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WELL-POSEDNESS OF WAVE EQUATION WITH A VARIABLE COEFFICIENT BY METHOD OF CHARACTERISTICS*

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Abstract This study proves well-posedness of wave equation with a variable coefficient in the Triebel-Lizorkin space $F^s_{q,p}$ using the method of characteristics. Fourier series or transform cannot typically provide an explicit solution formula for equations with variable coefficients. Moreover, the theory presented by [16] on well-posedness in the L_q space is not suitable for problems in the $F^s_{q,p}$ space. In this study, without using any solution formula and complex calculus, we describe the wave equation with variable coefficients as comprising ordinary differential equations in view of the theory of function spaces and method of characteristics.

Keywords Wave equation, transport equation, ordinary differential equation, function analysis, method of characteristics.

MSC(2010) 35L05, 34A12, 35A24.

1. Introduction

In this study, we prove well-posedness of the following problem:

$$\begin{aligned}
&\partial_t^2 u(x,t) - c(x)^2 \partial_x^2 u(x,t) = 0 && x \in \mathbb{R},\, t \in \mathbb{R} \\
&u(x,0) = u_1(x) && x \in \mathbb{R}, \\
&\partial_t u(x,0) = u_2(x) && x \in \mathbb{R}.
\end{aligned} \tag{1.1}$$

If $c(x)$ is constant, (1.1) is well posed, i.e., (1.1) has a unique solution called d'Alembert formula ([2, 10]) for initial values $u_1 \in C^2(\mathbb{R})$, $u_2 \in C^1(\mathbb{R})$. The solution can be written as

$$u(x,t) = \frac{u_1(x+c_0t) + u_1(x-c_0t)}{2} + \frac{1}{2c_0}\int_{x-c_0t}^{x+c_0t} u_2(\zeta)\,d\zeta, \tag{1.2}$$

and gain $u \in C^2(\mathbb{R}^2)$. However, if $c(x)$ is not constant, we could not generally obtain a explicit solution formula similar to (1.2). Therefore, we prove well-posedness in the Triebel-Lizorkin space ([5, 9, 18]) for (1.1) under certain conditions on $c(x)$.

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Let φ be a C^∞-function on $\mathbb{R}^n$ with

$$\operatorname{supp}\varphi \subset \{\xi \in \mathbb{R}^n \mid |\xi| \le 2\},\ \varphi(\xi) = 1 \text{ if } |\xi| \le 1. \tag{1.3}$$

Let $j \in \mathbb{N}$,

$$\varphi_j(\xi) = \varphi(2^{-j}\xi) - \varphi(2^{-j+1}\xi),\ \xi \in \mathbb{R}^n \tag{1.4}$$

and $\varphi_0 = \varphi$. Set

$$\varphi_k(D)f(x) = \mathcal{F}^{-1}[\varphi_k \mathcal{F}[f]](x), \tag{1.5}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote Fourier transform and inverse Fourier transform, respectively.

Definition 1.1. Let $1 \le p \le \infty,\ 1 \le q \le \infty$ and $s \in \mathbb{R}$. Then $F^s_{q,p}(\mathbb{R}^n)$ is defined as a whole set of slowly increasing distribution, f satisfying $||f||_{F^s_{q,p}} < \infty$, where

$$||f||_{F^s_{q,p}(\mathbb{R}^n)} := \begin{cases} ||\varphi_0(D)f||_{L_q(\mathbb{R}^n)} + \Big|\Big|\Big(\sum_{k=1}^{\infty} 2^{ksp}|\varphi_k(D)f(\cdot)|^p\Big)^{\frac{1}{p}}\Big|\Big|_{L_q(\mathbb{R}^n)} & (1 \le p < \infty) \\ ||\varphi_0(D)f||_{L_q(\mathbb{R}^n)} + \sup_{j\in\mathbb{N}} 2^{js}\Big|\Big|\varphi_k(D)f\Big|\Big|_{L_q(\mathbb{R}^n)} & (p = \infty), \end{cases}$$

and

$$||f||_{L_q(\mathbb{R}^n)} := \begin{cases} \Big(\int_{\mathbb{R}^n} |g(x)|^q\, dx\Big)^{\frac{1}{q}} & (q < \infty) \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |g(x)| & (q = \infty). \end{cases}$$

Even if we replace φ in defnition of the Triebel-Lizorkin space for $\tilde{\varphi}$ satisfying (1.3) and (1.4), these spaces are homeomorphic in the sense of norm.

