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TIME-DEPENDENT ASYMPTOTIC BEHAVIOR OF THE WAVE EQUATION WITH STRONG DAMPING ON $\mathbb{R}^{N}$*

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Abstract We study the longtime dynamics of non-autonomous wave equations with strong damping in the case of critical nonlinearity. First of all, when $1 \leq p \leq p^* = \frac{N+2}{(N-2)_+}$, we get the well-posedness of strong damped equation with dime-dependent decay coefficient in $\mathcal{H}_t = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, and prove the quasi-stability of weak solution in $\mathcal{H}_{t,-1} = H^1(\mathbb{R}^N) \times H^{-1}(\mathbb{R}^N)$. Then the time-dependent attractor is proved in $\mathcal{H}_t$. Finally, by using the quasi-stability of weak solution, we establish the existence the pullback exponential attractor for non-autonomous dynamical system $(U(t,\tau), \mathcal{H}_t, \mathcal{H}_{t,-1})$.

Keywords Wave equation, critical exponent, well-posedness, time-dependent attractor, pullback exponential attractor.

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1. Introduction

We consider the time-dependent attractor of the wave equation with strong damping on $\mathbb{R}^N$, that is,

$$\begin{cases} \varepsilon(t)u_{tt} - \triangle u - \triangle u_t + \lambda u + f(u) = g(x), \quad x \in \mathbb{R}^N,\ t > \tau,\ \tau \in \mathbb{R}, \\ u(x,\tau) = u_0(x),\ u_t(x,\tau) = u_1(x),\ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where the unknown variable $u = u(x,t) : \mathbb{R}^N \times [\tau, \infty) \to \mathbb{R},\ \lambda > 0$. We assume that $\varepsilon(\cdot) \in C^2(\mathbb{R})$, and

$$\lim_{t\to+\infty} \varepsilon(t) \to 0,\ \sup_{t\in\mathbb{R}}[|\varepsilon(t)| + |\varepsilon'(t)| + |\varepsilon''(t)|] \leq L, \tag{1.2}$$

here L is a proper constant.

The nonlinear term $f \in C^2(\mathbb{R}),\ f(0) = 0$, and

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{1.3}$$

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where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\triangle$, and the following conditions hold:

$(H1)$ if $N = 1$, then the growth of f is arbitrary;

$(H2)$ if $N = 2$, then

$$|f'(u) \leq C(1 + |u|^{p-1}) \quad \text{for some } p \geq 1; \tag{1.4}$$

$(H3)$ if $N \geq 3$, then

$$|f'(u)| \leq C(1 + |u|^{p-1}) \quad \text{with some } 1 \leq p \leq p^* = \frac{N+2}{(N-2)^+}, \tag{1.5}$$

where C is a positive constant and $s_+ = (s + |s|)/2$.

(1.1) is found a possible application within the theory of type III proposed by Green and Naghdi in the last two decades [13–15, 17, 24]. For more details about the derivation of the physical model we refer the reader to [10].

When $\varepsilon(t)$ is a positive constant independent of time t, the system is a classical autonomous strongly damped wave equation. In particular, for the IBVP of the type of equation (1.1) on a bounded domain $\Omega \subset \mathbb{R}^N$, there have been a lot of well-posedness results in the literatures (see for instance[5, 6, 9, 22]). The existence of regular global attractor for the nonlinear strongly damped wave equation (1.1) within the critical growth condition (1.5) on $f(u)$ was well known in the literatures such as [3, 7, 16]. In [25] the authors investigated non-autonomous Kirchhoff wave model with strong damping in a bounded domain Ω in $\mathbb{R}^N (N \geq 3)$, in which they showed that when the growth exponent p of the nonlinearity $f(u)$ is up to the critical range: $1 \leq p \leq p^* \equiv \frac{N+2}{N-2}$ $(N \geq 3)$, the related non-autonomous dynamical system possessed a pullback attractor $\mathcal{A}_\epsilon = \{A_\epsilon(t)\}_{t\in\mathbb{R}}$ for each $\epsilon \geq 0$, and then they proved the upper-semicontinuity of pullback attractor. But under the same growth of nonlinearity $f(u)$ as [25] and coefficients is dependent on time, there are no any results of asymptotic behavior for the wave equation with strong damping on $\mathbb{R}^N$.

When $\varepsilon(t)$ is a positive decreasing function and vanishes at positive infinity, the problem (1.1) becomes more complex and interesting. One of the reason is that the dynamical system associated with (1.1) is still understood under the non-autonomous framework even through the forcing term is not dependent on time t. In order to deal with these problems, in [8], Conti, Pata and Temam presented a notion of time-dependent global attractor exploiting the minimality with respect to the pullback attraction property, and gave a sufficient condition proving the existence of time-dependent attractor based on the theory established by Plinio, Duane and Temam ([11]); besides, they applied the abstract results into the following weak damped wave equations with time-dependent speed of propagation

$$\varepsilon(t)u_{tt} + \alpha u_t - \Delta u + f(u) = g(x).$$

After that, this method was applied to the damped wave equation with linear memory(see for instance[20]). But, when the domains are unbounded, such attractors are not yet well understood. The reason is that when Ω is unbounded, the compactness of the Sobolev embedding which is indispensable for constructing the global attractor is lost. In order to move this obstacle, several remedies for the evolution equation on an unbounded domain have been found. Babin and Vishik in [1] first showed the existence of attractors to the equations of parabolic type in weighted Sobolev spaces. In 1999, Wang introduced the method of ¡°tail estimate¡± to scrutinize the existence of global attractor for reaction-diffusion equations on unbounded domains([26]). It is worthy mentioned that the tail estimate method has been extensively used in dealing with case of unbounded domains. For example, Yang and Ding [28] studied the longtime dynamics of Kirchhoff equation with strong damping and critical nonlinearity. Liu and Ma [18] achieved the existence of time-dependent attractor for the plate equation on $H^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ by using the tail estimate along with the asymptotic contractive process on the time-dependent entire space. Inspired by the idea of [18, 19, 28], we are interested in existence of time-dependent global attractor and pullback exponential attractor for equation (1.1) in $\mathbb{R}^N$.

The main purpose of this paper is to solve the following questions. Firstly, we will show that problem (1.1) has a unique weak solution in natural energy space $\mathcal{H}_t$ when the growth exponent p of the nonlinearity $f(u)$ is up to the critical range: $1 \le p \le p^*$. Secondly, we need to overcome some difficulties caused by time dependent coefficient in the calculus process to obtain the quasi-stability in weaker space $\mathcal{H}_{t,-1}$ as well as the regularity of u_t. Thirdly, we will prove that the process $U(t,\tau)$ is pullback asymptotically compact in $\mathcal{H}_t$. Therefore, we can show that it has a time-dependent attractor $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$. Finally, based on the criterion of pullback exponential attractor developed in [27], and by using the quasi-stability of weak solution, we investigate the existence of pullback exponential attractor $\mathcal{M} = \{M(t)\}_{t\in\mathbb{R}}$ about the non-autonomous dynamical system $(U(t,\tau), \mathcal{H}_t, \mathcal{H}_{t,-1})$.

The paper is organized as follows. In Section 2, we make some preparations for our consideration. In Section 3, when $1 \le p \le p^*$, we give some results on the well-posedness of problem (1.1). In Section 4, we obtain the existence of time-dependent attractor in phase space $\mathcal{H}_t$. Finally, we establish the existence of pullback exponential attractor $\mathcal{M} = \{M(t)\}_{t\in\mathbb{R}}$ about the non-autonomous dynamical system $(U(t,\tau), \mathcal{H}_t, \mathcal{H}_{t,-1})$ in Section 5.

2. Preliminaries

In this section, we iterate some notations and abstract results.

Without loss of generality, set $H = L^2(\mathbb{R}^N)$, equipped with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. $L^p = L^p(\mathbb{R}^N)$, $W^{s,p} = W^{s,p}(\mathbb{R}^N)$, $\int = \int_{\mathbb{R}^N}$, $\|\cdot\|_p = \|\cdot\|_{L^p}$, $\|\cdot\| = \|\cdot\|_{L^2}$. We define the time-dependent space

$$\mathcal{H}_t = H^1 \times L^2, \quad \mathcal{H}_{t,-1} = H^1 \times H^{-1}, \quad \mathcal{H}_{t,1} = H^1 \times H^1$$

endowed with norm

$$\|\{a,b\}\|^2_{\mathcal{H}_t} = \|a\|^2_{H^1} + \varepsilon(t)\|b\|^2, \quad \|a\|^2_{H^1} = \|\nabla a\|^2 + \|a\|^2.$$

For every $t \in \mathbb{R}$, we introduce the R-ball of Banach space X_t

$$\mathbb{B}_t(R) = \{z \in X_t : \|z\|_{X_t} \le R\}.$$

For any given $\delta > 0$, the $\delta-$neighborhood of a set $B \subset X_t$ is defined as

$$\mathcal{O}_t^\delta(B) = \bigcup_{x\in B}\{y \in X_t : \|x-y\|_{X_t} \le \delta\} = \bigcup_{x\in B}\{x + \mathbb{B}_t(\delta)\}.$$

We denote the Hausdorff semidistance of two (nonempty) sets $B, C \subset X_t$ by

$$dist_{X_t}(B,C) = \sup_{x\in B} dist_{X_t}(x,C) = \sup_{x\in B}\inf_{y\in C} \|x-y\|_{X_t}.$$

Finally, given any set $B \subset X_t$, the symbol $\mathbb{B}$ stands for the closure of B in X_t.

In this paper any positive constant denoted by C (or c)which may be different from line to line and even in the same line and Q is a generic positive increasing function.

Definition 2.1 ([4]). A function $u(t)$ is said to be a weak solution to (1.1) on an interval $[0,T]$ if

$$u \in L^\infty(0,T;H^1(\mathbb{R}^N)), \quad u_t \in L^\infty(0,T;L^2(\mathbb{R}^N)) \cap L^2(0,T;H^1(\mathbb{R}^N)) \tag{2.1}$$

and (1.1) is satisfied in the sense of distributions.

