Journal of Applied Analysis and Computation
Volume 13, Number 4, August 2023, 2297–2314
Website:http://www.jaac-online.com
DOI:10.11948/20220501

GLOBAL DYNAMICS OF A MOSQUITO POPULATION SUPPRESSION MODEL UNDER A PERIODIC RELEASE STRATEGY*

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Abstract It has been proved that periodic releases of *Wolbachia*-infected or irradiation-treated mosquitoes is an effective way to suppress wild mosquitoes and prevent the prevalence of mosquito-borne diseases. We have discussed some cases in consideration of the release amount c and release period T, and in this paper we continue to explore the remaining complementary case and investigate the relevant stability of the origin and the exact number of periodic solutions in the switching model. Based on the release period threshold T^* introduced in the extant works, we define a new threshold T^{**} between the sexual lifespan $\bar{T}$ of sterile mosquitoes and T^*, and reveal the complete dynamics of the model, particularly, no T-periodic solutions when $T \in (\bar{T}, T^{**})$, a unique T-periodic solution when $T = T^{**}$, and exactly two T-periodic solutions when $T \in (T^{**}, T^*)$. Finally, we give some numerical simulations to seek the approximate value of T^{**} and demonstrate the global dynamical behaviors of wild mosquito population.

Keywords Sterile mosquitoes, mosquito population suppression, periodic solutions, asymptotic stability, impulsive and periodic releases.

MSC(2010) 34C25, 34D20, 34D23, 92D25, 93D20.

1. Introduction

Mosquitoes bring great harm to human beings by transmitting various diseases, such as dengue, malaria, chikungunya, yellow fever and Zika. About 390 million people are infected with dengue every year, and hundreds of thousands are affected by Zika, chikungunya and yellow fever [29]. Since there are neither effective specific drugs nor safe vaccines available, an effective way to prevent and control mosquito-borne diseases is to suppress or even eradicate mosquito vectors [28]. It has been found that *Wolbachia*-infected mosquitoes are less able to transmit some viruses, including dengue and Zika [4, 8]. In addition, we assume perfect mater-

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* The authors were supported by National Natural Science Foundation of China (12071095, 12171112) and Shanxi Scholarship Council of China (2022-176).

nal transmission: if the mother is infected by *Wolbachia*, then all the offspring are also infected [43]. In this way, it makes mosquito population replacement an environmental-friendly and efficient strategy to control mosquito-borne diseases. In 1959, Caspari and Watson [7] analyzed the potential of mosquito population replacement through a discrete mathematical model. Subsequently, many scholars established various mathematical models on mosquito population replacement by *Wolbachia* [9, 13, 15, 21].

Since only female mosquitoes bite humans, releasing male mosquitoes to suppress wild mosquito population is a more acceptable measure. Among the methods of suppressing mosquitoes, sterile insect technique (SIT) and incompatible insect technique (IIT) are two eco-friendly and sustainable potential new tools [26, 42]. Massive number of mosquitoes are reared in the laboratories or mosquito factories by using SIT and IIT and then treated with irradiation or the endosymbiotic bacteria *Wolbachia*, respectively. A wild female mosquito that mated with an irradiated or a *Wolbachia*-infected male mosquito (we call these two types of mosquitoes sterile mosquitoes hereafter), will not be capable of producing eggs or produce eggs that cannot hatch, which gradually reduces the densities of wild mosquito populations in the target areas [10, 11]. Actually, the cost of raising and releasing mosquitoes cannot be ignored. The strategy of releasing sterile mosquitoes, which mainly contains the release amount in a batch and the release period between two consecutive releases, needs to be carefully designed. Moreover, a body of mathematical models [1–3, 5, 22, 30, 33–36] have been formulated and studied in order to obtain cost-efficient strategies.

As long as the sterile mosquitoes are released, the interactive dynamics of wild and sterile mosquitoes could illustrate the suppression effect of wild mosquitoes. There have been some works studying the asymptotic behaviors of solutions of the switching system caused by the periodic releases of sterile male mosquitoes [2,31,39]. In [46], we studied the interactive dynamics governed by the following model

$$\frac{dw}{dt} = \frac{aw^2}{w+g}(1-\xi w) - \mu w, \tag{1.1}$$

in which $w = w(t), g = g(t)$ are the numbers of wild and sterile mosquitoes at time t, respectively, a denotes the number of survived offspring produced per wild mosquito through all matings, per unit time, $w/(w+g)$ represents the mating probability between wild mosquitoes. Moreover, μ and ξ characterize the density-independent death rate and the carrying capacity, respectively, and the term $1 - \xi w$ describes the density-dependent survival probability.

Before discussing the dynamics of model (1.1), we give the following two assumptions.

($A1$) The release begins at time $t = 0$, thus, $g(t) \equiv 0$ for $t < 0$. Furthermore, a constant number c of sterile mosquitoes is released after a constant waiting period T such that the sterile mosquitoes are released impulsively and periodically at discrete time points $T_i = iT, i = 0, 1, 2, \cdots$.

($A2$) All the released sterile mosquitoes have the same sexual lifespan, denoted by $\bar{T}$, and the waiting period T is larger than the sexual lifespan $\bar{T}$ of sterile mosquitoes, that is, $T > \bar{T}$.

Then the release function $g(t)$ takes the form

$$g(t) = \begin{cases} c, \ t \in [iT, iT + \bar{T}), \\ 0, \ t \in [iT + \bar{T}, (i+1)T), \end{cases} \quad i = 0, 1, 2, \cdots,$$

which divides equation (1.1) into the next two sub-equations:

$$\frac{dw}{dt} = \frac{aw^2}{w+c}(1 - \xi w) - \mu w, \ t \in [iT, iT + \bar{T}), \tag{1.2}$$

and

$$\frac{dw}{dt} = -a\xi w(w - A), \ t \in [iT + \bar{T}, (i+1)T), \tag{1.3}$$

where $i = 0, 1, 2, \cdots$ and $A = (a - \mu)/(a\xi)$.

