

STUDY OF A GENERALIZED WEIBULL DISTRIBUTION WITH OSCILLATORY FAILURE RATE*

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Abstract A four-parameter life distribution $ORS(\lambda, A, m, \beta)$ with oscillatory failure rate is given, which can be regarded as a generalization of the Weibull distribution. For the three-parameter oscillation type distribution $ORS(A, m, \beta)$, the image characteristics of the failure rate are theoretically proved, and the parameter estimation method is given in the whole sample case. Finally, the paper illustrates the feasibility of the method by simulating and practical examples.

Keywords Density function, failure rate, lifetime distribution, oscillating type, parameter estimation.

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1. Introduction

The two-parameter Weibull distribution is one of the most commonly used life distributions in reliability theory, and it is widely used in the analysis and processing of various life test failure data. Usually the first choice is the two-parameter Weibull distribution when the actual data to be fitted has a power function for the failure rate. However, in the study of failure data of complex systems and products such as brittle materials, it is found that the failure rate is not in the form of a power function (i.e. not monotonous), it might be in the shape of a bathtub curve. In this regard, many scholars have proposed different forms of generalized Weibull distributions to fit the life of the product and make corresponding statistical inferences, specifically can refer to the literatures [1–5, 7–10, 12, 14, 15, 17–21, 23, 24, 31–33]. For example, the exponential Weibull distribution is proposed in the literature [18], which is obtained by adding a shape parameter to the two-parameter Weibull distribution. It has many good statistical properties and intuitive physical interpretation, and many distributions in practical applications are also special cases of this distri-

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bution. Whereas the distribution also has some defects, the relevant elaboration in the literature [20] can be consulted.

It should be pointed out that some complex products in reliability engineering, many of their failures are cyclical (such as the product is subjected to periodic stress) and have oscillation characteristics. The failure rate function of its life distribution demonstrates the law of oscillation, and its density function also shows a local oscillation trend. H. Hecht and E. Fiorentio studied more than 300 spacecraft at the Rome Aviation Development Center in the literature [11] and argued that both the failure efficiency function of the overall system and the failure rate function of parts and mass failures exhibit strong oscillatory properties (or roller coaster characteristics). K. L. Wong argues in the literature [28–30] that the failure rate function for semiconductor devices and electronic systems is in the shape of a roller coaster curve. In addition, the failure rate image is oscillating, as well as the density function image. Literature [13] presents the lunar pool in the wave height and hydrophone power density function plot as the circular lunar pool in the wave height of 60mm, the flow rate of 2.0m/s, the wave period of 1.5s and 2s, respectively, as shown in Figure 1-4. The piston oscillation of the fluid in the lunar pool under the condition of a wave period of 1.5s exhibits a certain self-oscillation characteristic, while the piston oscillation of the fluid in the lunar pool under the condition of a wave period of 2s shows the characteristics of forced oscillation. The image of several surface tangent stress angle density functions given in the literature [25] is shown in Figure 5, the results of which show that the distribution density of the friction angle shows a trend of high in the middle and low on both sides without oscillation, and the distribution density of the normal distribution is well in line with the distribution density of the normal distribution near the center. In the case of oscillation on the wall surface, the shape of the distribution density changes significantly, showing a trend of high on both sides and low in the middle, and the peak position of the density function is about 80° to 90° . The reason for this phenomenon is that the amplitude of the oscillation spread velocity is significantly greater than the flow velocity near the wall, therefore the shear force of the extended direction is dominant. At this time, due to the sinusoidal distribution of the oscillation velocity, the distribution of the extended tangential stress is also similar to that of the sinusoidal type, so there will be discrete multiple maximum points of density functions.

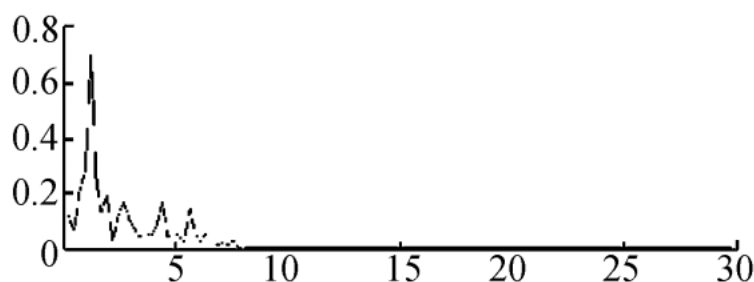


Figure 1. Wave high power spectral density function plot with wave height of 60mm, period of 1.5s and flow rate of 2.0m/s

Regarding the study of oscillatory failure rate, as early as 1951 literature [27] has proposed the concept of roller-coaster-shaped failure rate, which is actually the

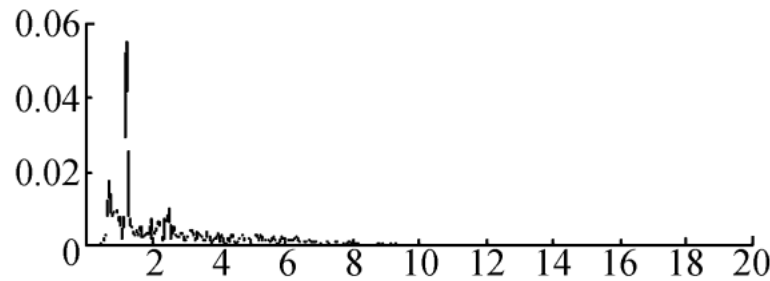


Figure 2. Hydrophone signal power spectral density function diagram with wave height of 60mm, period of 1.5s and flow rate of 2.0m/s

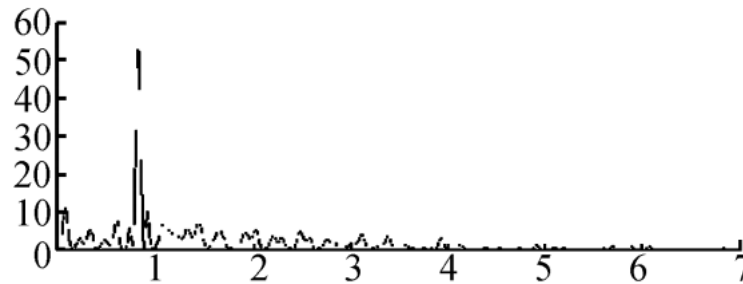


Figure 3. Wave high power spectral density function plot with wave height of 60mm, period of 2s and flow rate of 2.0m/s