Definition 2.2 ([8]). Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. A process is a two parameter family of mappings $U(t,\tau): X_\tau \to X_t,\ t \geq \tau,\ t,\tau \in \mathbb{R}$ with properties

(i) $U(\tau,\tau) = Id$ is the identity operator on $X_\tau,\ \tau \in \mathbb{R}$,

(ii) $U(t,s)U(s,\tau) = U(t,\tau), \quad \forall t \geq s \geq \tau, \tau \in \mathbb{R}$.

Definition 2.3 ([8]). A family $\mathcal{C} = \{C_t\}_{t\in\mathbb{R}}$ of bounded sets $C_t \subset \mathcal{H}_t$ is called uniformly bounded if there exists $R > 0$ such that

$$C_t \subset \{\zeta \in \mathcal{H}_t : \|\zeta\|_{\mathcal{H}_t} \leq R\}, \qquad \forall t \in \mathbb{R}.$$

Definition 2.4 ([8]). A family $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ is called pullback absorbing if it is uniformly bounded and, for every $R > 0$,there exists $t_0 = t_0(t,R) \leq t$ such that

$$\tau \leq t_0 \Rightarrow U(t,\tau)\mathbb{B}_\tau(R) \subset B_t, \tag{2.2}$$

the process $U(t,\tau)$ is called dissipative whenever it admits a pullback absorbing family.

Definition 2.5 ([8]). A(uniformly bounded)family $\mathcal{K} = \{k_t\}_{t\in\mathbb{R}}$ is called pullback attracting if for all $\varepsilon > 0$ the family $\{\mathcal{O}_t^\varepsilon(K_t)\}_{t\in\mathbb{R}}$ is pullback absorbing.

Corollary 2.1. *The attracting property can be equivalently stated in terms of the Hausdorff semidistance: $\mathcal{K} = \{K_t\}_{t\in\mathbb{R}}$ is pullback attracting if and only if it is uniformly bounded and the limit*

$$\lim_{\tau\to-\infty} dist_{\mathcal{H}_t}(U(t,\tau)C_\tau, K_t) = 0,$$

holds for every uniformly bounded family $\mathcal{C} = \{C_t\}_{t\in\mathbb{R}}$ and every fixed $t \in \mathbb{R}$.

Definition 2.6 ([8]). A time-dependent absorbing set for the process $U(t,\tau)$ is a uniformly bounded family $\mathcal{B} = \{B_t\}_{t\in\mathbb{B}}$ with the following property: for every $R \geq 0$ there exists $\theta_e = \theta_e(R) \geq 0$ such that

$$\tau \leq t - \theta_e \Rightarrow U(t,\tau)\mathbb{B}_\tau(R) \subset B_t.$$

Definition 2.7. ([8]) The time-dependent global attractor for $U(t,\tau)$ is the smallest family $\mathcal{U} = \{A_t\}_{t\in\mathbb{R}}$ such that

(i) each A_t is compact in $\mathcal{H}_t$;

(ii) A_t is pullback attracting. i.e.,it is uniformly bounded and the limit

$$\lim_{\tau\to-\infty} dist_{\mathcal{H}_t}(U(t,\tau)C_\tau, A_t) = 0$$

holds for every uniformly bounded family $\mathcal{C} = \{C_t\}_{t\in\mathbb{R}}$ and every fixed $t \in \mathbb{R}$.

Definition 2.8 ([8])**.** A family of uniformly bounded sets $\mathcal{C} = \{C_t\}_{t\in\mathbb{R}}$ is called invariant if

$$U(t,\tau)C_\tau = C_t, \quad \forall t \geq \tau,\ \tau \in \mathbb{R}.$$

Definition 2.9. ([22]) A function $z : t \mapsto u(t) \in \mathcal{H}_t$ is a complete bounded trajectory(CBT)of $U(t,\tau)$ if and only if

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{\mathcal{H}_t} < \infty$$

and

$$u(t) = U(t,\tau)u(\tau), \qquad \forall t \geq \tau,\ \tau \in \mathbb{R}.$$

Lemma 2.1 ([22])**.** *Let Φ be an absolutely continuous positive function on $\mathbb{R}^+$, which satisfies for some $\varepsilon > 0$ the differential inequality:*

$$\frac{d}{dt}\Phi(t) + 2\varepsilon\Phi(t) \leq g(t)\Phi(t) + h(t), \qquad t \in \mathbb{R}^+,$$

where $h \in L^1_{loc}(\mathbb{R}^+)$, $\int_\tau^t g(s)ds \leq \varepsilon(t-\tau) + m$ for $t > \tau$ and some $m > 0$. Then

$$\Phi(t) \leq e^m(\Phi(0)e^{-\varepsilon t} + \int_\tau^t |h(s)|e^{-\varepsilon(t-\tau)}ds), t > 0.$$

Lemma 2.2 ([22])**.** *Let X be a Banach space, and let $Z \subset C(\mathbb{R}^+, X)$, Let $\Phi : X \to \mathbb{R}$ be a function such that*

$$\sup_{t\in\mathbb{R}^+} \Phi(z(t)) \geq -\eta,\ \ \Phi(z(0)) \leq K,$$

for some $\eta, K > 0$ and every $z \in Z$. In addition, assume that for every $z \in Z$ the function $t \mapsto \Phi(z(t))$ is continuously differentiable, and satisfies the differential inequality

$$\frac{d}{dt}\Phi(z(t)) + \delta\|z(t)\|_X^2 \leq k$$

for some $\delta > 0$, and $k \geq 0$ independent of $z \in Z$. Then, for every $\gamma > 0$ there exists $t_0 = \frac{\eta+K}{\gamma} > 0$ such that

$$\Phi(z(t)) \leq \sup_{\zeta\in X}\{\Phi(\zeta) : \delta\|\zeta\|_X^2 \leq k + \gamma\},\ \ t \geq t_0.$$

Lemma 2.3 ([23])**.** *Let $X,\ B$ and Y be Banach spaces, $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$,*

$$W = \{u \in L^p(0,T;X) \mid u_t \in L^1(0,T;Y)\}, 1 \leq p < \infty,$$
$$W_1 = \{u \in L^\infty(0,T;X) \mid u_t \in L^r(0,T;Y)\}, r > 1.$$

Then

$$W \hookrightarrow\hookrightarrow L^p(0,T;B), W_1 \hookrightarrow\hookrightarrow C([0,T];B).$$

Lemma 2.4 ([24]). *Let X, Y be two Banach spaces such that $X \hookrightarrow Y$. If $\phi \in L^\infty(0,T;X) \cap C_w([0,T];Y)$, then $\phi \in C_w([0,T];X)$.*

Lemma 2.5 ([25]). *Let the family $\mathcal{D} = \{D(t)\}_{t\in\mathbb{R}}$ be a pullback $\mathcal{D}$-absorbing family of the process $U(t,\tau)$. And assume that for any $\delta > 0$ and $t \in \mathbb{R}$, there exist a $\tau = \tau(t,\delta,\mathcal{D}) > 0$ and a contractive functional $\Psi_{t,\tau}(\cdot,\cdot)$ defined on $D(t-\tau)\times D(t-\tau)$ such that*

$$\|U(t,t-\tau)x - U(t,t-\tau)y\|_X \leq \delta\Psi_{t,\tau}(x,y), \forall x,y \in D(t-\tau).$$

Then the process $U(t,\tau)$ is pullback $\mathcal{D}$-asymptotically compact in X.

3. Well-posedness

Theorem 3.1. *Let assumptions (1.2)-(1.5) be in force and $(u_0,u_1) \in \mathcal{H}_\tau$. Then for every $\tau \in \mathbb{R}$, and $\tau < t$, problem (1.1) has a unique weak solution $u(t)$. This solution possesses the following propertries:*
(i) The function $t \mapsto (u(t);u_t(t))$ is continuous in $\mathcal{H}_t$ and

$$u_{tt} \in L^2(\tau,t;H^{-1}) + L^\infty(\tau,t;L^{1+\frac{1}{p}}). \tag{3.1}$$

Moreover, there exists a constant $C(\rho) > 0$ such that

$$\varepsilon(t)\|u_t\|^2 + \|u(t)\|^2_{H^1} + \int_\tau^t \|u_t(r)\|^2_{H^1}dr \leq C(\rho), \tag{3.2}$$

for initial data $\|(u_0,u_1)\|_{\mathcal{H}_\tau} \leq \rho$. We also have the following additional regularity:

$$u_t \in L^\infty(\tau,t;H^1), \quad u_{tt} \in L^\infty(\tau,t;H^{-1}) \cap L^2(\tau,t;L^2),$$

for every $\tau < t$, we have

$$\varepsilon(t)\|u_{tt}\|^2 + \|u_t\|^2_{H^1} \leq \frac{C(\rho,T)}{(t-\tau)^2}, \tag{3.3}$$

where as above $\|(u_0,u_1)\|_{\mathcal{H}_\tau} \leq \rho$.
(ii) The following energy identity

$$E(\xi_u(t)) + \int_s^t (\|\nabla u_t(r)\|^2 - \frac{\varepsilon'(r)}{2}\|u_t(r)\|^2)dr = E(\xi_u(s)), \tag{3.4}$$

holds for every $t > s \geq \tau$, where $\xi_u(t) = (u,u_t)$ and

$$E(\xi_u(t)) = \varepsilon(t)\|u_t\|^2 + \|\nabla u\|^2 + \lambda\|u\|^2 + 2\langle F(u),1\rangle - 2\langle g(x),u\rangle. \tag{3.5}$$

(iii) If $u^1(t)$ and $u^2(t)$ are two weak solutions such that $\|(u^i(\tau),u^i_t(\tau))\|_{\mathcal{H}_\tau} \leq R$, $i = 1,2$, then there exists $b(\rho) > 0$ such that the difference $z(t) = u^1(t) - u^2(t)$ satisfies the relation

$$\|(z(t),z_t(t))\|^2_{\mathcal{H}_{t,-1}} + \int_\tau^t \|z_t(r)\|^2dr \leq b(\rho)(\|(z(\tau),z_t(\tau))\|^2_{\mathcal{H}_{\tau,-1}}), \tag{3.6}$$

for all $\tau < t$, and quasi-stability

$$\begin{aligned}\|(z(t),z_t(t))\|^2_{\mathcal{H}_{t,-1}} \leq& e^{-k(t-\tau)}\|(z(\tau),z_t(\tau))\|^2_{\mathcal{H}_{\tau,-1}} \\ &+ b(\rho)\int_\tau^t (\|z(s)\|^2 + \varepsilon(t)\|z_t(s)\|^2_{H^{-2}})ds,\end{aligned} \tag{3.7}$$

where $k > 0$ is a small constant.

