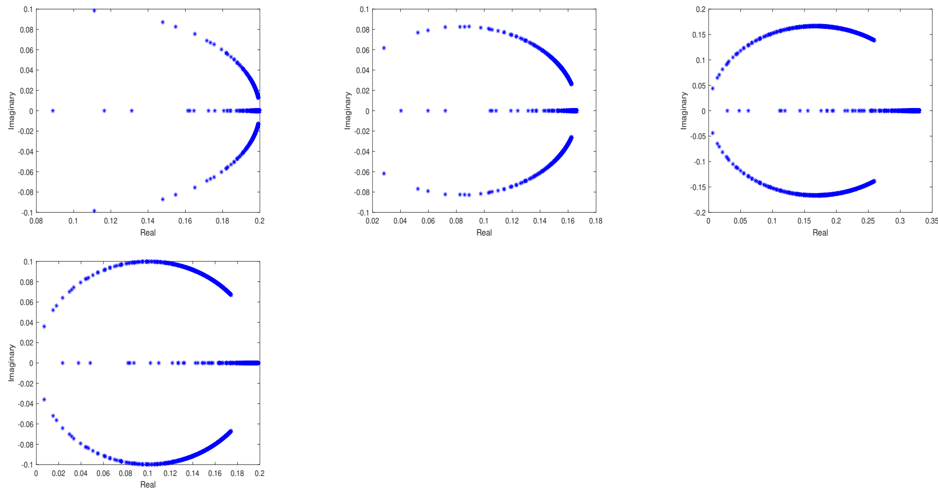

Figure 7. The eigenvalue distribution for the block preconditioned matrix $\mathcal{P}_{\alpha,\beta}^{-1}A$ when $\alpha = 0.1, \beta = 5$(the first), $\alpha = 0.3, \beta = 6$(the second),$\alpha = 0.5, \beta = 3$(the third) and $\alpha = 0.6, \beta = 5$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

Table 3. Iteration counts and time for preconditioned matrices $\mathcal{P}_{\alpha}^{-1}A$ and $\mathcal{P}_{\alpha,\beta}^{-1}A$ when choosing different parameters. Here, $h = \frac{\sqrt{2}}{12}$ denotes the size of the corresponding grid.

α	$It_{BICGSTAB(\mathcal{P}_{\alpha}^{-1}A)}$	$Res_{BICGSTAB(\mathcal{P}_{\alpha}^{-1}A)}$	α	β	$It_{BICGSTAB(\mathcal{P}_{\alpha,\beta}^{-1}A)}$	$Res_{BICGSTAB(\mathcal{P}_{\alpha,\beta}^{-1}A)}$
0.1	6.5	5.9457×10^{-7}	0.1	5	4	5.2489×10^{-7}
0.3	22	9.6384×10^{-7}	0.3	6	6.5	5.9457×10^{-7}
0.5	45.5	3.7234×10^{-7}	0.5	3	27.5	5.8299×10^{-7}
0.6	55.5	8.7840×10^{-7}	0.6	5	17	9.5169×10^{-7}
α	$It_{GMRES(\mathcal{P}_{\alpha}^{-1}A)}$	$Res_{GMRES(\mathcal{P}_{\alpha}^{-1}A)}$	α	β	$It_{GMRES(\mathcal{P}_{\alpha,\beta}^{-1}A)}$	$Res_{GMRES(\mathcal{P}_{\alpha,\beta}^{-1}A)}$
0.1	15(1)	4.9104×10^{-7}	0.1	5	9(1)	3.2480×10^{-7}
0.3	33(1)	5.5752×10^{-7}	0.3	6	15(1)	4.9104×10^{-7}
0.5	50(1)	7.9397×10^{-7}	0.5	3	36(1)	5.5689×10^{-7}
0.6	58(1)	7.9814×10^{-7}	0.6	5	28(1)	4.0712×10^{-7}

Table 4. Iteration counts and time for preconditioned matrices $\mathcal{P}_{\alpha}^{-1}A$ and $\mathcal{P}_{\alpha,\beta}^{-1}A$ when choosing different parameters. Here, $h = \frac{\sqrt{2}}{18}$ denotes the size of the corresponding grid.

α	$It_{BICGSTAB(\mathcal{P}_{\alpha}^{-1}A)}$	$Res_{BICGSTAB(\mathcal{P}_{\alpha}^{-1}A)}$	α	β	$It_{BICGSTAB(\mathcal{P}_{\alpha,\beta}^{-1}A)}$	$Res_{BICGSTAB(\mathcal{P}_{\alpha,\beta}^{-1}A)}$
0.1	13	9.2683×10^{-7}	0.1	5	6.5	1.4240×10^{-7}
0.3	66.5	2.6147×10^{-7}	0.3	6	13	9.2682×10^{-7}
0.5	111.5	9.6254×10^{-7}	0.5	3	72.5	5.5149×10^{-7}
0.6	141.5	5.7863×10^{-7}	0.6	5	49.5	1.7692×10^{-7}
α	$It_{GMRES(\mathcal{P}_{\alpha}^{-1}A)}$	$Res_{GMRES(\mathcal{P}_{\alpha}^{-1}A)}$	α	β	$It_{GMRES(\mathcal{P}_{\alpha,\beta}^{-1}A)}$	$Res_{GMRES(\mathcal{P}_{\alpha,\beta}^{-1}A)}$
0.1	27(1)	3.9182×10^{-7}	0.1	5	14(1)	4.7574×10^{-7}
0.3	66(1)	9.2262×10^{-7}	0.3	6	27(1)	3.9182×10^{-7}
0.5	102(1)	8.7194×10^{-7}	0.5	3	73(1)	7.5543×10^{-7}
0.6	118(1)	8.6680×10^{-7}	0.6	5	55(1)	6.4807×10^{-7}

5. Conclusions

In this paper, based on modified shift-splitting (denoted by MSSP) iteration technique by Huang and Su [30], we establish a accelerated (named after AMSSP) iterative method for nonsymmetric saddle point problems. Furthermore, we theoretically verify the AMSSP iteration method unconditionally converges to the unique solution of the saddle point problems, compute the spectral radius of the AMSSP iteration matrix. Finally, numerical examples show the spectrum of the new preconditioned matrix for the different parameters.

The authors declare that they have no competing interests.

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