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A PRIORI ESTIMATES FOR THE FIFTH-ORDER MODIFIED KDV EQUATIONS IN BESOV SPACES WITH LOW REGULARITY*

Mingjuan Chen[1] and Minjie Shan[2,†]

Abstract We get a priori estimates for the fifth-order modified KdV equations in Besov spaces with low regularity which cover the full subcritical range. These estimates are obtained from the power series expansion of the perturbation determinant associated to the Lax pair. More precisely, we get the global in time bounds of the $B^s_{2,r}$ norm of the solution for $-1/2 < s < 1$, $1 \leq r \leq \infty$. Then we can obtain the sharp global well-posedness in H^s for $s \geq 3/4$, which is the minimal regularity threshold for which the well-posedness problem can be solved via the contraction principle.

Keywords Fifth-order modified KdV equations, a priori estimates, Lax Pair, sharp global well-posedness.

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1. Introduction

We are interested in the fifth-order modified KdV equations (5th-mKdV)

$$q_t + q_{xxxxx} \pm 10q^2 q_{xxx} \pm 10(q_x)^3 \pm 40qq_xq_{xx} + 30q^4q_x = 0, \quad q(x,0) = q_0(x) \tag{1.1}$$

which are the second equations from the modified KdV hierarchy [5,11]. $q = q(x,t)$ is a real-valued function, and the signs $+$ and $-$ in front of the third order derivative nonlinearities represent the focusing and defocusing cases, respectively. Equations (1.1) can be used to describe nonlinear wave propagation in physical systems with polarity symmetry.

Like the KdV equation and the mKdV equation, the 5th-mKdV equations are completely integrable in the sense that they admit Lax pair formulations [10,13], thus one can show that the solution exists globally in time for any Schwartz initial data. For the low regularity initial data, one can apply the theory of dispersive PDEs to get the local well-posedness [9] and use I-method introduced by Colliander-Keel-Staffilani-Takaoka-Tao [3] to extend the local solution to a global one [10]. To be specific, the best and sharp local well-posedness theory was obtained by Kwon [9]

†The corresponding author.
Email: mjchen@jnu.edu.cn(M. Chen), smj@muc.edu.cn(M. Shan)
[1]Department of Mathematics, Jinan University, Guangzhou 510632, China
[2]College of Science, Minzu University of China, Beijing 100081, China
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via Bourgain method, where he showed that Equations (1.1) are locally well-posed in Sobolev space $H^s(\mathbb{R})$ for $s \geq 3/4$, and are ill-posed when $s < 3/4$ in the sense that the solution map fails to be uniformly continuous. Later the first author and her co-authors [10] showed that Equations (1.1) are globally well-posed in $H^s(\mathbb{R})$ for $s > 19/22$ by utilizing I-method. However, there is still a gap between 3/4 and 19/22, and the obstacle of getting global solutions in lower regularity spaces is the lack of conservation laws. The main goal of this paper is to solve this problem, and we even obtain that the Besov norm of the solution is essentially conserved in all subcritical cases.

Theorem 1.1. *Fix $-\frac{1}{2} < s < 1$ and $1 \leq r \leq \infty$. Let $q(t)$ be a Schwartz solution to 5th-mKdV (1.1). Then there exists a nondecreasing function $C_s : [0,\infty) \to [0,\infty)$ such that*

$$\sup_{t\in\mathbb{R}} \|q(t)\|_{B^s_{2,r}} \lesssim C_s(\|q_0\|_{B^s_{2,r}}). \tag{1.2}$$

Time translation symmetry yields that it also has a lower bound.

Remark 1.1. Equations (1.1) enjoy the scaling symmetry:

$$q \mapsto q_\lambda(x,t) = \lambda q(\lambda x, \lambda^5 t).$$

This implies that $\|q_\lambda(x,0)\|_{\dot{H}^{-1/2}} \sim \|q_0(x)\|_{\dot{H}^{-1/2}}$ is invariant, i.e. $s_c = -1/2$ is the critical regularity index. So the result in Theorem 1.1 covers the full subcritical range.

Therefore, the sharp local well-posedness in Sobolev space $H^s(\mathbb{R})$ for $s \geq 3/4$ yields the following global well-posedness result. Furthermore, the long-time asymptotic behavior of solutions to 5th-mKdV (1.1) in [10] can also be improved to hold in lower regularity weighted spaces $H^{3/4,1}$.

Corollary 1.1 (Sharp global well-posedness)**.** *Let $s \geq 3/4$, the initial value problems of 5th-mKdV (1.1) are globally well-posed from initial data $q_0 \in H^s(\mathbb{R})$.*

Remark 1.2. For $-1/2 < s < 3/4$, the well-posedness theory can not be solved via the contraction principle. One must loosen the continuous dependence of the solution on initial data, and may get the non-analytical well-posedness theory. This is an interesting problem just like the ones [2,4,6].