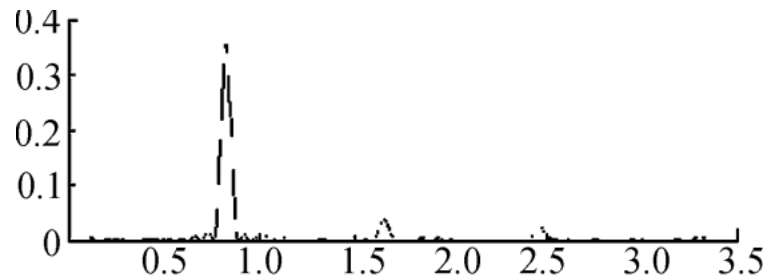


Figure 4. Hydrophone signal power spectral density function diagram with wave height of 60mm, period of 2s and flow rate of 2.0m/s

oscillatory failure rate of this paper.

Definition 1.1. Let $h(x)$ be the failure efficiency function of the life X of a product, which is valued on $[0, +\infty)$, assuming that there are points $0 < x_1 < x_2 < \dots < x_k < +\infty$ and intervals $[x_{j-1}, x_j]$, $j = 1, 2, \dots, k+1$, where $x_0 = 0, x_{k+1} = +\infty$, denote $[x_k, x_{k+1}] = [x_k, +\infty)$ such that the function is strictly monotonous on each interval and has opposite monotonicity on any two adjacent intervals, then $h(x)$ is called oscillating type on $[0, +\infty)$ and $\{x_1, x_2, \dots, x_k\}$ are called the points of variability of $h(x)$.

The literatures [26] and [6] examine the relationship between the oscillating deformation failure rate and its corresponding mean residual life function, and discuss the shape of the mean residual life function through the shape and variation of the failure rate. The literature [22] proposes a new life distribution with oscillatory

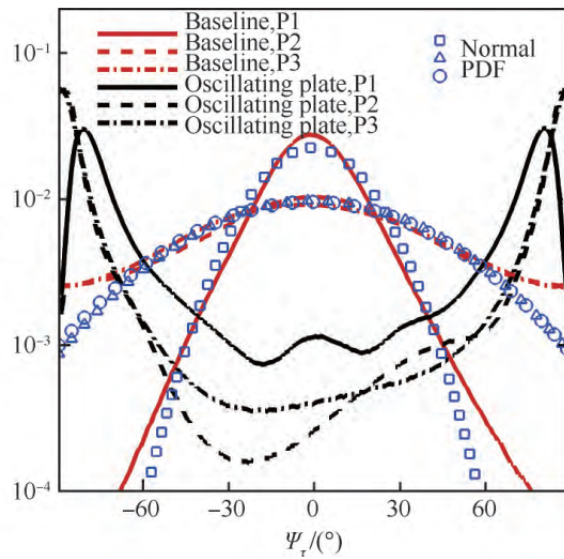


Figure 5. Density function of surface tangent stress angle

failure efficiency, which adds a sine function to the Weibull distribution. The properties of the failure efficiency function, cumulative failure function, density function and distribution function are studied, and the example results show that the distribution can well simulate the life with periodic and oscillatory failure efficiency. The literature [16] constructs a life distribution model with Exponential decay oscillation probability density distribution, and the corresponding calculation models of reliability, failure rate and average life are given. Taking the failure time data of hydraulic automatic transmission carried by a certain type of forklift as the research object, the established probability distribution model is used to fit the failure time data of this type of hydraulic automatic transmission. The validity of the developed probability distribution model is verified.

2. Oscillatory failure rate life distribution

Literature [22] proposes a three-parameter life distribution with oscillating failure efficiency. Its distribution function $F(x)$, density function $f(x)$ and failure function $h(x)$ are:

$$F(x) = 1 - \exp \left\{ -\lambda \left\{ \sin \left[\left(\frac{x}{\beta} \right)^m \right] + \left(\frac{x}{\beta} \right)^m \right\} \right\},$$

$$f(x) = \lambda \frac{m x^{m-1}}{\beta^m} \left\{ \cos \left[\left(\frac{x}{\beta} \right)^m \right] + 1 \right\} \exp \left\{ -\lambda \left\{ \sin \left[\left(\frac{x}{\beta} \right)^m \right] + \left(\frac{x}{\beta} \right)^m \right\} \right\},$$

$$h(x) = \lambda \frac{m x^{m-1}}{\beta^m} \left\{ \cos \left[\left(\frac{x}{\beta} \right)^m \right] + 1 \right\},$$

where $\lambda > 0$ is the attenuation coefficient, $m > 0$ is the shape parameter, $\beta > 0$ is the scale parameter, β^{-m} is the oscillation angle frequency.

However, literature [22] considers the above three-parameter life probability distribution model as an extension of the two-parameter Weibull distribution. Actually, that is not the case. This distribution does not contain the Weibull distribution.

The literature [16] proposes a three-parameter lifetime distribution model with Exponential decay oscillation probability density distribution, and its distribution function, density function and failure rate function are respectively

$$\begin{aligned} F(x) &= 1 - \exp \left\{ -\lambda [A \sin^2(\omega x) + x] \right\}, \\ f(x) &= \lambda [A\omega \sin(2\omega x) + 1] \exp \left\{ -\lambda [A \sin^2(\omega x) + x] \right\}, \\ h(x) &= \lambda [A\omega \sin(2\omega x) + 1], x > 0, \end{aligned}$$

where $\lambda > 0$ is the attenuation coefficient, A is the oscillating amplitude, ω is the oscillating angular frequency, and $|A\omega| < 1$.

The above three-parameter lifetime distribution can be regarded as an extension of the Exponential distribution, but it does not include the Weibull distribution.

The definition of the life distribution with oscillatory failure rate is given below (referred to as the oscillating life distribution), which can be regarded as an extended form of the Weibull distribution.