The dynamical feature of the single equation (1.2) or (1.3) is simple, but their combination shows rich and complex dynamics. With the introduction of the following three thresholds:

$$g^* = \frac{(a-\mu)^2}{4a\mu\xi}, \quad c^* = \frac{a-\mu}{\mu\xi}, \quad T^* = \frac{a}{a-\mu}\bar{T},$$

we obtained an almost complete depiction of dynamics of model (1.2)-(1.3) under the case $c > g^*$ in [46]. However, for the case when (c, T) lies in

$$\Gamma = \{(c, T) | g^* < c < c^*, \bar{T} < T < T^*\}, \tag{1.4}$$

we just proved the local asymptotic stability of the origin $E_0 = 0$, both the exact number of periodic solutions and the global asymptotic stability of E_0 are fuzzy.

In the current study, by assuming that the number of released sterile mosquitoes is kept in (g^*, c^*), we seek the other release period threshold T^{**} in interval $(\bar{T}, T^*)$. T^{**} determines the exact number of T-periodic solutions of model (1.2)-(1.3) according to its relationship with T, and benefits us greatly to further explore the model dynamics when $(c, T) \in \Gamma$. We first define a Poincaré map for obtaining periodic solutions, then we derive a crucial function generated by the Poincaré map, through analyzing the monotonicity and the number of zeros of the function, we prove the existence and uniqueness of T^{**} and present a useful separatrix

$$S = \{(c, T) | c \in (g^*, c^*), \ T = T^{**}\},$$

which splits Γ into two sub-regions

$$\Gamma_1 = \{(c, T) | c \in (g^*, c^*), \ T \in (\bar{T}, T^{**})\}$$

and

$$\Gamma_2 = \{(c, T) | c \in (g^*, c^*), \ T \in (T^{**}, T^*)\}.$$

Most importantly, we gain deeper insights into the long-term behaviors of solutions of model (1.2)-(1.3) when $(c, T) \in \Gamma$, and complete the characterization of dynamics of model (1.2)-(1.3) when $T > \bar{T}$. More precisely, we have the following theorem.

Theorem 1.1. *The three conclusions below hold for model* (1.2)-(1.3).

(i) If $(c, T) \in \Gamma_1$, *then the model has no* T*-periodic solutions, and* E_0 *is globally asymptotically stable.*

(ii) *If* $(c,T) \in S$*, then the model has a unique* T*-periodic solution, which is semi-stable: stable on the right-side but unstable on the left-side.*

(iii) *If* $(c,T) \in \Gamma_2$*, then the model has exactly two* T*-periodic solutions, one of which evolved from a larger initial value is asymptotically stable, while the other is unstable.*

We first give some preliminaries before sharing the detailed proof of the theorem.

2. Preliminaries

Let $w(t) = w(t;0,u)$ denote the solution of model (1.2)-(1.3) with initial value $w(0) = u > 0$. Set $\bar{h}(u) = w(\bar{T};0,u)$ and $h(u,T) = w(T;0,u)$. From (1.2) and $w(0) = u$, we have

$$P(\bar{h}(u)) = P(u)e^{-a\xi\bar{T}}, \tag{2.1}$$

where

$$P(u) = u^{\alpha}\left(\left(u-\frac{A}{2}\right)^2+b^2\right)^{\frac{\beta}{2}}\exp\left(\frac{\gamma}{b}\tan^{-1}\left(\frac{u-\frac{A}{2}}{b}\right)\right), \tag{2.2}$$

and

$$\alpha = \frac{a\xi}{\mu},\quad \beta = -\frac{a\xi}{\mu},\ \gamma = \frac{a+\mu}{2\mu},\ b = \sqrt{\frac{\mu}{a\xi}(c-g^*)}.$$

Moreover, (2.2) offers

$$P'(u) = \left(\frac{\alpha}{u}+\frac{\beta\left(u-\frac{A}{2}\right)+\gamma}{\left(u-\frac{A}{2}\right)^2+b^2}\right)P(u) = \frac{u+c}{u\left(\left(u-\frac{A}{2}\right)^2+b^2\right)}P(u), \tag{2.3}$$

which implies that P is a strictly increasing function with respect to u. Meanwhile, solving (1.3) in $[\bar{T},T)$ with $w(\bar{T}^-) = \lim_{t\to\bar{T}^-} w(t)$, we obtain

$$\frac{\bar{h}(u)}{A-\bar{h}(u)} = \frac{mh(u,T)}{A-h(u,T)}, \tag{2.4}$$

where $m = e^{-aA\xi(T-\bar{T})}$. Obviously, both $\bar{h}(u)$ and $h(u,T)$ are continuously differentiable with respect to u [12]. To depict the long-term behaviors of solutions of model (1.2)-(1.3), we define sequences $\{\bar{h}_n\}$ and $\{h_n\}$ by

$$\bar{h}_n(u) = w(nT+\bar{T};0,u),\ h_n(u,T) = w(nT;0,u), n = 0,1,2,\cdots,$$

which lead to

$$\bar{h}_0(u) = w(\bar{T};0,u),\ h_0(u,T) = u,\ \text{and}\ h_1(u,T) = h(u,T),$$

and by induction, we arrive at

$$\bar{h}_n(u)=\bar{h}(h_n(u,T)),\ h_{n+1}(u,T)=w(T;0,h_n(u,T))=h(h_n(u,T),T),\ n = 0,1,2,\cdots.$$

The following lemma, whose proof is similar to Lemma 3.1 in [33] and is omitted here, reveals the relation between the sign of $h(u,T)-u$ and the monotonicity of sequence $\{h_n(u,T)\}$ or $\{\bar{h}_n(u)\}$, and thus plays a central role in determining the number of periodic solutions of model (1.2)-(1.3).

Lemma 2.1. *For any given initial value u, the following conclusions hold.*

(i) Sequences $\{h_n(u,T)\}$ and $\{\bar{h}_n(u)\}$ are both strictly increasing if and only if $h(u,T) > u$.

