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# ANALYSIS OF A STOCHASTIC NONAUTONOMOUS HYBRID POPULATION MODEL WITH IMPULSIVE PERTURBATIONS

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**Abstract** In this paper, we propose a stochastic nonautonomous hybrid population model with Allee effect, Markovian switching and impulsive perturbations and investigate its stochastic dynamics. We first establish sufficient conditions for the extinction and permanence. Then, we study some asymptotic properties and the lower- and upper-growth rates of the positive solutions. Finally, by performing numerical simulations we verify the main results and analyze the impact on the system from the Allee effect, the Markovian switching and the impulsive perturbations.

**Keywords** Allee effect, markovian switching, impulsive perturbations, permanence, extinction.

**MSC(2010)** 34F05.

## 1. Introduction

As we all know, the significance of logistic model is that it is the basis of many two interacting population growth models. It is also the main model to study the maximum sustainable yield in the practical fields such as fishery, forestry and agriculture. Stochastic logistic model, as one of most important evolutionary models from classical logistic model, can reflect the random disturbance of the real environment. It is the most important evolutionary model of classical logistic model. Several scholars (see e.g. [6–8]) have studied the following stochastic logistic system

$$dx(t) = x(t)(a(t) - b(t)x(t))dt + \sigma(t)x(t)dB(t), \tag{1.1}$$

where $x(t)$ is the population size at time $t$, $a(t)$ is the growth rate, $b(t) > 0$ characterises the positive influences of cooperation and aggregation, $c(t) > 0$ stands for the intra-specific competition, $B(t)$ is a standard Brownian motion that arise from the environmental white noise, $\sigma(t)$ denotes the intensity of environment white noise. Lots of literature such as [9,12,17,18,23] have investigated extensively system (1.1) and obtained a large number of important results, for instance the global positivity of the solution, the asymptotic properties of the solution, the persistence, extinction and ergodicity of the equation and so on.

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Many species show an inverse density dependence phenomenon at low population density, called Allee effect. It can describe the negative population growth caused by the decline of reproductive success rate when the population level is lower than the agronomic threshold [3, 5, 19]. The general form of logistic model with Allee effect is

$$\frac{dx(t)}{dt} = x(t)(a + bx(t) - cx^2(t))$$

where $b$ and $c$ measures the intra-specific competition. In [28], the scholars have classified Allee effect into strong and weak Allee effect, and studied the influence of weak Allee effect on population dynamics. Global bifurcation analysis of a class of general predator-prey models with a strong Allee effect in prey population is carried out in details in [30]. Allee effect can interact with environmental random disturbance. Ji [11] explored a stochastic population model with Allee effect as follows

$$dx(t) = x(t)[a(r(t)) + b(r(t))x(t) - c(r(t))x^2(t)]dt + \sigma(r(t))dB(t), \tag{1.2}$$

where $r(t)$ is a right-continuous Markov chain on a given state space and it is often used to describe the color noise of environmental random disturbance [13,33]. They investigated the existence of global positive solution, the sufficient conditions for permanence and extinction. Liu [22] improved model (1.2) and researched the permanence, the extinction and the existence of a unique ergodic invariant measure of the presented model.

The impulse term in mathematical model is often used to describe the instantaneous change of variables under certain conditions. Affected by natural or human factors (such as flood, picking, hunting, sowing, etc.), the internal laws of many biological species or ecological environment usually occur discrete changes with relatively short duration in some fixed time or critical state. The biological system with this mutation can be modeled by a mathematical model with pulse term. In recent years, many stochastic biological models with impulsive disturbances have been proposed and studied [2, 4, 20, 24, 29, 31]. Zhang [34] proposed an impulsive stochastic non-autonomous Lotka-Volterra predator-prey model and investigated the existence and global attraction of the positive periodic solution of the subsystem, the thresholds for stochastic persistence and extinction. Jiang [14] studied a stochastic Gilpin-Ayala model with regime switching and impulsive perturbations

$$\begin{cases} dx(t) = x(t)r(i)[1 - \dfrac{x^{\theta(i)}(t)}{K(i)}]dt + \sigma(i)x(t)dB(t), & t \neq t_k, \\ x(t_k^+) = x(t_k) + hx(t_k), & h_k \in (-1, \infty), k \in N. \end{cases}$$

They provided the sufficient conditions for extinction, nonpersistence in the mean, weak persistence, and stochastic permanence of the solutions. Their results demonstrate that the dynamics of the model have close relations with the impulses and the Markov switching.

As is known to all that no creature can be separated from a specific living environment. When considering the influence of living environment on population number or growth law, it is impractical to assume that the parameters such as population growth rate and environmental capacity are constant. Therefore, it is necessary to consider the non autonomous model in which the parameters of the population change with time. To this end, this paper presents and studies the following stochastic non-autonomous hybrid population model with Allee effect, Markovian switching and impulsive perturbations

$$\begin{cases} dx(t) = x(t)[a(t,r(t)) + b(t,r(t))x(t) - c(t,r(t))x^2(t)]dt \\ \qquad\qquad + \sigma(t,r(t))x(t)dB(t), \quad t \neq t_k, \\ x(t_k^+) = g_k x(t_k), \quad k \in Z_+ = \{1,2,...\}, \\ x(0^+) = x(0) > 0, \quad r(0^+) = r_0 \in S = \{1,2,...,m\}, \end{cases} \tag{1.3}$$

where $g_k$ is the impulse coefficients. Obviously, $g_k > 0$ for all $k \in Z_+$ from the biological meanings. Moreover, $g_k < 1$ means that the population size undergoes an instantaneous decline at the pulse time, such as harvesting, hunting and so on and $g_k > 1$ implies a dramatic increase at the pulse time such as planting, supplement etc.

The model is presented for the following two improvements:

(i) Compared with the model studied in [34] and [32], the present model (1.3) takes the Allee effects into consideration.

(ii) Compared with most papers which study the hybrid system with Allee effect, Markov switching and impulsive effects (such as [11,14,20,25,33]), this paper considers the nonautonomous situation.

This paper will focus on deriving sufficient conditions on the extinction and permanence of system (1.3) and exploring the effects of the Allee effect, the Markovian switching and the impulsive perturbations on the extinction and persistence of the system.

The rest of the paper is organized as follows. Section 2 presents some related preliminaries. Sufficient conditions on the extinction and permanence of the system are established in Section 3. Section 4 investigates some asymptotic properties. In Section 5, we carry out numerical examples to verify the validity of the results and to discuss the effects of the Allee effect, the Markovian switching and the impulsive perturbations on the extinction and persistence of the system. Finally, we conclude the paper by a brief discussion in Section 6.

## 2. Preliminary

Let $(\Omega, F, \{F_t\}_{t \geqslant 0}, P)$ be a complete probability space. Let $r(t), t \geqslant 0$ be a continuous-time Markov chain with finite space $S = \{1, 2, \ldots, m\}$. Suppose that $r(t)$ is generated by $Q = (q_{ij})_{m \times m}$, i.e.,

$$P\{r(t+\Delta t) = j | r(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), & \text{if } j \neq i, \\ 1 + q_{ii}\Delta t + o(\Delta t), & \text{if } j = i, \end{cases}$$

where $q_{ij} \geqslant 0$ for $i, j = 1, 2, ..., m$ with $j \neq i$ and $\sum_{j=1}^{m} q_{ij} = 0$ for every $i = 1, 2, ..., m$.