Definition 2.1. If the distribution function $F(x)$ of the life X of a product has the following form, then X follows the four-parameter oscillation life distribution, which is denoted by $X \sim \text{ORS}(\lambda, A, m, \beta)$,

$$F(x) = 1 - \exp \left\{ -\lambda \left\{ A \sin \left[\left(\frac{x}{\beta} \right)^m \right] + \left(\frac{x}{\beta} \right)^m \right\} \right\}, x \geq 0,$$

where $\lambda > 0$ is the attenuation coefficient, $|A| \leq 1$ is the oscillation amplitude, $m > 0$ is the shape parameter, $\beta > 0$ is the scale parameter, β^{-m} is the oscillation angle frequency.

It's easy to get the density function $f(x)$ and the failure rate function $h(x)$ of X , respectively

$$\begin{aligned} f(x) &= \lambda \frac{mx^{m-1}}{\beta^m} \left\{ A \cos \left[\left(\frac{x}{\beta} \right)^m \right] + 1 \right\} \exp \left\{ -\lambda \left\{ A \sin \left[\left(\frac{x}{\beta} \right)^m \right] + \left(\frac{x}{\beta} \right)^m \right\} \right\}, \\ h(x) &= \lambda \frac{mx^{m-1}}{\beta^m} \left\{ A \cos \left[\left(\frac{x}{\beta} \right)^m \right] + 1 \right\}, \end{aligned}$$

In particular,

- (1) the oscillation life distribution can be seen as an extension of the Weibull distribution.

In fact, when $A = 0$, let $\beta' = \beta\lambda^{-1/m}$, then the life distribution model degenerates into a two-parameter Weibull distribution whose distribution function is

$$F(x) = 1 - \exp \left\{ -\left(\frac{x}{\beta'} \right)^m \right\}, x \geq 0.$$

Or when $A = 0, \lambda = 1$, the life distribution model also degenerates to a two-parameter Weibull distribution whose distribution function is

$$F(x) = 1 - \exp \left\{ -\left(\frac{x}{\beta} \right)^m \right\}, x \geq 0.$$

- (2) When $\lambda = 1$, it is said that this life distribution follows a three-parameter oscillation life distribution, denoted as $X \sim \text{ORS}(A, m, \beta)$, and its corresponding distribution function is

$$F(x) = 1 - \exp \left\{ - \left\{ A \sin \left[\left(\frac{x}{\beta} \right)^m \right] + \left(\frac{x}{\beta} \right)^m \right\} \right\}, x \geq 0.$$

- (3) When $A = 1$, the distribution function is

$$F(x) = 1 - \exp \left\{ - \lambda \left\{ \sin \left[\left(\frac{x}{\beta} \right)^m \right] + \left(\frac{x}{\beta} \right)^m \right\} \right\}, x \geq 0.$$

At this point, the life distribution model degenerates into the distribution model proposed in the literature [22].

Definitions of oscillatory rise, uniform oscillation and oscillation fall of failure rate function are given below.

Definition 2.2. Let the failure efficiency function $h(x)$ of the life X of a product be oscillating,

- (1) if the maximum and minimum values of $h(x)$ increase over time x , then $h(x)$ is called the oscillation rising function;
- (2) if the maximum and minimum values of $h(x)$ remain unchanged with time x , $h(x)$ is called the uniform oscillation function;
- (3) if the maximum and minimum values of $h(x)$ show a downward trend with time x , then $h(x)$ is called the oscillation decline function.

Set the life of a product $X \sim \text{ORS}(\lambda, A, m, \beta)$, its failure rate function consists of two parts: the first part $\lambda m x^{m-1} / \beta^m$ is to control the failure rate function $h(x)$ in what form of oscillation, that is, when $m < 1$, the oscillation of $h(x)$ rises; when $m = 1$, $h(x)$ oscillates evenly; whereas $m > 1$, the oscillation of $h(x)$ goes down. The second part $A \cos(x^m / \beta^m) + 1$ is the oscillation part.

Theorem 2.1. Let the life X of a product obey a four-parameter oscillation life distribution, that is $X \sim \text{ORS}(\lambda, A, m, \beta)$, the following conclusions are drawn:

(1)

$$\lim_{x \rightarrow 0^+} f(x) = \begin{cases} +\infty, & m < 1 \\ \lambda \frac{A+1}{\beta}, & m = 1, \quad \lim_{x \rightarrow +\infty} f(x) = 0; \\ 0, & m > 1 \end{cases}$$

(2)

$$\lim_{x \rightarrow 0^+} h(x) = \begin{cases} +\infty, & m < 1 \\ \lambda \frac{A+1}{\beta}, & m = 1, \\ 0, & m > 1 \end{cases}, \quad \lim_{x \rightarrow +\infty} h(x) = \begin{cases} 0, & m < 1 \\ \text{inexistent}, & m = 1; \\ \text{inexistent}, & m > 1 \end{cases}$$

(3) The k order moment of X exists for positive integers k .

Proof. (1) and (2) are obvious, let's prove (3).

$$E(X^k) = \int_0^{+\infty} x^k \lambda \frac{mx^{m-1}}{\beta^m} \left\{ A \cos \left[\left(\frac{x}{\beta} \right)^m \right] + 1 \right\} \\ \times \exp \left\{ -\lambda \left\{ A \sin \left[\left(\frac{x}{\beta} \right)^m \right] + \left(\frac{x}{\beta} \right)^m \right\} \right\} dx.$$

Let $t = x^m/\beta^m$, $dt = mx^{m-1}/\beta^m dx$, then

$$E(X^k) = \lambda \beta^k \int_0^{+\infty} t^{k/m} [A \cos(t) + 1] \exp \{ -\lambda [A \sin(t) + t] \} dt.$$

Since $\left| t^{k/m} [A \cos(t) + 1] \exp \{ -\lambda [A \sin(t) + t] \} \right| \leq (|A| + 1) e^{\lambda|A|} t^{k/m} e^{-\lambda t}$ and $(|A| + 1) e^{\lambda|A|}$ is a constant, the integral $\int_0^{+\infty} t^{k/m} e^{-\lambda t} dt$ converges, and thus $E(X^k)$ converges. $\square$