(ii) $h_n(u,T) \equiv u, n = 1,2,3,\cdots,$ if and only if $h(u,T) = u$. Hence, $w(t) = w(t;0,u)$ is a T-periodic solution of model (1.2)-(1.3).

(iii) Sequences $\{h_n(u,T)\}$ and $\{\bar{h}_n(u)\}$ are both strictly decreasing if and only if $h(u,T) < u$.

Generally speaking, it is not easy to figure out the exact number of periodic solutions for a given model due to the lack of a broadspectrum method. Nevertheless, Lemma 2.1 tells us that the fixed points of h are indeed the initial values of periodic solutions of model (1.2)-(1.3). Hence, in the following, we concentrate on seeking the fixed points of h. Obviously, $h(0,T) = \bar{h}(0) = 0$, and the fact $c > g^*$ yields

$$\bar{h}(u) < u, u > 0, \tag{2.5}$$

since the right hand side function of (1.2) is negative in this case. Furthermore, (1.3) implies

$$\begin{aligned}
&h(u,T) < \bar{h}(u),\ \bar{h}(u) \in (A,+\infty),\\
&h(u,T) = \bar{h}(u),\ \bar{h}(u) = A,\\
&h(u,T) > \bar{h}(u),\ \bar{h}(u) \in (0,A),
\end{aligned}$$

which derive, along with (2.5),

$$h(u,T) < u, u \in [A,+\infty).$$

Thus, the potential initial values of the T-periodic solutions of model (1.2)-(1.3) stay in $(0,A)$, this discovery enables us to narrow the range of the fixed points of h to $(0,A)$. Moreover, Lemma 2.1 also indicates that the stability of E_0 is determined by the sign of $h(u,T) - u$, for sufficiently small positive real number u. From (2.1)-(2.2), we obtain

$$\lim_{u\to 0} \frac{\bar{h}(u)}{u} = e^{-\frac{a\xi\bar{T}}{\alpha}} = e^{-\mu\bar{T}},$$

since $\bar{h}(u) \to 0$ and $h(u,T) \to 0$ as $u \to 0$. Similarly, from (2.4), we have

$$\lim_{u\to 0} \frac{h(u,T)}{\bar{h}(u)} = \lim_{u\to 0} \frac{A - h(u,T)}{m(A - \bar{h}(u))} = \frac{1}{m} = e^{aA\xi(T-\bar{T})} = e^{(a-\mu)(T-\bar{T})}.$$

Thus, we reach

$$h'(0) = \lim_{u\to 0} \frac{h(u,T)}{u} = \lim_{u\to 0} \left(\frac{h(u,T)}{\bar{h}(u)} \cdot \frac{\bar{h}(u)}{u} \right) = e^{(a-\mu)\left(T - \frac{a}{a-\mu}\bar{T}\right)} = e^{(a-\mu)(T-T^*)}.$$

Hence, if $T < T^*$, then $\lim_{u\to 0} \frac{h(u,T)}{u} < 1$, which tells us that there exists sufficiently small $\delta > 0$ such that

$$h(u,T) < u, u \in (0,\delta). \tag{2.6}$$

3. Proof of the theorem

Before proceeding to the detailed proof of the theorem, we first make a detour into the following four lemmas, which are essential for deriving the theorem.

3.1. Four springboard lemmas

Lemma 3.1. *Assume that* $(c,T)\in\Gamma$, *which is defined in* (1.4). *Then model* (1.2)-(1.3) *has at most two* T*-periodic solutions.*

Proof. Clearly, the sign of $\partial h(u,T)/\partial u-1$ determines the number of fixed points of $h(u,T)$, and hence determines the number of periodic solutions of model (1.2)-(1.3). We begin the proof with calculating the sign of $\partial h(u,T)/\partial u-1$ at the fixed point of $h(u,T)$. At any fixed point of $h(u,T)$, that is, at any point u which belongs to the set $\Omega=\{u|u>0,h(u,T)=u\}$, we have, from (2.1), (2.3) and (2.4),

$$\begin{aligned}\frac{\partial h(u,T)}{\partial u}&=\frac{h(u,T)\left\{\frac{A^2}{4}\left[A-(1+m)h(u,T)\right]^2+b^2\left[A-(1-m)h(u,T)\right]^2\right\}(u+c)}{Au\left[\left(u-\frac{A}{2}\right)^2+b^2\right]\left\{cA+\left[mA-(1-m)c\right]h(u,T)\right\}}\\&=\frac{\left\{\frac{A^2}{4}\left[A-(1+m)u\right]^2+b^2\left[A-(1-m)u\right]^2\right\}(u+c)}{A\left[\left(u-\frac{A}{2}\right)^2+b^2\right]\left\{cA+\left[mA-(1-m)c\right]u\right\}}.\end{aligned}\tag{3.1}$$

In [46], we proved that $\partial h(u,T)/\partial u<1(=1,\text{or}>1)$ if and only if

$$Q(u,T)=B(T)u^2+C(T)u+D<0(=0,\text{or}>0),\tag{3.2}$$

with

$$B(T)=\frac{a-m\mu}{a\xi},\ C(T)=\frac{(1-m)c\mu\xi-2(a-\mu)}{a\xi^2},\ D=-\frac{\mu(c-c^*)}{a\xi}A,\tag{3.3}$$

where $m=e^{-aA\xi(T-\bar{T})}$. Obviously, $B(T)>0$ is always true, and $C(T)<0,D=Q(0,T)>0$ hold when $c\in(g^*,c^*)$. These facts, coupled with

$$Q(A,T)=B(T)A^2+C(T)A+D=-\frac{a-\mu+ac\xi}{a^2\xi^2}m\mu A<0,$$

imply that $Q(u,T)=0$ has a unique root, denoted by u^*, for $u\in(0,A)$. Then we obtain