Proof. Let $\Omega = \Omega_R$ be a ball in $\mathbb{R}^N$ with radius R. We first consider problem (1.1) on Ω:

$$\begin{cases} \varepsilon(t)u_{tt} - \triangle u - \triangle u_t + \lambda u + f(u) = g(x), \quad x \in \Omega,\ t > \tau,\ \tau \in \mathbb{R}, \\ u \mid_{\partial\Omega} = 0,\ t > \tau, \\ u(x,\tau) = \bar{u}_0^R(x),\ u_t(x,\tau) = \bar{u}_1^R(x),\ x \in \Omega, \end{cases} \tag{3.8}$$

where functions $\bar{u}_i^R (i = 0, 1)$ are the forms: $\bar{u}_i^R(x) = \theta(|x|)u_i(x)$, and $\theta(x)$ is a smooth function:

$$\theta(x) = \begin{cases} 1, \quad |x| \le R - 1, \quad x \in \mathbb{R}^N, \\ 0, \quad |x| \ge R, \quad x \in \mathbb{R}^N, \end{cases}$$

and

$$0 \le \theta(x) \le 1,\ |\nabla\theta(x)| \le C, \quad x \in \mathbb{R}^N.$$

Now, we formally give some a priori estimates to the solutions of problem (3.8). Multiplying (3.8) by $u_t + \delta u$, we obtain

$$\frac{d}{dt}\Lambda_1(\xi_u(t)) + K(\xi_u(t)) = 0, \tag{3.9}$$

where

$$\begin{aligned} \Lambda_1(\xi_u(t)) = {} & \frac{1}{2}\varepsilon(t)\|u_t\|^2_{L^2(\Omega)} + (\frac{1}{2} + \delta)\|u\|^2_{H_0^1(\Omega)} + \frac{1}{2}\lambda\|u\|^2_{L^2(\Omega)} + \varepsilon(t)\delta\langle u, u_t\rangle \\ & - 2\langle g(x), u\rangle + 2\langle F(u), 1\rangle, \\ K(\xi_u(t)) = {} & -((\frac{1}{2} - \delta)\varepsilon'(t) + \delta\varepsilon(t))\|u_t\|^2_{L^2(\Omega)} + \|u_t\|^2_{H_0^1(\Omega)} - \delta\varepsilon'(t)\langle u, u_t\rangle \\ & + \delta\|u\|^2_{H_0^1(\Omega)} + \delta\lambda\|u\|^2_{L^2(\Omega)} + \langle f(u), \delta u\rangle - \langle g(x), \delta u\rangle, \end{aligned}$$

and $\xi_u(t) = (u, u_t),\ F(u) = \int_0^u f(r)dr$. Obviously, $\Lambda_1 : \mathcal{H}_t(\Omega) \mapsto \mathbb{R}$ is a continous function. Making use of (1.2)-(1.5) we can infer that

$$\Lambda_1(\xi_u(t)) \ge k(\varepsilon(t)\|u_t\|^2_{L^2(\Omega)} + \|u\|^2_{H_0^1(\Omega)}) - C(\|g\|_{L^2(\Omega)}, L), \tag{3.10}$$

$$K(\xi_u(t)) \ge \|u_t\|^2_{H_0^1(\Omega)} + k(\varepsilon(t)\|u_t\|^2_{L^2(\Omega)} + \|u\|^2_{H_0^1(\Omega)}) - C(\|g\|_{L^2(\Omega)}, L), \tag{3.11}$$

for $\delta > 0$ suitably small, where and in the flowing k stands for a small positive constant. Obviously,

$$\Lambda_1(\xi_u(0)) \le C(\|\bar{u}_1^R\|^2_{L^2(\Omega)} + \|\bar{u}_0^R\|^2_{H_0^1(\Omega)}) \le C(\rho, \|g\|_{L^2(\Omega)}, L). \tag{3.12}$$

Inserting (3.11) into (3.9) we have

$$\frac{d}{dt}\Lambda_1(\xi_u(t)) + k\|\xi_u(t)\|^2_{\mathcal{H}_t(\Omega)} \le C(\|g\|_{L^2(\Omega)}, L). \tag{3.13}$$

Applying Lemma 2.2 to (3.13) we have

$$\Lambda_1(\xi_u(t)) \le \sup_{\zeta \in \mathcal{H}_t(\Omega)} \{\Phi(\zeta) \mid \|\zeta\|^2_{\mathcal{H}_t(\Omega)} \le \frac{C(\|g\|_{L^2(\Omega)}, L) + 1}{k}\},$$

$$t \geq t_0 = C(\|g\|_{L^2(\Omega)}, L, \rho),$$

the above estimate is established with $\eta = C(\|g\|_{L^2(\Omega)}, L),\ k = C(\|g\|_{L^2(\Omega)}, L),\ K = C(\|g\|_{L^2(\Omega)}, L, \rho),\ \delta = k,\ \gamma = 1$. Therefore,

$$\varepsilon(t)\|u_t\|^2_{L^2(\Omega)} + \|u\|^2_{H^1_0(\Omega)} \leq C(\|g\|_{L^2(\Omega)}, L, \rho),\ t \geq t_0 = C(\|g\|_{L^2(\Omega)}, L, \rho). \quad (3.14)$$

Integrating (3.13) on (τ, t) with $t \leq t_0$, we get

$$\varepsilon(t)\|u_t\|^2_{L^2(\Omega)} + \|u\|^2_{H^1_0(\Omega)} \leq C(\|g\|_{L^2(\Omega)}, L, \rho). \quad (3.15)$$

Letting $\delta = 0$ in (3.9) then integrating it over (τ, t), together with (3.14)-(3.15) we have

$$\int_\tau^t \|u_t(r)\|^2_{H^1_0(\Omega)} dr \leq C(\|g\|_{L^2(\Omega)}, L, \rho). \quad (3.16)$$

We infer from (1.1) and (3.14)-(3.16) that

$$\int_\tau^t \varepsilon(t)\|u_{tt}(t)\|^2_{H^{-1}(\Omega)} dt \leq C(\|g\|_{L^2(\Omega)}, L, \rho). \quad (3.17)$$

Formal differentiation gives that $v(t) = u_t(t)$ solves the equation

$$\varepsilon(t)v_{tt} + \varepsilon'(t)v_t(t) - \triangle v - \triangle v_t + \lambda v + f'(u)v = 0. \quad (3.18)$$

Multiplying (3.18) by $v_t + \delta v$, we arrive that

$$\begin{aligned}
&\frac{d}{dt}[\varepsilon(t)\|v_t\|^2_{L^2(\Omega)} + 2\delta\varepsilon(t)\langle v, v_t\rangle + (\lambda + \delta\varepsilon'(t))\|v\|^2_{L^2(\Omega)} + (1+\delta)\|\nabla v\|^2_{L^2(\Omega)}] \\
&+ [\varepsilon'(t) - 2\delta\varepsilon(t)]\|v_t\|^2_{L^2(\Omega)} - 2\delta\varepsilon'(t)\langle v, v_t\rangle + (-\delta\varepsilon''(t) + 2\delta\lambda)\|v\|^2_{L^2(\Omega)} \\
&+ 2\delta\|\nabla v\|^2_{L^2(\Omega)} + 2\|\nabla v_t\|^2_{L^2(\Omega)} + 2\langle f'(u)v, v_t + \delta v\rangle = 0.
\end{aligned}$$

We introduce now the functional

$$\Lambda_2(t) = \varepsilon(t)\|v_t\|^2_{L^2(\Omega)} + 2\delta\varepsilon(t)\langle v, v_t\rangle + (\lambda + \delta\varepsilon'(t))\|v\|^2_{L^2(\Omega)} + (1+\delta)\|\nabla v\|^2_{L^2(\Omega)},$$

if δ is small enough we have

$$2\delta\varepsilon(t)\langle v, v_t\rangle \leq 2\delta\varepsilon(t)\|v\|^2_{L^2(\Omega)} + \frac{\delta\varepsilon(t)}{2}\|v_t\|^2_{L^2(\Omega)}, \quad (3.19)$$

hence, we can choose proper constants a_ρ, b_ρ such that

$$a_\rho\delta[\varepsilon(t)\|v_t\|^2_{L^2(\Omega)} + \|v\|^2_{H^1_0(\Omega)}] \leq \Lambda_2(t) \leq b_\rho[\varepsilon(t)\|v_t\|^2_{L^2(\Omega)} + \|v\|^2_{H^1_0(\Omega)}].$$

By (1.5), we have

$$\begin{aligned}
&|\langle f'(u)v, v_t + \delta v\rangle| \\
\leq& C(\|v\|_{L^2(\Omega)}\|v_t\|_{L^2(\Omega)} + \delta\|v\|^2_{L^2(\Omega)} \\
&+ \|u\|^{p-1}_{L^{p+1}(\Omega)}(\delta\|v\|^2_{L^{p+1}(\Omega)} + \|v\|_{L^{p+1}(\Omega)}\|v_t\|_{L^{p+1}(\Omega)})) \\
\leq& \delta\|v_t\|^2_{L^2(\Omega)} + C\|v\|^2_{L^2(\Omega)} + C\|u\|^{p-1}_{H^1_0(\Omega)}(\delta\|v\|^2_{H^1_0(\Omega)} + \|v\|_{H^1_0(\Omega)}\|v_t\|_{H^1_0(\Omega)})
\end{aligned}$$

$$\leq 2\delta\|v_t\|^2_{H^1_0(\Omega)} + C\|v\|^2_{H^1_0(\Omega)},$$

in the above formula, we have used the embedding $\|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_{H^1_0(\Omega)}$ and C is a positive constant. Similar to the estimate of (3.19), we have

$$-2\delta\varepsilon'(t)\langle v, v_t\rangle \geq -2L\|v\|^2_{L^2(\Omega)} - \frac{\delta^2}{2}\|v_t\|^2_{L^2(\Omega)},$$

from the above estimates, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Lambda_1(t) + k\Lambda_1(t) + 2\|\nabla v_t\|^2_{L^2(\Omega)}\\
\leq &[-\varepsilon'(t) + 2\delta\varepsilon'(t) + \varepsilon(t)]\|v_t\|^2_{L^2(\Omega)} + [4L^2 - \delta\varepsilon''(t)\\
&+ (2\delta+1)\lambda + \delta\varepsilon'(t)]\|v\|^2_{L^2(\Omega)} + (2+\delta)\|\nabla v\|^2_{L^2(\Omega)}.
\end{aligned}$$