Throughout of the paper, we always assume that $b(t,i) \geq 0$, $c(t,i) > 0$, $a(\cdot,i)$ and $\sigma(\cdot,i)$ are bounded integrable functions defined on $[0, +\infty)$ for every $i \in S$. Define

$$\check{f} = \max_{(t,i) \in [0,+\infty) \times S} f(\cdot,i), \quad \check{f}(i) = \max_{t \in [0,+\infty)} f(\cdot,i),$$

$$\hat{f} = \min_{(t,i) \in [0,+\infty) \times S} f(\cdot,i), \quad \hat{f}(i) = \min_{t \in [0,+\infty)} f(\cdot,i),$$

where $f(t,i)$ can be any of the $a(\cdot,\cdot), b(\cdot,\cdot), c(\cdot,\cdot)$ or $\sigma(\cdot,\cdot)$. Suppose that $r(t)$ is independent of $B(t)$. Further, we assume that $r(t)$ is irreducible, i.e., the system can switch from any regime to any other regime. The irreducibility implies that the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2, ..., \pi_m)$ which can be determined by solving the following equation (see e.g. [1, 16, 21] )

$$\pi Q = 0 \tag{2.1}$$

subject to $\sum_{i=1}^{m} \pi_i = 1$ and $\pi_i > 0$, $i \in S$.

Let $(u(t), r(t))$ be a diffusion process given by the following scalar equation:

$$du(t) = f(t, u(t), r(t))dt + g(t, u(t), r(t))dB(t).$$

For arbitrary twice continuously differentiable function $V(t,u,i)$, define

$$\begin{aligned} LV(t,u,i) =& \frac{\partial V(t,u,i)}{\partial t} + \frac{\partial V(t,u,i)}{\partial u} f(t,u,i) + \frac{1}{2}\frac{\partial^2 V(t,u,i)}{\partial u^2} g^2(t,u,i) \\ &+ \sum_{j=1}^{m} q_{ij} V(t,u,j). \end{aligned}$$

Before we state and prove the main results, let us prepare some definitions and lemmas.

**Definition 2.1.** The solution of the system (1.3) is a stochastic process $x(t)$, $t \in [0, +\infty)$ such that
(i) $x(t)$ is $F_t$-adapted and continuous on $[t_{k-1}, t_k), k \in Z_+$;
(ii) for each $[t_{k-1}, t_k), k \in Z_+$, $x(t)$ exists and $x(t_k^+) = g_k x(t_k)$ a.s.;
(iii) for each $[t_{k-1}, t_k), k \in Z_+$, $x(t)$ obeys the integral equation

$$\begin{aligned} x(t) =& x(t_{k-1}) + \int_{t_{k-1}}^{t} \Big[a(s, r(s))x(s) + b(s, r(s))x^2(s) - c(s, r(s))x^3(s)\Big] ds \\ &+ \int_{t_{k-1}}^{t} \sigma(s, r(s))x(s)dB(s). \end{aligned}$$

**Definition 2.2** ( [25]). Let $x(t)$ be the solution of (1.3). $x(t)$ is said to be extinct if $\lim_{t\to\infty} x(t) = 0$.

**Definition 2.3** ( [32]). (i) The solution of (1.3) is called stochastically ultimately upper bounded, if for any $\varepsilon \in (0,1)$, there exists a positive constant $\kappa = \kappa(\varepsilon)$ such that the solution $x(t)$ with any positive initial value satisfies that

$$\limsup_{t\to+\infty} P\{x(t) > \kappa\} < \varepsilon.$$

(ii) The solution of (1.3) is called stochastically ultimately lower bounded, if for any $\varepsilon \in (0,1)$, there exists a positive constant $\lambda = \lambda(\varepsilon)$ such that the solution $x(t)$ with any positive initial value satisfies that

$$\limsup_{t\to+\infty} P\{x(t) < \lambda\} < \varepsilon.$$

(iii) The solution of (1.3) is called stochastically permanent if the solutions both stochastically ultimately upper bounded and stochastically ultimately lower bounded.

**Lemma 2.1** ( [15], Lemma 2.3). *Let $Q$ be irreducible and $v = (v_1, ..., v_m)^T \in R^m$. Equation $Qx = v$ has a solution if and only if $\pi^T v = 0$.*

**Lemma 2.2** ( [33], Lemma 2.2). *The positive recurrent process $y(t, u) = (x(t), r(t))$ has a unique stationary distribution $v(\cdot, \cdot)$ which is ergodic, that is, if $f$ is a function integrable with respect to the measure $v(\cdot, \cdot)$, then*

$$P\Big[\lim_{t\to+\infty} \frac{1}{t}\int_0^t f(x(s), r(s))ds = \sum_{i=1}^{m}\int_{R^n} f(y, i)dv(y, i)\Big] = 1.$$

**Lemma 2.3.** *For any initial data $(x(0), r(0)) \in (0, +\infty) \times S$, (1.3) admits a unique global positive solution $x(t)$ on $t \geqslant 0$ a.s. (almost surely).*

**Proof.** Considering the following stochastic system without impulse:

$$\begin{aligned} dy(t) =& y(t)\Big[a(t, r(t)) + b(t, r(t)) \prod_{0<t_k<t} g_k y(t) - c(t, r(t))\big(\prod_{0<t_k<t} g_k y(t)\big)^2\Big]dt \\ &+ \sigma(t, r(t))y(t)dB(t), \end{aligned} \tag{2.2}$$

where the initial value is defined as $y(0) = x(0)$. Utilizing the same method with [26], the system (2.2) has a unique global positive solution $y(t)$ when $t \geq 0$. Let $x(t) = \prod_{0<t_k<t} g_k y(t)$ be the solution of system (1.3), it is easy to see that $x(t)$ is continuous on each interval $(t_k, t_{k+1}) \in R_+, k \in N$ and for any $t \neq t_k$,

$$\begin{aligned} dx(t) =& d\big(\prod_{0<t_k<t} g_k y(t)\big) \\ =& \prod_{0<t_k<t} g_k dy(t) \\ =& \prod_{0<t_k<t} g_k y(t)\Big[a(t, r(t)) + b(t, r(t)) \prod_{0<t_k<t} g_k y(t) - c(t, r(t)) \\ & \times \big(\prod_{0<t_k<t} g_k y(t)\big)^2\Big]dt + \sigma(t, r(t))y(t)dB(t) \\ =& x(t)[a(t, r(t)) + b(t, r(t))x(t) - c(t, r(t))x^2(t)]dt + \sigma(t, r(t))x(t)dB(t). \end{aligned}$$

On the other hand, for each $k \in N$ and $t_k \in [0, +\infty)$,

$$x(t_k^+) = \lim_{t\to t_k^+} \prod_{0<t_j<t} g_j y(t) = \prod_{0<t_j<t_k} g_j y(t_k^+) = g_k \prod_{0<t_j<t_k} g_k y(t) = g_k x(t_k).$$

And

$$x(t_k^-) = \lim_{t\to t_k^-} \prod_{0<t_j<t} g_j y(t) = \prod_{0<t_j<t_k} g_j y(t_k^-) = g_k \prod_{0<t_j<t_k} g_k y(t) = g_k x(t_k).$$

□

## 3. Extinction and permanence

Firstly, let's study the extinction of system (1.3).

