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ATTRACTORS FOR RANDOM LATTICE DYNAMICAL SYSTEMS WITH INFINITE MULTIPLICATIVE COLORED NOISE

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Abstract In this paper, we establish the existence and uniqueness of random attractor for the first-order random lattice differential equation with a nonlinear colored noise at each node. We first rewrite the equation as a random evolution system and then prove the existence of a unique weak solution. Finally, we obtain the existence of a unique random attractor for the underlying random dynamical system.

Keywords Random lattice differential equation, random dynamical system, random attractor, colored noise.

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1. Introduction

Lattice systems can be considered as the spatial discretation of partial differential equations or be coupled of infinite ordinary differential equations or difference equations. These lattice models have been applied in many fields, such as image processing, pattern recognition, neural pulse and material science, see, e.g., [10,11,15,17,18]. There are many related studies of the deterministic lattice dynamical systems, see, e.g., [3, 9, 24] and the references therein. Also, lattice systems are often subject to random influences, see e.g., [4, 5, 7, 8, 13, 23] and the references therein for the stochastic (random) lattice dynamical systems.

In this paper, we consider the long-term behavior for the following random lattice differential equation with a diffusive nearest neighbor interaction, a dissipative nonlinear reaction term and a different multiplicative colored noise at each node:

$$\frac{du_i(t)}{dt} = (u_{i-1} - 2u_i + u_{i+1}) - f_i(u_i) + \eta(\theta_t\omega)g_i + \sigma_i(u_i)\zeta_\delta(\theta_t\omega_i), \quad i \in \mathbb{Z}, \tag{1.1}$$

where $\mathbb{Z}$ denotes the integer set, $u_i \in \mathbb{R}$, $g_i \in \mathbb{R}$, η is a random variable, f_i and σ_i are smooth nonlinear functions that satisfy some growth and dissipative conditions, ζ_δ is the colored noise with correlation time $\delta > 0$.

There are two features involved in system (1.1). One is that the nonlinear functions σ_i appear in the diffusion term. As we all know that the studying of the long-term dynamics for Itô-type stochastic partial differential equations driven by

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a nonlinear noise term is still open. The main reason is that Kolmogorov's test theorem fails for random fields parameterized by infinite-dimensional Hilbert spaces. Recently, several methods such as replacing the white noise by the additive fractional noise [5] or by some smooth approximations [13,23] are implemented to partly give an answer to the counterpart problem in lattice differential equations. Here, we use the colored noise, which was originally constructed in [21, 22] to approximately describe the stochastic behavior of the velocity and hence it can be further used to determine the position of the particle. The other feature is that there are different multiplicative noises at each node, which is different from the stochastic models first considered in [4], [8] and even in [5] and [13]. Only when lattice system perturbed by additive white noise, there should be different noises at each node. When the system driven by the multiplicative one, the problem was first proposed in [8] and then was solved in [7]. In [7], due to the linear diffusion term, the classical Doss-Sussmann-type transformation relied on Ornstein-Uhlenbeck (OU) process are used to transform the stochastic lattice differential equations into a random lattice system, which can be reformulated as an abstract random evolution equation over a Gelfand evolution triplet.

In order to study the long-term dynamics of the lattice differential equations driven by the nonlinear noise term, we introduce the colored noise (see [14, 19, 21, 22]). Let $\zeta_\delta : \Omega \to \mathbb{R}$ be a random variable given by $\zeta_\delta(\omega) = \frac{1}{\delta}\int_{-\infty}^{0} e^{\frac{s}{\delta}} dW$ ($\forall \omega \in \Omega$), then $\zeta_\delta(\theta_t\omega)$ is a special stationary Gaussian OU process, which satisfies the stochastic differential equation $d\zeta_\delta + \frac{1}{\delta}\zeta_\delta = \frac{1}{\delta}dW$. Here W is a two-sided real-valued Wiener process defined on the classical Wiener space $(\Omega, \mathscr{F}, \mathbb{P})$ with $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$. The Wiener shift $\theta_t : \Omega \to \Omega$ for $t \in \mathbb{R}$ is given by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$, $\forall \omega \in \Omega$. Note that $\zeta_\delta(\theta_t\omega) \in C^1(\mathbb{R}, \mathbb{R})$, we can release the nonlinear functions σ_i to be global Lipschitz continuous with small enough Lipschitz constants. Based on [7] and [13], we first reformulate the random system (1.1) as an abstract evolution equation, and then use its abstract theory on the existence of weak solutions of general random differential equations defined in Gelfand triples in Hilbert spaces to prove that the system possesses a global random attractor.

The structure of this paper is as follows. In Section 2, we introduce some basic concepts related to random dynamical systems and global random attractors. We also deal with the noise term and give some properties of the colored noise. In Section 3, we recall the theorem on the existence and uniqueness of weak solutions for general abstract random evolution equations. Later, we prove that system (1.1) generates a continuous random dynamic system, and then the existence of the global random attractor for (1.1) is obtained in Section 4.

2. Random Dynamical Systems and Preliminaries

In this section, we first recall some basic concepts related to random attractors for random dynamical systems (see more in [2, 6, 12]). Let $(H, \|\cdot\|_H)$ be a separable Banach space and $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition 2.1. We call $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ a metric dynamical system if

(i) $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathscr{F}, \mathscr{F})$-measurable,

(ii) $\theta_0 = \mathrm{id}$,

(iii) $\theta_{t+s} = \theta_t \circ \theta_s$, $\forall s, t \in \mathbb{R}$,

(iv) $\theta_t\mathbb{P} = \mathbb{P}$, $\forall t \in \mathbb{R}$.

Let $(\Omega, \mathscr{F}, \mathbb{P}) = (C_0, \mathcal{B}(C_0), \mathbf{P})$, where $\Omega = C_0 = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ with the open compact topology, $\mathbf{P}$ is the Wiener measure on $\mathcal{B}(C_0)$. Consider the measure-preserving transformation θ_t on Ω by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for $(\omega, t) \in \Omega \times \mathbb{R}$, then $(C_0, \mathcal{B}(C_0), \mathbf{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

Definition 2.2. A stochastic process $\Phi(t)$ is called a continuous random dynamical system over $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ if Φ is $(\mathcal{B}([0,\infty)) \times \mathscr{F} \times \mathcal{B}(H), \mathcal{B}(H))$-measurable, and for all $\omega \in \Omega$,

(i) the mapping $\Phi(t, \omega, \cdot) : H \to H$ is continuous for $(t, \omega) \in \mathbb{R}^+ \times \Omega$,

(ii) $\Phi(0, \omega, \cdot)$ is the identity on H,

(iii) $\Phi(s + t, \omega, \cdot) = \Phi(t, \theta_s\omega, \cdot) \circ \Phi(s, \omega, \cdot)$ for all $s, t \geq 0$ (cocycle property).