$$Q(u,T)>0,u\in(0,u^*)\quad\text{and}\quad Q(u,T)<0,u\in(u^*,A).\tag{3.4}$$

Now, suppose by contradiction that model (1.2)-(1.3) has at least three T-periodic solutions. We first assume that model (1.2)-(1.3) has exactly three T-periodic solutions, and denote their initial values by u_0,u_1,u_2 with $u_0<u_1<u_2$. It then follows from

$$h(u,T)<u,u\in(0,\delta)\quad\text{and}\quad h(A,T)<A,$$

that $\frac{\partial h(u,T)}{\partial u}\big|_{u=u_0} \geq 1$ and $\frac{\partial h(u,T)}{\partial u}\big|_{u=u_2} \leq 1$, which consists of the following four cases:

$$\begin{aligned}
&(a)\quad \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_0} \geq 1,\ \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_1} \leq 1,\ \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_2} = 1;\\
&(b)\quad \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_0} = 1,\ \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_1} \geq 1,\ \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_2} \leq 1;\\
&(c)\quad \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_0} = 1,\ \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_1} = 1,\ \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_2} = 1;\\
&(d)\quad \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_0} \geq 1,\ \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_1} = 1,\ \frac{\partial h(u,T)}{\partial u}\Big|_{u=u_2} \leq 1.
\end{aligned}$$

Next, we exclude cases (a)-(d) one by one. First, if case (a) holds, then we have

$$Q(u_0,T) \geq 0,\ Q(u_1,T) \leq 0,\ Q(u_2,T) = 0,$$

a contradiction to (3.4). Consequently, if case (b) is true, we must have

$$Q(u_0,T) = 0,\ Q(u_1,T) \geq 0,\ Q(u_2,T) \leq 0,$$

which also contradicts (3.4). Panels (A) and (B) in Fig.1 illustrate the two cases.

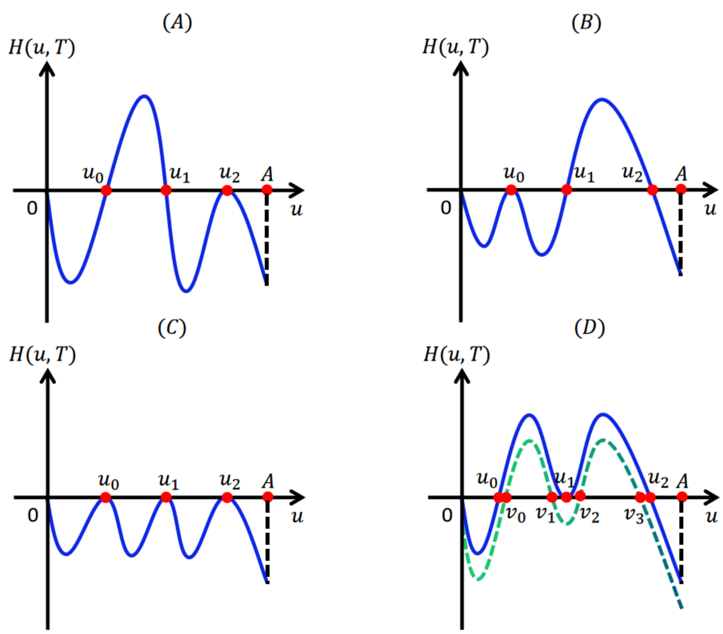

Figure 1. Let $H(u,T) = h(u,T) - u$. Panels (A)-(D) are four schematic diagrams for excluding cases (a)-(d), respectively. Among them, panels (A) and (B) are inconsistent with (3.4). Panel (C) is inconsistent with the fact that $Q(u,T) = 0$ has exactly one root for $u \in (0, A)$, and panel (D) is inconsistent with the facts that $B_k(T) > 0$ and $Q_k(u,T)$ is a quadratic polynomial.

Next, if case (c) holds, then we derive

$$Q(u_0,T) = Q(u_1,T) = Q(u_2,T) = 0,$$

which contradicts the fact $Q(u,T) = 0$ has exactly one root for $u \in (0, A)$. See Fig.1(C).

Finally, if case (d) is true, then we turn to perturbation technique for obtaining a contradiction. Let $k-1>0$ be small enough such that $h(u,T)-ku=0$ has exactly four roots, denoted by v_0, v_1, v_2 and v_3, with $v_0<v_1<v_2<v_3$, then we have

$$\frac{\partial h(u,T)}{\partial u}\Big|_{u=v_0}\geq k,\ \ \frac{\partial h(u,T)}{\partial u}\Big|_{u=v_1}\leq k,\ \ \frac{\partial h(u,T)}{\partial u}\Big|_{u=v_2}\geq k,\ \frac{\partial h(u,T)}{\partial u}\Big|_{u=v_3}\leq k. \tag{3.5}$$

See Fig.1(D). From (3.1), the inequalities in (3.5) can be transformed to

$$\Psi(v_0)\geq k,\ \ \Psi(v_1)\leq k,\ \ \Psi(v_2)\geq k,\ \ \Psi(v_3)\leq k, \tag{3.6}$$

respectively, where

$$\Psi(u)=\frac{k\left\{\frac{A^2}{4}\left[A-(1+m)ku\right]^2+b^2\left[A-(1-m)ku\right]^2\right\}(u+c)}{A\left[\left(u-\frac{A}{2}\right)^2+b^2\right]\left\{cA+\left[mA-(1-m)c\right]ku\right\}}.$$

Set

$$Q_k(u,T)=B_k(T)u^2+C_k(T)u+D_k(T),$$

where

$$\begin{aligned}
B_k(T)&=k\left(\frac{k(1-m)^2}{\alpha}c+(k-1)mA^2+(1-m)cA\right)(>0),\\
C_k(T)&=\frac{k^2(1-m)^2}{\alpha}c^2+(k+1)(mk-1)cA^2+\frac{2k(m-1)}{\alpha}cA,\\
D_k(T)&=-\frac{k(1-m)}{\alpha}cA\left(c-\frac{1-mk}{k(1-m)}c^*\right).
\end{aligned}$$

Then, (3.6) gives

$$Q_k(v_0,T)\geq 0,\ \ Q_k(v_1,T)\leq 0,\ \ Q_k(v_2,T)\geq 0,\ \ Q_k(v_3,T)\leq 0,$$

which imply that the function $Q_k(u,T)$ changes monotonicity at least twice for $u\in(v_0,v_3)$, a contradiction to the facts that $B_k(T)>0$ and $Q_k(u,T)$ is a quadratic polynomial.