**Theorem 3.1.** *Denote by*

$$\check{h}(i) = \check{a}(i) - 0.5\hat{\sigma}^2(i),\ \bar{h} = \sum_{i=1}^{m} \pi_i \check{h}(i), \quad \bar{b} = \sum_{i=1}^{m} \pi_i \frac{\check{b}^2(i)}{4\hat{c}(i)},\ i \in S$$

*and*

$$\bar{g} = \limsup_{t\to\infty} t^{-1} \sum_{0<t_k<t} \ln g_k.$$

*If* $\bar{h} + \bar{b} + \bar{g} < 0$*, then the population* $x(t)$ *goes to extinction a.s..*

**Proof.** Applying the generalized Itô's formula (see, e.g., [27]) to Eq. (2.2) gives

$$\begin{aligned} d\ln y(t) =& \frac{1}{y(t)} dy(t) - \frac{1}{2y^2(t)} (dy(t))^2 \\ =& \Big[a(t,r(t)) - 0.5\sigma^2(t,r(t)) + b(t,r(t)) \prod_{0<t_k<t} g_k y(t) - c(t,r(t)) \\ & \times \big(\prod_{0<t_k<t} g_k y(t)\big)^2\Big] dt + \sigma(t,r(t)) dB(t). \end{aligned}$$

Integrating both sides from 0 to $t$, we have

$$\begin{aligned} \ln y(t) - \ln y(0) = \int_0^t \Big[a(s,r(s)) - 0.5\sigma^2(s,r(s)) + b(s,r(s)) \prod_{0<t_k<s} g_k y(s) \\ - c(s,r(s)) \big(\prod_{0<t_k<s} g_k y(s)\big)^2\Big] ds + \int_0^t \sigma(s,r(s)) dB(s), \end{aligned} \quad (3.1)$$

that is

$$\begin{aligned} \ln y(t) - \ln y(0) \le \int_0^t \Big[\check{a}(r(s)) - 0.5\hat{\sigma}^2(r(s)) + \check{b}(r(s)) \prod_{0<t_k<s} g_k y(s) - \hat{c}(r(s)) \\ \times \big(\prod_{0<t_k<s} g_k y(s)\big)^2\Big] ds + \int_0^t \sigma(s,r(s)) dB(s). \end{aligned}$$

Thus

$$\ln y(t) \le \ln y(0) + \int_0^t \Big[\check{h}(r(s)) + \frac{\check{b}^2(r(s))}{4\hat{c}(r(s))}\Big] ds + M(t), \quad (3.2)$$

where $M(t) = \int_0^t \sigma(s,r(s)) dB(s)$. Note that $M(t)$ is local martingale, whose quadratic variation is

$$\langle M(t), M(t) \rangle = \int_0^t \sigma^2(s,r(s)) ds \leqslant \check{\sigma}^2 t.$$

Making use of the strong law of large numbers for martingales (see, e.g., [27], P. 16) leads to

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0, \quad a.s.. \quad (3.3)$$

According $x(t) = \prod_{0<t_k<t} g_k y(t)$ and (3.2), we have

$$\begin{aligned}\ln x(t) &= \sum_{0<t_k<t} \ln g_k + \ln y(t) \\ &\leqslant \sum_{0<t_k<t} \ln g_k + \ln x(0) + \int_0^t \Big[\check{h}(r(s)) + \frac{\check{b}^2(r(s))}{4\hat{c}(r(s))}\Big] ds + M(t). \qquad (3.4)\end{aligned}$$

According to lemma 2.2 and (3.3) that

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \leqslant \limsup_{t\to\infty} t^{-1} \sum_{0<t_k<t} \ln g_k + \sum_{i=1}^{m} \pi_i \Big[\check{h}(i) + \frac{\check{b}^2(i)}{4\hat{c}(i)}\Big] = \bar{g} + \bar{h} + \bar{b} < 0.$$

Therefore, $\lim_{t\to\infty} x(t) = 0$, a.s.. □

Now, let's discuss the permanence of system (1.3).

**Assumption 3.1.** *(i) For some $\nu \in S$, $q_{i\nu} > 0$, $\forall i \neq \nu$.*
*(ii)( [32]) There exist a pair of positive constants $l$ and $L$ such that*

$$l \leqslant \prod_{0<t_k<t} g_k \leqslant L, \qquad (3.5)$$

*for $t \in R_+$.*

**Theorem 3.2.** *Let Assumption 3.1 hold. If $\bar{h} > 0$, then solutions of (1.3) are stochastically ultimately upper bounded, where*

$$\bar{h} = \sum_{i=1}^{m} \pi_i \check{h}(i), \;\; \check{h}(i) = \check{a}(i) - 0.5\hat{\sigma}^2(i).$$

**Proof.** Applying the same method that used in [32] to equation (3.1), we have

$$\begin{aligned}\ln x(t) - \ln x(0) = \sum_{0<t_k<t} \ln g_k + \int_0^t \Big[&a(s, r(s)) - 0.5\sigma^2(s, r(s)) \\ &+ b(s, r(s))x(s) - c(s, r(s))x^2(s)\Big] ds + M(t), \qquad (3.6)\end{aligned}$$

where $M(t) = \int_0^t \sigma(s, r(s)) dB(s)$. According equation (3.3), we get that

$$\lim_{t\to+\infty} \frac{M(t)}{t} = 0.$$

Direct calculation reveals that

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \Big[a(s, r(s)) - 0.5\sigma^2(s, r(s))\Big] ds \leq \lim_{t\to+\infty} \frac{1}{t} \int_0^t \Big[\check{a}(r(s)) = \bar{h}$$

and

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \Big[b(s, r(s))x(s) - c(s, r(s))x^2(s)2\Big] ds \leq \lim_{t\to+\infty} \frac{1}{t} \int_0^t \frac{\check{b}^2(r(s))}{4\hat{c}(r(s))} ds = \bar{b}.$$

For arbitrary positive number $\epsilon > 0$, let $\zeta := \bar{h} + \bar{b} + \theta\epsilon \neq 0$, where $\theta$ is a positive constant. Thus, there exists a positive constant $t_\omega$ for all $\omega \in \Omega$ such that

$$\ln x(0) < \frac{1}{4}\theta\epsilon t, \ M(t) < \frac{1}{4}\theta\epsilon t$$

and

$$\int_0^t \Big[a(s, r(s)) - 0.5\sigma^2(s, r(s))\Big] ds < (\bar{h} + \frac{1}{4}\theta\epsilon)t,$$
$$\int_0^t \Big[b(s, r(s))x(s) - c(s, r(s))x^2(s)2\Big] ds < (\bar{b} + \frac{1}{4}\theta\epsilon)t$$

for $t > t_\omega$. Substituting these inequalities in (3.6), we obtain

$$\ln x(t) \leq \sum_{0<t_k<t} \ln g_k + \zeta t \tag{3.7}$$

on $t \in (t_\omega, +\infty)$.

Case1: $\zeta < 0$.

By taking $t_k = k, K > 1, \ g_k = e^{\frac{\ln K}{2^k} - \zeta}, k \in Z_+$, it is easy to verify that

$$\sum_{0<t_k<t} \ln g_k \leq \ln K - \zeta t.$$

From (3.7), we have $\ln x(t) \leq \ln K$ on $t \in (t_\omega, +\infty)$. So

$$\limsup_{t\to+\infty} x(t) \leq K \quad a.s.,$$

which implies that solutions of (1.3) are stochastically ultimately upper bounded.

Case2: $\zeta > 0$.