By (1.2) and $\delta > 0$ suitably small, we can easily get $-\varepsilon'(t) + \frac{1}{2}\delta\varepsilon'(t) + \varepsilon(t) > 0, 4L^2 - \delta\varepsilon''(t) + (2\delta+1)\lambda + \delta\varepsilon'(t) > 0$. So,

$$\frac{d}{dt}\Lambda_1(t) + k\Lambda_1(t) + \frac{1}{2}\|\nabla v_t\|^2_{L^2(\Omega)} \leq C\Lambda_1(t) + C\|\nabla v\|^2_{L^2(\Omega)}. \tag{3.20}$$

When $\tau < t \leq 1$, multiplying (3.20) by $(t-\tau)^2$, we get

$$\begin{aligned}
&\frac{d}{dt}[(t-\tau)^2\Lambda_1(t)] + (t-\tau)^2 k\Lambda_1(t) + \frac{1}{2}(t-\tau)\|v_t\|^2_{H^1_0(\Omega)}\\
&\leq C(t-\tau)^2\Lambda_1(t) + C(t-\tau)\|v\|^2_{H^1_0(\Omega)} + C(t-\tau)(\|v_t\|^2_{L^2(\Omega)} + \|v\|^2_{H^1_0(\Omega)}) \qquad (3.21)\\
&\leq C(t-\tau)^2\Lambda_1(t) + \frac{1}{2}(t-\tau)^2\|v_t\|^2_{H^1_0(\Omega)} + C(\|v_t\|^2_{H^{-1}(\Omega)} + \|v\|^2_{H^1_0(\Omega)}),
\end{aligned}$$

here, we use the interpolation inequality

$$\begin{aligned}
C(t-\tau)\|v_t\|^2_{L^2(\Omega)} &\leq C(t-\tau)\|v_t\|^2_{H^1_0(\Omega)}\|v_t\|^2_{H^{-1}(\Omega)}\\
&\leq \frac{1}{2}(t-\tau)^2\|v_t\|^2_{H^1_0(\Omega)} + C\|v_t\|^2_{H^{-1}(\Omega)}.
\end{aligned}$$

Because of

$$C\int_\tau^t \|\nabla u_t(s)\|^2_{L^2(\Omega)}ds \leq C(\int_\tau^t \|\nabla u_t(s)\|^2_{L^2(\Omega)}ds)^{\frac{1}{2}} + (t-\tau)^{\frac{1}{2}} \leq \frac{k}{2}(t-\tau) + m$$

for $t > \tau$ and some $m > 0$, using Lemma 2.1 to (3.21), we obtain

$$(t-\tau)^2\Lambda_1(t) \leq C(\rho, L),\ \|u_t(t)\|^2_{H^1_0(\Omega)} + \varepsilon(t)\|u_{tt}(t)\|^2_{L^2(\Omega)} \leq \frac{C(L,\rho)}{(t-\tau)^2}, \tau < t \leq 1. \tag{3.22}$$

When $t \geq 1$, applying Lemma 2.1 to (3.20) on $(1, t)$, we have

$$\|u_t(t)\|^2_{H^1_0(\Omega)} + \varepsilon(t)\|u_{tt}(t)\|^2_{L^2(\Omega)} \leq C(\rho, L)e^{-kt} \leq C(\rho, L, t). \tag{3.23}$$

Therefore, together with (3.22) and (3.23), we get

$$\|u_t(t)\|^2_{H^1_0(\Omega)} + \varepsilon(t)\|u_{tt}(t)\|^2_{L^2(\Omega)} \leq \frac{C(\rho, L)(1 + (t-\tau)^2)}{2(t-\tau)^2},\ t > \tau. \tag{3.24}$$

Now, we look for the approximate solutions of (3.8) with the form

$$u^n(t)=\sum_{k=1}^{n} g_k(t)e_k, n=1,2,\dots,$$

where $-\triangle e_k=\lambda_k e_k,\ k=1,2,...,\ e_k\mid_{\partial\Omega}=0$, satisfying

$$\begin{aligned}&\langle\varepsilon(t)u^n_{tt},e_k\rangle+\langle-\triangle u^n,e_k\rangle+\langle-\triangle u^n_t,e_k\rangle+\langle\lambda u^n,e^k\rangle+\langle f(u^n),e_k\rangle=\langle g(x),e_k\rangle,\ t>\tau,\\&u^n(0)=\bar{u}_{0n},\qquad u^n_t(0)=\bar{u}_{1n},\end{aligned}$$

where $(\bar{u}_{0n},\bar{u}_{1n})\to(\bar{u}^R_0,\bar{u}^R_1)$ in $\mathcal{H}_t(\Omega)$. Obviously, the estimates (3.3), (3.15) and (3.17) hold for u^n. So we can extract a subsequence, still denoted by $\{u^n\}$, such that

$$u^n\to u \qquad \text{weakly}^* \text{ in } L^\infty(\tau,t;H^1_0(\Omega)); \tag{3.25}$$

$$u^n_t\to u_t \qquad \text{weakly}^* \text{ in } L^\infty(\tau,t;L^2(\Omega))\cap L^2(\tau,t;H^1_0(\Omega)); \tag{3.26}$$

$$u^n_{tt}\to u_{tt} \qquad \text{weakly in } L^2(\tau,t;H^{-1}(\Omega)). \tag{3.27}$$

Due to Lemma 2.3 it's easy to know that (u^n,u^n_t) is compact in

$$C(\tau,t;H^{1-\delta}(\Omega))\times[C(\tau,t;H^{-\delta}(\Omega))\cap L^2(\tau,t;H^{1-\delta}(\Omega))],$$

for every $0<\delta<1$. Moreover, we also have

$$f(u^n)\to f(u)\ \text{ weakly in } L^\infty(\tau,t;L^{1+\frac{1}{p}}(\Omega)). \tag{3.28}$$

Letting $n\to\infty$ we get that the limiting function $u\in L^\infty(\tau,t;H^1_0(\Omega))$ solves (3.8).

Now, we show the existence of solutions for the cauchy problem (1.1). For brevity, in the following, we use the abbreviations as is shown in the beginning of this section.

Let $u^R\in L^\infty(\tau,t;H^1_0(\Omega))$ be the solution of (3.8). Define the natural extension of u^R on $\mathbb{R}^N$

$$\bar{u}^R=\begin{cases}u^R, & |x|\le R,\\ 0, & |x|>R,\end{cases}\quad g_R=\begin{cases}g, & |x|\le R,\\ 0, & |x|>R,\end{cases}\quad \nabla\bar{u}^R=\begin{cases}\nabla u^R, & |x|\le R,\\ 0, & |x|>R,\end{cases}$$

and for any $\phi\in C^\infty_0(\mathbb{R}^N)$, noticing that $u^R\mid_{\partial\Omega}=0$, we have

$$\int\bar{u}^R\nabla\phi dx=\int_\Omega u^R\nabla\phi dx=-\int_\Omega\nabla u^R\phi dx=-\int\nabla\bar{u}^R\phi dx.$$

Hence, $\bar{u}^R\in L^\infty(\tau,t;H^1)$ solves the following problem

$$\begin{cases}\varepsilon(t)\bar{u}^R_{tt}-\triangle\bar{u}^R-\triangle\bar{u}^R_t+\lambda\bar{u}^R+f(\bar{u}^R)=g_R(x),\ x\in\mathbb{R}^N,\ t>\tau,\ \tau\in\mathbb{R},\\ \bar{u}^R(\tau)=\bar{u}^R_0,\ \bar{u}^R_t(\tau)=\bar{u}^R_1,\end{cases} \tag{3.29}$$

and the estimates (3.2)-(3.3) and (3.17) hold for $\bar{u}^R$. Since

$$f(\bar{u}^R)=(f'(k\bar{u}^R)-f'(0))\bar{u}^R+f'(0)\bar{u}^R=f''(k\delta\bar{u}^R)k|\bar{u}^R|^2+f'(0)\bar{u}^R, \tag{3.30}$$

where $0 < k, \delta < 1$, from assumption $(H3)$ and Sobolev embedding: $L^{1+\frac{1}{p}} \hookrightarrow H^{-1},\ H^2 \hookrightarrow H^{-1}$, we have

$$
\begin{aligned}
&|f(\bar{u}^R) - f'(0)\bar{u}^R| = |f''(k\delta\bar{u}^R)k|\bar{u}^R|^2| \le C(1+|\bar{u}^R|^{p-2})|\bar{u}^R|^2,\\
&\|f(\bar{u}^R) - f'(0)\bar{u}^R\|_{H^{-1}} \le C\|f''(k\delta\bar{u}^R)k|\bar{u}^R|^2\|_{1+\frac{1}{p}} \le C(\|\bar{u}^R\|^2_{\frac{2(p+1)}{p}} + \|\bar{u}^R\|^p_{p+1}) \le C,\\
&\|f(\bar{u}^R)\|_{H^{-1}} \le \|f'(0)\bar{u}^R\|_{H^{-1}} + C \le C(\|\bar{u}\|+1) \le C, t \ge \tau.
\end{aligned}
$$

Hence, there exists a limiting function defined on $\mathbb{R}^N$, still denoted by u, such that

$$
\begin{aligned}
&\bar{u}^R \to u && \text{weakly}^* \text{ in } L^\infty(\tau, t; H^1);\\
&\bar{u}^R_t \to u_t && \text{weakly}^* \text{ in } L^\infty(\tau, t; L^2) \cap L^2(\tau, t; H^1);\\
&\bar{u}^R_{tt} \to u_{tt} && \text{weakly in } L^2(\tau, t; H^{-1});\\
&f(\bar{u}^R) \to \zeta && \text{weakly in } L^\infty(\tau, t; H^{-1}).
\end{aligned}
$$

So, we can easily get that

$$
\|\bar{u}^R_0 - u_0\|^2_{H^1} + \|\varepsilon(t)\bar{u}^R_1 - \varepsilon(t)u_1\|^2 + \|g_R - g\|^2 \to 0 \ \text{ as } R \to \infty. \tag{3.31}
$$