The main idea is from Killip-Visan-Zhang [7] where they obtained the conservation laws for KdV, NLS and mKdV equations. We know that Equations (1.1) belong to the classical ZS-AKNS system [1,14] and admit Lax pairs of the following form:

$$\frac{d}{dt}L(t;\kappa) = [P(t;\kappa), L(t;\kappa)] \quad \text{with} \quad L(t;\kappa) = \begin{bmatrix} -\partial_x + \kappa & q(x) \\ \mp q(x) & -\partial_x - \kappa \end{bmatrix}$$

and some operator pencil $P(t;\kappa)$ which will play no role in this paper. We define the perturbation determinant

$$\alpha(\kappa;q) := \sum_{\ell=1}^{\infty} \frac{(\pm 1)^{\ell-1}}{\ell} \mathrm{tr}\Big\{ \big[(\kappa-\partial_x)^{-1/2} q (\kappa+\partial_x)^{-1} q (\kappa-\partial_x)^{-1/2}\big]^{\ell} \Big\} \tag{1.3}$$

which formally represents

$$\mp \log \det \left(\begin{bmatrix} (-\partial_x+\kappa)^{-1} & 0 \\ 0 & (-\partial_x-\kappa)^{-1} \end{bmatrix} \begin{bmatrix} -\partial_x+\kappa & q(x) \\ \mp q(x) & -\partial_x-\kappa \end{bmatrix} \right). \tag{1.4}$$

We will show that $\alpha(\kappa;q)$ is conserved under the 5th-mKdV flow in section 3. Then the error terms can be controlled by the main term of $\alpha(\kappa;q)$, which gives the result (1.2) for $-1/2<s<0$. For $0\le s<1/2$, we consider a difference of $\alpha(\kappa)$ and $\alpha(\kappa/2)$ to extend the regularity range. Finally, by a more precise estimate (2.8) the result for $1/2\le s<1$ can be obtained from the result $s=1/4$.

2. Notations and Preliminaries

We denote $\mathscr{S}(\mathbb{R})$ the Schwartz space, and $\widehat{\phi}$ the Fourier transform of a distribution ϕ. We write $a\lesssim b$ to mean that $a\le Cb$, and analogous for $a\gtrsim b$. We use the notation $a\sim b$ if $a\lesssim b\lesssim a$. Define the L^2-based Besov spaces via the norms

$$\|f\|_{B^s_{2,r}}=\Big(\|\hat f\|^r_{L^2(|\xi|\le 1)}+\sum_{N\in 2^{\mathbb{N}}}N^{rs}\|\hat f\|^r_{L^2(N<|\xi|\le 2N)}\Big)^{1/r},\quad s\in\mathbb{R},\ 1\le r\le\infty$$

where the sum in N is taken over dyadic number $2^{\mathbb{N}}:=\{1,2,4,8,\dots\}$, and with the usual interpretation when $r=\infty$.

We recall the ideas and notations which the researchers used in [7, 12]. Let A be an operator on $L^2(\mathbb{R})$ with continuous integral kernel $K(x,y)$. The trace of A is defined by

$$\mathrm{tr}(A):=\int K(x,x)dx.$$

If A is a Hilbert-schmidt operator with integral kernel $K(x,y)\in L^2(\mathbb{R}\times\mathbb{R})$, then

$$\mathrm{tr}(A^2)=\iint_{\mathbb{R}^2}K(x,y)K(y,x)dxdy,$$

and

$$\|A\|^2_{\mathfrak{I}_2}:=\mathrm{tr}(AA^*)=\iint_{\mathbb{R}^2}|K(x,y)|^2dxdy.$$

Nevertheless, we find it is more convenient to consider our problem on the Fourier transform form as the following lemma.

Lemma 2.1. *Suppose that the operator A is given on the Fourier side by*

$$\widehat{Af}(\xi)=\int m(\xi,\eta)\hat f(\eta)d\eta,$$

then the following results hold

$$\widehat{A^*f}(\xi)=\int \overline{m(\eta,\xi)}\hat f(\eta)d\eta, \tag{2.1}$$

$$\mathrm{tr}(A)=\int m(\xi,\xi)d\xi,\qquad \|A\|^2_{\mathfrak{I}_2}=\iint_{\mathbb{R}^2}|m(\xi,\eta)|^2d\xi d\eta. \tag{2.2}$$

Moreover, if $A_1, A_2, \cdots A_n$ are Hilbert-Schmidt operators with Fourier kernels $m_1, m_2, \cdots m_n$, then

$$\mathrm{tr}(A_1 A_2 \cdots A_n) = \int_{\mathbb{R}^n} m_1(\xi_1, \xi_2) \cdots m_n(\xi_n, \xi_1) \; d\xi_1 \cdots d\xi_n. \tag{2.3}$$

Proof. By Plancherel and Fubini's theorem, we know (2.1) follows from

$$\langle Af, g\rangle = \iint m(\xi,\eta)\hat{f}(\eta)d\eta \cdot \overline{\hat{g}(\xi)}d\xi = \iint \hat{f}(\eta) \cdot \overline{\overline{m(\xi,\eta)}\hat{g}(\xi)}d\xi d\eta = \langle \hat{f}, \widehat{A^* g}\rangle.$$

By Fubini's theorem, we have

$$Af(x) = \int \mathscr{F}_\xi^{-1}\big(m(\xi,\eta)\big)\hat{f}(\eta)d\eta = \int_{\mathbb{R}^3} m(\xi,\eta)e^{i(x\xi - y\eta)}d\xi d\eta \; f(y)dy,$$

so the trace of A is

$$\begin{aligned} \mathrm{tr}(A) = \int K(x,x)dx &= \int_{\mathbb{R}^3} m(\xi,\eta)e^{ix(\xi-\eta)}d\xi d\eta \; dx \\ &= \iint m(\xi,\eta)\delta(\xi-\eta)d\xi d\eta = \int m(\xi,\xi)d\xi. \end{aligned}$$