3. Image features of the three-parameter oscillation lifetime distribution ORS(A, m, β)

Let the life X of a product obey the three-parameter oscillation life distribution ORS(A, m, β), and its distribution function $F(x)$, density function $f(x)$ and failure rate function $h(x)$ are:

$$F(x) = 1 - \exp \left\{ - \left\{ A \sin \left[\left(\frac{x}{\beta} \right)^m \right] + \left(\frac{x}{\beta} \right)^m \right\} \right\}, x \geq 0, \\ f(x) = \frac{mx^{m-1}}{\beta^m} \left\{ A \cos \left[\left(\frac{x}{\beta} \right)^m \right] + 1 \right\} \exp \left\{ - \left\{ A \sin \left[\left(\frac{x}{\beta} \right)^m \right] + \left(\frac{x}{\beta} \right)^m \right\} \right\}, \\ h(x) = \frac{mx^{m-1}}{\beta^m} \left\{ A \cos \left[\left(\frac{x}{\beta} \right)^m \right] + 1 \right\},$$

where $|A| \leq 1$, $m > 0$, $\beta > 0$.

In order to investigate the image characteristics of the distribution function $F(x)$, the density function $f(x)$ and the failure function $h(x)$, without losing generality, it can be discussed on the basis of assumption $\beta = 1$, at this time:

$$F(x) = 1 - \exp \{ - [A \sin(x^m) + x^m] \}, x \geq 0, \\ f(x) = mx^{m-1} [A \cos(x^m) + 1] \exp \{ - [A \sin(x^m) + x^m] \}, \\ h(x) = mx^{m-1} [A \cos(x^m) + 1].$$

Take the parameters $A = 0.8, -0.8$, $m = 0.5, 1, 2$ as well, the images of the distribution function $F(x)$ are shown in Figures 6 and 7, respectively.

Take the parameters $A = 0.8, -0.8$ and $m = 0.8, 1, 2$, the images of the density function $f(x)$ are shown in Figures 8 and 9, from which it can be seen: when $A > 0$ or $A < 0$,

(1) If $m < 1$, then the oscillation of $f(x)$ starts from infinity and drops to 0.

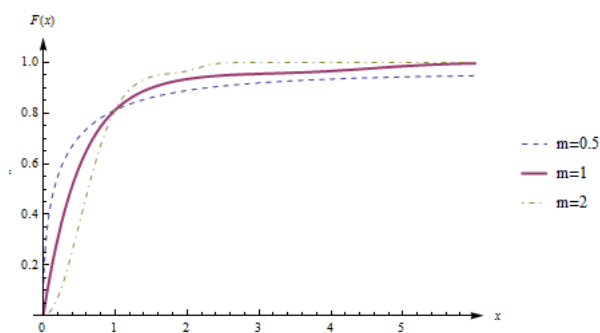


Figure 6. Image of distribution function $F(x)$ when $A = 0.8$

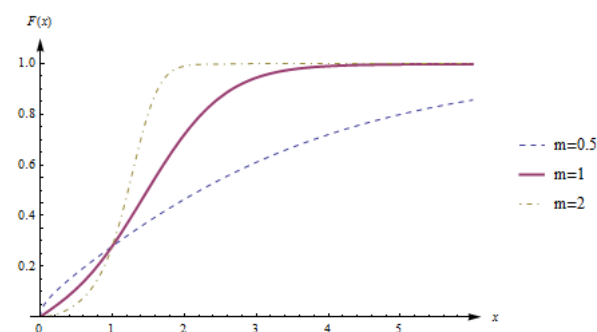


Figure 7. Image of the distribution function $F(x)$ when $A = -0.8$

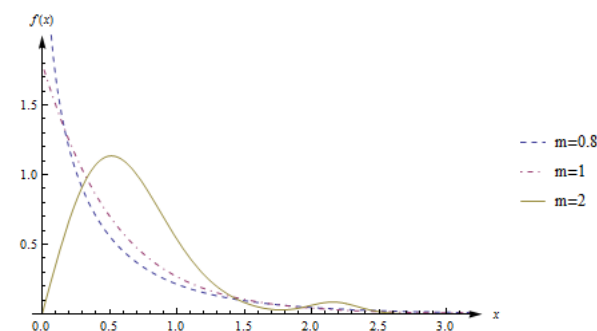


Figure 8. Image of the density function $f(x)$ when $A = 0.8$

- (2) If $m = 1$, then the oscillation of $f(x)$ begins from $A + 1$ and drops to 0.
- (3) If $m > 1$, then its oscillation starts to rise from 0 and falls to 0.

Take the parameters $A = 0.8, -0.8$, and $m = 0.8, 1, 1.2$ as well, the images of the failure efficiency function $h(x)$ are shown in Figures 10 and 11, from which it can be seen that the oscillation of $h(x)$ starts at infinity and drops to 0 as $m < 1$, it oscillates evenly as $m = 1$, and the oscillation of $h(x)$ rises from 0 as $m > 1$.

Theorem 3.1. For $m > 0$, $|A| \leq 1$, the function $h(x) = mx^{m-1} [A \cos(x^m) + 1]$, $x > 0$ has the following characteristics:

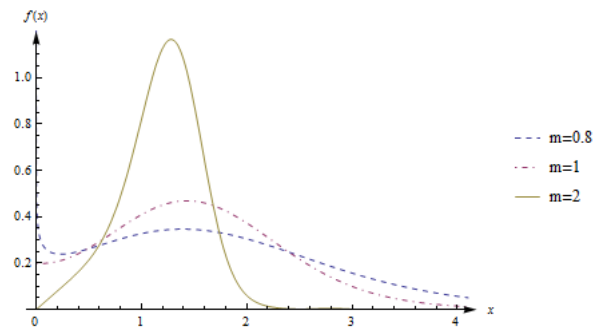


Figure 9. Image of the density function $f(x)$ when $A = -0.8$

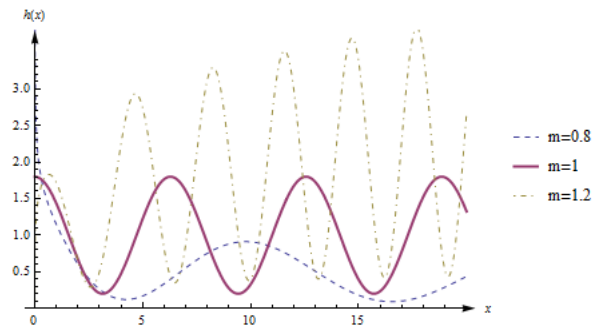


Figure 10. Image of the failure rate function $h(x)$ when $A = 0.8$

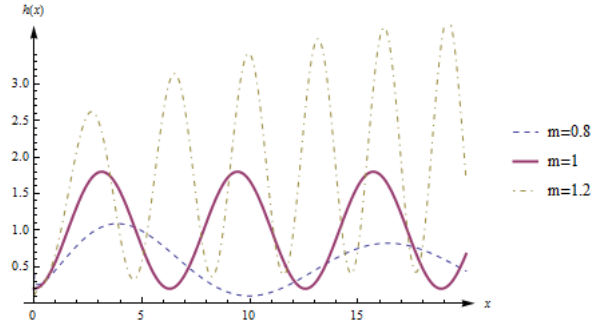


Figure 11. Image of the failure rate function $h(x)$ when $A = -0.8$

- (1) When $m = 1$, the maximum value of the function $h(x)$ is unchanged, and the minimum value is also unchanged (referred to as the uniform oscillation shape);
- (2) When $m > 1$, the maximum value of the function $h(x)$ shows an upward trend, and the minimum value also shows an upward trend (referred to as the oscillatory ascending shape);
- (3) When $m < 1$, the maximum value of the function $h(x)$ shows a downward trend, and the minimum value also shows a downward trend (referred to as the oscillatory descent shape).

Proof. (1) When $m = 1$, the image of $h(x) = A \cos x + 1$ presents a uniform oscillation shape.

(2) If $m \neq 1$,

$$h(x) = mx^{m-1} [A \cos(x^m) + 1], x > 0.$$

Let $t = x^m$, $\alpha = (m-1)/m < 1$, the $h(x) = mt^\alpha (A \cos t + 1)$. We have $h(x) = m\varphi(t)$ by denoting $\varphi(t) = t^\alpha (A \cos t + 1)$, moreover $h'(x) = m^2 x^{m-1} \varphi'(t)$.

If $m < 1$, $\varphi(t)$ shows an oscillatory decline trend, and since x^{m-1} is strictly monotonous decline, $h(x)$ also shows an oscillatory decline.

If $m > 1$, $\varphi(t)$ demonstrates an oscillating up trend, meanwhile $h(x)$ presents oscillatory rise because x^{m-1} is strictly monotonous rise.

The following proof is only for $m > 1, 0 < A < 1$, other cases can be derived similarly. Obviously

$$\varphi'(t) = \alpha t^{\alpha-1} (A \cos t + 1) - At^\alpha \sin t = t^{\alpha-1} (\alpha A \cos t - At \sin t + \alpha).$$

Let $g(t) = \alpha A \cos t - At \sin t + \alpha, t > 0$, then

$$g'(t) = -\alpha A \sin t - A \sin t - At \cos t = -A [(\alpha + 1) \sin t + t \cos t].$$

Given a non-negative positive number k , it is desirable to note the following two conclusions when $k > (\alpha - \pi A/2)/(2A\pi)$:

Conclusion 1:

The function $g(t)$ contains only one minimal point $t_{1k} \in (2k\pi + \pi/2, 2k\pi + \pi)$ and one maximal value $t_{2k} \in (2k\pi + 3\pi/2, 2k\pi + 2\pi)$ with $g(t_{1k}) < 0, g(t_{2k}) > 0$ in the interval $(2k\pi, 2k\pi + 2\pi)$ and its monotonicity is shown in the following table (where $0^* = 2k\pi, (2\pi)^* = 2k\pi + 2\pi$).

t	0^*	$(0^*, t_{1k})$	t_{1k}	(t_{1k}, t_{2k})	t_{2k}	$(t_{2k}, (2\pi)^*)$	$(2\pi)^*$
$g'(t)$		$-$	0	$+$	0	$-$	
$g(t)$	$(A+1)\alpha$	$\searrow$	$g(t_{1k}) < 0$	$\nearrow$	$g(t_{2k}) > 0$	$\searrow$	$(A+1)\alpha$

In fact,

$$g'(t) = -A [(\alpha + 1) \sin t + t \cos t].$$

Let $g'(t) = 0$, solve the equivalent equation $g_0(t) = 0$ to find the stationary point, where $g_0(t) = \tan t + t/(\alpha + 1)$.

Also

$$g'_0(t) = \sec^2 t + 1/(\alpha + 1) > 0,$$

$$\lim_{t \rightarrow (2k\pi + \pi/2)^+} g_0(t) = -\infty, g_0(2k\pi + \pi) = \frac{2k\pi + \pi}{\alpha + 1} > 0.$$

That is, there is a unique point $t_{1k} \in (2k\pi + \pi/2, 2k\pi + \pi)$ such that $g_0(t_{1k}) = 0$.

$$\lim_{t \rightarrow (2k\pi + 3\pi/2)^+} g_0(t) = -\infty, g_0(2k\pi + 2\pi) = \frac{2k\pi + 2\pi}{\alpha + 1} > 0.$$

In similar, it means there is a unique point $t_{2k} \in (2k\pi + 3\pi/2, 2k\pi + 2\pi)$ such that $g_0(t_{2k}) = 0$.

As for functions $y = \tan t$ and $y = -t/(\alpha + 1)$, there are two points of intersections at t_{1k}, t_{2k} and we can determine the monotonicity on the corresponding interval. It's easy to see that $g'(t) < 0$ when $2k\pi < t < t_{1k}$ or $t_{2k} < t < 2k\pi + 2\pi$ and $g'(t) > 0$ when $t_{1k} < t < t_{2k}$.