Let $\bar{u}^{R_i}$ be weak solutions to (3.29) with different initial data $(\bar{u}^{R_i}_0, \bar{u}^{R_i}_1) \in \mathcal{H}_t$ such that $\|\bar{u}^{R_i}_t\|^2 + \|\nabla u^{R_i}(t)\| \le R^2$ for all $t \ge \tau$, and $z(t) = \bar{u}^{R_1} - \bar{u}^{R_2}$ solves the equation

$$
\varepsilon(t)z_{tt} - \triangle z - \triangle z_t + \lambda z + f(\bar{u}^{R_1}) - f(\bar{u}^{R_2}) = 0. \tag{3.32}
$$

Since $f(\bar{u}^{R_i}) \in L^2(\tau, t; H^{-1}) + L^\infty(\tau, t; L^{1+\frac{1}{p}})$ and $z \in L^\infty(\tau, t; H^1)$ for any couple $\bar{u}^{R_1}$ and $\bar{u}^{R_2}$. Hence, multiplying (3.32) by $A^{-1}z_t + \delta z$, we get

$$
\begin{aligned}
&\frac{d}{dt}[\varepsilon(t)\|z_t\|_{H^{-1}} + \|z\|^2 + \lambda\|z\|_{H^{-1}} + \delta(2\varepsilon(t)\langle z, z_t\rangle + \|\nabla z\|^2)] - \varepsilon'(t)\|z_t\|^2_{H^{-1}}\\
&- 2\delta\varepsilon(t)\|z_t\|^2 - 2\delta\varepsilon'(t)\langle z, z_t\rangle + 2\delta\|\nabla z\|^2 + 2\|z_t\|^2 + 2\lambda\delta\|z\|^2\\
&+ \langle f(\bar{u}^{R_1} - f(\bar{u}^{R_2}), A^{-1}z_t + \delta z\rangle = 0.
\end{aligned} \tag{3.33}
$$

We set

$$
\Lambda_3(t) = \varepsilon(t)\|z_t\|_{H^{-1}} + \|z\|^2 + \lambda\|z\|_{H^{-1}} + \delta(2\varepsilon(t)\langle z, z_t\rangle + \|\nabla z\|^2),
$$

for $\delta > 0$ small enough, we get

$$
a_\rho\delta[\varepsilon(t)\|A^{-\frac{1}{2}}z_t\|^2 + \|z\|^2_{H^1}] \le \Lambda_3(t) \le b_\rho[\varepsilon(t)\|A^{-\frac{1}{2}}z_t\|^2 + \|z\|^2_{H^1}], \tag{3.34}
$$

by (1.2), we have

$$
-2\delta\varepsilon'(t)\langle z, z_t\rangle \ge -2L^2\|z\|^2 - \frac{\delta^2}{2}\|z_t\|^2,
$$

we note that in the non-supercritical case by the embedding $H^1 \hookrightarrow L^r$ for $r = \infty$ in the case $N = 1$, for arbitrary $1 \le r \le \infty$ when $N = 2$ and for $r = \frac{2N}{N-2}$ in the case $N \ge 3$ we have that

$$
\|f(\bar{u}^{R_1}) - f(\bar{u}^{R_2})\|_{H^{-1}} \le C(\rho)\|\nabla(\bar{u}^{R_1} - \bar{u}^{R_2})\|,\ \bar{u}^{R_1},\ \bar{u}^{R_2} \in H^1,\ \|\nabla\bar{u}^{R_i}\| \le R, \tag{3.35}
$$

which implies that $|\langle (f(\bar{u}^{R_1})-f(\bar{u}^{R_2})), z\rangle| \leq C(\rho)\|\nabla z\|^2$, by (1.5) and interpolation formula,

$$\begin{aligned}
&\int |f(\bar{u}^{R_1}) - f(\bar{u}^{R_2})||A^{-1}z_t|dx \\
\leq & k\int (1+|\bar{u}^{R_1}|^{p-1}+|\bar{u}^{R_2}|^{p-1})|z|^2 dx + C(k)\int (1+|\bar{u}^{R_1}|^{p-1}+|\bar{u}^{R_2}|^{p-1})|A^{-1}z_t|^2 dx \\
\leq & k\int (1+|\bar{u}^{R_1}|^{p-1}+|\bar{u}^{R_2}|^{p-1})|z|^2 dx \\
&+ C(k)[\int (1+|\bar{u}^{R_1}|^{p+1}+|\bar{u}^{R_2}|^{p+1})dx)]^{\frac{p-1}{p+1}}\|A^{-1}z_t\|^2_{L^{p+1}} \\
\leq & k\|z_t\|^2 + C(k)\|z_t\|^2_{H^{-2}}.
\end{aligned} \tag{3.36}$$

Combining with (3.33)-(3.36), we get

$$\frac{d}{dt}\Lambda_2(t) + k\Lambda_2(t) \leq C(\|z\|^2 + \|z_t\|^2_{H^{-2}}), \tag{3.37}$$

$$\begin{aligned}
\|(z(t), z_t(t))\|^2_{\mathcal{H}_{t,-1}} \leq & e^{-k(t-\tau)}\|(z(\tau), z_t(\tau))\|^2_{\mathcal{H}_{t,-1}} \\
&+ b(\rho)\int_\tau^t (\|z(s)\|^2 + \varepsilon(t)\|z_t(s)\|^2_{H^{-2}})ds,
\end{aligned} \tag{3.38}$$

next, multiplying (3.32) by $z_t + \delta z$ we get

$$\begin{aligned}
&\frac{d}{dt}[\varepsilon(t)\|z_t\|^2 + 2\delta\varepsilon(t)\langle z, z_t\rangle + (1+\delta)\|\nabla z\|^2 + \lambda\|z\|^2] - [\varepsilon'(t) + 2\delta\varepsilon(t)]\|z_t\|^2 \\
&- 2\delta\varepsilon'(t)\langle z, z_t\rangle + 2\|\nabla z_t\|^2 + 2\delta\|\nabla z\|^2 + 2\delta\lambda\|z\|^2 + 2\langle f(\bar{u}^{R_1}) - f(\bar{u}^{R_1}), z_t + \delta z\rangle = 0,
\end{aligned} \tag{3.39}$$

and set

$$H_1(t) = \varepsilon(t)\|z_t\| + 2\delta\varepsilon(t)\langle z, z_t\rangle + (1+\delta)\|\nabla z\|^2 + \lambda\|z\|^2 \sim \|z\|^2_{H^1} + \varepsilon(t)\|z_t\|^2,$$

obviously,

$$\begin{aligned}
&- 2\delta\varepsilon'(t)\langle z, z_t\rangle \geq -2L^2\|z\|^2 - \frac{\delta^2}{2}\|z_t\|^2, \\
&|\langle f(\bar{u}^{R_1}) - f(\bar{u}^{R_2}), z_t + \delta z\rangle| \\
\leq & C\int (1+|f(\bar{u}^{R_1})|^{p-1} - |f(\bar{u}^{R_2})|^{p-1})|z|(|z_t| + \delta|z|)dx \\
\leq & C\|z\|(\|z_t\| + \delta\|z\|) + C(\|\bar{u}^{R_1}\|^{p-1}_{p+1} + \|\bar{u}^{R_2}\|^{p-1}_{p+1})\|z\|_{p+1}(\|z_t\|_{p+1} + \delta\|z\|_{p+1}) \\
\leq & \frac{1}{8}\|z_t\|^2 + c\|\nabla z\|^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{d}{dt}H_1(t) + [-\varepsilon'(t) - 2\delta\varepsilon(t) - \frac{\delta^2}{2}]\|z_t\|^2 + 2\|\nabla z_t\|^2 \\
\leq & (2L^2 - 2\delta\lambda)\|z\|^2 - 2\delta\|\nabla z\|^2 + \frac{1}{8}\|z_t\|^2 + c\|\nabla z\|^2,
\end{aligned}$$

from (1.2), we can see the following estimate

$$\min\{2, \frac{\delta^2}{2}\}\|z_t\|^2_{H^1} \leq 2\|\nabla z_t\|^2 + [-\varepsilon'(t) - 2\delta\varepsilon(t) - \frac{\delta^2}{2}]\|z_t\|^2$$
$$\leq \max\{2, (1+\delta)L + \frac{\delta^2}{2}\}\|z_t\|^2_{H^1}.$$

Then, we choose a proper constant L such that $-\varepsilon'(t)-2\delta\varepsilon(t)-\frac{\delta^2}{2} > 0, 2L^2-2\delta\lambda > 0$ and use the equivalent norm theorem

$$\|z(t)\|^2_{H^1} + \varepsilon(t)\|z_t(t)\|^2 + \int_\tau^t \|z_t(s)\|^2_{H^1} ds \leq Ce^{kt}(\|z_0\|^2_{H^1} + \|z_1\|^2). \tag{3.40}$$

Like in [28], we also have $f(\bar{u}^R) = \zeta$, therefore, together with (3.31) we obtain $u \in L^\infty(\tau, t; H^1)$, with $u_t \in L^\infty(\tau, t; L^2) \cap L^2(\tau, t; H^1)$ is the solution of Cauchy problem (1.1). By the lower semi-continuity of the norm of the weak* limit, the estimates (3.2)-(3.3) hold for u. An argument similar to the one used in [4](Appendix A) show that the function $t \mapsto (u(t); u_t(t))$ is (strong) continuous in $\mathcal{H}_t$.

Finally, we establish estimate (3.4). In space $[H^{-1} + L^{1+\frac{1}{p}}]$, one can see that estimate (3.3) is satisfied on any interval $[a, b]$, $\tau < a < b \leq t$, furthermore we note that $f(u)u_t \in L^1([a, b] \times \mathbb{R}^N)$, which implies that we can multiply (1.1) by u_t and prove (3.4) for $t \geq s > \tau$. Next, we prove energy equation (3.4) holds for $s = \tau$, the limit $E(\xi_u(s))$ as $s \to \tau$ exists and

$$E_* \equiv \lim_{s\to\tau} E(\xi_u(s)) = E(\xi_u(t)) + \int_\tau^t \|u_t(r)\|^2_{H^1} - \frac{\varepsilon'(t)}{2}\|u_t(r)\|^2 dr.$$

Because $u(t)$ is continuous in H^1 on $[\tau, t]$, we can find a sequence $\{s_n\}$, $s_n \to 0$, such that $u(x, s_n) \to u_\tau(x)$ almost surely. Due to $F(u) \geq -C$, by Fatou's lemma we obtain

$$\int F(u_\tau(x))dx \leq \liminf_{s\to\tau} \int (F(x, s))dx.$$

The weak continuity of $u_t(t)$ at time τ means that

$$\|u_1\|^2 \leq \liminf_{s\to\tau} \|u_t(s)\|^2.$$

Hence, we have the relation $E(\xi_u(\tau)) \leq E_*$. Therefore from the energy inequality for weak solutions we have (3.4) for all $t \geq s \geq \tau$.