By taking $t_k = k, K > 1, \ g_k = e^{\frac{\ln K - \zeta}{2^k} - \zeta}, k \in Z_+$, it is easy to verify that

$$\sum_{0<t_k<t} \ln g_k \leq \ln K - \sum_{0<t_k<t} \frac{\zeta}{2^k} - \zeta(t-1).$$

From (3.7), we have

$$\ln x(t) \leq \ln K - \sum_{0<t_k<t} \frac{\zeta}{2^k} + \zeta$$

on $t \in (t_\omega, +\infty)$. So

$$\limsup_{t\to+\infty} x(t) \leq K \quad a.s.,$$

which also implies that the solution of (1.3) is stochastically ultimately upper bounded. □

**Theorem 3.3.** *Let Assumption 3.1 hold. If $\tilde{h} > 0$, then the solution of (1.3) is stochastically ultimately lower bounded, where*

$$\tilde{h} = \sum_{i=1}^m \pi_i \hat{h}(i) \text{ and } \hat{h}(i) = \hat{a}(i) - 0.5\check{\sigma}^2(i).$$

**Proof.** From equation (3.6), a direct calculation reveals that

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t\Big[a(s,r(s))-0.5\sigma^2(s,r(s))\Big]ds \geq \lim_{t\to+\infty}\frac{1}{t}\int_0^t\Big[\hat{a}(r(s))-0.5\check{\sigma}^2(r(s))\Big]ds$$
$$=\sum_{i=1}^m \pi_i(\hat{a}(i)-0.5\check{\sigma}^2(i))=\tilde{h}$$

and

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t\Big[b(s,r(s))x(s)-c(s,r(s))x^2(s)\Big]ds \geq \lim_{t\to+\infty}\frac{1}{t}\int_0^t\Big[\hat{b}(r(s))x(s)$$
$$-\check{c}(r(s))x^2(s)\Big]ds$$
$$\geq \lim_{t\to+\infty}\frac{1}{t}\int_0^t\frac{\hat{b}^2(r(s))}{4\check{c}(r(s))}ds$$
$$=\sum_{i=1}^m \pi_i\frac{\hat{b}^2(i)}{4\check{c}(i)}=\tilde{b}.$$

For arbitrary positive number $\epsilon>0$, let $\varsigma:=\tilde{h}+\tilde{b}-\vartheta\epsilon\neq 0$, where $\vartheta$ is a positive constant. Thus, there exists a positive constant $t_\omega$ for all $\omega\in\Omega$ such that

$$\ln x(0)>-\frac{1}{4}\vartheta\epsilon t,\ M(t)>-\frac{1}{4}\vartheta\epsilon t$$

and

$$\tilde{h}>\tilde{h}-\frac{1}{4}\vartheta\epsilon t,\ \tilde{b}>\tilde{b}-\frac{1}{4}\vartheta\epsilon t$$

for $t>t_\omega$. Substituting these inequalities in (3.6), we abtain

$$\ln x(t)\geq\sum_{0<t_k<t}\ln g_k-\varsigma t \tag{3.8}$$

on $t\in(t_\omega,+\infty)$.

Let $K$ be an arbitrary positive constant.

Case 1: $\varsigma>0$.

By taking $t_k=k$, $g_k=e^{\frac{\ln K}{2^k}-\varsigma}, k\in Z_+$, it is easy to verify that

$$\sum_{0<t_k<t}\ln g_k\geq\sum_{0<t_k<t}\frac{\ln K}{2^k}-\varsigma t.$$

From (3.8),

$$\ln x(t)\geq\sum_{0<t_k<t}\frac{\ln K}{2^k}$$

on $t\in(t_\omega,+\infty)$. So

$$\liminf_{0\to+\infty}x(t)\geq K,\quad a.s.,$$

which implies that the solution of (1.3) is stochastically ultimately lower bounded.

Case 2: $\varsigma<0$.

By taking $t_k = k$, $g_k = e^{\frac{\ln K - \varsigma}{2^k} - \varsigma}, k \in Z_+$, it is easy to verify that

$$\sum_{0<t_k<t} \ln g_k \geq \sum_{0<t_k<t} \frac{\ln K}{2^k} - \sum_{0<t_k<t} \frac{\varsigma}{2^k} - \varsigma t + \varsigma.$$

From (3.8),

$$\ln x(t) \geq \sum_{0<t_k<t} \frac{\ln K}{2^k} - \sum_{0<t_k<t} \frac{\varsigma}{2^k} + \varsigma$$

on $t \in (t_\omega, +\infty)$. So

$$\liminf_{0\to+\infty} x(t) \geq K, \quad a.s.,$$

which implies that the solution of (1.3) is stochastically ultimately lower bounded. □

**Theorem 3.4.** *Let Assumption 3.1 hold. If $\tilde{h} > 0$, then the solution of (1.3) is stochastically permanent a.s..*

**Proof.** Theorems 3.2 and 3.3 show that solutions of (1.3) are both stochastically ultimately upper bounded and stochastically ultimately lower bounded. An easy calculation reveals that

$$\bar{h} > \tilde{h} > 0.$$

That is to say, as long as $\tilde{h} > 0$, solutions of (1.3) are always stochastically permanent a.s.. □

**Remark 3.1.** The autonomous hybrid system of (1.3) without impulsive perturbations has been studied in [11]. Comparing theorems 3.1-3.4 with theorems 1-2 in [11], it can be seen that under Assumption 3.1 and the condition "there exist a pair of positive constants $l$ and $L$ such that $l \leq \prod_{0<t_k<t} g_k \leq L$", solutions of the system with impulsive perturbations in this paper and system without impulsive perturbations in [11] have the same stochastically permanent. That is to say, bounded impulses do not affect the stochastically permanent of the solution.

## 4. Asymptotic properties

**Theorem 4.1.** *If $\bar{h} + \bar{g} > 0$, then the solution of system (1.3) obeys*

$$\limsup_{t\to\infty} x(t) > 0 \quad a.s..$$

**Proof.** If the assertion is not true, then $P\{\omega : \limsup_{t\to\infty} x(t,\omega) = 0\} > 0$. By $x(t) = \prod_{0<t_k<t} g_k y(t)$ and (3.1), we can get

$$\begin{aligned}\frac{\ln x(t)}{t} =& \frac{\ln x(0)}{t} + \frac{1}{t}\sum_{0<t_k<t} \ln g_k + \frac{1}{t}\int_0^t \Big(a(s,r(s)) - 0.5\sigma^2(s,r(s))\Big)ds \\ &+ \frac{1}{t}\int_0^t b(s,r(s))x(s)ds - \int_0^t c(s,r(s))x^2(s)ds + \frac{M(t)}{t}. \end{aligned} \tag{4.1}$$

Note that for $\forall \bar{\omega} \in \{\omega : \limsup_{t\to\infty} x(t,\omega) = 0\}$, $\lim_{t\to\infty} x(t,\bar{\omega}) = 0$. Therefore,

$$\limsup_{t\to\infty} \frac{\ln x(t,\bar{\omega})}{t} \leqslant 0, \ \lim \frac{1}{t}\int_0^t b(s,r(s))x(s,\bar{\omega})ds = 0,$$

$$\lim \frac{1}{t}\int_0^t c(s,r(s))x^2(s,\bar{\omega})ds = 0,\ \lim_{t\to\infty}\frac{M(t)}{t} = 0.$$

Substituting above inequalities into (3.5) and applying (3.3), we can get a contradiction

$$0 \geq \limsup_{t\to\infty}\frac{\ln x(t,\bar{\omega})}{t} = \bar{h} + \bar{g} > 0.$$

The proof is completed. □

**Theorem 4.2.** *The solution $x(t)$ of (1.3) satisfies that*

$$\liminf_{t\to+\infty}\frac{\ln x(t)}{t} \geq \tilde{g} + \tilde{h} + \tilde{b} \quad a.s.,$$

*where*

$$\tilde{g} = \liminf_{t\to+\infty} t^{-1}\sum_{0<t_k<t}\ln g_k,\ \tilde{h} = \sum_{i=1}^{m}\pi_i(\hat{a}(i) - 0.5\check{\sigma}^2(i)),$$