Therefore, combining this with (2.1), we immediately get

$$\|A\|_{\mathfrak{I}_2}^2 = \mathrm{tr}(AA^*) = \iint_{\mathbb{R}^2} |m(\xi,\eta)|^2 d\xi d\eta.$$

By the definition, we know

$$\mathscr{F}(A_1 A_2 \cdots A_n f)(\xi_1) = \int_{\mathbb{R}^n} m_1(\xi_1,\xi_2) \cdots m_n(\xi_n, \xi_{n+1})\hat{f}(\xi_{n+1}) \; d\xi_2 \cdots d\xi_n d\xi_{n+1},$$

thus (2.3) follows from (2.2). □

In the following, we summarize some important lemmas.

Lemma 2.2 (Lemma 1.5, [7])**.** *Let $t \mapsto A(t)$ define a C^1 curve in $\mathfrak{I}_2$. Suppose*

$$\|A(t_0)\|_{\mathfrak{I}_2} < \frac{1}{3}.$$

Then there is a closed neighborhood I of t_0 on which the series

$$\alpha(t) := \sum_{\ell=1}^{\infty} \frac{(\pm 1)^{\ell-1}}{\ell} \mathrm{tr}\{A(t)^\ell\} \tag{2.4}$$

converges and defines a C^1 function with

$$\frac{d}{dt}\alpha(t) := \sum_{\ell=1}^{\infty} (\pm 1)^{\ell-1} \mathrm{tr}\Big\{A(t)^{\ell-1}\frac{d}{dt}A(t)\Big\}. \tag{2.5}$$

Lemma 2.3. *For any $\kappa > 0$ and $q \in \mathscr{S}(\mathbb{R})$, the following estimates are true:*

(i) *(Lemma 4.1, [7]): The Hilbert-Schmidt norm of the generator in* (1.3) *is as follows*

$$\left\|(\kappa-\partial_x)^{-1/2}q(\kappa+\partial_x)^{-1/2}\right\|^2_{\mathfrak{I}_2(\mathbb{R})}=\left\|(\kappa+\partial_x)^{-1/2}q(\kappa-\partial_x)^{-1/2}\right\|^2_{\mathfrak{I}_2(\mathbb{R})}$$
$$\sim\int_{\mathbb{R}}\log\big(4+\tfrac{\xi^2}{\kappa^2}\big)\frac{|\hat{q}(\xi)|^2\,d\xi}{\sqrt{4\kappa^2+\xi^2}}. \tag{2.6}$$

(ii) *(Lemma 4.2, [7]): The leading term in the series* (1.3) *is given by*

$$\operatorname{tr}\big\{(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q\big\}=\int\frac{2\kappa|\hat{q}(\xi)|^2\,d\xi}{4\kappa^2+\xi^2}. \tag{2.7}$$

(iii) *(Lemma 3.2, [8]): Let $\ell\geq 2$ and $s\geq 1/4$, then the following trace estimates hold*

$$\left|\operatorname{tr}\Big\{\big[(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q\big]^{\ell}\Big\}\right|\lesssim(\kappa^{-3/4}\|q\|_{H^s})^{2\ell}. \tag{2.8}$$

Lemma 2.4 (Lemma 3.2 and Lemma 3.5, [7]). *For $\kappa\geq\kappa_0\geq 1$, define*

$$w_1(\xi,\kappa):=\frac{\kappa^2}{4\kappa^2+\xi^2},\quad w_2(\xi,\kappa):=w_1(\xi,\kappa)-w_1\big(\xi,\frac{\kappa}{2}\big)=\frac{3\kappa^2\xi^2}{4(\kappa^2+\xi^2)(4\kappa^2+\xi^2)},$$
$$\text{and}\quad \|f\|_{Z^{s,r}_{\kappa_0,j}}:=\Big(\sum_{N\in 2^{\mathbb{N}}}N^{rs}\langle f,\,w_j(-i\partial_x,\kappa_0N)f\rangle^{r/2}\Big)^{1/r},\quad j=1,2. \tag{2.9}$$

Then for $-1<\sigma<0$, $-1<s<1$ and $1\leq r\leq\infty$, we have

$$\|f\|_{B^{\sigma}_{2,r}}\lesssim\|f\|_{Z^{\sigma,r}_{\kappa_0,1}}\lesssim\kappa_0^{-\sigma}\|f\|_{B^{\sigma}_{2,r}}; \tag{2.10}$$
$$\text{and}\quad \|f\|_{B^{s}_{2,r}}\lesssim\|f\|_{H^{\sigma}}+\kappa_0\|f\|_{Z^{s,r}_{\kappa_0,2}},\qquad \|f\|_{Z^{s,r}_{\kappa_0,2}}\lesssim\kappa_0^{-s}\|f\|_{B^{s}_{2,r}}. \tag{2.11}$$

3. Conservation of the perturbation determinant

Proposition 3.1 (Conservation of $\alpha(\kappa;q(t))$). *Let $q(t)$ be a Schwartz solution to 5th-mKdV* (1.1). *Then*

$$\frac{d}{dt}\alpha(\kappa;q(t))=0$$

as soon as κ is large enough such that

$$\int_{\mathbb{R}}\log\big(4+\tfrac{\xi^2}{\kappa^2}\big)\frac{|\hat{q}(\xi)|^2\,d\xi}{\sqrt{4\kappa^2+\xi^2}}<c \tag{3.1}$$

holds for some absolute constant $c>0$.