Then the function $g(t)$ contains only one minimum point $t_{1k} \in (2k\pi + \pi/2, 2k\pi + \pi)$ and one maximum point $t_{2k} \in (2k\pi + 3\pi/2, 2k\pi + 2\pi)$ in the interval $(2k\pi, 2k\pi + 2\pi)$.

Substitute $\tan t = -t/(\alpha + 1)$ into the expression of $g(t)$ at the points t_{1k}, t_{2k} , then the corresponding function values $g(t_{1k}), g(t_{2k})$ on the points t_{1k}, t_{2k} are obtained that

$$\begin{aligned} g(t_{1k}) &< g\left(2k\pi + \frac{\pi}{2}\right) = -A\left(2k\pi + \frac{\pi}{2}\right) + \alpha < 0, \\ g(t_{2k}) &> g(2k\pi + \pi) = -\alpha A + \alpha > 0. \end{aligned}$$

Conclusion 2:

There is only one maximal point $t_{Mk} \in (2k\pi, t_{1k})$ and one minimal point $t_{mk} \in (t_{1k}, 2k\pi + \pi)$ in $(2k\pi, 2k\pi + 2\pi)$ for the function $\varphi(t)$.

In fact, from Conclusion 1 we may have the following results:

$$g(2k\pi) = \alpha(A + 1) > 0, g(t_{1k}) < 0, g(t_{2k}) > 0, g(2k\pi + 2\pi) = \alpha(A + 1) > 0.$$

So there is only one $t_{Mk} \in (2k\pi, t_{1k})$ which makes $g(t_{Mk}) = 0$, that is $\varphi'(t_{Mk}) = 0$. Besides $\varphi'(t) > 0$ when $t \in (2k\pi, t_{Mk})$ and $\varphi'(t) < 0$ when $t \in (t_{Mk}, t_{1k})$, hence t_{Mk} is the maximum point of $\varphi(t)$.

In similar, we can prove that there is a unique minimum point $t_{mk} \in (t_{1k}, t_{2k})$ for the function $\varphi(t)$. The monotonicity of the corresponding interval is listed as follows.

t	0^*	$(0^*, t_{Mk})$	t_{Mk}	(t_{Mk}, t_{1k})	t_{1k}	(t_{1k}, t_{mk})	t_{mk}	(t_{mk}, t_{2k})	t_{2k}	$(t_{2k}, (2\pi)^*)$
$g(t)$	+	+	0	-	-	-	0	+	+	+
$\varphi'(t)$	+	+	0	-	-	-	0	+	+	+
$\varphi(t)$		$\nearrow$	maxi- mum	$\searrow$		$\searrow$	mini- mum	$\nearrow$		$\nearrow$

Also

$$g(t_{1k}) < 0, g(2k\pi + \pi) = \alpha(1 - A) > 0.$$

Therefore there is only one maximal point $t_{Mk} \in (2k\pi, t_{1k})$ and one minimal point $t_{mk} \in (t_{1k}, 2k\pi + \pi)$ in the interval $(2k\pi, 2k\pi + 2\pi)$.

The maximum and minimum values of the function $\varphi(t)$ in the interval $(2k\pi, 2k\pi + 2\pi)$ are denoted by $M_k = \varphi(t_{Mk}), m_k = \varphi(t_{mk})$ respectively.

Afterwards we prove the sequences $\{M_k\}, \{m_k\}$ are strictly monotonous. In fact, let $\bar{\varphi}(t) = (A + 1)t^\alpha$, $\underline{\varphi}(t) = (1 - A)t^\alpha$, $t > 0$ and notice that $\bar{\varphi}(t), \underline{\varphi}(t)$ are both monotonic increasing functions. For any given $t > 0$, we have

$$\underline{\varphi}(t) \leq \varphi(t) \leq \bar{\varphi}(t).$$

Moreover

$$\bar{\varphi}(2k\pi) = \varphi(2k\pi), \underline{\varphi}(2k\pi + \pi) = \varphi(2k\pi + \pi).$$

And then

$$\begin{aligned} M_k &= \varphi(t_{Mk}) \leq \bar{\varphi}(t_{Mk}) \leq \bar{\varphi}(2k\pi + 2\pi) = \varphi(2k\pi + 2\pi) < \varphi(t_{M(k+1)}) = M_{k+1}, \\ m_k &= \varphi(t_{mk}) \leq \varphi(2k\pi + \pi) = \underline{\varphi}(2k\pi + \pi) \leq \underline{\varphi}(t_{m(k+1)}) \leq \varphi(t_{m(k+1)}) = m_{k+1}. \end{aligned}$$

□

4. Parameter estimation of the three-parameter oscillation life distribution $\text{ORS}(A, m, \beta)$

Let the life X of a product follow a three-parameter oscillation life distribution $\text{ORS}(A, m, \beta)$. $X_1, X_2, \dots, X_n$ is a sample from X with size of n . We consider the moment estimations of the parameters because the likelihood function is quite complex.

Let $Y = \ln X$, $Y_i = \ln X_i, i = 1, 2, \dots, n$ and denote $\sigma = 1/m$, $\mu = \ln \beta$. Hence

$$\begin{aligned} F_Y(y) &= P(X \leq e^y) = 1 - \exp \left\{ - \left\{ A \sin \left[\left(\frac{e^y}{\beta} \right)^m \right] + \left(\frac{e^y}{\beta} \right)^m \right\} \right\} \\ &= 1 - \exp \left\{ - \left\{ A \sin \left[\exp \left(\frac{y - \mu}{\sigma} \right) \right] + \exp \left(\frac{y - \mu}{\sigma} \right) \right\} \right\}, \end{aligned}$$

for $-\infty < y < +\infty$.

Let $Z = (Y - \mu)/\sigma$, then

$$\begin{aligned} F_Z(z) &= P(Y \leq \mu + \sigma z) = 1 - \exp \{ - [A \sin(e^z) + e^z] \}, \\ f_Z(z) &= e^z [A \cos(e^z) + 1] \exp \{ - [A \sin(e^z) + e^z] \}, \end{aligned}$$

for $-\infty < z < +\infty$.