Moreover, by the relation between the signs of $\partial h(u,T)/\partial u-1$ and $Q(u,T)$, along with (3.4), we can similarly exclude the case that model (1.2)-(1.3) has at least four T-periodic solutions. This completes the proof. □

Next, we prove the existence and uniqueness of T^{**} for any given $c\in(g^*,c^*)$. To begin with, we determine the signs of $dB(T)/dT$ and $dC(T)/dT$. From (3.3), we obtain

$$\frac{dB(T)}{dT}=-\frac{\mu}{a\xi}\frac{dm}{dT}=-\frac{\mu}{a\xi}(-aA\xi)e^{-aA\xi(T-\bar{T})}=\mu Am>0,$$

and

$$\frac{dC(T)}{dT}=-\frac{c\mu\xi}{a\xi^2}\frac{dm}{dT}=-\frac{c\mu}{a\xi}(-aA\xi)e^{-aA\xi(T-\bar{T})}=c\mu Am>0,$$

which lead to the following lemma.

Lemma 3.2. *Let $c\in(g^*,c^*)$ and $Q(u,T)$ be defined in* (3.2). *Then $Q(u,T)$ and its unique zero $u^*=u^*(T)$ are strictly increasing with respect to T.*

To complete the proof of the theorem, we also need the subsequent and crucial lemma.

Lemma 3.3. *Let $c \in (g^*, c^*)$. Then there is a unique $T^{**} \in (\bar{T}, T^*)$ such that model* (1.2)-(1.3) *has*

(i) no T-periodic solutions, if $T \in (\bar{T}, T^{**})$;

(ii) a unique T-periodic solution, if $T = T^{**}$;

(iii) exactly two T-periodic solutions, if $T \in (T^{**}, T^*)$.

Proof. When $g^* < c < c^*$ and $T = T^*$, model (1.2)-(1.3) has a unique T-periodic solution, which is globally asymptotically stable [46]. Let $u_T(T^*)$ be the initial value with $u_T(T^*) \in (0, A)$. Then

$$h(u, T^*) > u, \ 0 < u < u_T(T^*) < A,$$

which, compounded by the fact that $h(u, T) - u$ is strictly increasing with respect to T for any $u \in (0, A)$, imply that there is a $\bar{v} \in (\delta, u_T(T^*))$ such that $h(\bar{v}, T) > \bar{v}$, provided $T < T^*$ and T is sufficiently close to T^*, where δ is defined in (2.6). Moreover, we already know

$$h(u, \bar{T}) = \bar{h}(u) < u, \ 0 < u < A$$

holds for $c \in (g^*, c^*)$. Hence, the existence and uniqueness of T^{**} are ensured by the continuity and monotonicity of $h(u, T) - u$ with respect to T. For $u \in (0, A)$, we have

$$\begin{aligned}
&h(u, T) < u, \ \bar{T} < T < T^{**}, \\
&|\{u | h(u, T^{**}) = u\}| = 1, \\
&\{u | h(u, T) > u\} \neq \varnothing, \ T^{**} < T < T^*,
\end{aligned}$$

where $|M|$ denotes the number of the elements of the set M, see Fig.2 for illustration. Thus, if $T \in (\bar{T}, T^{**})$, then model (1.2)-(1.3) has no T-periodic solutions. If $T = T^{**}$, then model (1.2)-(1.3) has a unique T-periodic solution, and we assume that its initial value is u_0^*. If $T \in (T^{**}, T^*)$, then the facts that $h(u, T) < u$ holds for $u \in (0, \delta) \cup [A, +\infty)$, $h(\bar{v}, T) > \bar{v}$, along with Lemma 3.1, manifest that model (1.2)-(1.3) has exactly two T-periodic solutions. This concludes the proof. □

If $(c, T) \in \Gamma_2$, then Lemma 3.3 indicates that model (1.2)-(1.3) has exactly two T-periodic solutions. We further assume that the initial values of the two T-periodic solutions are u_1 and u_2, satisfying $u_1 < u_0^* < u_2$. Since $w(t) = w(t; 0, u)$, for convenience, we set

$$w_1(t) = w(t; 0, u_1) \quad \text{and} \quad w_2(t) = w(t; 0, u_2)$$

be the two periodic solutions. Then we obtain

$$h(u, T) < u, u \in (0, u_1) \cup (u_2, +\infty), h(u, T) > u, u \in (u_1, u_2). \tag{3.7}$$

For the stabilities of $w_1(t)$ and $w_2(t)$, we have the following lemma.

Lemma 3.4. *Assume that $(c, T) \in \Gamma_2$. Then model* (1.2)-(1.3) *generates bistability: both the origin and $w_2(t)$ are asymptotically stable, while $w_1(t)$ is unstable.*

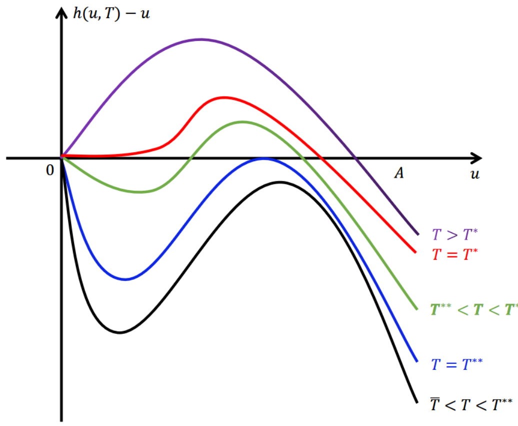

Figure 2. The schematic diagram to depict the monotone increasing property of $h(u,T)-u$ with respect to T. We mention here that the blue curve manifests that the set $\{u|h(u,T^{**})=u\}$ contains only one element.