Proof. Note that we can rewrite (1.1) as

$$q_t+q_{xxxxx}\pm 5(q(q^2)_{xx})_x+6(q^5)_x=0.$$

Since (2.6) and the condition (3.1) hold, by Lemma 2.2 the series of $\alpha(\kappa;q(t))$ converges and can be differentiated term by term, that is

$$\frac{d}{dt}\alpha(\kappa;q(t))=\sum_{\ell=1}^{\infty}(\pm1)^{\ell-1}\operatorname{tr}\Big\{\big((\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q\big)^{\ell-1}$$

$$\times\Big[(\kappa-\partial_x)^{-1}q_t(\kappa+\partial_x)^{-1}q+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q_t\Big]\Big\}$$
$$=\sum_{\ell=1}^{\infty}-(\pm1)^{\ell-1}\mathrm{tr}\Big\{\big((\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q\big)^{\ell-1}$$
$$\times\Big[(\kappa-\partial_x)^{-1}\big(q_{xxxxx}\pm5(q(q^2)_{xx})_x+6(q^5)_x\big)(\kappa+\partial_x)^{-1}q$$
$$+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}\big(q_{xxxxx}\pm5(q(q^2)_{xx})_x+6(q^5)_x\big)\Big]\Big\}.$$

For convenience, denote $A(q):=(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q$. It suffices to prove that

$$\begin{aligned}&\mathrm{tr}\Big\{A^\ell(q)\Big[(\kappa-\partial_x)^{-1}q_{xxxxx}(\kappa+\partial_x)^{-1}q+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q_{xxxxx}\Big]\Big\}\\&+\ 5\ \mathrm{tr}\Big\{A^{\ell-1}(q)\Big[(\kappa-\partial_x)^{-1}(q(q^2)_{xx})_x(\kappa+\partial_x)^{-1}q\\&+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}(q(q^2)_{xx})_x\Big]\Big\}\\=&-6\ \mathrm{tr}\Big\{A^{\ell-2}(q)\Big[(\kappa-\partial_x)^{-1}(q^5)_x(\kappa+\partial_x)^{-1}q\\&+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}(q^5)_x\Big]\Big\},\quad\forall\ell\geq2;\end{aligned}\tag{3.2}$$

$$\begin{aligned}&\mathrm{tr}\Big\{A(q)\Big[(\kappa-\partial_x)^{-1}q_{xxxxx}(\kappa+\partial_x)^{-1}q+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q_{xxxxx}\Big]\Big\}\\=&-5\ \mathrm{tr}\Big\{\Big[(\kappa-\partial_x)^{-1}(q(q^2)_{xx})_x(\kappa+\partial_x)^{-1}q\\&+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}(q(q^2)_{xx})_x\Big]\Big\},\end{aligned}\tag{3.3}$$

and

$$\mathrm{tr}\Big\{\Big[(\kappa-\partial_x)^{-1}q_{xxxxx}(\kappa+\partial_x)^{-1}q+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q_{xxxxx}\Big]\Big\}=0.\tag{3.4}$$

To prove (3.2), we recall the identity

$$q_{xxxxx}=\partial_x^5q-5\partial_x^4q\partial_x+10\partial_x^3q\partial_x^2-10\partial_x^2q\partial_x^3+5\partial_xq\partial_x^4-q\partial_x^5.$$

In order to absorb $(\kappa-\partial_x)^{-1}$ and $(\kappa+\partial_x)^{-1}$ from the left or right side, we rewrite

$$\begin{aligned}q_{xxxxx}=&\big(\partial_x^5+5\kappa\partial_x^4+10\kappa^2\partial_x^3+10\kappa^3\partial_x^2+5\kappa^4\partial_x-15\kappa^5\big)q\\&-q\big(\partial_x^5-5\kappa\partial_x^4+10\kappa^2\partial_x^3-10\kappa^3\partial_x^2+5\kappa^4\partial_x+15\kappa^5\big)\\&+5(\kappa-\partial_x)\Big[\partial_x^3q-q\partial_x^3-2\partial_x^2q\partial_x+2\partial_xq\partial_x^2\\&+3\kappa q_{xx}+2\kappa\partial_xq\partial_x+5\kappa^2q_x+6\kappa^3q\Big](\kappa+\partial_x);\\q_{xxxxx}=&-q\big(\partial_x^5+5\kappa\partial_x^4+10\kappa^2\partial_x^3+10\kappa^3\partial_x^2+5\kappa^4\partial_x-15\kappa^5\big)\\&+\big(\partial_x^5-5\kappa\partial_x^4+10\kappa^2\partial_x^3-10\kappa^3\partial_x^2+5\kappa^4\partial_x+15\kappa^5\big)q\\&+5(\kappa+\partial_x)\Big[\partial_x^3q-q\partial_x^3-2\partial_x^2q\partial_x+2\partial_xq\partial_x^2\\&-3\kappa q_{xx}-2\kappa\partial_xq\partial_x+5\kappa^2q_x-6\kappa^3q\Big](\kappa-\partial_x).\end{aligned}$$