Definition 2.3. A set-valued map $A : \Omega \to 2^H \setminus \emptyset$, $\omega \mapsto A(\omega)$, where $A(\omega)$ is closed for all $\omega \in \Omega$, is called a random set if for each $x \in H$ the map $\omega \mapsto \mathrm{dist}(x, A(\omega))$ is measurable.

Definition 2.4. A random bounded set $B(\omega) \in H$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for $\omega \in \Omega$

$$\lim_{t\to\pm\infty} \frac{\log^+ d(B(\theta_{-t}\omega))}{|t|} = 0,$$

where $d(B) = \sup_{x\in B} \|x\|_H$.

Now let $\mathcal{D}$ denote the collection of random tempered sets in H.

Definition 2.5. A random set $K \in \mathcal{D}$ is called an absorbing set in $\mathcal{D}$ if for $B \in \mathcal{D}$ and $\omega \in \Omega$ there exists $t_B(\omega) > 0$ such that

$$\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega) \text{ for all } t \geq t_B(\omega).$$

Definition 2.6. A random set $\mathscr{A}$ is called a global $\mathcal{D}$ random attractor for Φ if the following conditions hold:

(I1) $\mathscr{A} \in \mathcal{D}$ is compact set for $\omega \in \Omega$;

(I2) $\mathscr{A}$ is strictly invariant, i.e. for $\omega \in \Omega$ and all $t \geq 0$ it holds

$$\Phi(t, \omega, \mathscr{A}(\omega)) = \mathscr{A}(\theta_t\omega);$$

(I3) $\mathscr{A}$ attracts all sets in $\mathcal{D}$, i.e., for all $B \in \mathcal{D}$ and a.e. $\omega \in \Omega$ it holds

$$\lim_{t\to\infty} \mathrm{d}(\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathscr{A}(\omega)) = 0,$$

where $d(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_H$ is the Hausdorff semi-metric (here $X \subset H, Y \subset H$).

Now we give the abstract result of the existence of global random attractors for continuous random dynamical systems.

Proposition 2.1 (see [12])**.** *Let $\Phi(t)$ be a continuous random dynamical system over $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. Suppose that $\Phi(t)$ has a random absorbing set $K \in \mathcal{D}$, $K(\omega)$ compact for $\omega \in \Omega$, then Φ possesses unique a $\mathcal{D}$-random attractor $\mathscr{A} = \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ with its element given by*

$$\mathscr{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

Next, we give some properties of colored noise. Let $\mathcal{P} = (C_0, \mathcal{B}(C_0), \mathbf{P})$ be a Wiener space. Define the product space $(\Omega, \mathscr{F}, \mathbb{P}) := \prod_i \mathcal{P}$ in a usual way. Since C_0 is a Polish space, $\mathscr{F}$ can also be generated by the product topology of C_0 (see e.g. [16]). Also, Ω is a Fréchet-space, the convergence is understood in the component-wise sense. Now we expand the Wiener-shift from $\mathbb{R} \times C_0$ to $\mathbb{R} \times \Omega$ by

$$\theta_t \omega = (\dots, \theta_t \omega_i, \dots), \quad \omega = (\omega_i)_{i \in \mathbb{Z}} \in \Omega.$$

Note that

$$\begin{aligned} &t \mapsto \theta_t \omega_i \ \text{ is continuous for any } \omega_i \in C_0, \\ &\omega_i \mapsto \theta_t \omega_i \ \text{ is continuous for any } t \in \mathbb{R}, \end{aligned}$$

we obtain that the continuity of the mappings $\theta_t \cdot$ on Ω with respect to the metric of the Fréchet-space and $\theta.\omega$ on $\mathbb{R}$. By [1] we get the measurability of

$$\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R} \times \Omega)) = (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathscr{F}) \to (\Omega, \mathscr{F}).$$

The measures $\mathbf{P}$ obtained by the projections of $\mathbb{P}$ to $\mathcal{B}(C_0)$ are still θ-ergodic.

Now we introduce the noise terms used in this paper. For $\delta > 0$ and each $i \in \mathbb{Z}$, denote

$$\zeta_\delta(\omega_i) := \zeta_{\delta,i}(\omega) = -\frac{1}{\delta^2} \int_{-\infty}^0 e^{\frac{s}{\delta}} \omega_i(s) ds, \quad \omega \in \Omega.$$

Then the process $\zeta_\delta(\theta_t \omega_i) := \zeta_{\delta,i}(\theta_t \omega)$ satisfies the one-dimensional stochastic equation:

$$d\zeta_\delta + \frac{1}{\delta} \zeta_\delta dt = \frac{1}{\delta} dw_i(t), \tag{2.1}$$

where $w_i(t)(\omega) = w_i(t, \omega) = \omega_i(t)$ for any $\omega \in \Omega$ and $t \in \mathbb{R}$.

In addition, the colored noise ζ_δ has the following properties.

Lemma 2.1 (see [14])**.** *Let $0 < \delta \leq 1$. Then there exists a $(\theta_t)_{t \in \mathbb{R}}$-invariant subset of full measure (still denoted by Ω), such that for any $\omega \in \Omega$,*

(i) for each $i \in \mathbb{Z}$,

$$\lim_{t \to \pm\infty} \frac{|\omega_i(t)|}{t} = 0;$$

(ii) for each $i \in \mathbb{Z}$, the mapping $(t, \omega) \mapsto \zeta_\delta(\theta_t \omega_i)$ is a stationary solution of (2.1) with continuous trajectories satisfying

$$\begin{aligned} &\lim_{t \to \pm\infty} \frac{|\zeta_\delta(\theta_t \omega_i)|}{t} = 0 && \text{for every } 0 < \delta \leq 1, \\ &\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t \zeta_\delta(\theta_s \omega_i) ds = 0 && \text{uniformly for } 0 < \delta \leq 1; \end{aligned}$$

(iii) for each $i \in \mathbb{Z}$ and arbitrary $T > 0$, $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\omega, T, \varepsilon) > 0$ such that for all $0 < \delta < \delta_0$ and $t \in [0, T]$,

$$\left| \int_0^t \zeta_\delta(\theta_s \omega_i) ds - \omega_i(t) \right| < \varepsilon.$$

3. Abstract Theory on Weak Solutions to General Random Evolution Equations

In this section, we recall the framework of the existence and uniqueness of weak solutions for general random evolution equations with specific types of operators in [7].