The proof Theorem 3.1 is complete. □

4. Existnece of time-dependent attractor

By Theorem 3.1 the problem (1.1) generates a process $U(t, \tau)$ in the space $\mathcal{H}_t$:

$$U(t, \tau)z(\tau) = \{u(t), u_t(t)\},$$

where $z(\tau) = \{u_0, u_1\} \in \mathcal{H}_\tau$. Moreover, we can easily get the following result.

Theorem 4.1. *Under the assumptions* (1.2)-(1.5)*, the process* $U(t, \tau)$ *is continuous in phase space* $\mathcal{H}_t$.

Proposition 4.1. *Let assumptions* (1.2)-(1.5) *be valid, then there exists* $\rho_0 > 0$ *such that*

$$\|U(t,\tau)z(\tau)\|^2_{\mathcal{H}_t} \le \rho_0, \quad \forall t > \tau.$$

Proof. Estimate (3.2) shows that the dynamical system $(U(t,\tau), \mathcal{H}_t)$ is dissipative. Hence, we can get the time-dependent absorbing set

$$B_t = \bigcup_{t \ge \tau} U(t,\tau) B_0,$$

where, $B_0 = \{(u_0, u_1) \in \mathcal{H}_\tau : \ \|u_0\|^2_{H^1} + \varepsilon(\tau)\|u_1\|^2 \le \rho_0\}$. □

Next, we prove asymptotic compactness of the process $U(t,\tau)$ using method introduced in [19]. It is worth noting that the space discussed in this paper is not only dependent on time but also unbounded. To this end, we need to use the tail estimate method in order to obtain the corresponding results. One of main results is as follows.

Theorem 4.2. *In addition to assumption* (1.2)-(1.5)*, if also* $f'(u) > -l$ *for some constant* $l > 0$*. Then the dynamical system* $(U(t,\tau), \mathcal{H}_t)$ *possesses time-dependent attractors, moreover, for each* $\tau < t$*,* $A(t)$ *is bounded in* $\mathcal{H}_{t,1}$*.*

In order to prove this theorem, we need the following preparations.

Define the functions

$$M_0(s) = \begin{cases} 0, & 0 \le s \le 1, \\ s-1, & 1 < s \le 2, \\ 1, & s > 2, \end{cases}$$

$$M_\delta(s) = (\chi_\delta * M_0)(s) = \int_{\mathbb{R}} \chi_\delta(s-y) M_0(y) dy,$$

where $\chi_\delta(s)$ is the standard mollifier on $\mathbb{R}$ with $\mathrm{supp}\chi_\delta \subset [-\delta, \delta]$. Obviously,

$$\begin{aligned} &M_\delta \in C^\infty(\mathbb{R}), \ 0 \le M_\delta(s) \le 1, \\ &M_\delta(s) = 0 \ \text{ as } 0 \le s < 1; \quad M_\delta(s) = 1 \ \text{ as } s > 2, \end{aligned}$$

with $0 < \delta \ll 1$. Let $\varphi(x) = M_\delta(\frac{|x|}{R})$, with $R > r_0$. We have the following results

$$\varphi(x) = 0 \text{ as } |x| < R, \ 0 \le \varphi(x) \le 1 \text{ and } |\nabla\varphi(x)| \le 1 \text{ and } |\nabla\varphi(x)|^2 \le \frac{C}{R^2}, \ x \in \mathbb{R}^N. \tag{4.1}$$

Lemma 4.1 ([28])**.** *Let assumption* (1.2)-(1.5) *be in force and* $f'(u) > -l$*. Then*

$$\int \varphi^2 F(u) dx \ge \frac{C}{2}\|\varphi u\|^2,$$

$$\int \varphi^2 (f(u)u - kF(u)) dx \ge k\|\varphi u\|^2,$$

for $k : 0 < k \le \frac{2C}{2C+l+2}(< 1)$.

Lemma 4.2. *Under the assumptions* (1.2)-(1.5)*, let* $U(t,\tau)(u_0,u_1) = (u(t),u_t(t))$ *holds true, with* $(u_0,u_1) \in B_t$*. Then for any* $\delta > 0$*, there exist positive constants* $R_1 = R_1(\rho_0) > \tau$ *and* $T_0 = T_0(\rho_0)$ *such that*

$$\int_{\Omega_R^C} |u|^2 + |\nabla u|^2 + \varepsilon(t)|u_t|^2 dx < \delta \text{ as } R > R_1,\ \tau < T_0 < t,$$

where and the following Ω_R *is a ball in* $\mathbb{R}^N$ *with radius* R*,* $\Omega_R^C = \{x \in \mathbb{R}^N : |x| \geq R\}$*.*

Proof. Multiplying (1.1) by $\varphi^2(u_t+\delta u)$, together with (3.15) and (4.1), we have

$$\begin{aligned}
&\frac{d}{dt}[\frac{1}{2}\varepsilon(t)\|\varphi u_t\|^2 + \langle\varepsilon(t)u_t,\varphi^2\delta u\rangle + \frac{1+\delta}{2}\|\varphi\nabla u\|^2 + \frac{\lambda}{2}\|\varphi u\|^2 + \int\varphi^2F(u)dx \\
&- \langle g(x),\varphi^2u\rangle] - \frac{1}{2}\varepsilon'(t)\|\varphi u_t\|^2 - \langle\varepsilon'(t)u_t,\varphi^2\delta u\rangle - \varepsilon(t)\delta\|\varphi u_t\|^2 \\
&+ \|\varphi\nabla u_t\|^2 + \delta\|\varphi\nabla u\|^2 + \lambda\delta\|\varphi u\|^2 + \delta\langle f(u),\varphi^2u\rangle - \delta\langle g(x),\varphi^2u\rangle \\
=&-2\int\varphi\nabla\varphi(u_t+\delta u)\nabla u_t dx - 2\int\varphi\nabla\varphi(u_t+\delta u)\nabla u dx \\
\leq&\frac{1}{2}(\|\varphi u_t\|^2 + \|\varphi\nabla u_t\|^2) + \delta^2\|\varphi\nabla u\|^2 + \frac{C}{R^2}.
\end{aligned} \tag{4.2}$$

Set

$$\begin{aligned}
\Lambda_5(t) =&\frac{1}{2}\varepsilon(t)\|\varphi u_t\|^2 + \langle\varepsilon(t),u_t,\varphi^2\delta u\rangle + \frac{1+\delta}{2}\|\varphi\nabla u\|^2 + \frac{\lambda}{2}\|\varphi u\|^2 \\
&+ \int\varphi^2F(u)dx - \langle g(x),\varphi^2u\rangle,
\end{aligned}$$

so

$$\frac{d}{dt}\Lambda_5(t) + \delta\Lambda_5(t) + N(t) \leq C(\|\varphi u_t\|^2\Lambda_5(t) + \frac{C}{R^2}, \tag{4.3}$$

where

$$\begin{aligned}
N(t) =&[-\frac{1}{2}\varepsilon'(t) - \frac{1}{2}]\|\varphi u_t\|^2 - \delta\varepsilon'(t)\langle u_t,\varphi^2u\rangle - \varepsilon(t)\delta\|\varphi u_t\|^2 + \|\varphi\nabla u_t\|^2 \\
&+ [\lambda-\delta^2]\|\varphi\nabla u\|^2 + \lambda\delta\|\varphi u\|^2 + \delta\langle f(u),\varphi^2u\rangle - \frac{\delta\varepsilon(t)}{2}\|\varphi u_t\|^2 \\
&- \delta^2\varepsilon(t)\langle u_t,\varphi^2u\rangle - \frac{\delta(1+\delta)}{2}\|\varphi\nabla u\|^2 - \frac{\lambda\delta}{2}\|\varphi u\|^2 - \delta\int\varphi^2F(u)dx.
\end{aligned} \tag{4.4}$$

By (1.2), (4.1) Young and Hölder inequality, we have

$$-\delta\varepsilon'(t)\langle u_t,\varphi^2u\rangle \geq -\frac{\delta L}{2}\|\varphi u_t\|^2 - \frac{\delta L}{2}\|\varphi u\|^2,$$

hence, combining with the above estimate and Lemma 4.1, we choose the proper positive constants $\lambda, L(L>1)$ and $\delta(\delta \ll 1)$ small enough such that

$$\begin{aligned}
N(t) =&(-\frac{1}{2}\varepsilon'(t) - \frac{1}{2} - \frac{\delta L}{2} - \frac{\delta^2L}{2} - \varepsilon(t)\delta)\|\varphi u_t\|^2 + \|\varphi\nabla u_t\|^2 \\
&+ \frac{\delta}{2}(\lambda - \delta L - \delta^2L)\|\varphi u\|^2 + (\lambda-\delta^2)\|\varphi\nabla u\|^2 \geq 0,
\end{aligned} \tag{4.5}$$

after, inserting (4.5) to (4.3), we obtain

$$\frac{d}{dt}\Lambda_5(t) + \delta\Lambda_5(t) \leq C\|\varphi u_t\|^2\Lambda_5(t) + \frac{C}{R^2}. \tag{4.6}$$

Since $0 < \varphi(x) < 1$ and

$$|\langle\varepsilon(t)u_t, \varphi^2\delta u\rangle| \leq \frac{1}{2}\varepsilon(t)\|u_t\|^2 + 2L\delta^2\|u\|^2,$$

combining with (1.2)-(1.5) we can easily get that

$$\Lambda_5(t) \geq \delta(\|\varphi u_t\|^2 + \|\varphi u\|^2 + \|\varphi\nabla u\|^2) - C(\|\varphi g\|^2).$$