The contribution of the first two terms from each identity cancel each other when they were inserted into (3.2), then

$$\begin{aligned}&\mathrm{tr}\Big\{A^{\ell}(q)\Big[(\kappa-\partial_x)^{-1}q_{xxxxx}(\kappa+\partial_x)^{-1}q+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q_{xxxxx}\Big]\Big\}\\ &=5\mathrm{tr}\Big\{A^{\ell-1}(q)\Big[(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}T_1(q)\\ &\quad+(\kappa-\partial_x)^{-1}T_2(q)(\kappa+\partial_x)^{-1}q\Big]\Big\},\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}T_1(q)=&q\partial_x^3q^2-q^2\partial_x^3q-2q\partial_x^2q\partial_xq+2q\partial_xq\partial_x^2q\\ &+3\kappa q^2q_{xx}+2\kappa q\partial_xq\partial_xq+5\kappa^2q^2q_x+6\kappa^3q^3;\\ T_2(q)=&q\partial_x^3q^2-q^2\partial_x^3q-2q\partial_x^2q\partial_xq+2q\partial_xq\partial_x^2q\\ &-3\kappa q^2q_{xx}-2\kappa q\partial_xq\partial_xq+5\kappa^2q^2q_x-6\kappa^3q^3.\end{aligned}$$

Note that

$$\begin{aligned}q\partial_x^3q^2-2q\partial_x^2q\partial_xq&=q\partial_x^2(2qq_x+q^2\partial_x)-2q\partial_x^2q(q_x+q\partial_x)\\ &=-\partial_xq\partial_xq^2\partial_x+q_x\partial_xq^2\partial_x;\\ -q^2\partial_x^3q+2q\partial_xq\partial_x^2q&=-(\partial_xq^2-2qq_x)\partial_x^2q+2(\partial_xq-q_x)q\partial_x^2q\\ &=\partial_xq^2\partial_xq_x+\partial_xq^2\partial_xq\partial_x,\end{aligned}$$

then

$$q\partial_x^3q^2-q^2\partial_x^3q-2q\partial_x^2q\partial_xq+2q\partial_xq\partial_x^2q=\partial_xq^2q_x\partial_x+(q^2q_{xx})_x.\tag{3.6}$$

To absorb $(\kappa-\partial_x)^{-1}$ and $(\kappa+\partial_x)^{-1}$ from the left or right side again, we can rewrite

$$\begin{aligned}\partial_xq^2q_x\partial_x&=-(\kappa+\partial_x)q^2q_x(\kappa-\partial_x)+\kappa(q^2q_x)_x+\kappa^2q^2q_x;\\ \partial_xq^2q_x\partial_x&=-(\kappa-\partial_x)q^2q_x(\kappa+\partial_x)-\kappa(q^2q_x)_x+\kappa^2q^2q_x,\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}(q^2q_{xx})_x&=(\kappa+\partial_x)q^2q_{xx}+q^2q_{xx}(\kappa-\partial_x)-2\kappa q^2q_{xx};\\ (q^2q_{xx})_x&=-(\kappa-\partial_x)q^2q_{xx}-q^2q_{xx}(\kappa+\partial_x)+2\kappa q^2q_{xx}.\end{aligned}\tag{3.8}$$

Similarly, we can obtain

$$\begin{aligned}2\kappa q\partial_xq\partial_xq&=-2\kappa(\kappa+\partial_x)q^3(\kappa-\partial_x)+\kappa(q^2q_x)_x+\kappa q^2q_{xx}+2\kappa^3q^3+6\kappa^2q^2q_x,\\ -2\kappa q\partial_xq\partial_xq&=2\kappa(\kappa-\partial_x)q^3(\kappa+\partial_x)-\kappa(q^2q_x)_x-\kappa q^2q_{xx}-2\kappa^3q^3+6\kappa^2q^2q_x.\end{aligned}\tag{3.9}$$

Therefore, from (3.6)-(3.9) we have

$$\begin{aligned}T_1(q)=&(\kappa+\partial_x)(q^2q_{xx}+4\kappa^2q^3)+(q^2q_{xx}+4\kappa^2q^3)(\kappa-\partial_x)\\ &-(\kappa+\partial_x)q^2q_x(\kappa-\partial_x)-2\kappa(\kappa+\partial_x)q^3(\kappa-\partial_x)+2\kappa(q^2q_x)_x+2\kappa q^2q_{xx};\\ T_2(q)=&-(\kappa-\partial_x)(q^2q_{xx}+4\kappa^2q^3)-(q^2q_{xx}+4\kappa^2q^3)(\kappa+\partial_x)\end{aligned}$$

$$-(\kappa-\partial_x)q^2q_x(\kappa+\partial_x)+2\kappa(\kappa-\partial_x)q^3(\kappa+\partial_x)-2\kappa(q^2q_x)_x-2\kappa q^2q_{xx}.$$