And then for positive integers k ,

$$\begin{aligned} E(Z^k) &= \int_{-\infty}^{+\infty} z^k e^z [A \cos(e^z) + 1] \exp \{ - [A \sin(e^z) + e^z] \} dz \\ &= \int_0^{+\infty} (\ln t)^k [A \cos(t) + 1] \exp \{ - [A \sin(t) + t] \} dt. \end{aligned}$$

By denoting $g_1(A) = E(Z)$, $g_2(A) = E(Z^2)$, $g_3(A) = E(Z^3)$, we have

$$\begin{aligned} E(Y) &= E(\mu + \sigma Z) = \mu + \sigma E(Z), \\ E(Y^2) &= E[(\mu + \sigma Z)^2] = \mu^2 + 2\mu\sigma E(Z) + \sigma^2 E(Z^2), \\ E(Y^3) &= E[(\mu + \sigma Z)^3] = \mu^3 + 3\mu^2\sigma E(Z) + 3\mu\sigma^2 E(Z^2) + \sigma^3 E(Z^3). \end{aligned}$$

Let $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$, $\bar{Y}^2 = \frac{1}{n} \sum_{i=1}^n Y_i^2$, $\bar{Y}^3 = \frac{1}{n} \sum_{i=1}^n Y_i^3$, $S_{ny}^2 = \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2$, and establish the following three moment equations:

$$\mu + \sigma g_1(A) = \bar{Y}, \quad (4.1)$$

$$\mu^2 + 2\mu\sigma g_1(A) + \sigma^2 g_2(A) = \bar{Y}^2, \quad (4.2)$$

$$\mu^3 + 3\mu^2\sigma g_1(A) + 3\mu\sigma^2 g_2(A) + \sigma^3 g_3(A) = \bar{Y}^3. \quad (4.3)$$

From (4.1), we get

$$\mu = \bar{Y} - \sigma g_1(A) \quad (4.4)$$

(4.2) - (4.1)²:

$$\sigma^2 [g_2(A) - g_1^2(A)] = S_{ny}^2,$$

$$\sigma = \frac{S_{ny}}{\sqrt{g_2(A) - g_1^2(A)}}. \quad (4.5)$$

Substitute (4.4) into (4.3), we obtain

$$\begin{aligned} & [\bar{Y} - \sigma g_1(A)]^3 + 3[\bar{Y} - \sigma g_1(A)]^2 \sigma g_1(A) + 3[\bar{Y} - \sigma g_1(A)] \sigma^2 g_2(A) + \sigma^3 g_3(A) = \bar{Y}^3, \\ & \bar{Y}^3 - 3\sigma \bar{Y}^2 g_1(A) + 3\sigma^2 \bar{Y} g_1^2(A) - \sigma^3 g_1^3(A) + 3\sigma \bar{Y}^2 g_1(A) - 6\sigma^2 \bar{Y} g_1^2(A) + 3\sigma^3 g_1^3(A) \\ & + 3\sigma^2 \bar{Y} g_2(A) - 3\sigma^3 g_1(A) g_2(A) + \sigma^3 g_3(A) = \bar{Y}^3, \\ & \bar{Y}^3 + 3\sigma^2 \bar{Y} [g_2(A) - g_1^2(A)] + \sigma^3 [g_3(A) + 2g_1^3(A) - 3g_1(A)g_2(A)] = \bar{Y}^3. \end{aligned} \quad (4.6)$$

Substituting (4.5) into (4.6) yields:

$$\begin{aligned} & \bar{Y}^3 + 3\bar{Y} S_{ny}^2 + \frac{g_3(A) + 2g_1^3(A) - 3g_1(A)g_2(A)}{[g_2(A) - g_1^2(A)]^{3/2}} S_{ny}^3 = \bar{Y}^3, \\ & \frac{[g_2(A) - g_1^2(A)]^{3/2}}{g_3(A) + 2g_1^3(A) - 3g_1(A)g_2(A)} = \frac{S_{ny}^3}{\bar{Y}^3 - \bar{Y}^3 - 3\bar{Y} S_{ny}^2}. \end{aligned}$$

That is

$$\frac{[g_2(A) - g_1^2(A)]^{3/2}}{g_3(A) + 2g_1^3(A) - 3g_1(A)g_2(A)} = \frac{S_{ny}^3}{\bar{Y}^3 + 2\bar{Y}^3 - 3\bar{Y} \bar{Y}^2}. \quad (4.7)$$

Denote the function

$$G(A) = \frac{[g_2(A) - g_1^2(A)]^{3/2}}{g_3(A) + 2g_1^3(A) - 3g_1(A)g_2(A)}, |A| \leq 1.$$

Clearly

$$G(-1) = -1.26838, G(0) = -0.87754, G(1) = -1.34608.$$

The image of $G(A)$ is obtained by tracing points roughly as shown in Figure 12 below, from which can be seen that the function monotonically increases from the beginning and then monotonically decreases when $A = -1(0.05)1$. Meanwhile the image of function $G(A)$ obtained by tracing the dots is roughly as shown in Figure 13 below, which strictly increases monotonically when $A = -1(0.01)(-0.9)$. Thus the point estimate of the parameter A usually can't be obtained by equation (4.7).

Write the equations (4.5) and (4.4) as $\sigma(A) = S_{ny}/\sqrt{g_2(A) - g_1^2(A)}$, $\mu(A) = \bar{Y} - \sigma(A)g_1(A)$ respectively, and denote the observations $y_1, y_2, \dots, y_n$ from least to greatest as $y_{(1)}, y_{(2)}, \dots, y_{(n)}$.

$$\begin{aligned} & F_i(A) = 1 - \exp \left\{ - \left\{ A \sin \left[\exp \left(\frac{y_{(i)} - \mu(A)}{\sigma(A)} \right) \right] + \exp \left(\frac{y_{(i)} - \mu(A)}{\sigma(A)} \right) \right\} \right\}, \\ & i = 1, 2, \dots, n, |A| \leq 1, \\ & Q(A) = \sum_{i=1}^n \left| F_i(A) - \frac{i}{n} \right|, |A| \leq 1. \end{aligned} \quad (4.8)$$

Since the range of A is $[-1, 1]$, set the step size to 0.0002, denote $a_j = -1 + 0.0002j$, $j = 0, 1, \dots, 10000$, calculate the values of $Q(a_j)$, $j = 0, 1, \dots, 10000$.