Applying Lemma 2.1 to (4.6), we conclude

$$\begin{aligned}
&\Lambda_5(t) \leq C\Lambda_5(\tau)e^{-\delta(t-\tau)} + C(\frac{1}{R^2} + \|\varphi g\|^2),\\
&\|(u(t), u_t(t))\|_{\mathcal{H}_t(\Omega_R^C)} \leq Ce^{-\delta(t-\tau)} + C(\frac{1}{R^2} + \|\varphi g\|^2_{L^2(\Omega_R^C)}).
\end{aligned} \tag{4.7}$$

Lemma 4.1 is proved. □

Proof of Theorem 4.2. For any fixed $t_1 \in \mathbb{R}$, let sequences $\tau_m \to \infty$ as $m \to \infty$, and $\xi_m \in B(t_1 - \tau_m)$, we set

$$\xi_u^m(t) = (u^m(t), u_t^m(t)) = U(t, t_1 - \tau_m)\xi_m,\ t \geq t_1 - \tau_m,\ m \geq 1. \tag{4.8}$$

One can see that for any $T \in \mathbb{N}$, it satisfies (3.2) and (3.3), when $m > N$, there exists a constant $N > 0$ such that

$$\begin{aligned}
&\xi_u^m(t) = U(t, t_1 - T - 1)U(t_1 - T - 1, t_1 - \tau_m)\xi_m\\
&\qquad \in U(t, t_1 - T - 1)B(t_1 - T - 1),\ t \in [t_1 - T, t_1],\\
&\{(u^m, u_t^m)\}_{m \geq N} \text{ is bounded in } L^\infty(t_1 - T, t_1; H^1 \times H^1),\\
&\{u_{tt}^m\}_{m \geq N} \text{ is bounded in } L^\infty(t_1 - T, t_1; H^{-2}).
\end{aligned}$$

By Lemma 2.3, one can see that $\xi_u = (u, u_t) \in L^\infty(t_1 - T, t_1; \mathcal{H}_{t,1})$ such that

$$\xi_u^m \to \xi_u \text{ weakly}^* \text{ in } L^\infty(t_1 - T, t_1; \mathcal{H}_{t,1}), \tag{4.9}$$

$$\xi_u^m \to \xi_u \text{ in } C(t_1 - T, t_1; H^{1-\delta} \times H^{1-\delta}),\ \delta \in (0, 1), \tag{4.10}$$

$$\xi_u^m \to \xi_u \text{ weakly in } \mathcal{H}_{t,1}, t \in [t_1 - T, t_1]. \tag{4.11}$$

It follows from (3.7) that, for every $\tau \geq 0$ and $x, y \in B(t - \tau)$,

$$\begin{aligned}
&\|U(t, t - \tau)x - U(t, t - \tau)y\|^2_{\mathcal{H}_{t,-1}}\\
\leq &e^{-k\tau}\|x - y\|^2_{\mathcal{H}_{t,-1}} + C\int_{t-\tau}^t \|U(s, t - \tau)x - U(s, t - \tau)y\|^2_{L^2 \times H^{-2}} ds.
\end{aligned} \tag{4.12}$$

According to (3.2) and Lemma 2.3 we have

$$U(\cdot, t - \tau)B(t - \tau) \text{ is precompact in } L^2(t - \tau, t; L^2 \times H^{-2}).$$

Hence, we can define a contractive functional

$$\psi_{t,\tau}(x,y)=[C\int_{t-\tau}^{t}\|U(s,t-\tau)x-U(s,t-\tau)y\|^2_{L^2\times H^{-2}}ds]^{\frac{1}{2}}$$

on $B(t-\tau)\times B(t-\tau)$ such that

$$\|U(t,t-\tau)x-U(t,t-\tau)y\|_{\mathcal{H}_{t,-1}}\le \delta+\psi_{t,\tau}(x,y),\ \forall x,y\in B(t-\tau),$$

where $\delta>0,\tau>0$. From Lemma 2.5, one can see that the process $U(t,\tau)$ is pullback $\mathcal{D}$-asymptotically compact in the topology of $\mathcal{H}_{t,-1}$. So

$$\xi_u^m(t)(t_1-T)=U(t_1-T,t_1-\tau_n)\xi^m\to\xi_u(t_1-T)\text{ in }\mathcal{H}_{t,-1}. \tag{4.13}$$

Next, by Lemma 4.1 and the interpolation between H^{-1} and H^1 we obtain that $U(t,\tau)$ is pullback $\mathcal{D}$-asymptotically compact in $\mathcal{H}_t$.

Combining with (4.9)-(4.11) and (4.13) we get

$$\xi_u^m(t_1-T)\to\xi_u(t_1-T)\text{ in }\mathcal{H}_t.$$

From the continuity of the operator $U(t,\tau)$ in $\mathcal{H}_t$ and the uniqueness of limit,

$$\begin{aligned}\xi_u^m(t)&=U(t,t_1-T)\xi_u^m(t_1-T)\to U(t,t_1-T)\xi_u(t_1-T)\\&=\xi_u(t)\text{ in }\mathcal{H}_t\end{aligned} \tag{4.14}$$

for every $t\in[t_1-T,t_1]$. Hence,

$$\xi_u^m(t_1)=U(t_1,t_1-\tau_n)\xi_u\to\xi_u(t_1)\text{ in }H^1\times L^2. \tag{4.15}$$

That is to say that $U(t,\tau)$ is pullback $\mathcal{D}$-asymptotically compact in $\mathcal{H}_t$. We now see the process $U(t,\tau)$ has a time-dependent attractor $\mathcal{A}=\{A(t)\}_{t\in\mathbb{R}}$, and by the boundedness of B_t in $\mathcal{H}_{t,1}$, $A(t)\subset\mathcal{H}_{t,1}$ is bounded.

5. Existnece of pullback exponential attractors

In this section, we show the existence of pullback exponential attractor under the non-autonomous case, contrary to the general non-autonomous situation, our external force term is independent of time but the coefficient $\varepsilon(t)$ depends on time. Hence, we obtain the existence of pullback exponential attractor without additional boundedness assumptions for the external force term in time-dependent space.

Definition 5.1 ([19]). Let E be a Banach space, M be a subset of E, which is a metric space equipped with the distance $d(x,y)=\|x-y\|_E$, and $\{U(t,\tau)\}_{t\ge\tau}$ be a process acting on M. Then the triple $(U(t,\tau),M,E)$ is said to be a non-autonomous dynamical system, M and E are said to be the phase space and the universal space, respectively.

Definition 5.2 ([19]). A family of subsets $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ of M is said to be a pullback exponential attractor of the non-autonomous dynamical system $(U(t,\tau),M,E)$, if
(i) it is semi-invariant, i.e., $U(t,\tau)\mathcal{M}(\tau)\subset\mathcal{M}(t)$ for all $t\ge\tau,\ \tau\in\mathbb{R}$;
(ii) each section $\mathcal{M}(t)$ is a compact subset of E and its fractal dimension in E is uniformly bounded, i.e.,

$$\sup_{t\in\mathbb{R}} dim_f(\mathcal{M}(t),E)<+\infty;$$

(iii) it pullback attracts every bounded subset B of M at an exponential rate, i.e.,

$$\sup_{t\in\mathbb{R}} dist_E\{U(t,t-s)B,\mathcal{M}(t)\} \leq C(B)e^{-\beta s}, s \geq T(B),$$

for some $\beta > 0$.

Remark 5.1. In particular, when $M = E$, Definition 5.2 coincides with the standard definition of pullback exponential attractor.

Lemma 5.1 ([27]). *Let $(U(t,\tau),M,E)$ be a non-autonomous dynamical system. Assume that*
(i) there exist positive constants T and L_T such for any $\tau \in \mathbb{R}$ and $x_1, x_2 \in M$,

$$U(t+\tau,\tau)M \subset M,\ t \geq T,$$
$$\sup_{t\in[0,T]} \|U(t+\tau,\tau)x_1 - U(t+\tau,\tau)x_2\|_E \leq L_T\|x_1 - x_2\|_E;$$

(ii) there exist a Banach space Z and a compact seminorm $n_Z(\cdot)$ on Z, and there exists a mapping $K_n : M \to Z$ and $n \in \mathbb{Z}$ such that for any $x_1, x_2 \in M$,

$$\sup_{n\in\mathbb{Z}} \|K_n x_1 - K_n x_2\|_Z \leq L\|x_1 - x_2\|_E,$$
$$\|U((n+1)T,nT)x_1 - U((n+1)T,nT)x_2\|_E \leq \eta\|x_1 - x_2\|_E + n_Z(K_n x_1 - K_n x_2),$$

where $\eta \in [0,1]$, $L > 0$ are constants independent of n. Then the non-autonomous dynamical system $(U(t,\tau),M,E)$ has a pullback exponential attractor $\mathcal{M} = \{M(t)\}_{t\in\mathbb{R}}$.