*and*

$$\tilde{b} = \sum_{i=1}^{m}\pi_i\frac{\hat{b}^2(i)}{4\check{c}(i)}.$$

**Proof.** According to equation (4.1), we have

$$\begin{aligned}\frac{\ln x(t)}{t} \geq & \frac{\ln x(0)}{t} + \frac{1}{t}\sum_{0<t_k<t}\ln g_k + \frac{1}{t}\int_0^t(\hat{a}(r(s)) - 0.5\check{\sigma}^2(r(s)))ds\\ & + \frac{1}{t}\int_0^t\hat{b}(r(s))x(s)ds + \frac{1}{t}\int_0^t(\check{c}(r(s))x^2(s)ds + M(t).\end{aligned}$$

Thus

$$\begin{aligned}\liminf_{t\to+\infty}\frac{\ln x(t)}{t} \geq & \liminf_{t\to+\infty}\Big\{\frac{\ln x(0)}{t} + \frac{1}{t}\sum_{0<t_k<t}\ln g_k + \frac{1}{t}\int_0^t[\hat{a}(r(s))\\ & - 0.5\check{\sigma}^2(r(s))]ds + \frac{1}{t}\int_0^t\hat{b}(r(s))x(s)ds\\ & + \frac{1}{t}\int_0^t(\check{c}(r(s))x^2(s)ds + \frac{M(t)}{t}\Big\}.\end{aligned}$$

By using Lemma (2.2) and equation (3.3) we can get that

$$\liminf_{t\to+\infty}\frac{\ln x(t)}{t} \geq \tilde{g} + \tilde{h} + \tilde{b} \quad a.s..$$

The proof is completed. □

**Remark 4.1.** From theorem (3.1) we can get that

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{t} \leq \bar{g} + \bar{h} + \bar{b},$$

where

$$\bar{h}=\sum_{i=1}^{m}\pi_i(\breve{a}(i)-0.5\hat{\sigma}^2(i)),\quad \bar{b}=\sum_{i=1}^{m}\pi_i\frac{\breve{b}^2(i)}{4\hat{c}(i)},\ i\in S$$

and

$$\bar{g}=\limsup_{t\to\infty}t^{-1}\sum_{0<t_k<t}\ln g_k.$$

**Theorem 4.3.** *The solution $x(t)$ of (1.3) satisfies that*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x^2(s)ds\geq\frac{\underline{\kappa}}{\check{c}},\quad a.s.,$$

*where*

$$\underline{\kappa}=\liminf_{t\to\infty}\frac{1}{t}\Big\{\sum_{0<t_k<t}\ln g_k+\int_0^t[a(s,r(s))-0.5\sigma^2(s,r(s))]ds\Big\}.$$

**Proof.** For arbitrary $\epsilon>0$, there exists a positive constant $t_\omega$ for all $\omega\in\Omega$ such that $\ln x(0)>-\frac{1}{3}\epsilon t$, $M(t)>-\frac{1}{3}\epsilon t$ and

$$\sum_{0<t_k<t}\ln g_k+\int_0^t[a(s,r(s))-0.5\sigma^2(s,r(s))]ds>(\underline{\kappa}-\frac{1}{3}\epsilon)t$$

for all $t>t_\omega$. By $x(t)=\prod_{0<t_k<t}g_k y(t)$ and (3.1), we obtain

$$\ln x(t)\geq(\underline{\kappa}-\epsilon)t-\check{c}\int_0^t x^2(s)ds\quad t\in(t_\omega,+\infty).\tag{4.2}$$

Let $u(t)=\int_0^t x^2(s)ds$. It follows from (4.2) that

$$e^{2\check{c}u(t)}u^{'}(t)\geq e^{2(\underline{\kappa}-\epsilon)t}\quad t\in(t_\omega,+\infty).$$

Therefore,

$$e^{2\check{c}u(t)}\geq e^{2\check{c}u(t_\omega)}+\frac{\check{c}}{\underline{\kappa}-\epsilon}\Big(e^{2\underline{\kappa}-\epsilon)t}-e^{2(\underline{\kappa}-\epsilon)t_\omega}\Big).$$

Thus

$$u(t)\geq\frac{1}{2\check{c}}\ln\Big[e^{2\check{c}ut_\omega}+\frac{\check{c}}{\underline{\kappa}-\epsilon}\Big(e^{2(\underline{\kappa}-\epsilon)t}-e^{2(\underline{\kappa}-\epsilon)t_\omega}\Big)\Big].$$

Consequently,

$$\begin{aligned}\liminf_{t\to+\infty}\frac{1}{t}\int_0^t x^2(s)ds&\geq\frac{1}{2\check{c}}\liminf_{t\to+\infty}\frac{1}{t}\ln\Big[e^{2\check{c}ut_\omega}+\frac{\check{c}}{\underline{\kappa}-\epsilon}\Big(e^{2(\underline{\kappa}-\epsilon)t}-e^{2(\underline{\kappa}-\epsilon)t_\omega}\Big)\Big]\\&=\frac{\underline{\kappa}-\epsilon}{\check{c}}.\end{aligned}$$

By virtue of the arbitrariness of $\epsilon$, we easily get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x^2(s)ds\geq\frac{\underline{\kappa}}{\check{c}}.$$

This completes the proof. □

**Theorem 4.4.** *The solution $x(t)$ of (1.3) satisfies that*

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x^2(s)ds \le \frac{2\bar{\kappa}}{\hat{c}}, \quad a.s.,$$

*where*

$$\bar{\kappa}=\limsup_{t\to\infty}\frac{1}{t}\Big\{\sum_{0<t_k<t}\ln g_k+\int_0^t[a(s,r(s))-0.5\sigma^2(s,r(s))]ds+\int_0^t\frac{\check{b}^2(r(s))}{2\hat{c}(r(s))}ds\Big\}.$$

**Proof.** For arbitrary $\epsilon>0$, there exists a positive constant $t_\omega$ for all $\omega\in\Omega$ such that

$$\ln x(0)<\frac{1}{3}\epsilon t,\ \ M(t)<\frac{1}{3}\epsilon t$$

and

$$\sum_{0<t_k<t}\ln g_k+\int_0^t[a(s,r(s))-0.5\sigma^2(s,r(s))]ds+\int_0^t\frac{\check{b}^2(r(s))}{2\hat{c}(r(s))}ds<(\bar{\kappa}+\frac{1}{3}\epsilon)t$$

for all $t>t_\omega$. By $x(t)=\prod_{0<t_k<t}g_k y(t)$ and (3.1), we obtain

$$\ln x(t)\le(\bar{\kappa}+\epsilon)t-\frac{1}{2}\hat{c}\int_0^t x^2(s)ds \quad t\in(t_\omega,+\infty). \tag{4.3}$$

Let $u(t)=\int_0^t x^2(s)ds$. It follows from (4.3) that

$$e^{\hat{c}u(t)}u'(t)\le e^{2(\bar{\kappa}+\epsilon)t}, \quad t\in(t_\omega,+\infty).$$

Therefore,

$$e^{\hat{c}u(t)}\le e^{\hat{c}ut_\omega}+\frac{\hat{c}}{2(\bar{\kappa}+\epsilon)}\Big(e^{2(\bar{\kappa}+\epsilon)t}-e^{2(\bar{\kappa}+\epsilon)t_\omega}\Big).$$

Thus

$$u(t)\le\frac{1}{\hat{c}}\ln\Big[e^{\hat{c}ut_\omega}+\frac{\hat{c}}{2(\bar{\kappa}+\epsilon)}\Big(e^{2(\bar{\kappa}+\epsilon)t}-e^{2(\bar{\kappa}+\epsilon)t_\omega}\Big)\Big].$$