In case $p = 2,\ s \ge 0$, it is well known that $F^s_{q,2}$ is homeomorphic to the Bessel pottential space H^s_q. In particular, for a non ngative integer s, $F^s_{q,2}$ is homeomorphic to the Sobolev space W^s_q.

[15] used technique employed by Shibata and Shimizu ([14]) on the basis of operator valued Fourier multiplier theorem in [19] and proved maximal regularity ([6]) in the L_q space for a linear Stokes Equation with a constant coefficient. If we make use of the result for a constant coefficient, we can prove maximal regularity in the L_q space for a problem with variable coefficient. In [16], we assume bounded uniformly continuous for a variable coefficient and suppose that the variable coefficient is a constant in the outside of a ball $B_R(0)$, where we define $B_R(0)$ as

$$B_R(0) = \{x \in \mathbb{R}^n \mid |x| \le R\}.$$

$B_R(0)$ is a bounded closed set in $\mathbb{R}^n$, so we use localizing method for the problem in $B_R(0)$ and should analyze the problem with a constant coefficient in the outside of $B_R(0)$. Thus, we obtain a result for variable coefficients.

However, we could not utilize the theory of maximal regularity in the L_q theory for the problem considered in this study, which made it difficult to prove the maximal regularity of (1.1). In fact, we don't know a relatively simple condition like method by Shibata and Shimizu ([14]) for the Triebel-Lizorkin space, $F^s_{q,p}$.

Also, we could not immediately see whether or not the derivative operator in the left-hand side of (1.1) generates a semi-group ([1, 17]).

Moreover, if we do not utilize method by Shibata and Shimizu, we know a lot of results for wave equation in the L_2 space, beause energy method is available ([8, 11–13]). However, it is difficult for us to analyze (1.1) except for L_2-frame.

Thus, we used a simpler method to solve (1.1) with theory of function spaces and function analysis. As a result, we need not make use of localizing method in this paper.

Essentially, (1.1) can be considered a problem relative to ordinary equations by method of characteristics and technical deformation,

$$\partial_t^2 u(x,t) - c(x)^2 \partial_x^2 u(x,t) = (\partial_t - c(x)\partial_x)(\partial_t + c(x)\partial_x)u(x,t) + c(x)c'(x)\partial_x u(x,t) \tag{1.6}$$

for the left-hand side of (1.1). In view of (1.6), the equation (1.1) can be divided into the following equations:

$$\begin{aligned} &(\partial_t - c(x)\partial_x)v(x,t) = -c'(x)c(x)\partial_x u(x,t) && x \in \mathbb{R},\, t \in \mathbb{R}, \\ &v(x,0) = u_2(x) + c(x)u_1'(x) && x \in \mathbb{R}, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} &(\partial_t + c(x)\partial_x)u(x,t) = v(x,t) && x \in \mathbb{R},\, t \in \mathbb{R}, \\ &u(x,0) = u_1(x) && x \in \mathbb{R}. \end{aligned} \tag{1.8}$$

Definition 1.2 ([7]). For the surface $S = \{(x,t,z) \in \mathbb{R}^3 \mid z = u(x,t)\}$, we consider first order linear equations:

$$a(x,t)\partial_x u + b(x,t)\partial_t u = c(x,t). \tag{1.9}$$

We call the curve ${}^t(x(s), t(s), z(s)) \in \mathbb{R}^3$ parameterized by s "**characteristic curve**" (1.9), which satisfies

$$\begin{aligned} x'(s) &= a(x(s), t(s)) \\ t'(s) &= b(x(s), t(s)), \\ z'(s) &= c(x(s), t(s)). \end{aligned} \tag{1.10}$$

The equations in (1.10) are called the "**characteristic equations**" for (1.9). By solving (1.10), we obtain a solution to (1.9).

We display a numerical example as a graph of the solution for the next problem:

$$\begin{aligned} &\partial_t u(x,t) + \partial_x u(x,t) = 0 && x \in \mathbb{R},\, t > 0, \\ &u(x,0) = \sin x && x \in \mathbb{R}. \end{aligned} \tag{1.11}$$

In view of the method of characteristics, (1.9), and (1.10), we solve the following:

$$\begin{aligned} t'(s) &= 1, \quad & t(0) &= 0, \\ x'(s) &= 1, \quad & x(0) &= x_0, \\ z'(s) &= 0, \quad & z(0) &= \sin x_0. \end{aligned}$$

Then, we have an explicit description of the solution to (1.11), $u(x,t) = \sin(x-t)$. The graph of surface $z = \sin(x-t)$ is shown in Firgure 1.