Theorem 5.1. *Let assumption (1.2)-(1.5) be valid, then the non-autonomous dynamical system $(U(t,\tau),\mathcal{H}_t,\mathcal{H}_{t,-1})$ related to problem (1.1) has a pullback exponential attractor $\mathcal{M} = \{M(t)\}_{t\in\mathbb{R}}$.*

Proof. We set B_0 as follows

$$B_0 = \{\xi \in \mathcal{H}_t \mid \|\xi\|^2_{\mathcal{H}_t} \leq R_0\} \text{ with } R_0 = C(\|g\|^2), \tag{5.1}$$

by Theorem 3.1, it is easy to see that B_0 is a uniform pullback absorbing set of $U(t,\tau)$ and there exists a $T_0 > 1$ such that

$$\bigcup_{\tau\in\mathbb{R}} U(t+\tau,\tau)B_0 \subset B_0, \text{ for } t \geq T_0 - 1. \tag{5.2}$$

Then, we construct

$$\mathcal{B}_t = [\bigcup_{\tau\in\mathbb{R}} \bigcup_{t\geq T_0} U(t+\tau,\tau)B_0]_{\mathcal{H}_t} \subset B_0. \tag{5.3}$$

So, $\mathcal{B}_t$ is also a uniform pullback absorbing set and

$$\bigcup_{\tau\in\mathbb{R}} U(t+\tau,\tau)\mathcal{B}_t \subset \bigcup_{\tau\in\mathbb{R}} U(t+\tau,\tau)B_0 \subset \mathcal{B}_t,\ \forall t \geq T_0. \tag{5.4}$$

By Theorem 4.2, we know that $\mathcal{B}_t$ is bounded in $\mathcal{H}_{t,1}$. Taking account of (3.7), we have

$$\sup_{t\in[0,T]} \|U(t+\tau,\tau)\xi_1 - U(t+\tau,\tau)\xi_2\|_{\mathcal{H}_{t,-1}} \leq L_T\|\xi_1 - \xi_2\|_{\mathcal{H}_{t,-1}},\ \forall\tau \in \mathbb{R}, \tag{5.5}$$

where $L_T = C(t, \mathcal{B}_t)$. We construct the following space

$$\Sigma = \{(u, u_t) \in L^2(0, T; \mathcal{H}_{t,-1}) \mid u_{tt} \in L^2(0, T; H^{-r})\}, \text{ with } r > \max\{\frac{N}{2}, 3\}, \quad (5.6)$$

equipped with the norm

$$\|(u, u_t)\|_\Sigma = [\int_0^T (\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{H^{-1}} + \varepsilon(t)\|u_{tt}(t)\|_{H^{-r}}^2)dt]^{\frac{1}{2}}.$$

Clearly, Σ is a Banach space, and

$$n_\Sigma(u, u_t) = \|(u, u_t)\|_{L^2(0,T;L^2 \times H^{-2})} \text{ is a compact seminorm on } \Sigma.$$

After, we define the mapping

$$K_n : \mathcal{B}_t \to \Sigma, \; K_n\xi = (u(\cdot + nT), u_t(\cdot + nT)), \; \xi \in \mathcal{B}_t, n \in \mathbb{N}^+, \quad (5.7)$$

where $(u(\cdot+nT), u_t(\cdot+nT)) = U(\cdot+nT, nT)\xi$, and $u(\cdot+nT)$ means $u(s+nT)$, $s \in [0, T]$. For every $n \in \mathbb{N}^+$, $\xi_1, \xi_2, \in \mathcal{B}_t$, let

$$\begin{aligned}
&(z(t + nT), z_t(t + nT)) \\
=&U(t + nT, nT)\xi_1 - U(t + nT, nT)\xi_2 \\
=&(u^1(t + nT), u_t^1(t + nT)) - (u^2(t + nT), u_t^2(t + nT)), \; \forall t \in [0, T].
\end{aligned}$$

Then z solves the following equation

$$\begin{aligned}
&\varepsilon(t)z_{tt} - \triangle z - \triangle z_t + \lambda z + f(u^1) - f(u^2) = g(x), \\
&(z(nT), z_t(nT)) = \xi_1 - \xi_2.
\end{aligned} \quad (5.8)$$

By estimate (3.7), (5.5), (5.8) and the fact: $r > \max\{\frac{N}{2}, 3\}$, which implies $H^{-1} \hookrightarrow H^{2-r}$ and $L^1 \hookrightarrow H^{-r}$, we obtain

$$\begin{aligned}
&\int_{nT}^{(n+1)T} \varepsilon(t)\|z_{tt}\|_{H^{-r}} ds \\
\leq&C(L) \int_{nT}^{(n+1)T} (\|z(s)\|_{H^{2-r}}^2 + \|z(s)\|_{H^1}^2 + \|z_t(s)\|_{H^{2-r}}^2 + \|f(u^1) - f(u^2)\|_{H^{-r}}^2)ds \\
\leq&C(L) \int_{nT}^{(n+1)T} (\|z(s)\|_{H^1}^2 + \|z_t(s)\|_{H^{-1}}^2 + b_0 \int (|u^1|^{p-1} + |u^2|^{p-1})|z|^2 dx)dx \\
\leq&C(L)\|\xi_1 - \xi_2\|_{\mathcal{H}_{t,-1}}^2.
\end{aligned} \quad (5.9)$$

In bine with (5.5) and (5.9), we have

$$\begin{aligned}
\|K_n\xi_1 - K_n\xi_2\|_\Sigma^2 =& \int_0^T (\|z(t + nT)\|_{H^1}^2 + \|z_t(t + nT)\|_{H^{-1}}^2 + \|z_{tt}(t + nT)\|_{H^{-r}}^2)dt \\
\leq& L_1^2\|\xi_1 - \xi_2\|_{\mathcal{H}_{t,-1}}^2,
\end{aligned} \quad (5.10)$$

where $L_1 = C(\mathcal{B}_t, L)$ is a constant. Choosing $T > T_0$ such that $\eta^2 = e^{-kT} < 1$ and combining with (3.7) and (5.7), it follows that

$$\begin{aligned}
&\|U((n + 1)T, nT)\xi_1 - U((n + 1)T, nT)\xi_2\|_{\mathcal{H}_{t,-1}} \\
\leq&\eta\|\xi_1 - \xi_2\|_{\mathcal{H}_{t,-1}} + n_\Sigma(K_n\xi_1 - K_n\xi_2), \; \xi_1, \xi_2 \in \mathcal{B}_t, \; n \in \mathbb{N}^+.
\end{aligned} \quad (5.11)$$

Theretore, the non-autonomous dynamical system $(U(t, \tau), \mathcal{H}_t, \mathcal{H}_{t,-1}))$ has a pullback exponential attractor $\mathcal{M} = \{M(t)\}_{t \in \mathbb{R}}$. □

References

[1] A. V. Babin and M. I. Vishik, *Attractors of partial differential evolution equation in an unbounded domain*, Proc. Roy. Soc. Edinburgh Sect. A, 1990, 116, 221–243.

[2] J. M. Ball, *Global attractors for damped semilinear wave equations*, Discret. Contin. Dyn. Syst., 2004, 10, 31–52.

[3] A. N. Carvalho and J. W. Cholewa, *Attractors for strongly damped wave equations with critical nonlinearities*, Pacific J. Math., 2002, 207, 287–310.

[4] I. Chueshov, *Long-time dynamics of Kirchhoff wave models with strong nonlinear damping*, J. Diff. Equ., 2012, 252, 1229–1262.

[5] M. Conti and V. Pata, *Asymptotic structure of the attractor for processes on time-dependent spaces*, Nonlinear Analysis RWA, 2014, 19, 1–10.

[6] M. Conti and V. Pata, *On the time-dependent cattaneo law in space dimension one*, Applied Mathematic and Computation, 2015, 259, 32–44.

[7] M. Conti and V. Pata, *On the regularity of global attractors*, Discrete Cont. Dyn. Sys., 2009, 25, 1209–1217.

[8] M. Conti, V. Pata and R. Temam, *Attractors for processes on time-dependent spaces, Applications to wave equations,* J. Differential Equations, 255, 2013, 1254–1277.

[9] V. V. Chepyzhov, M. Conti and V. Pata, *A minimal approach to the theory of global attractor*, Discrete Contin. Dyn. Syst., 2012, 32, 2079–2088.

[10] F. DellOro and V. Pata, *Long-term analysis of strongly damped nonlinear wave equations*, Nonlinearity, 2011, 24, 3413–3435.

[11] F. Di Plinio, G. S. Duane and R. Temam, *Time-Dependent attractor for the oscillon equation*, Discrete Contin. Dyn. Syst., 2011, 29, 141–167.

[12] P. Ding, Z. Yang and Y. Li, *Global attractor of the Kirchhoff wave models with strong nonlinear damping*, Appl. Math. Lett., 2018, 76, 40–45.

[13] A. E. Green and P. M. Naghdi, *On undamped heat waves in an elastic solid*, J. Thermal Stresses, 1992, 15, 253–264.

[14] A. E. Green and P. M. Naghdi, *Thermoelasticity without energy dissipation*, J. Elasticity, 1993, 31, 189–208.

[15] A. E. Green and P. M. Naghdi, *A unified procedure for construction of theories of deformable media, I. Classical continuum physics*, Proc. Roy. Soc. London A, 1995, 448, 335–356.

[16] A. E. Green and P. M. Naghdi, *A unified procedure for construction of theories of deformable media, II. Generalized continua*, Proc. Roy. Soc. London A, 1995, 448, 357–377.

[17] A. E. Green and P. M. Naghdi, *A unified procedure for construction of theories of deformable media, III. Mixtures of interacting continua*, Proc. Roy. Soc. London A, 1995, 448, 379–388.

[18] T. Liu and Q. Ma, *Time-dependent attractor for plate equations on* $\mathbb{R}^N$, J. Math. Anal. Appl., 2019, 479, 315–332.

[19] Y. Li and Z. Yang, *Robustness of attractors for non-autonomous Kirchhoff wave models with strong nonlinear damping*, Appl. Math. Opt., 2019.

[20] Q. Ma, J. Wang and T. Liu, *Time-dependent asymptotic behavior of the solution for wave equations with linear memory*, Comput. Math. Appl., 2018, 76, 1372–1387.

[21] H. Ma and C. Zhong, *Attractors for the Kirchhoff equations with strong nonlinear damping*, Appl. Math. Lett., 2017, 74 , 127–133.

[22] H. F. Smith and C. D. Sogge, *Global Strichartz estimates for non-trapping perturbations of the Laplacian*, Comm. Partial Differential Equations, 2000, 25, 2171–2183.

[23] J. Simon, *Compact sets in the space* $L^p(0,T;B)$*,* Ann. Mat. Pur. Appl., 1986, 146, 65–96.

[24] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd edition. App. Math. Sci., Springer-Verlag, New York, 1997, 68.

[25] Y. Wang and C. Zhong, *Upper semicontinuity of pullback attractors for nonautonomous Kirchhoff wave models*, Discret. Contin. Dyn. Syst., 2013, 7, 3189–3209.

[26] B. Wang, *Attractors for reaction-diffusion equations in unbounded domains*, Phys. D, 1999, 128, 41–52.

[27] Z. Yang and Y. Li, *Criteria on the existence and stability of pullback exponential attractors and their application to non-autonomous Kirchhoff wave models*, Discret, Contin. Dyn. Syst., 2018, 38, 2629–2653.

[28] Z. Yang and P. Ding, *Longtime dynamics of Kirchhoff equation with strong damping and critical nonlinearity on* $\mathbb{R}^N$, J. Math. Anal. Appl., 2016, 434, 1826–1851.