Let $\mathcal{H}$ be a separable Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Let V be a dense subspace of $\mathcal{H}$ with the inner product $(\cdot,\cdot)_V$ and the norm $\|\cdot\|_V$, and assume that V has a topological vector space structure, which includes continuous mappings. And V' is the dual space of V with the norm $\|\cdot\|_{V'}$. Define $\langle\cdot,\cdot\rangle$ a duality map between V and V' by

$$\langle u, v\rangle = (u, v), \quad \forall u \in V \subset \mathcal{H},\ v \in \mathcal{H} = \mathcal{H}' \subset V'.$$

Let $(e^k)_{k\in\mathbb{N}} \in \mathcal{H}$ be a complete orthonormal basis of $\mathcal{H}$, and consider a sequence of finite dimensional linear subspaces $\mathcal{H}_n \subset \mathcal{H}_{n+1} \subset V \subset \mathcal{H}$ given by

$$\mathcal{H}_n = \operatorname{span}\{e_1, \dots, e_n\}.$$

Define the projection $P_n : V \to \mathcal{H}_n$ by

$$P_n = \sum_{j=1}^{n} (\cdot, e_j)e_j,$$

then $P_n : \mathcal{H} \to \mathcal{H}$ is an orthonormal projection. We assume that

$$\overline{\bigcup_n \mathcal{H}_n}^{\mathcal{H}} = \mathcal{H}, \quad \overline{\bigcup_n \mathcal{H}_n}^{V} = V,$$

where $\overline{\ \cdot\ }^{\mathcal{H}}$ and $\overline{\ \cdot\ }^{V}$ denote the closures in the norm topology of $\mathcal{H}$ and V, respectively. Obviously, P_n can be extended to V'.

Define a linear continuous operator $\tilde{A} : V \to V'$, which satisfies

$$\langle \tilde{A}u, u\rangle \geq \alpha\|u\|_V^2, \qquad \|\tilde{A}u\|_{V'} \leq \alpha'\|u\|, \quad \forall u \in V.$$

We will study the following evolution system in a weak sense:

$$\frac{du(t)}{dt} + \tilde{A}u(t) = \tilde{F}(\theta_t\omega, u(t)) + \tilde{G}(\theta_t\omega), \quad u(0) = u_0 \in \mathcal{H}. \tag{3.1}$$

Definition 3.1. The element $u \in L^2(0, T; V)$ has a weak derivative $\frac{du}{dt} \in L^2(0, T; V')$ and is called a weak solution of (3.1) if for every $\xi \in V$ and $\phi \in C_0^\infty(0, T)$,

$$-\int_0^T (u(t), \xi)\phi'(t)dt = -\int_0^T \langle \tilde{A}u(t), \xi\rangle\phi(t)dt + \int_0^T \langle \tilde{F}(\theta_t\omega, u(t)) + \tilde{G}(\theta_t\omega), \xi\rangle\phi(t)dt.$$

In order to prove the existence of a weak solution to (3.1), we impose the following assumptions on the mappings $\tilde{F} : \Omega \times V \to V'$ and $\tilde{G} : \Omega \to V'$:

(F1) $(\omega, t) \mapsto \langle \tilde{F}(\omega, u(t)), \xi\rangle$ is measurable for all $u \in L^2(0, T; V)$ and $\xi \in V$, and for every $\omega \in \Omega$, $\phi \in C_0^\infty(0, T)$, $\xi \in \cup_{m\in\mathbb{N}}\mathcal{H}^m$, and any sequence $u^{(n)}$ such that

$$u^{(n)} \to u \text{ strongly in } L^2(0, T; \mathcal{H}),$$

we have

$$\lim_{n\to\infty}\int_0^T \langle \tilde F(\theta_t\omega, u^{(n)}(t)), \xi\rangle \phi(t)dt = \int_0^T \langle \tilde F(\theta_t\omega, u(t)), \xi\rangle \phi(t)dt.$$

(F2) Linear boundedness with respect to the V'-norm: for any $u \in V$,

$$\|\tilde F(\omega, u)\|_{V'}^2 \le \tilde M_1(\omega) + \tilde M_2(\omega)\|u\|_{\mathcal H}^2 + \tilde M_3\|u\|_V^2,$$
$$\langle \tilde F(\omega, u), u\rangle \le \tilde K_1(\omega) + \tilde K_2(\omega)\|u\|_{\mathcal H}^2 + \frac{\alpha}{2}\|u\|_V^2,$$

where $t \mapsto \tilde M_j(\theta_t\omega),\ t \mapsto \tilde K_j(\theta_t\omega) \in L^1_{\rm loc}(\mathbb R),\ j = 1, 2,$ for all ω in a $(\theta_t)_{t\in\mathbb R}$-invariant set of full measure, and $\tilde M_3 > 0$.

(F3) $\tilde F$ is semi-Lipschitz continuous: there exists a positive random variable $\tilde M(\omega)$ such that

$$\langle -\tilde A(x-y) + \tilde F(\omega, x) - \tilde F(\omega, y), x-y\rangle \le \tilde M(\omega)\|x-y\|_{\mathcal H}^2 \quad \text{for any } x, y \in V,$$

where $t \mapsto \tilde M(\theta_t\omega) \in L^1_{\rm loc}(\mathbb R)$ for all ω in a $(\theta_t)_{t\in\mathbb R}$-invariant set of full measure.

(F4) $\tilde G$ takes values in V', $\omega \mapsto \langle \tilde G(\omega), \xi\rangle$ is measurable and satisfies

$$t \mapsto \|\tilde G(\theta_t\omega)\|_{V'}^2 \in L^1_{\rm loc}(\mathbb R),$$

for all ω in a $(\theta_t)_{t\in\mathbb R}$-invariant set of full measure.

The above conditions ensure that $u \in L^2(0, T; V)$ with its weak derivative in $L^2(0, T; V')$. Now, we give the main theorems of this section.