Proof. Combining (3.7) and Lemma 2.1, we know that the sequence $\{h_n(u,T)\}$ is strictly decreasing for $u\in(0,u_1)$, which results in $\lim\limits_{n\to+\infty} w(nT)=0$. Since each $w(nT)$ is the maximum value of $w(t)$ for $t\in[nT,(n+1)T), n=0,1,2,\cdots$, we get $\lim\limits_{t\to+\infty} w(t)=0$. This implies the stability of the origin and the instability of $w_1(t)$.

Next, we prove that $w_2(t)$ is asymptotically stable. We first show that $w_2(t)$ is stable, i.e., for every $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that

$$|w(t)-w_2(t)|<\varepsilon, t\geq 0, \tag{3.8}$$

provided that $|u-u_2|<\delta$. Without loss of generality, we consider the case $w(t)>w_2(t)$. Set

$$F(t)=w(t)-w_2(t).$$

Let $\varepsilon\in(0,A-u_2)$. It suffices to find a $\delta>0$ such that $F(t)<\varepsilon$ for all $t>0$ when $0<u-u_2<\delta$.

From (3.7), Lemma 2.1 and the fact that (1.2)-(1.3) is T-periodic, we only need to prove

$$F(t)<\varepsilon, 0\leq t\leq T. \tag{3.9}$$

In the following, we consider two cases $t\in(0,\bar{T})$ and $t\in[\bar{T},T]$ to prove (3.9).

On one hand, if $t\in(0,\bar{T})$, then we have

$$\begin{aligned}
F'(t) &= w'(t)-w_2'(t)\\
&= \frac{aw^2}{w+c}(1-\xi w)-\mu w-\left(\frac{aw_2^2}{w_2+c}(1-\xi w_2)-\mu w_2\right)\\
&= \frac{aw^2}{w+c}(1-\xi w)-\frac{aw^2}{w+c}(1-\xi w_2)+\frac{aw^2}{w+c}(1-\xi w_2)\\
&\quad -\frac{aw_2^2}{w_2+c}(1-\xi w_2)+\mu(w_2-w)\\
&= (w-w_2)\frac{(a-\mu)[w_2(w+c)+cw]-a\xi[w(w_2+w)(w_2+c)+cw_2^2]-\mu c^2}{(w+c)(w_2+c)}.
\end{aligned}$$

Since $w(t), w_2(t)$ and c are bounded on $(0, \bar{T})$, there exists some $M_1 > 0$ such that

$$\frac{(a-\mu)[w_2(w+c)+cw] - a\xi[w(w_2+w)(w_2+c)+cw_2^2] - \mu c^2}{(w+c)(w_2+c)} \leq M_1.$$

Then we obtain $F'(t) \leq M_1(w - w_2) = M_1 F(t)$, which gives

$$F(t) \leq F(0)e^{M_1 t} \leq F(0)e^{M_1 \bar{T}}. \tag{3.10}$$

By choosing $\delta_1 = \frac{\varepsilon}{e^{M_1 \bar{T}}}$, it then follows from (3.10) and the fact $F(0) = u - u_2 < \delta_1$ that $F(t) < \varepsilon$. Thus (3.9) holds for the case $t \in (0, \bar{T})$.

On the other hand, if $t \in [\bar{T}, T]$, then we get

$$\begin{aligned} F'(t) &= w'(t) - w_2'(t) \\ &= -a\xi w(w-A) - (-a\xi w_2(w_2 - A)) \\ &= (w - w_2)\left(a - \mu - a\xi(w_2 - w)\right). \end{aligned}$$

Similarly, because of the boundednesses of $w(t)$ and $w_2(t)$ on $[\bar{T}, T]$, there is a $M_2 > 0$ such that $a - \mu - a\xi(w_2 - w) \leq M_2$. Hence, $F'(t) \leq M_2(w - w_2) = M_2 F(t)$, which offers

$$\begin{aligned} F(t) &\leq F(\bar{T})e^{M_2 t} \leq F(\bar{T})e^{M_2(T-\bar{T})} \\ &\leq F(0)e^{M_1\bar{T}}e^{M_2(T-\bar{T})} = F(0)e^{M_2(T-\bar{T})+M_1\bar{T}} \\ &\leq F(0)e^{M_2(T^*-\bar{T})+M_1\bar{T}}. \end{aligned} \tag{3.11}$$

Selecting $\delta_2 = \frac{\varepsilon}{e^{M_2(T^*-\bar{T})+M_1\bar{T}}}$. Then (3.11) and the fact $F(0) = u - u_2 < \delta_2$ tell us that $F(t) < \varepsilon$ is also true. Set $\delta = \min\{\delta_1, \delta_2\} = \delta_2$. Then the above procedure proves (3.9) and hence (3.8).

Finally, we prove the attractivity of $w_2(t)$. This is equivalent to prove that

$$\lim_{t\to\infty}[w(t) - w_2(t)] = 0, \tag{3.12}$$

where $u \in (u_1, \infty)$. From (3.7) and Lemma 2.1, we can easily see that (3.12) is true. The proof is completed. □

With the above pieces in place, now, we move to the proof of our theorem.

3.2. Proof of the theorem

When $T \in (\bar{T}, T^{**})$, part (i) of Lemma 3.3 shows that model (1.2)-(1.3) has no T-periodic solutions, which implies the global asymptotic stability of E_0 and proves part (i) of the theorem. Furthermore, when $T \in (T^{**}, T^*)$, part (iii) of Lemma 3.3 and Lemma 3.4 prove part (iii) of the theorem. Next, we prove part (ii) of the theorem.

When $T = T^{**}$, the uniqueness of the T-periodic solution of model (1.2)-(1.3) follows from part (ii) of Lemma 3.3 directly. For the stability, we set $w^*(t) = w(t; 0, u_0^*)$ be the unique T-periodic solution of model (1.2)-(1.3). Subsequently, we manifest that $w^*(t)$ is asymptotically stable on the right-side and unstable on the left-side.