Consequently,

$$\begin{aligned}\limsup_{t\to+\infty}\frac{1}{t}\int_0^t x^2(s)ds&\le\frac{1}{\hat{c}}\limsup_{t\to+\infty}\frac{1}{t}\ln\Big[e^{\hat{c}ut_\omega}+\frac{\hat{c}}{2(\bar{\kappa}+\epsilon)}\Big(e^{2(\bar{\kappa}+\epsilon)t}-e^{2(\bar{\kappa}+\epsilon)t_\omega}\Big)\Big]\\&=\frac{2(\bar{k}+\epsilon)}{\hat{c}}.\end{aligned}$$

By virtue of the arbitrariness of $\epsilon$, we easily get

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x^2(s)ds\le\frac{2\bar{\kappa}}{\hat{c}}.$$

This completes the proof. □

**Theorem 4.5.** *Assume that (3.5) holds, then the solution of system (1.3) obeys*

$$\limsup_{t\to\infty}\frac{\ln x(t)}{\ln t}\leqslant 1 \quad a.s..$$

**Proof.** Applying Itô's formula to Eq. (2.2), one can obtain that

$$
\begin{aligned}
d[e^t \ln y(t)] =& e^t \ln y(t)dt + e^t d\ln y(t) \\
=& e^t\Big[\ln y(t) + a(t,r(t)) - 0.5\sigma^2(t,r(t)) + b(t,r(t)) \prod_{0<t_k<t} g_k y(t) \\
& - c(t,r(t))\Big(\prod_{0<t_k<t} g_k y(t)\Big)^2\Big]dt + e^t\sigma(t,r(t))dB(t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
e^t \ln y(t) - \ln y(0) = \int_0^t e^s\Big[&\ln y(s) + a(s,r(s)) - 0.5\sigma^2(s,r(s)) + b(s,r(s)) \\
&\prod_{0<t_k<s} g_k y(s) - c(s,r(s))\Big(\prod_{0<t_k<s} g_k y(s)\Big)^2\Big]ds + \tilde{M}(t), \quad (4.4)
\end{aligned}
$$

where $\tilde{M}(t) = \int_0^t e^s\sigma(s,r(s))dB(s)$ is a martingale. The quadratic form of $\tilde{M}(t)$ is

$$
\langle \tilde{M}(t), \tilde{M}(t)\rangle = \int_0^t e^{2s}\sigma^2(s,r(s))ds.
$$

By virtue of the exponential martingale inequality (see, e.g., [27], P. 44), we have

$$
P\Big\{\sup_{0\le t\le T}[\tilde{M}(t) - \frac{\alpha}{2}\langle \tilde{M}(t), \tilde{M}(t)\rangle] > \beta\Big\} \le \exp^{-\alpha\beta}.
$$

Choose $T = \mu k, \alpha = e^{-\mu k}, \beta = \rho e^{\mu k}\ln k$. Then it follows that

$$
P\Big\{\sup_{0\leqslant t\leqslant \mu k}[\tilde{M}(t) - 0.5e^{-\mu k}\langle \tilde{M}(t), \tilde{M}(t)\rangle] > \rho e^{\mu k}\ln k\Big\} \leqslant k^{-\rho},
$$

where $\rho > 1$ and $\mu > 0$ are arbitrary. It follows from Borel-Cantelli lemma (see, e.g., [27], P. 7) that for almost all $\omega \in \Omega$, there exists $k_0(\omega)$ such that for every $k \geqslant k_0(\omega)$,

$$
\tilde{M}(t) \leqslant 0.5e^{-\mu k}\langle \tilde{M}(t), \tilde{M}(t)\rangle + \rho e^{\mu k}\ln k, \quad 0 \leqslant t \leqslant \mu k.
$$

That is to say,

$$
\tilde{M}(t) \leqslant 0.5e^{-\mu k}\int_0^t e^{2s}\sigma^2(s,r(s))ds + \rho e^{\mu k}\ln k
$$

for $k \geqslant k_0(\omega)$, $0 \leqslant t \leqslant \mu k$. Substituting this inequality into (4.4) results in

$$
\begin{aligned}
e^t \ln y(t) - \ln y(0) \leqslant \int_0^t e^s\Big[&\ln y(s) + a(s,r(s)) - 0.5\sigma^2(s,r(s)) \\
&+ b(s,r(s))\prod_{0<t_k<s} g_k y(s) - c(s,r(s))\Big(\prod_{0<t_k<s} g_k y(s)\Big)^2\Big]ds \\
&+ 0.5\int_0^t e^s e^{s-\mu k}\sigma^2(s,r(s))ds + \rho e^{\mu k}\ln k
\end{aligned}
$$

$$\leqslant \int_0^t e^s \big[\ln y(s) + h(r(s)) + \check{b}(r(s))Ly(s) - \hat{c}(r(s))l^2y^2(s) + 0.5\check{\sigma^2}(r(s))\big]ds + \rho e^{\mu k}\ln k$$

for all $0 \leqslant s \leqslant \mu k$. Note that for $y > 0$, there is a positive constant $\bar{C}$ independent of $k$ such that

$$\ln y(s) + h(r(s)) + 0.5\check{\sigma}^2(r(s)) + \check{b}(r(s))Ly(s) - \hat{c}(r(s))l^2y^2(s) \leqslant \bar{C}.$$

That is to say, for any $0 \leqslant t \leqslant \mu k$, we get

$$e^t \ln y(t) - \ln y(0) \leqslant \bar{C}(e^t - 1) + \rho e^{\mu k}\ln k.$$

Therefore, if $\mu(k-1) \leqslant t \leqslant \mu k$ and $k \geqslant k_0(\omega)$, one can see that

$$\frac{\ln y(t)}{\ln t} \leqslant e^{-t}\frac{\ln y(0)}{\ln t} + \frac{\bar{C}(1-e^{-t})}{\ln t} + \rho e^{-\mu(k-1)}e^{\mu k}\frac{\ln k}{\ln t}.$$

Consequently,

$$\limsup_{t\to\infty}\frac{\ln y(t)}{\ln t} \leqslant \rho e^{\mu}.$$

Letting $\rho \to 1$ and $\mu \to 0$, one can derive that

$$\limsup_{t\to\infty}\frac{\ln y(t)}{\ln t} \leqslant 1.$$

Moreover, it follows from condition (3.5) that

$$\lim_{t\to\infty}\frac{\prod_{0<t_k<t} g_k}{\ln t} = 0. \tag{4.5}$$

Finally, we have

$$\limsup_{t\to\infty}\frac{\ln x(t)}{\ln t} = \limsup_{t\to\infty}\frac{\sum_{0<t_k<t}\ln g_k + \ln y(t)}{\ln t} = \limsup_{t\to\infty}\frac{\ln y(t)}{\ln t} \leqslant 1.$$

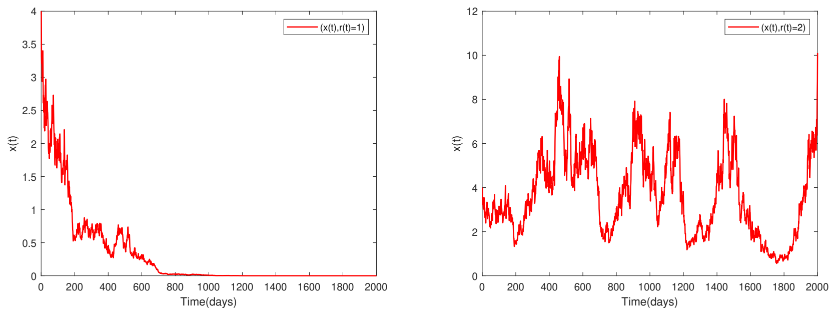
This completes the proof. □

# 5. Numerical simulations

In this section, we will present computer simulations to support our results and illustrate the effects of the Allee effect, Markovian switching and impulsive perturbations on survival of species by several examples. Without loss of generality, we hypothesize the switching regimes $S = \{1, 2\}$.