Theorem 3.1 (see [7]). *Let $\tilde A \in L(V, V')$ be the linear operator defined in* (3.1), *and assume that $\tilde F$ and $\tilde G$ satisfy assumptions (F1)-(F4). Then*

(i) For any ω in a $(\theta_t)_{t\in\mathbb R}$-invariant set of full measure and $u_0 \in \mathcal H$, system (3.1) *possesses a unique global solution u such that, for any $T > 0$, we have $u \in C([0, T]; \mathcal H) \cap L^2(0, T; V)$ and its weak derivative $\frac{du}{dt} \in L^2(0, T; V')$.*

(ii) The solution of (3.1) *generates a continuous random dynamical system.*

4. Existence of Global Random Attractors for the Random Lattice Dynamical Systems

In this section, we reformulate (1.1) to an evolution equation, and prove that it generates a random dynamic system and hence possesses a unique global random attractor.

Denote

$$\mathcal H := \ell^2 = \left\{ u = (u_i)_{i\in\mathbb Z} : \sum_{i\in\mathbb Z} u_i^2 = \|u\|_{\mathcal H}^2 < \infty \right\}$$

with inner product

$$(u, v) := \sum_{i\in\mathbb Z} u_i v_i, \quad \text{for any } u, v \in \mathcal H.$$

Let $(\lambda_i)_{i\in\mathbb{Z}}$ be a sequence of positive numbers. Moreover, we assume that

$$\begin{aligned}
i\in\mathbb{Z}^+ \mapsto \lambda_i &\quad \text{is increasing,}\\
i\in\mathbb{Z}^- \mapsto \lambda_i &\quad \text{is decreasing,}
\end{aligned}$$

and in addition that $\sum_{i\in\mathbb{Z}}\lambda_i^{-1+\kappa}<\infty$ for some positive $\kappa\in(0,1)$, which implies that $\sum_{i\in\mathbb{Z}}\lambda_i^{-1}<\infty$.

Denote

$$V=\left\{u\in\mathcal{H}:\sum_{i\in\mathbb{Z}}\lambda_i u_i^2:=\|u\|_V^2<\infty\right\},$$

where $\|\cdot\|_V$ is associated with the inner product given by

$$(u,v)_V:=\sum_{i\in\mathbb{Z}}\lambda_i u_i v_i,\quad \text{for any } u,v\in V.$$

Denote

$$V'=\left\{u\in\mathcal{H}:\sum_{i\in\mathbb{Z}}\lambda_i^{-1} u_i^2:=\|u\|_{V'}^2<\infty\right\},$$

which is exactly the dual space of V. Then $(V,\mathcal{H},V')$ is a Gelfand triple.

Define $A_2:V\to V'$ by

$$(A_2u)_i=\lambda_i u_i,\quad i\in\mathbb{Z},$$

then system (1.1) can be rewrite as

$$\frac{du(t)}{dt}+Au(t)=F(\theta_t\omega,u(t))+G(\theta_t\omega), \tag{4.1}$$

where

$$\begin{aligned}
&A=A_1+A_2,\quad A_1=2\mathrm{id}_{\mathcal{H}},\quad (A_2u)=(\lambda_iu_i)_{i\in\mathbb{Z}},\\
&F(\omega,u)=\left(u_{i+1}+u_{i-1}+\lambda_iu_i-f_i(u_i)+\sigma_i(u_i)\zeta_\delta(\omega_i)\right)_{i\in\mathbb{Z}}, && (4.2)\\
&G(\omega)=\left(\eta(\omega)g_i\right)_{i\in\mathbb{Z}}. && (4.3)
\end{aligned}$$

In order to prove that these operators F and G satisfy the assumptions (F1)-(F4) with $\alpha=\frac{1}{2}$, we assume that f_i,σ_i and g_i satisfy the following conditions:

(A0) $f_i:\mathbb{R}\to\mathbb{R}$ is continuous for each $i\in\mathbb{Z}$.

(A1) There exists $\beta=(\beta_i)_{i\in\mathbb{Z}}\in\mathcal{H}$ such that

$$f_i^2(s)\le\lambda_is^2+\beta_i^2,\quad \forall s\in\mathbb{R}.$$

(A2) There exists $\gamma=(\gamma_i)_{i\in\mathbb{Z}}\in V$ such that

$$sf_i(s)\ge-\gamma_i^2+\frac{3\lambda_i}{4}s^2,\quad \forall s\in\mathbb{R}.$$

(A3) For each $i\in\mathbb{Z}$, there exists $L_f>0$ such that

$$f_i'(s)\ge-L_f,\quad \forall s\in\mathbb{R}.$$

(A4) For each $i \in \mathbb{Z}$, σ_i is a global Lipschitz continuous function with small enough Lipschitz constant $L_{\sigma,i}$ and $L_\sigma = (L_{\sigma,i})_{i\in\mathbb{Z}} \in \ell^\infty$. This further means that there exist $\psi = (\psi_i)_{i\in\mathbb{Z}} \in \ell^\infty$ with $\|\psi\|_\infty < 1^*$ and $\varphi = (\varphi_i)_{i\in\mathbb{Z}} \in V_{1+\kappa}$ ($\subset \mathcal{H}$) such that

$$\sigma_i^2(s) \le \psi_i s^2 + \varphi_i^2, \quad \forall s \in \mathbb{R}.$$

(A5) $g = (g_i)_{i\in\mathbb{Z}} \in \mathcal{H}$, and $\eta(\omega) \in L^1(\Omega)$ such that $\eta(\theta_t\omega) \in L^1_{\mathrm{loc}}(\mathbb{R})$.

Theorem 4.1. *Suppose the assumptions (A1)-(A5) hold. Then for any $u_0 \in \mathcal{H}$ and $T > 0$, there exists a $(\theta_t)_{t\in\mathbb{R}}$-invariant set of full measure such that system (4.1) has a unique weak solution $u = (u_i)_{i\in\mathbb{Z}} \in C([0,T];\mathcal{H}) \cap L^2(0,T;V)$ on $[0,T]$, with initial condition $u(0) = u_0$ and its weak derivative $\frac{du}{dt} \in L^2(0,T;V')$.*

Proof. In order to check (F1)-(F4) in Theorem 3.1, we divide the proof into four steps.