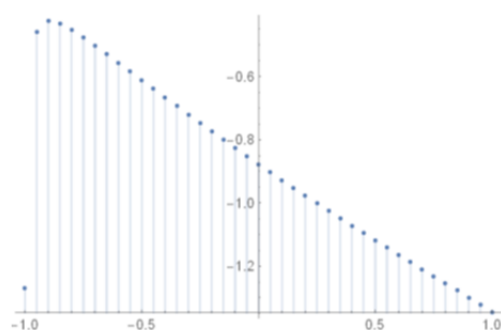


Figure 12. Image of the function $G(A)$ ($A = -1(0.05)1$)

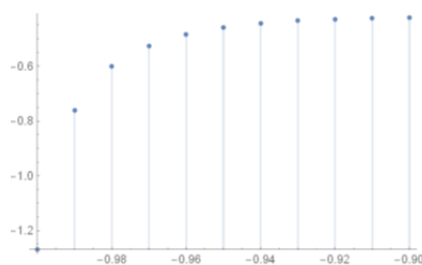


Figure 13. Image of the function $G(A)$ ($A = -1(0.01)(-0.9)$)

Take its minimum value, and its corresponding a_j can be considered as the point estimator $\hat{A}$ of the parameter A .

With the point estimator $\hat{A}$ of the parameter A , we have the point estimators $\hat{\sigma}, \hat{m}$ of the parameters σ, m as follows:

$$\hat{\sigma} = \frac{S_{ny}}{\sqrt{g_2(\hat{A}) - g_1^2(\hat{A})}}, \quad \hat{m} = \frac{1}{\hat{\sigma}}.$$

The point estimators $\hat{\mu}, \hat{\beta}$ of the parameters μ, β are estimated to be

$$\hat{\mu} = \bar{Y} - \hat{\sigma}g_1(\hat{A}), \quad \hat{\beta} = e^{\hat{\mu}}.$$

Example 4.1. Take the sample size $n = 100$ and the truth values of parameters to be $A = 0.5, m = 1.2, \beta = 1.5$, respectively. Generate a set of samples from the three-parameter oscillation lifetime distribution $ORS(A, m, \beta)$ by Monte Carlo simulations as follows: 1.31000, 0.61944, 1.63486, 1.88778, 0.47722, 2.28013, 0.38259, 0.78273, 0.28874, 0.37027, 2.17687, 1.44713, 0.16140, 2.18462, 0.55976, 0.22268, 0.04384, 1.93388, 0.24852, 1.22257, 2.69652, 1.02005, 0.35261, 1.54096, 0.62959, 1.39978, 0.96958, 1.09663, 0.26027, 2.93294, 0.38427, 0.69248, 0.94430, 0.03419, 0.46970, 2.10916, 2.14711, 0.57446, 1.29544, 1.71342, 0.78357, 0.23324, 0.55041, 0.39452, 1.62742, 2.16067, 0.24820, 1.36177, 0.86386, 0.26977, 0.81342, 0.18044, 1.69390, 1.20355, 0.67630, 3.04729, 0.04803, 0.60118, 0.45235, 0.30552, 7.05133, 0.50302, 0.37488, 4.55278, 0.80559, 0.73934, 4.69231, 0.64510, 1.57311, 0.15866, 0.45778, 0.70659, 0.20459, 4.84306, 0.08196, 0.72266, 0.74355, 0.46036, 0.85225, 0.38748,

0.35608, 0.18149, 0.86645, 0.30898, 5.38428, 0.84365, 2.45736, 0.18189, 0.58963, 4.29123, 1.54904, 2.13365, 0.16768, 0.96855, 0.23196, 4.95233, 0.29932, 0.51420, 0.63465, 1.38791.

By calculating the minimum value of equation (4.8), we get the point estimator of parameter A as $\hat{A} = 0.54$, and then the point estimators of the parameters m, β are $\hat{m} = 1.27973$, $\hat{\beta} = 1.51955$. The image of its empirical distribution function and distribution function is shown in Figure 14.

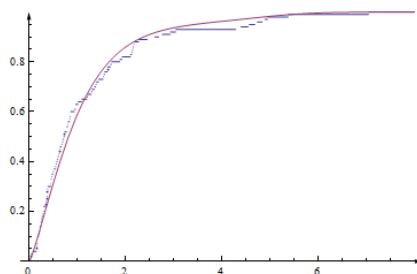


Figure 14. Plot of empirical distribution function and distribution function for Example 4.1

Example 4.2. Taking a certain type of forklift in literature [16] as an example, this type of forklift sold 30,000 units in 2018. After one year's after-sales maintenance records were tracked, the failure time of some mounted hydraulic automatic transmission recorded by the dashboard was arranged in ascending order as follows: 0.7, 5, 10, 15, 18, 23.8, 30, 35, 39.3, 48, 56.8, 60, 62, 90, 95.3, 107.2, 110.7, 112, 116.5, 131.8, 134.7, 135, 147, 156, 166.9, 213, 218.1, 227.1, 234.3, 240, 247, 250, 259, 260, 281, 285, 300, 320, 326, 330, 338, 350, 360, 360.8, 370, 390, 413, 430, 468.1, 477.1, 484, 500, 510, 512, 520, 523, 560, 570, 576, 577.

By calculating the minimum value of equation (4.8), we get the point estimator of parameter A as $\hat{A} = -0.5$, and then the point estimators of the parameters m, β are $\hat{m} = 0.84276$, $\hat{\beta} = 181.455$. The image of its empirical distribution function and distribution function is shown in Figure 15.

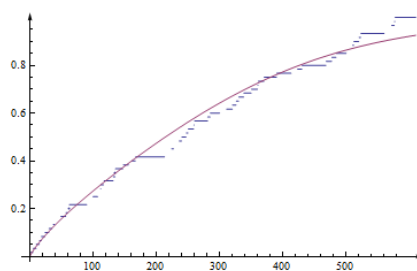


Figure 15. Plot of empirical distribution function and distribution function for Example 4.2

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