We first discuss the effects of Markovian switching on the survival of population. Consider subsystems

$$\begin{cases} dx(t) = x(t)[a(t,1) + b(t,1)x(t) - c(t,1)x^2(t)]dt + \sigma(t,1)x(t)dB(t), \\ \qquad t \neq t_k, k \in Z_+, \\ x(t_k^+) = g_k x(t_k), k \in Z_+ = \{1, 2, \dots\}, \\ x(0^+) = x(0) > 0 \end{cases} \tag{5.1a}$$




(a) (b)

**Figure 1.** Figure (a) represents the trajectory of the solution of subsystem (5.1a). It is shown that population $x$ will die out about at t=700. Figure (b) represents the trajectory of the solution of subsystem(5.1b) and shows that the population will be permanent. All parameters are taken from Table 1.

and

$$\begin{cases} dx(t) = x(t)[a(t,2) + b(t,2)x(t) - c(t,2)x^2(t)]dt + \sigma(t,2)x(t)dB(t), \\ \qquad t \neq t_k, k \in Z_+, \\ x(t_k^+) = g_k x(t_k), k \in Z_+ = \{1, 2, \dots\}, \\ x(0^+) = x(0) > 0. \end{cases} \tag{5.1b}$$

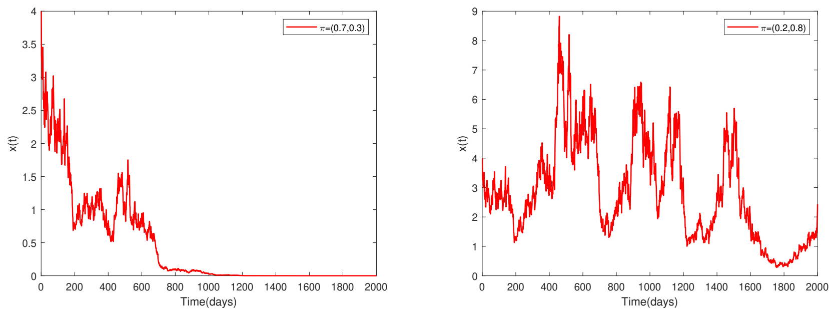
**Table 1.** Parameter values.

| Parameters | Values of state 1 | Values of state 2 |
|---|---|---|
| $a(t)$ | $a(t,1) = 0.35 + 0.01\sin t$ | $a(t,2) = 0.35 + 0.01\sin t$ |
| $b(t)$ | $b(t,1) = 0.5 + 0.01\sin t$ | $b(t,2) = 0.6 + 0.05\sin t$ |
| $c(t)$ | $c(t,1) = 0.1 + 0.01\sin t$ | $c(t,2) = 0.1 + 0.01\sin t$ |
| $\sigma(t)$ | $\sigma(t,1)=\sqrt{0.802} + 0.2\cos 2t$ | $\sigma(t,2)=\sqrt{0.802} + 0.2\cos 2t$ |
| $g_k$ | $g_k(1)$=0.9 | $g_k(2)$=0.9 |

**Example 5.1.** Table 1 lists values of parameters in model (5.1). We set the initial value as $x(0) = 4$, and let $t_k = 10k, k \in Z_+$. By simple calculation, we obtain that subsystem (5.1a) is extinct (see Fig.1(a)) and subsystem (5.1b) is permanent (see Fig.1(b)). It shows that state 1 is an extinct state and state 2 is a permanent state.

**Case 5.1.1.** Let the generator of $r(t)$ be $Q = \begin{pmatrix} -3 & 3 \\ 7 & -7 \end{pmatrix}$, by the irreducible property, we can get that the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2) = (0.7, 0.3)$. Then by simple calculation, we get $\bar{h} + \bar{b} + \bar{g} < 0$. In view of Theorem 3.1 we obtain that population $x$ will go extinct.

**Case 5.2.2.** Let the generator of $r(t)$ be $Q = \begin{pmatrix} -4 & 4 \\ 1 & -1 \end{pmatrix}$, by the irreducible

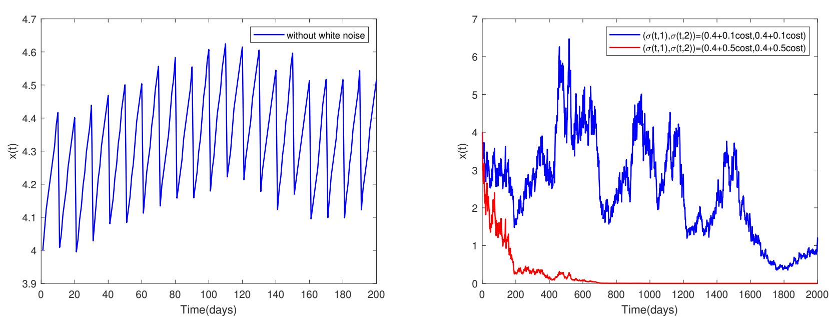



(a) (b)

**Figure 2.** Trajectories of system (5.1) switching between states 1 and 2. Figure (a) represents the trajectory of the stationary distribution $\pi = (\pi_1, \pi_2) = (0.7, 0.3)$. It is shown that shows population $x$ will die out about at t=1000. Figure (b) represents the trajectory of the solution of the stationary distribution $\pi = (\pi_1, \pi_2) = (0.2, 0.8)$ and shows that the population will be permanent. All parameters are taken from Table 1.

property, we can get that the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2) = (0.2, 0.8)$. Then by simple calculation, we get $\tilde{h} > 0$. In view of Theorem 3.4 we obtain that population $x$ will go permanent.

Obviously, the population $x$ of the hybrid system in **Case 5.1.1** will be extinct. However, the population will survive longer than the system without Markov switching (see Fig. 1(a) and Fig. 2(a)). **Case 5.1.2** shows that if the population survives in a persistent state for a longer time, the population will go permanent (see Fig.1(a) and Fig.2 (b)). That is to say, the Markov switching is conducive to the survival of the population.




(a) (b)

**Figure 3.** Trajectories of system (5.1) switching between states 1 and 2, the stationary distribution of Markov chain is $\pi = (\pi_1, \pi_2) = (0.7, 0.3)$. Figure (a) shows the trajectory without the white noise. Figure (b) represents the trajectory with the white noise intensity is $(\sigma(t,1), \sigma(t,2)) = (0.5 - 0.1\cos t, 0.4 + 0.1\cos t)$ and $(\sigma(t,1), \sigma(t,2)) = (0.8 - 0.1\cos t, 0.8 + 0.1\cos t)$. All parameters are taken from Table 1.

**Example 5.2.** Set $(\sigma(t,1), \sigma(t,2)) = (0,0)$ (i.e., there is no white noise), we obtain $\bar{h}+\bar{b}+\bar{g} > 0$. Hence, population $x$ will be permanent (see Fig. 3 (a)). However, if we take the white noise into account by setting$(\sigma(t,1),\sigma(t,2)) = (0.4+0.1\cos t, 0.4+0.1\cos t)$ $((\sigma(t,1),\sigma(t,2)) = (0.4+0.5\cos t, 0.4+0.5\cos t))$ and other parameters are chosen from Table 1. It can be easily calculated we see that the white noise is not conducive to the survival of the population (see Fig.2 (a) and Fig.3 (a)). The greater the intensity of white noise interference, the more unfavorable it is to the survival of the population (see Fig. 3(b)).