Step 1: For any $u \in V$, we have

$$\begin{aligned}\|F(\omega,u)\|_{V'}^2 &= \sum_{i\in\mathbb{Z}} \frac{1}{\lambda_i}[(u_{i-1}+u_{i+1}) + \lambda_i u_i - f_i(u_i) + \sigma_i(u_i)\zeta_\delta(\omega_i)]^2 \\ &\le \frac{4}{\lambda_i}\sum_{i\in\mathbb{Z}}(u_{i-1}+u_{i+1})^2 + 4\sum_{i\in\mathbb{Z}}\lambda_i u_i^2 \\ &\quad + \sum_{i\in\mathbb{Z}}\frac{4}{\lambda_i}f_i^2(u_i) + \sum_{i\in\mathbb{Z}}\frac{4}{\lambda_i}\sigma_i^2(u_i)\zeta_\delta^2(\omega_i). \end{aligned} \tag{4.4}$$

Now, we estimate each term in (4.4). First, we have

$$4\sum_{i\in\mathbb{Z}}\frac{1}{\lambda_i}(u_{i-1}+u_{i+1})^2 \le 8\|u\|_{\mathcal{H}}^2 \cdot \sup_{i\in\mathbb{Z}}\frac{1}{\lambda_i}, \tag{4.5}$$

$$4\sum_{i\in\mathbb{Z}}\lambda_i u_i^2 = 4\|u\|_V^2. \tag{4.6}$$

By (A1) and (A4), we obtain

$$\sum_{i\in\mathbb{Z}}\frac{4}{\lambda_i}f_i^2(u_i) \le 4\sum_{i\in\mathbb{Z}}u_i^2 + \sum_{i\in\mathbb{Z}}\frac{4}{\lambda_i}\beta_i^2 \le 4\|u\|_{\mathcal{H}}^2 + 4\sup_{i\in\mathbb{Z}}\frac{1}{\lambda_i}\|\beta\|_{\mathcal{H}}^2, \tag{4.7}$$

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\frac{4}{\lambda_i}\sigma_i^2(u_i)\zeta_\delta^2(\omega_i) &\le \sum_{i\in\mathbb{Z}}\frac{4}{\lambda_i}(|\psi_i|u_i^2 + \varphi_i^2)\zeta_\delta^2(\omega_i) \\ &\le 4\sup_{i\in\mathbb{Z}}\frac{\zeta_\delta^2(\omega_i)}{\lambda_i}\|\psi\|_\infty\|u\|_{\mathcal{H}}^2 + 4\sup_{i\in\mathbb{Z}}\frac{\zeta_\delta^2(\omega_i)}{\lambda_i}\|\varphi\|_{\mathcal{H}}^2. \end{aligned} \tag{4.8}$$

Let

$$M_1(\omega) = 4\sup_{i\in\mathbb{Z}}\frac{1}{\lambda_i}\|\beta\|_{\mathcal{H}}^2 + 4\sup_{i\in\mathbb{Z}}\frac{\zeta_\delta^2(\omega_i)}{\lambda_i}\|\varphi\|_{\mathcal{H}}^2,$$

$$M_2(\omega) = 8\sup_{i\in\mathbb{Z}}\frac{1}{\lambda_i} + 4 + 4\|\psi\|_\infty\sup_{i\in\mathbb{Z}}\frac{\zeta_\delta^2(\omega_i)}{\lambda_i}.$$

Since $\zeta_\delta(\omega_i)$ is an $\mathcal{N}(0,\frac{1}{2\delta})$-distributed Guassian random variable and $\sum_{i\in\mathbb{Z}}\frac{1}{\lambda_i} < \infty$, we have that $\sum_{i\in\mathbb{Z}}\frac{\zeta_\delta^2(\omega_i)}{\lambda_i} \in L^1(\Omega)$. Then by ergodic theorem we obtain $t \mapsto$

*We should remark here that the constant 1 is optional, which depends on the coefficient of the Young inequality used in (4.31).

$M_i(\theta_t\omega) \in L^1_{\text{loc}}(\mathbb{R}), i = 1, 2$, on a $(\theta_t)_{t\in\mathbb{R}}$-invariant set of full measure. Collecting (4.5)-(4.8), we obtain the first inequality in (F2), that is,

$$\|F(\omega, u)\|^2_{V'} \leq M_1(\omega) + M_2(\omega)\|u\|^2_{\mathcal{H}} + 4\|u\|^2_V.$$

Let $u \in L^2(0, T; V)$, due to the continuity of f_i,

$$t \mapsto f_i(u_i(t)) \in \mathbb{R},$$

is measurable. Then the sum of measurable mappings

$$\sum_{i\in\mathbb{Z}} f_i(u_i(t))\xi_i, \quad \xi = (\xi_i)_{i\in\mathbb{Z}} \in V,$$

is measurable. Also

$$(t, \omega_i) \mapsto \sigma_i(u_i)\zeta_\delta(\theta_t\omega_i) \in \mathbb{R}$$

is measurable, and hence

$$\sum_{i\in\mathbb{Z}} \sigma_i(u_i)\zeta_\delta(\theta_t\omega_i)\xi_i, \quad \xi = (\xi_i)_{i\in\mathbb{Z}} \in V,$$

is measurable. Similarly, the other terms in $\langle F(\theta_t\omega, u(t)), \xi\rangle$ are measurable. Thus $(\omega, t) \mapsto \langle F(\theta_t\omega, u(t)), \xi\rangle$ is measurable for all $u \in L^2(0, T; V)$ and $\xi \in V$. We get that $\|F(\theta_t\omega, u)\|_{V'}$ is finite for almost all t and $t \mapsto F(\theta_t\omega, u(t)) \in L^2(0, T; V')$ for all $\omega \in \Omega$.