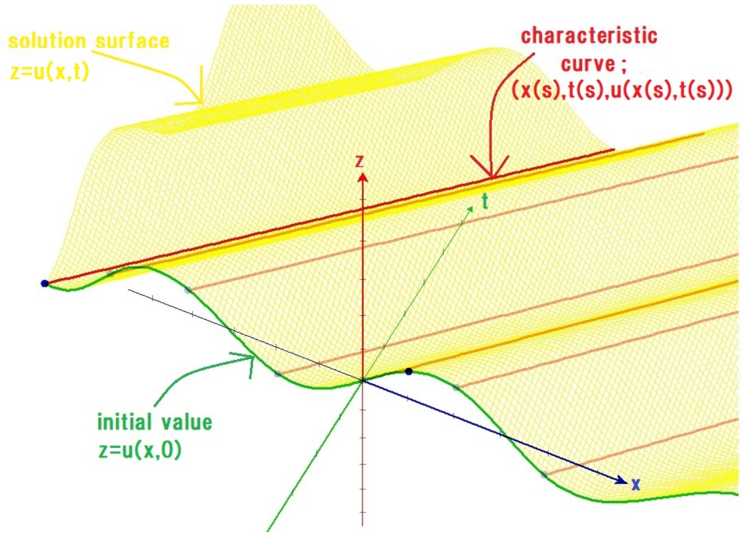

Figure 1. Solution Surface: $z = u(x,t) = \sin(x-t)$

In this paper, first, solving (1.7) for v, we substitue the solution of (1.7) in (1.8). If $c(x)$ is a constant, it hold that $c'(x) = 0$. Hence, we obtain (1.2) easily in the same way as aregument for (1.11). $c(x)$ is not a constant, therefore we need device. We define the Hölder-Zygmund space as the following and describe a main tool for proving well-posedness of (1.1) with a variable coefficient.

Definition 1.3 (Hölder-Zygmund Space)**.** For $s > 0$, we denote a universal set of f satisfying $||f||_{\mathcal{C}^s(\mathbb{R})} < \infty$ as $\mathcal{C}^s(\mathbb{R})$, where

$$||f||_{\mathcal{C}^s(\mathbb{R})} = ||f||_{L_\infty(\mathbb{R})} + \sup_{x,y\in\mathbb{R},y\neq0} \frac{|f(x+y)-2f(x)+f(x-y)|}{|y|^s}.$$

Theorem 1.1. *Suppose that* $1 \le p,q \le \infty$*,* $u_1 \in F^0_{q,p}(\mathbb{R})$*,* c*,* $1/c \in \mathcal{C}^\rho(\mathbb{R})$*satisfies* $c(x) \neq 0$ *and*

$$|c(x_1) - c(x_2)| \le C|x_1 - x_2| \quad \textit{for any} \;\; x_1, x_2 \in \mathbb{R}, \tag{1.12}$$

where C *is independent of* x_1, x_2*.*

Then, there exists a unique solution, $u \in C^1(\mathbb{R}; F^0_{q,p}(\mathbb{R})) \cap C(\mathbb{R}; F^1_{q,p}(\mathbb{R}))$*, to*

$$\begin{aligned} &(\partial_t + c(x)\partial_x)u(x,t) = Lu(x,t) && x \in \mathbb{R},\; t \in \mathbb{R} \\ &u(x,0) = u_1(x) && x \in \mathbb{R}, \end{aligned} \tag{1.13}$$

where $L : F^0_{q,p}(\mathbb{R}) \to F^0_{q,p}(\mathbb{R})$ *is a continuous map satisfying*

$$||Lu - Lv||_{F^0_{q.p}(\mathbb{R})} \le C||u - v||_{F^0_{q.p}(\mathbb{R})} \quad \textit{for any} \;\; u, v \in F^0_{q.p}(\mathbb{R}). \tag{1.14}$$

If we gain a solution, v to (1.7) for given u and describe $v = Fu$ with a continuous map F satisfying (1.14), we can eliminate v in(1.8). Namely, in case there exists the continuous map F, we can deform (1.1) to a problem similar to (1.13). Thus, Theorem 1.1 is available for analyzing (1.1).