The contribution of the first two terms from each identity cancel each other again. For the second term of (3.2), rewriting

$$\begin{aligned}(q(q^2)_{xx})_x &= -(\kappa-\partial_x)q(q^2)_{xx}-q(q^2)_{xx}(\kappa+\partial_x)+2\kappa q(q^2)_{xx};\\ (q(q^2)_{xx})_x &= (\kappa+\partial_x)q(q^2)_{xx}+q(q^2)_{xx}(\kappa-\partial_x)-2\kappa q(q^2)_{xx},\end{aligned}$$

and bearing $q(q^2)_{xx}=(q^2q_x)_x+q^2q_{xx}$ in mind, then we can get from (3.5) that

$$\begin{aligned}\text{LHS of (3.2)} = &-\operatorname{tr}\Big\{A^{\ell-2}(q)\Big[(\kappa-\partial_x)^{-1}\big((q^5)_x+10\kappa q^5\big)(\kappa+\partial_x)^{-1}q\\ &+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}\big((q^5)_x-10\kappa q^5\big)\Big]\Big\}.\end{aligned}\tag{3.10}$$

Note that

$$(q^5)_x=-(\kappa-\partial_x)q^5-q^5(\kappa+\partial_x)+2\kappa q^5=(\kappa+\partial_x)q^5+q^5(\kappa-\partial_x)-2\kappa q^5,$$

we obtain from (3.10) that

$$\begin{aligned}\text{LHS of (3.2)} = &-6\ \operatorname{tr}\Big\{A^{\ell-2}(q)\Big[(\kappa-\partial_x)^{-1}\big(2\kappa q^5\big)(\kappa+\partial_x)^{-1}q\\ &+(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}\big(-2\kappa q^5\big)\Big]\Big\}\\ =&\ \text{RHS of (3.2)}.\end{aligned}$$

For (3.3), its proof is similar and easier, we can get that

$$\begin{aligned}\text{LHS of (3.3)} &= 5\operatorname{tr}\Big\{\Big[-(\kappa-\partial_x)^{-1}(q^3q_x+2\kappa q^4)(\kappa-\partial_x)-q^3q_x+2\kappa q^4\Big]\Big\}\\ &=-\frac{5}{2}\operatorname{tr}\{(q^4)_x\}=0,\end{aligned}$$

where $(q^4)_x$ is a complete derivative and $(\widehat{(q^4)_x f})(\xi)=\int i(\xi-\eta)\hat{q^4}(\xi-\eta)\hat{f}(\eta)d\eta$, so its trace is zero (see (2.2)). For (3.4), we consider it on the Fourier transform point and use (2.3), then

$$\text{LHS of (3.4)}=\iint\frac{\big(i(\xi_1-\xi_2)^5+i(\xi_2-\xi_1)^5\big)\hat{q}(\xi_1-\xi_2)\hat{q}(\xi_2-\xi_1)}{(\kappa-i\xi_1)(\kappa+i\xi_2)}d\xi_1d\xi_2=0.$$

The proof of Proposition 3.1 is completed. □

4. Proof of Theorem 1.1

(1) $s\in(-\frac{1}{2},0)$. The proof of this case is due to the work in [7] considering the NLS and mKdV equations. We still give the details for the sake of completeness. From (2.6) and (2.10), we know that for $\kappa\geq\kappa_0$, $-\frac{1}{2}<s<0$ and $1\leq r\leq\infty$,

$$\begin{aligned}&\Big\|(\kappa-\ \partial_x)^{-1/2}q(\kappa+\partial_x)^{-1/2}\Big\|^2_{\mathfrak{I}_2(\mathbb{R})}\\ &\lesssim\frac{1}{\kappa}\int_{|\xi|\leq\kappa_0}|\hat{q}(\xi)|^2\,d\xi+\sum_{N\in 2^{\mathbb{N}}}\frac{\log(2+\kappa_0^2N^2\kappa^{-2})}{\kappa+\kappa_0N}\int_{\kappa_0N\leq|\xi|\leq2\kappa_0N}|\hat{q}(\xi)|^2\,d\xi\end{aligned}$$

$$
\begin{aligned}
&\lesssim \Big(\sum_{N\in 2^{\mathbb{N}}} \frac{\log(2+\kappa_0^2 N^2\kappa^{-2})}{\kappa+\kappa_0 N} \int \frac{\kappa_0^2 N^2}{4\kappa_0^2 N^2+|\xi|^2}|\hat{q}(\xi)|^2\,d\xi \Big) \\
&\lesssim \Big(\sum_{N\in 2^{\mathbb{N}}} \sqrt{\frac{\log(2+\kappa_0^2 N^2\kappa^{-2})}{\kappa+\kappa_0 N}} \cdot \Big(\int \frac{\kappa_0^2 N^2}{4\kappa_0^2 N^2+|\xi|^2}|\hat{q}(\xi)|^2\,d\xi \Big)^{1/2} \Big)^2 \\
&\lesssim \Big(\Big(\sum_{N\in 2^{\mathbb{N}}} N^{-sr'} \Big(\frac{\log(2+\kappa_0^2 N^2\kappa^{-2})}{(\kappa+\kappa_0 N)} \Big)^{r'/2} \Big)^{1/r'} \cdot \|q\|_{Z^{s,r}_{\kappa_0,1}} \Big)^2 \\
&\lesssim \kappa^{2|s|-1}\kappa_0^{-2|s|}\|q\|^2_{Z^{s,r}_{\kappa_0,1}} \lesssim \kappa^{2|s|-1}\|q\|^2_{B^{s,2}_r}, \qquad (4.1)
\end{aligned}
$$