For our purpose, we choose $\mathbb{R}^m, m \in \mathbb{N}$, as the finite-dimensional spaces $\mathcal{H}^m$. The complete orthonormal basis of $\mathcal{H}$ generating $\mathcal{H}^m$ is by

$$e_k = \begin{cases} \varepsilon^{\frac{k}{2}}, & \text{if } k \text{ is even,} \\ \varepsilon^{-\frac{k-1}{2}}, & \text{if } k \text{ is odd,} \end{cases}$$

where ε^i ($i \in \mathbb{Z}$) denotes the vector in $\mathcal{H}$, whose i-th element is 1 and 0 otherwise. Let $(u^{(n)})_{n\in\mathbb{N}}$ be a sequence, and $u^{(n)} \to u$ strongly in $L^2(0, T; \mathcal{H})$ as $n \to \infty$. Since f_i is continuous, we have that

$$f_i(u_i^{(n)}(t)) \to f_i(u_i(t)),$$

for all i, and almost all t. Let $\xi \in \cup_{m\in\mathbb{N}}\mathcal{H}^m$. For some $\overline{m} \in \mathbb{N}$, it is obvious that $\xi \in \mathcal{H}^{\overline{m}}$. And without loss of generality, we can choose that $\overline{m}$ is even. According to (A1), we get

$$\begin{aligned} \left| \sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} f_i(u_i^{(n)}(t))\xi_i \right|^2 &\leq \sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} f_i^2(u_i^{(n)}(t)) \sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} \xi_i^2 \\ &\leq \left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} \lambda_i(u_i^{(n)}(t))^2 + \sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} \beta_i^2 \right) \sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} \xi_i^2 \\ &\leq C_{1,\overline{m}} \left(\left\| u^{(n)}(t) \right\|^2_{\mathcal{H}} + 1 \right), \end{aligned}$$

for almost all $t \in [0, T]$ and all n.

Since $u^{(n)}$ is convergent in $L^2(0,T;\mathcal{H})$, by the Lebesgue theorem, we obtain for every $\phi \in C_0^\infty(0,T)$ that

$$\lim_{n\to\infty}\int_0^T\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} f_i(u_i^{(n)})\xi_i\right)\phi(t)dt = \int_0^T\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} f_i(u_i)\xi_i\right)\phi(t)dt.$$

Similarly, we have

$$\sigma_i(u_i^{(n)}(t)) \to \sigma_i(u_i(t)),$$

for all i and almost all t. Since $\zeta_\delta(\theta_t\omega_i)$ is continuous with respect to $t\in[0,T]$, it holds

$$\begin{aligned}
&\left|\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}} \sigma_i(u_i^{(n)}(t))\zeta_\delta(\theta_t\omega_i)\xi_i\right|^2\\
&\leq \sup_{\substack{i\in\{-\frac{\overline{m}}{2}+1,\cdots,\frac{\overline{m}}{2}\}\\ t\in[0,T]}} \zeta_\delta^2(\theta_t\omega_i)\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\sigma_i^2(u_i^{(n)}(t))\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\xi_i^2\\
&\leq \sup_{\substack{i\in\{-\frac{\overline{m}}{2}+1,\cdots,\frac{\overline{m}}{2}\}\\ t\in[0,T]}} \zeta_\delta^2(\theta_t\omega_i)\left(\|\psi\|_\infty\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}(u_i^{(n)}(t))^2+\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\varphi_i^2\right)\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\xi_i^2\\
&\leq C_{2,\overline{m}}\left(\left\|u^{(n)}(t)\right\|_{\mathcal{H}}^2 + 1\right)
\end{aligned}$$

for almost all $t\in[0,T]$ and all n. Then, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^T\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\sigma_i(u_i^{(n)}(t))\zeta_\delta(\theta_t\omega_i)\xi_i\right)\phi(t)dt\\
&=\int_0^T\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\sigma_i(u_i(t))\zeta_\delta(\theta_t\omega_i)\xi_i\right)\phi(t)dt,
\end{aligned}$$

for every $\phi\in C_0^\infty(0,T)$.

In the same way, we have

$$u_{i-1}^{(n)}(t)+u_{i+1}^{(n)}(t)\to u_{i-1}(t)+u_{i+1}(t),$$

for all i, and almost all $t\in[0,T]$, and hence

$$\begin{aligned}
&\left|\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\left(u_{i-1}^{(n)}(t)+u_{i+1}^{(n)}(t)\right)\xi_i\right|^2\\
&\leq\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\left(u_{i-1}^{(n)}(t)+u_{i+1}^{(n)}(t)\right)^2\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\xi_i^2
\end{aligned}$$

$$\leq 2\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\left((u_{i-1}^{(n)}(t)\right)^2+\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\left((u_{i-1}^{(n)}(t)\right)^2\right)\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\xi_i^2$$
$$\leq C_{3,\overline{m}}\left\|u^{(n)}(t)\right\|_{\mathcal{H}}^2.$$

This indicates that

$$\lim_{n\to\infty}\int_0^T\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\left(u_{i-1}^{(n)}(t)+u_{i+1}^{(n)}(t)\right)\xi_i\right)\phi(t)dt$$
$$=\int_0^T\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\left(u_{i-1}(t)+u_{i+1}(t)\right)\xi_i\right)\phi(t)dt,$$

for every $\phi\in C_0^\infty(0,T)$.

Finally, for every $\phi\in C_0^\infty(0,T)$ we have

$$\lim_{n\to\infty}\int_0^T\left(\sum_{i=-\frac{\overline{m}}{2}+1}^{\frac{\overline{m}}{2}}\lambda_i u_i^{(n)}(t)\xi_i\right)\phi(t)dt=\lim_{m\to\infty}\int_0^T\left(u^{(n)}(t),\xi_i\right)_V\phi(t)dt$$
$$=\int_0^T\left(u(t),\xi_i\right)_V\phi(t)dt.$$

Now, collecting all these terms of F we have

$$\lim_{n\to\infty}\int_0^T\left\langle F(\theta_t\omega,u^{(n)}(t)),\xi\right\rangle\phi(t)dt=\int_0^T\left\langle F(\theta_t\omega,u(t)),\xi\right\rangle\phi(t)dt, \tag{4.9}$$

for every $\phi\in C_0^\infty(0,T)$ and $\xi\in\cup_{m\in\mathbb{N}}\mathcal{H}^m$. Thus, (F1) holds.