2. Main Result

Theorem 2.1. *Suppose that* $1 \leq q, p \leq \infty$, $u_1 \in F^1_{q,p}(\mathbb{R})$, $u_2 \in F^0_{q,p}(\mathbb{R})$, *and* $c \in \mathcal{C}^{\rho+1}(\mathbb{R})$, $1/c \in \mathcal{C}^{\rho}(\mathbb{R})$ *which satisfies* $c(x) \neq 0$, $c'(x) \neq 0$, *(1.12) and* $\rho > 0$. *Additionally,*

$$u_2(x) + c(x)u_1'(x) = 0 \quad x \in \mathbb{R}.$$

Then, there exists a unique solution to (1.1),

$$u \in C^2(\mathbb{R}; F^0_{q,p}(\mathbb{R})) \cap C^1(\mathbb{R}; F^1_{q,p}(\mathbb{R})) \cap C(\mathbb{R}; F^2_{q,p}(\mathbb{R})).$$

If we analyze (1.1) in L_q-frame, we may suppose $c(x)$, $c'(x)$ are bounded uniformly conitious instead of $c \in \mathcal{C}^{\rho+1}(\mathbb{R})$. In view of Hölder inequlity for the Hölder-Zygmund space, we should assume $c \in \mathcal{C}^{\rho+1}(\mathbb{R})$ in order to obtain result in the Triebel-Lizorkin space.

Theorem 2.1 makes no assertion about the behavior of a solution to (1.1). Analysis the behavior of a solution considering the characteristic curve is a future research direction.

3. Proof of Theorem 1.1

The next theorem gives proof of Theorem 1.1 and becomes essential part in tihs paper.

Theorem 3.1 ([3]). *Let X be a Banach space, and a continuous map $R : X \to X$ satisfies*

$$||Ru - Rv||_X \leq C||u - v||_X \quad \textit{for any } u, v \in X, \tag{3.1}$$

where C is independent of u, v. Then, for any initial value $u_0 \in X$, the next equation:

$$u'(t) = Ru(t) \quad t \in \mathbb{R}$$

has a unique solution $u \in C^1(\mathbb{R}; X)$.

Making use of Theorem 3.1 and the fact that $c(x)$ satisfies (1.12) i.e. (3.1), we have a unique solution $x(t)$ to the following equation:

$$\begin{aligned} x'(t) &= c(x(t)), \\ x(0) &= x_0 \end{aligned}$$

from Theorem 3.1.

In the same way, a continuous operator L satisfies (1.14) i.e. (3.1), hence, there exists a unique solution $z(t)$ to

$$\begin{aligned} \frac{dz}{dt} &= Lz(t), \\ z(0) &= u_1(x(0)), \end{aligned}$$

where $z(t) = u(x(t), t)$ and $u_1 \in F^0_{q,p}(\mathbb{R})$. So, we see that $u \in C^1(\mathbb{R}; F^0_{q,p}(\mathbb{R}))$ and it holds that $u \in C(\mathbb{R}; F^1_{q,p}(\mathbb{R}))$ from (1.13). Thus, Theorem 1.1 is proved in view of Definition 1.2.

4. Proof of Theorem 2.1

For the right hand side of (1.7), we use Hölder inequlity for the Hölder-Zygmund space.

Definition 4.1. Let $1 \le p \le \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$ then the Besov space, $B^s_{q,p}(\mathbb{R}^n)$ is defined as a whole set of slowly increasing distribution, f satisfying $||f||_{B^s_{q,p}} < \infty$, where

$$||f||_{B^s_{q,p}(\mathbb{R}^n)} := \begin{cases} ||\varphi_0(D)f||_{L_q(\mathbb{R}^n)} + \Big(\sum_{k=1}^{\infty} 2^{ksp}||\varphi_k(D)f(\cdot)||^p_{L_q(\mathbb{R}^n)}\Big)^{\frac{1}{p}} & (1 \le p < \infty) \\ ||\varphi_0(D)f||_{L_q(\mathbb{R}^n)} + \sup_{j\in\mathbb{N}} 2^{js}\Big|\Big|\varphi_k(D)f\Big|\Big|_{L_q(\mathbb{R}^n)} & (p = \infty), \end{cases}$$

with (1.3), (1.4) and (1.5).