where we used $l^1 \hookrightarrow l^2$ and Hölder's inequality in the fourth line and fifth line, respectively. Besides, by breaking the sum into the cases $N \le \kappa/\kappa_0$ and $N > \kappa/\kappa_0$, one readily sees that

$$
\Big(\sum_{N\in 2^{\mathbb{N}}} N^{-sr'} \Big(\frac{\log(2+\kappa_0^2 N^2\kappa^{-2})}{(\kappa+\kappa_0 N)} \Big)^{r'/2} \Big)^{1/r'} \lesssim \kappa_0^{-|s|}\kappa^{|s|-\frac{1}{2}}.
$$

In order to ensure (3.1), we can choose

$$
\kappa_0 := C\big(1+\|q(0)\|^2_{B^s_{2,r}}\big)^{\frac{1}{1-2|s|}} \qquad (4.2)
$$

with some large absolute constant C, such that the series of $\alpha(\kappa; q(t))$ converges and Proposition 3.1 holds, then there is a closed interval I containing 0 on which the following estimate holds

$$
\Big|\alpha(\kappa, q(t)) - \mathrm{tr}\{(\kappa-\partial_x)^{-1}q(\kappa+\partial_x)^{-1}q\}\Big| \lesssim \kappa_0^{-4|s|}\kappa^{4|s|-2}\|q(t)\|^4_{Z^{s,r}_{\kappa_0,1}}.
$$

Then by the conservation of $\alpha(\kappa, q(t))$, we have from (2.7) that

$$
\begin{aligned}
\int \frac{2\kappa|\hat{q}(\xi)|^2}{4\kappa^2+|\xi|^2}d\xi &\lesssim \alpha(\kappa, q(t)) + \kappa_0^{-4|s|}\kappa^{4|s|-2}\|q(t)\|^4_{Z^{s,r}_{\kappa_0,1}} \\
&= \alpha(\kappa, q(0)) + \kappa_0^{-4|s|}\kappa^{4|s|-2}\|q(t)\|^4_{Z^{s,r}_{\kappa_0,1}} \\
&\lesssim \int \frac{2\kappa|\widehat{q(0)}(\xi)|^2}{4\kappa^2+|\xi|^2}d\xi + \kappa_0^{-4|s|}\kappa^{4|s|-2}\Big(\|q(t)\|^4_{Z^{s,r}_{\kappa_0,1}} + \|q(0)\|^4_{Z^{s,r}_{\kappa_0,1}}\Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\kappa_0^{-\frac{1}{2}}\|q(t)\|_{Z^{s,r}_{\kappa_0,1}} \lesssim& \kappa_0^{-\frac{1}{2}}\Big(\sum_{N\in 2^{\mathbb{N}}} N^{rs}\Big(\int \frac{\kappa_0^2 N^2|\widehat{q(0)}(\xi)|^2}{4\kappa_0^2 N^2+|\xi|^2}d\xi \\
&+ \kappa_0^{-1}N^{4|s|-1}\big(\|q(t)\|^4_{Z^{s,r}_{\kappa_0,1}} + \|q(0)\|^4_{Z^{s,r}_{\kappa_0,1}}\big)\Big)^{r/2}\Big)^{1/r} \\
\lesssim& \kappa_0^{-\frac{1}{2}}\|q(0)\|_{Z^{s,r}_{\kappa_0,1}} + \Big[\kappa_0^{-\frac{1}{2}}\|q(t)\|_{Z^{s,r}_{\kappa_0,1}}\Big]^2 + \Big[\kappa_0^{-\frac{1}{2}}\|q(0)\|_{Z^{s,r}_{\kappa_0,1}}\Big]^2.
\end{aligned}
$$

For κ_0 as in (4.2), we have from (2.10) that

$$
\kappa_0^{-\frac{1}{2}}\|q(0)\|_{Z^{s,r}_{\kappa_0,1}} \lesssim \kappa_0^{-\frac{1}{2}+|s|}\|q(0)\|_{B^s_{2,r}} \lesssim C^{|s|-\frac{1}{2}},
$$

then by choosing C large enough and a simple continuity argument, we can get

$$\|q(t)\|_{Z^{s,r}_{\kappa_0,1}} \lesssim \|q(0)\|_{Z^{s,r}_{\kappa_0,1}} \tag{4.3}$$

uniformly in time. Then from (2.10) and (4.2) we obtain

$$\|q(t)\|_{B^s_{2,r}} \lesssim \|q(0)\|_{B^s_{2,r}}\Big(1+\|q(0)\|^2_{B^s_{2,r}}\Big)^{\frac{|s|}{1-2|s|}}, \quad -\frac{1}{2} < s < 0. \tag{4.4}$$