Step 2: We first prove the second inequality in (F2). For any $u\in V$, we have

$$\sum_{i\in\mathbb{Z}}(u_{i-1}+u_{i+1}+\lambda_i u_i)u_i=2\|u\|_{\mathcal{H}}^2+\|u\|_V^2, \tag{4.10}$$
$$-\sum_{i\in\mathbb{Z}}f_i(u_i)u_i\leq\sum_{i\in\mathbb{Z}}\gamma_i^2-\sum_{i\in\mathbb{Z}}\frac{3\lambda_i}{4}u_i^2\leq\|\gamma\|_V^2-\frac{3}{4}\|u\|_V^2. \tag{4.11}$$

By (A4), we have

$$\sum_{i\in\mathbb{Z}}\sigma_i(u_i)\zeta_\delta(\omega_i)u_i\leq\frac{1}{2}\sum_{i\in\mathbb{Z}}|\zeta_\delta(\omega_i)|(\sigma_i^2(u_i)+u_i^2)$$
$$\leq\frac{1}{2}\sup_{i\in\mathbb{Z}}|\zeta_\delta(\omega_i)|\big(\|u\|_{\mathcal{H}}^2+\sum_{i\in\mathbb{Z}}\sigma_i^2(u_i)\big)$$
$$\leq\frac{1}{2}\sup_{i\in\mathbb{Z}}|\zeta_\delta(\omega_i)|\big(\|u\|_{\mathcal{H}}^2+\|\psi\|_\infty\|u\|_{\mathcal{H}}^2+\|\varphi\|_{\mathcal{H}}^2\big). \tag{4.12}$$

Let

$$K_1(\omega):=\|\gamma\|_V^2+\frac{1}{2}\sup_{i\in\mathbb{Z}}|\zeta_\delta(\omega_i)|\|\varphi\|_{\mathcal{H}}^2,$$
$$K_2(\omega):=2+\frac{1}{2}\sup_{i\in\mathbb{Z}}|\zeta_\delta(\omega_i)|(1+\|\psi\|_\infty).$$

Since $|\zeta_\delta(\omega)| \in L^1(\Omega)$, by the ergodic theorem again we know that $t \mapsto K_j(\theta_t\omega) \in L^1_{\text{loc}}(\mathbb{R})$ for $j = 1, 2$ on a $(\theta_t)_{t\in\mathbb{R}}$-invariant set of full measure. Collecting (4.10)-(4.12), we have

$$\langle F(\theta_t\omega, u), u\rangle \leq K_1(\omega) + K_2(\omega)\|u\|^2_{\mathcal{H}} + \frac{1}{4}\|u\|^2_V, \quad \forall\, u \in V. \tag{4.13}$$

Now, (F2) holds.

Step 3: We need to prove (F3). For any $x, y \in V$ we note that

$$\sum_{i\in\mathbb{Z}}[(x_{i-1} + x_{i+1}) - (y_{i-1} + y_{i+1})](x_i - y_i) \leq 2\|x - y\|^2_{\mathcal{H}}, \tag{4.14}$$

$$-\sum_{i\in\mathbb{Z}}(f_i(x_i) - f_i(y_i))(x_i - y_i) \leq L_f\|x - y\|^2_{\mathcal{H}}, \tag{4.15}$$

$$\sum_{i\in\mathbb{Z}}\|\sigma_i(x) - \sigma_i(y)\|^2 \leq \|L_\sigma\|^2_\infty\|x - y\|^2_{\mathcal{H}}. \tag{4.16}$$

Then we have

$$\sum_{i\in\mathbb{Z}}\zeta_\delta(\omega_i)(\sigma_{n,i}(x_i) - \sigma_{n,i}(y_i))(x_i - y_i) \leq \|L_\sigma\|_\infty \sup_{i\in\mathbb{Z}}|\zeta_\delta(\omega_i)|\|x - y\|^2_{\mathcal{H}}. \tag{4.17}$$

Collecting (4.14)-(4.17), we obtain

$$\begin{aligned}
&\langle -A(x-y) + F(\omega, x) - F(\omega, y), x - y\rangle \\
&\leq -\sum_{i\in\mathbb{Z}}\lambda_i(x_i - y_i)^2 + \sum_{i\in\mathbb{Z}}\Big([(x_{i-1} + x_{i+1}) - (y_{i-1} + y_{i+1})](x_i - y_i) \\
&\quad + \lambda_i(x_i - y_i)^2 - (f_i(x_i) - f_i(y_i))(x_i - y_i) + (\sigma_i(x_i) - \sigma_i(y_i))(x_i - y_i)\zeta_\delta(\omega_i)\Big) \\
&\leq M_3(\omega)\|x - y\|^2_{\mathcal{H}}, \quad \forall x, y \in V,
\end{aligned}$$

where

$$M_3(\omega) := 2 + L_f + \|L_\sigma\|_\infty \sup_{i\in\mathbb{Z}}|\zeta_\delta(\omega_i)|.$$

By the ergodic theorem again, we have that $t \mapsto M_3(\theta_t\omega) \in L^1_{\text{loc}}(\mathbb{R})$ on a $(\theta_t)_{t\in\mathbb{R}}$-invariant set of full measure.

Step 4: Obviously,

$$\|G(\omega)\|^2_{V'} = \eta^2(\omega)\sum_{i\in\mathbb{Z}}\frac{1}{\lambda_i}g_i^2 \leq \eta^2(\omega)\|g\|^2_{\mathcal{H}}\sup_{i\in\mathbb{Z}}\frac{1}{\lambda_i} := K_3(\omega) \in L^1(\Omega), \tag{4.18}$$

which implies that G satisfies (F4).

Now, by Theorem 3.1, we get the results. □

In the rest of the section, we need the properties of the random variables K_1 and K_3.

Lemma 4.1. *The random variables $K_1(\omega)$ and $K_3(\omega)$ are tempered.*

Proof. In order to obtain the temperedness of K_i $(i = 1, 3)$, we have to prove

$$\mathbb{E}\sup_{t\in[0,1]}\log^+ K_i(\theta_t\omega) \leq \mathbb{E}\sup_{t\in[0,1]} K_i(\theta_t\omega) < \infty \text{ for } i = 1, 3.$$

First we note that

$$\mathbb{E}\sup_{t\in[0,1]} K_1(\theta_t\omega) = \mathbb{E}\sup_{t\in[0,1]}\left(\|\gamma\|_V^2 + \frac{1}{2}\sup_{i\in\mathbb{Z}}|\zeta_\delta(\omega_i)|\|\varphi\|_{\mathcal{H}}^2\right)$$
$$= \|\gamma\|_V^2 + \frac{1}{2}\mathbb{E}\sup_{t\in[0,1]}|\zeta_\delta(\theta_t\omega_i)|\|\varphi\|_V^2.$$

We know that

$$\zeta_\delta(\theta_t\omega_i) = -\frac{1}{\delta^2}\int_{-\infty}^0 e^{\frac{s}{\delta}}(\omega_i(s+t)-\omega_i(t))ds$$
$$= -\frac{1}{\delta^2}\int_{-\infty}^0 e^{\frac{s}{\delta}}\omega_i(s+t)ds + \frac{1}{\delta}\omega_i(t).$$