For every $\varepsilon \in (0, A - u_0^*)$, by an argument similar to that described in Lemma 3.4, there exists $\delta = \delta(\varepsilon)$, such that

$$w(t;0,u) - w^*(t) < \varepsilon, \;\; t \geq t_0,$$

provided $0 < u - u_0^* < \delta$. Moreover, when $u \in (u_0^*, \infty)$, the fact that $h(u,T) < u$ implies

$$\lim_{t\to\infty} [w(t;0,u) - w^*(t)] = 0.$$

This shows that $w^*(t)$ is asymptotically stable on the right-side. Meanwhile, when $u \in (0, u_0^*)$, then $h(u,T) < u$, and thus the sequence $\{h_n(u,T)\}$ is strictly decreasing, which yields

$$\lim_{t\to\infty} w(t;0,u) = 0.$$

This proves that $w^*(t)$ is unstable on the left-side, and completes the proof. □

Since it is impossible for us to get the analytical solution of $H(u,T) = 0$ when $u \in (0, A)$ and $T \in (\bar{T}, T^*)$, we need to seek the approximate value of T^{**}. To this end, we give a numerical example below to determine the approximate value of T^{**} and support the theoretical results for the number of T-periodic solutions mentioned in the theorem as well.

Example. Let

$$a = 2, \;\; \mu = 0.05, \;\; \xi = 0.001, \;\; \bar{T} = 14. \tag{3.13}$$

Then we have $A = 975, g^* = 9506.25, c^* = 39000.00$ and $T^* \approx 14.3590$. Set $c = 20,000 \in (g^*, c^*)$ and $u \in (0, A)$. To numerically calculate T^{**}, we have made many trials by limiting $T \in (\bar{T}, T^*)$. More precisely, we plot the following two figures of $H(u,T)$ with respect to u by uniformly selecting 800 values, where the initial values u belong to intervals $(0, 1000)$ and $(0, 540)$ in panels (A) and (B), respectively. Then we find that $H(u,T) = 0$ has a unique root when $T \approx 14.3044$, no roots when $T = 14.2000 \in (\bar{T}, 14.3044)$, and exactly two roots when $T = 14.3400 \in (14.3044, T^*)$. Thus, we deduce that $T^{**} \approx 14.3044$. The two graphs are in accordance with the theorem and support our theoretical results.

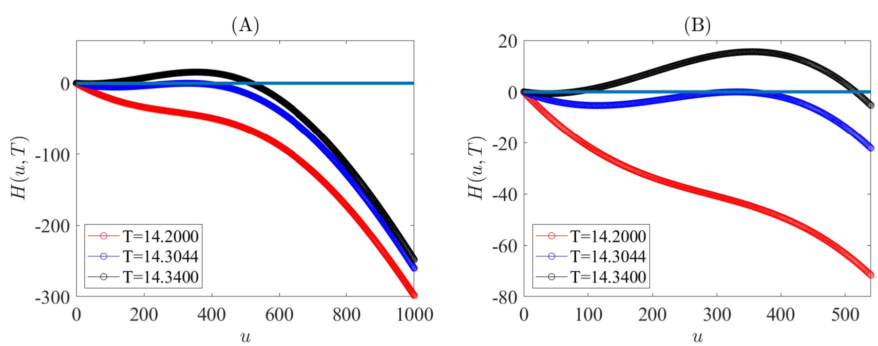

Figure 3. Let the parameters be defined in (3.13), and set $c = 20,000 \in (g^*, c^*)$. (A) shows the relation between $H(u,T)$ and u when exploring the approximate value of T^{**}, where T is evaluated at $14.2000, 14.3044$ and 14.3400 separately. (B) shows the details of $H(u,T) = 0$ when $u \in [0, 540]$.

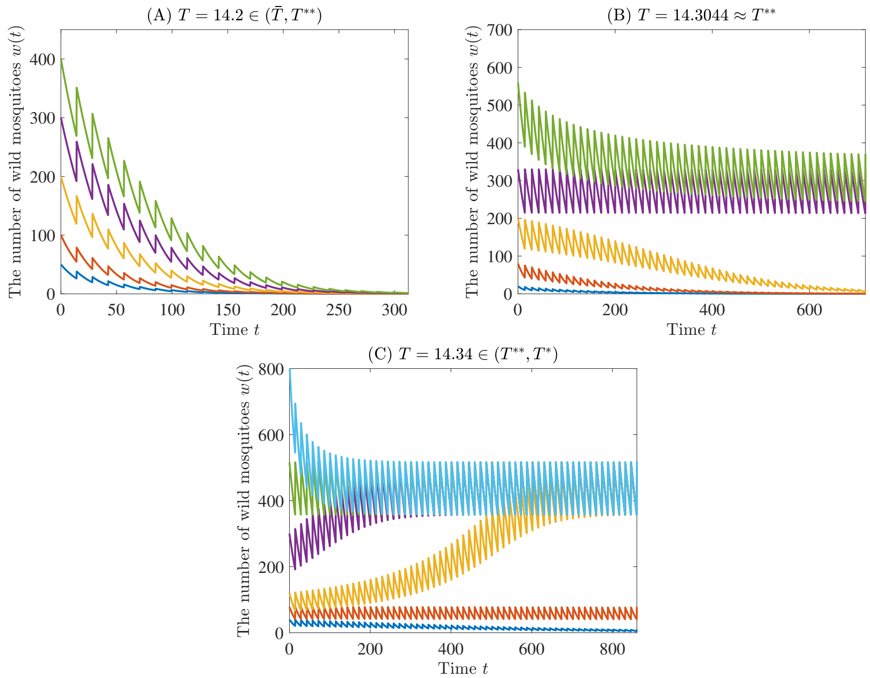

Figure 4. Let a, μ, ξ and $\bar{T}$ be set as in (3.13), and $c = 20,000 \in (g^*, c^*)$. By taking $T = 14.2000 \in (\bar{T}, T^{**})$, $T = 14.3044 \approx T^{**}$ and $T = 14.3400 \in (T^{**}, T^{*})$ in panels (A), (B) and (C), respectively, we obtain the above graphs of $w(t)$ against t for examining the long-term behaviors of solutions of model (1.2)-(1.3). (A) indicates that the mosquito-free equilibrium is globally stable, (B) confirms that the system has a local stable equilibrium $w(t) = 0$ and a positive periodic solution, (C) displays two positive periodic solutions and a local stable equilibrium $w(t) = 0$.