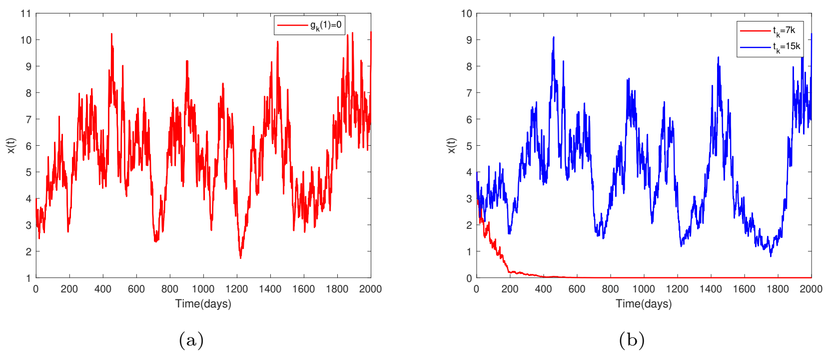



**Figure 4.** Trajectories of system (5.1). Figure (a) represents the trajectory of the solution of subsystem (5.1a). It shows that population $x$ will be persistent without impulsive perturbations. Figure (b) represents the trajectory of the system (5.1) switching between state 1 and state 2 at different pulse time intervals. The stationary distribution of Markov chain is $\pi = (\pi_1, \pi_2) = (0.7, 0.3)$ and all parameters are taken from Table 1.

Then, we discuss the effect of impulsive perturbations on population dynamics through numerical simulation.

**Example 5.3.** We set $g_k(1) = 0$, $x(0) = 4$, $t_k = 10k$ for all $k \in Z^+$ in subsystem (5.1a) and other parameters are chosen from Table 1 (i.e., there is no impulsive perturbations), which has been studied in [11]. By simple calculation we can obtain $x$ will go permanent (see Fig.4 (a)).

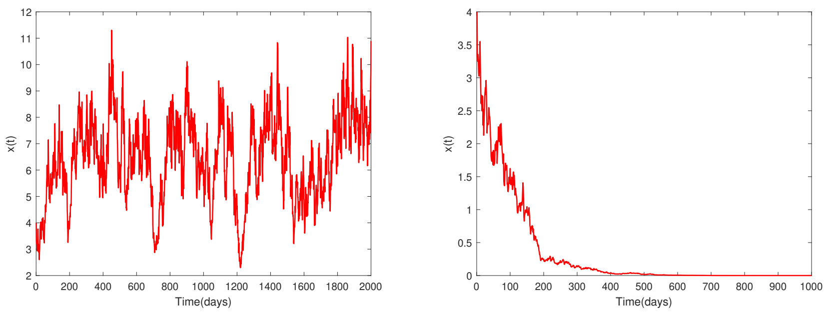
**Case 5.3.1.** Set $g_k(1) = 1.05$, $g_k(2) = 0.8$, $x(0) = 4$, $t_k = 10k$ for all $k \in Z^+$ in system (5.1) and other parameters are chosen from Table 1. By computation, we get the population $x$ in subsystem (5.1a) will be permanent (see Fig.5(a)). It shows that the extinction state has become a permanent state. On the other hand, the population $x$ in subsystem (5.1b) will be extinct and represents that the persistent state has become an extinct state (see Fig.5(b)).

That is to say, impulsive perturbations has a great influence effect on the dynamic behavior of the population.

Finally, we analyze the impact of Allee effect on population survival.

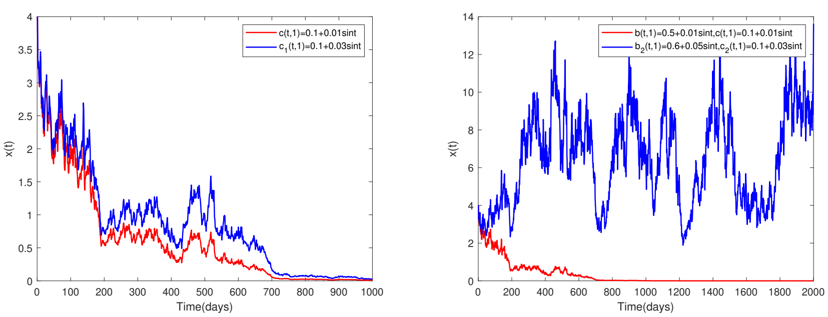
**Example 5.4.** Set $b_1(t,1) = 0.5+0.01\sin t$, $c_1(t,1) = 0.1+0.03\sin t$ and $b_2(t,1) = 0.6+0.05\sin t$, $c_2(t,1) = 0.1+0.03\sin t$, $x(0) = 4$, $t_k = 10k$ for all $k \in Z^+$ in system (5.1a) and other parameters are chosen from Table 1.

That is to say, when the population density is low, intra-specific competition among populations will decrease and intra-specific cooperation will increase. On the one hand, the role of intra-specific competition is reduced, and individuals get more survival resources, which is conducive to survival, (see Fig. 6(a)). On the other hand, the interspecific cooperation is enhanced, and the population is easier to survive the crisis (see Fig. 6(b)). Therefore, Allee effect can promote the survival of the population to a certain extent.




(a) (b)

**Figure 5.** Figure (a) represents the trajectory of the solution of subsystem (5.1a) with $g_k(1) = 1.05$. It is shown that population $x$ will go permanent. Figure (b) represents the trajectory of the solution of subsystem(5.1b) with $g_k(2) = 0.8$ and shows that the population will be extinct. All parameters are taken from Table 1.




(a) (b)

**Figure 6.** Figures represent the trajectory of the solution of subsystem (5.1a).

# 6. Discussion

This paper devotes to studying the dynamic behavior of a new stochastic nonautonomous hybrid population model with Allee effect, Markovian switching and impulsive perturbations. We first discuss the threshold of extinction and permanence. Then we investigate the asymptotic properties of the system. Finally, we illustrate numerically the correction of the results and the effects of the Allee effect, Markovian switching and impulsive perturbations on the survival of the population.

(1) **Markov chain $r(t)$ in this stochastic system can contribute to the survival of the population and reduce the risk of extinction.** (i) At least one of the two switched States needs to be persistent state; (ii) As a result of Markov

switching, on the one hand, extinction can be delayed (inhibited), see Fig.1 (a) and Fig.2 (a). On the other hand, if the species live longer in a good state, the whole population can survive for a long time, as shown in Fig.2 (b). That is to say, the species can obtain more survival time by switching between different states.
(2) **White noise is harmful to the survival of species.** It follows from Figs.3 (a) and 3 (b) that white noise has a great influence on the survival of population. White noise is harmful to the survival of the population (see Figs.3 (a) and 3 (b)), and the larger the intensity of the white noise, the faster of the extinction of the population.
(3) **Impulse intensity affect the survival of the species.** We can find that when the impulse coefficients $g_k < 1$, it is harmful to the survival of the population (see Fig.4 (a) and Fig.5 (b)). Otherwise, the impulse coefficients $g_k > 1$, it is beneficial to the survival of the population (see Fig.5 (a)). On the other hand, the time interval of impulse also affects the dynamic behavior of the population. When the pulse coefficient $g_k < 1$, the larger the impulse interval is, the better the survival of the population is (see Fig.4 (b)).
(4) **Allee effect can promote the survival of the population in a certain range.** On the one hand, the role of intra-specific competition is reduced, and individuals get more survival resources, which is conducive to survival, (see Fig.6 (a)). On the other hand, the interspecific cooperation is enhanced, and the population is easier to survive the crisis (see Fig.6 (b)).

Although all parameters are time-dependent, this will be more in line with the real survival situation of the population. However, there are still deficiencies in the process of mathematical modeling and mathematical theory derivation. The difficulty in dealing with time-dependent parameters in the process of theoretical proof is that the parameters can only take the upper and lower bounds directly. It is better to have conditions similar to the integral form [10], which is also the direction of our efforts.

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**Data availability statements.** All data generated or analysed during this study are included in this article.

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