Theorem 4.1 (Hölder inequlity for Hölder-Zygmund space, [18])**.** *Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $\rho > \max(s, \sigma_q - s)$, then it holds that*

$$||f \cdot g||_{F^s_{q,p}(\mathbb{R}^n)} \le C||f||_{B^\rho_{\infty,\infty}(\mathbb{R}^n)}||g||_{F^s_{q,p}(\mathbb{R}^n)}$$

where $\sigma_q = n \max\big(\frac{1}{q} - 1, 0\big)$. Moreover, Besov space, $B^\rho_{\infty,\infty}$ is homeomorphic to $\mathcal{C}^\rho$ in the sense of norm. Hence, it holds that

$$||f \cdot g||_{F^s_{q,p}(\mathbb{R}^n)} \le C||f||_{\mathcal{C}^\rho(\mathbb{R}^n)}||g||_{F^s_{q,p}(\mathbb{R}^n)}.$$

Making use of Theorem 4.1 with $n = 1$, $s = 0$, we have

$$||c'(\cdot)c(\cdot)\partial_x u||_{F^0_{q,p}(\mathbb{R})} \le C||\partial_x u||_{F^0_{q,p}(\mathbb{R})}.$$

Hence, we obtain a unique solution to (1.7), $\upsilon \in C^1(\mathbb{R}; F^0_{q,p}(\mathbb{R})) \cap C(\mathbb{R}; F^1_{q,p}(\mathbb{R}))$ for a given $u \in F^1_{q,p}(\mathbb{R})$ from Theorem 3.1.

Owing to the uniqueness of the solution to (1.7) and $u_2(x)+c(x)u_1'(x) = 0$, $\upsilon = 0$ is in the solution class if $u = 0$. Hence, we can regard the map of the left-hand side of (1.7) as a surjective map using the solvability of (1.7). Therefore, there exists a bounded linear map, $R : F^0_{q,p}(\mathbb{R}) \to F^1_{q,p}(\mathbb{R})$, that satisfies $\upsilon = R(\partial_x u)$ from the open mapping theorem with

$$\begin{aligned} ||R(\partial_x u_1) - R(\partial_x u_2)||_{F^1_{q,p}(\mathbb{R})} &\le C||\partial_x u_1 - \partial_x u_2||_{F^0_{q,p}(\mathbb{R})} \\ &\le C||u_1 - u_2||_{F^1_{q,p}(\mathbb{R})}. \end{aligned} \tag{4.1}$$

Here, we utlize the fact that a bounded linear map, $T : X \to Y$ satisfying

$$\|Tu\|_Y \le C\|u\|_X$$

is a Lipschitz continuus map on X.

Setting $F = R\partial_x$. we obtain the following problem by using (1.8):

$$(\partial_t + c(x)\partial_x)u(x,t) = Fu(x,t) \quad x \in \mathbb{R},\, t > 0,$$

$$u(x,0) = u_1(x) \qquad\qquad x \in \mathbb{R}. \tag{4.2}$$

From (4.1), the map F satisfies is Lipschitz continuous; that is, it holds that

$$||F\zeta_1 - F\zeta_2||_{F^1_{q,p}(\mathbb{R})} \leq C||\zeta_1 - \zeta_2||_{F^1_{q,p}(\mathbb{R})} \quad \text{for any } \zeta_1, \zeta_2 \in F^1_{q,p}(\mathbb{R}).$$

Thus, we obtain a unique solution to (4.2), $u \in C^1(\mathbb{R}; F^1_{q,p}(\mathbb{R}))$ with Theorem 1.1. From (4.2), we derive

$$\partial_t u = -c(x)\partial_x u + Fu \in C^1(\mathbb{R}; F^0_{q,p}(R)),$$
$$\partial_x u = \frac{1}{c(x)}(Fu - \partial_t u) \in C(\mathbb{R}; F^1_{q,p}(R)).$$

Hence, making use of Theorem 4.1, we obtain $\partial_t^2 u \in C(\mathbb{R}; F^0_{q,p}(\mathbb{R}))$ and $u \in C(\mathbb{R}; F^2_{q,p}(\mathbb{R}))$. Thus, Theorem 2.1 is proved with Theorem 1.1.

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