(2) $s \in [0, \frac{1}{2})$. We will use the results of the first step to extend the range of s. For $-\frac{1}{2} < \sigma < 0$, we obtain from the conservation of α, (2.7), (4.1) and (4.3) that

$$\begin{aligned}
&|\langle q(t),\, w_2(-i\partial_x,\kappa)q(t)\rangle - \langle q(0),\, w_2(-i\partial_x,\kappa)q(0)\rangle| \\
\le&|\langle q(t),\, w_1(-i\partial_x,\kappa)q(t)\rangle - \langle q(0),\, w_1(-i\partial_x,\kappa)q(0)\rangle| \\
&+|\langle q(t),\, w_1(-i\partial_x,\kappa/2)q(t)\rangle - \langle q(0),\, w_1(-i\partial_x,\kappa/2)q(0)\rangle| \\
\lesssim&\kappa\Big(\big\|(\kappa-\partial_x)^{-1/2}q(t)(\kappa+\partial_x)^{-1/2}\big\|^4_{\mathfrak{I}_2(\mathbb{R})}+\big\|(\kappa-\partial_x)^{-1/2}q(0)(\kappa+\partial_x)^{-1/2}\big\|^4_{\mathfrak{I}_2(\mathbb{R})}\Big) \\
\lesssim&\kappa^{4|\sigma|-1}\kappa_0^{-4|\sigma|}\|q(0)\|^4_{Z^{\sigma,r}_{\kappa_0,1}} \lesssim \kappa^{4|\sigma|-1}\|q(0)\|^4_{B^\sigma_{2,r}}.
\end{aligned}$$

Therefore for any $0 \le s < \frac{1}{2}$, we may choose some $\sigma \in (-\frac{1}{5}, 0)$ so that $s < 2\sigma + \frac{1}{2}$, from (2.11) and the result (4.4) in the first step we know that

$$\begin{aligned}
\|q(t)\|_{B^s_{2,r}} &\lesssim \|q(t)\|_{H^\sigma} + \kappa_0\Big(\sum_{N\in 2^{\mathbb{N}}} N^{rs}\langle q(t),\, w_2(-i\partial_x,\kappa_0 N)q(t)\rangle^{r/2}\Big)^{1/r} \\
&\lesssim \|q(t)\|_{H^\sigma} + \kappa_0\|q(0)\|_{Z^{s,r}_{\kappa_0,2}} + \kappa_0^{\frac{1}{2}+2|\sigma|}\Big(\sum_{N\in 2^{\mathbb{N}}} N^{r(s-2\sigma-\frac{1}{2})}\Big)^{1/r}\|q(0)\|^2_{B^\sigma_{2,r}} \\
&\lesssim \|q(0)\|_{H^\sigma}\Big(1+\|q(0)\|^2_{H^\sigma}\Big)^{\frac{|\sigma|}{1-2|\sigma|}} + \kappa_0^{1-s}\|q(0)\|_{B^s_{2,r}} + \kappa_0^{\frac{1}{2}+2|\sigma|}\|q(0)\|^2_{B^\sigma_{2,r}} \\
&\lesssim \|q(0)\|_{B^s_{2,r}}\Big(1+\|q(0)\|^2_{B^s_{2,r}}\Big)^2,
\end{aligned}$$

where $\kappa_0 := C\big(1+\|q(0)\|^2_{B^\sigma_{2,r}}\big)^{\frac{1}{1-2|\sigma|}}$ as in the first step. Thus the desired result (1.2) for $0 \le s < \frac{1}{2}$ is obtained.

(3) $s \in [\frac{1}{2}, 1)$. From (2.8), in order to ensure that $\kappa^{-3/4}\|q(t)\|_{H^{1/4}} < c$ uniformly in time, from the result in the second step we can choose

$$\kappa_0 := C\big(1+\|q(0)\|^2_{H^{1/4}}\big)^{10/3}.$$

Then for $\kappa > \kappa_0$, we have

$$\langle q(t),\, w_2(-i\partial_x,\kappa)q(t)\rangle \lesssim \langle q(0),\, w_2(-i\partial_x,\kappa)q(0)\rangle + \kappa^{-2}\big(\|q(t)\|^4_{H^{1/4}} + \|q(0)\|^4_{H^{1/4}}\big).$$

So for any $\frac{1}{2} \le s < 1$,

$$\begin{aligned}
&\|q(t)\|_{B^s_{2,r}} \\
\lesssim&\|q(t)\|_{H^{1/4}} + \kappa_0\|q(0)\|_{Z^{s,r}_{\kappa_0,2}} + \Big(\sum_{N\in 2^{\mathbb{N}}} N^{r(s-1)}\Big)^{1/r}\big(\|q(t)\|^2_{H^{1/4}} + \|q(0)\|^2_{H^{1/4}}\big)
\end{aligned}$$

$$\lesssim \|q(0)\|_{H^{1/4}}\Big(1+\|q(0)\|_{H^{1/4}}^2\Big)^2 + \kappa_0^{1-s}\|q(0)\|_{B^s_{2,r}} + \big(\|q(t)\|_{H^{1/4}}^2 + \|q(0)\|_{H^{1/4}}^2\big)$$
$$\lesssim \|q(0)\|_{B^s_{2,r}}\Big(1+\|q(0)\|_{B^s_{2,r}}^2\Big)^5.$$

Therefore we get the desired result (1.2) for $\frac{1}{2} \leq s < 1$.

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