4. Summary and discussion

Mosquitoes have been regarded as the source of many kinds of afflicting and life-threatening diseases [27]. Various attempts have shown that the most effective method for the prevention and control of mosquito-borne diseases is to suppress the number of mosquitoes below the risk threshold [18, 41]. The traditional method of mosquito control mainly employs pesticides, which kills mosquitoes quickly, but also shows some drawbacks, such as environmental pollution and drug resistance [17,25].

Fortunately, some biological mosquito control methods can be used as good supplements to chemical measures. SIT and IIT are two innovative and promising weapons for fighting pests including mosquitoes. Their application processes involve releasing "special" mosquitoes reared in laboratories or mosquito factories to disturb the natural reproductive process of wild mosquitoes, and hence reduces the number of their offspring [14, 19, 20, 32, 37, 38, 40]. Many countries and regions have begun to promote such mosquito control methods. For instance, the United States of America and Italy have conducted field trials based on IIT for limiting the population sizes of *Aedes albopictus* and *Aedes aegypti* [6, 23, 24]. Moreover, China and Thailand have tested the SIT-IIT strategy in open-field trials against *Aedes albopictus* and *Aedes aegypti*, respectively, and both projects have achieved obvious suppression results [16, 42]. Taking the release cost into account, the release strategy, including the release amount in a batch and the release period between two consecutive releases, requires careful planning [5, 31, 36].

In [33], Yu and Li introduced the idea of considering the sexual lifespan of sterile mosquitoes when modeling the suppression dynamics of wild mosquitoes, and assumed that all the released mosquitoes have the same sexual lifespan, denoted by $\bar{T}$. Then the relation between the release period T and $\bar{T}$ determines the number of sterile mosquitoes with mating competitiveness, which is essential to the suppression dynamics. This idea makes the main model switching among several simple submodels [30,39,44,45]. In [46], we discussed the dynamics of model (1.2)-(1.3) for the case with $T > \bar{T}$ and $g^* < c < c^*$, and derived the local asymptotic stability of the origin. However, for this case, the explorations of the global dynamics of the origin and the exact number of periodic solutions are very challenging and complicated, and so the relevant conclusions remains unclear.

In this paper, we continued to investigate the dynamics of model (1.2)-(1.3) for $T > \bar{T}$ and $g^* < c < c^*$. In order to overcome the difficulties encountered before, we employed some new constructive techniques to consider the number of zeros of $H(u,T) = h(u,T) - u$ for $u \in (0, A)$, and by analyzing the monotonicity of $H(u,T)$ with respect to u and T, respectively, we found that $H(u,T)$ has no zeros for $T \in (\bar{T}, T^{**})$, a unique zero for $T = T^{**}$, and exactly two zeros for $T \in (T^{**}, T^*)$. Thus, we say T^{**} is a new release waiting period threshold besides T^*. In particular, the above results imply that the origin is globally asymptotically stable whenever $T \in (\bar{T}, T^{**})$ and $g^* < c < c^*$. Next, from a more visual perspective, we present Fig.4 to show the number of wild mosquitoes $w(t)$ changing with the time t. Like Fig.3, this figure also depicts that our model has no T-periodic solutions when $T \in (\bar{T}, T^{**})$, a unique T-periodic solution when $T = T^{**}$, and exactly two T-periodic solutions when $T \in (T^{**}, T^*)$, which supports our main results. Compared with Theorem 3.2 in [46], where the global asymptotic stability of the origin can only be ensured when $\bar{T} < T \leq T^*$ and $c \geq c^*$, our findings in this paper may pave the way to provide more economical guidance for practitioners of releasing sterile mosquitoes.

Model (1.1) looks simple, but it involves some key factors in mosquito reproduction and suppression. Therefore, this model can well describe the dynamical features of mosquito population. Up to now, under different release strategies, we have obtained rich and complex dynamical behaviors of solutions of (1.1), including the global and local asymptotic stability of the origin, the existences, stabilities, semi-stabilities and instabilities of periodic solutions (See Table 1). For the strategy of $T < \bar{T}$, we defined two release amount thresholds G^* and G^{**} with $G^* < G^{**}$, and found that the dynamics of (1.1) is rather complicated when $c \in (G^*, G^{**})$. For this situation we only derived few theoretical results and left the detailed investigations of this issue for follow-up works. We hope that the existing discussions on the model can provide effective guidances and suggestions for the control of mosquito populations and mosquito-borne diseases.

Acknowledgements

The authors wish to thank the anonymous referees for their careful reading and useful comments of the paper.

Table 1. The number of periodic solutions of model (1.1) and their corresponding stabilities under different release strategies of sterile mosquitoes, where "GAS" stands for "globally asymptotically stable", "LAS" represents "locally asymptotically stable".

Strategies	Conditions	Results	
		Number	Stability
$T > \bar{T}$, $c > g^*$ [46]	$c \geq c^*$, $T \leq T^*$	0	/
	$g^* < c < c^*$, $T = T^*$, or $T > T^*$	1	GAS
	$g^* < c < c^*$, $T < T^*$	2	The larger one is LAS, the other is unstable
$T > \bar{T}$, $c \leq g^*$ [44]	E_0 is unstable	1	GAS
	E_0 is stable	2	The larger one is LAS, the other is unstable
$T < \bar{T}$ [45]	$c \leq G^*$	2	The larger one is LAS, the other is unstable

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