Then

$$\mathbb{E}\sup_{t\in[0,1]}|\zeta_\delta(\theta_t\omega_i)| \le \mathbb{E}\left(\sup_{t\in[0,1]}\frac{1}{\delta^2}\int_{-\infty}^0 e^{\frac{s}{\delta}}|\omega_i(s+t)|ds\right) + \mathbb{E}\left(\sup_{t\in[0,1]}\frac{1}{\delta}|\omega_i(t)|\right)$$
$$\le \mathbb{E}\left(\frac{1}{\delta^2}\int_{-\infty}^0 e^{\frac{s}{\delta}}|\omega_i(s)|ds\right) + \mathbb{E}\left(\sup_{t\in[0,1]}\frac{1}{\delta}|\omega_i(t)|\right).$$

Since $\int_{-\infty}^0 e^{\frac{s}{\delta}}|s|ds < \infty$,

$$\mathbb{E}\left(\frac{1}{\delta^2}\int_{-\infty}^0 e^{\frac{s}{\delta}}|\omega_i(s)|ds\right) < \infty.$$

Due to the properties of $\omega_i(t)$ we have

$$\mathbb{E}\left(\sup_{t\in[0,1]}\frac{1}{\delta}|\omega_i(t)|\right) \le \mathbb{E}\left(\sup_{t\in[0,1]}\frac{1}{\delta}\omega_i^2(t)\right)^{\frac{1}{2}} \le \sup_{t\in[0,1]}\frac{1}{\delta}t^{\frac{1}{2}} < \infty, \tag{4.19}$$

which shows that

$$\mathbb{E}\sup_{t\in[0,1]} K_1(\theta_t\omega) < \infty.$$

Next, we know that $\sum_{i\in\mathbb{Z}}\lambda_i^{-1} < \infty$ and $\eta(\theta_t\omega) \in L^1_{\mathrm{loc}}(\mathbb{R})$, then we have

$$\mathbb{E}\sup_{t\in[0,1]} K_3(\theta_t\omega) < \infty.$$

The proof is complete. □

Furthermore, we obtain some useful estimates for the solution u for later purpose.

Lemma 4.2. *Suppose (A1)-(A5) hold. For any $\omega \in \Omega$ and $T > 0$, the solution u satisfies*

$$\|u\|_{C([0,T],\mathcal{H})} \le M_4(\|u_0\|_{\mathcal{H}}, T, \omega), \qquad \int_0^T \|u\|_V^2 dt \le M_5(\|u_0\|_{\mathcal{H}}, T, \omega),$$

where $M_4(B,T,\omega)$ and $M_5(B,T,\omega)$ are bounded functions for any (B,T) in bounded sets.

Proof. For $i \in \mathbb{Z}$, we have

$$\frac{du_i(t)}{dt} + Au_i(t) = F(\theta_t\omega, u_i(t)) + G(\theta_t\omega).$$

Now, due to $\langle A_1 u, u\rangle = 2\|u\|^2_{\mathcal{H}}$, $\langle A_2 u, u\rangle = \|u\|^2_V$, $\langle G(\theta_t\omega), u\rangle \le K_3(\theta_t\omega) + \frac{1}{4}\|u\|^2_V$ and the estimate of $\langle F(\omega, u), u\rangle$ in (4.13) we have

$$\begin{aligned}
&\frac{1}{2}\|u\|^2_{\mathcal{H}} + 2\int_0^t \|u(s)\|^2_{\mathcal{H}} ds + \int_0^t \|u(s)\|^2_V ds \\
&\le \frac{1}{2}\|u_0\|^2_{\mathcal{H}} + \int_0^t (K_1(\theta_s\omega) + K_3(\theta_s\omega) ds \\
&\quad + \int_0^t K_2(\theta_s\omega)\|u(s)\|^2_{\mathcal{H}} ds + \frac{1}{2}\int_0^t \|u(s)\|^2_V,
\end{aligned}$$

that is

$$\begin{aligned}
&\|u(t)\|^2_{\mathcal{H}} + 4\int_0^t \|u(s)\|^2_{\mathcal{H}} ds + \int_0^t \|u(s)\|^2_V ds \\
&\le \|u_0\|^2_{\mathcal{H}} + 2\int_0^t (K_1(\theta_s\omega) + K_3(\theta_s\omega) ds + 2\int_0^t K_2(\theta_s\omega)\|u(s)\|^2_{\mathcal{H}} ds.
\end{aligned}$$

Notice that

$$\begin{aligned}
\|u(t)\|^2_{\mathcal{H}} \le &- 4\int_0^t \|u(s)\|^2_{\mathcal{H}} ds - \int_0^t \|u(s)\|^2_V ds + \|u_0\|^2_{\mathcal{H}} \\
&+ 2\int_0^t (K_1(\theta_s\omega) + K_3(\theta_s\omega)) ds + 2\int_0^t (K_2(\theta_s\omega)\|u(s)\|^2_{\mathcal{H}} ds. \qquad (4.20)
\end{aligned}$$

Let

$$\mathcal{J}(\omega) := -\lambda_0 - 4 + 2K_2(\omega).$$

Obviously, the mapping $t \mapsto \mathcal{J}(\theta_t\omega)$ is locally integrable for any $\omega \in \Omega$. Then, by the Gronwall Lemma, we get

$$\|u(t)\|^2_{\mathcal{H}} \le e^{\int_0^t \mathcal{J}(\theta_r\omega)dr}\|u_0\|^2_{\mathcal{H}} + 2\int_0^t e^{\int_s^t \mathcal{J}(\theta_r\omega)dr}(K_1(\theta_s\omega) + K_3(\theta_s\omega))ds. \qquad (4.21)$$

Define

$$M_4(\|u_0\|_{\mathcal{H}}, T, \omega) := e^{\int_0^T |\mathcal{J}(\theta_r\omega)|dr}\|u_0\|^2_{\mathcal{H}} + 2\int_0^T e^{\int_s^T |\mathcal{J}(\theta_r\omega)|dr}(K_1(\theta_s\omega) + K_3(\theta_s\omega))ds.$$

Then, we have

$$\|u\|_{C([0,T],\mathcal{H})} \le M_4(\|u_0\|_{\mathcal{H}}, T, \omega).$$

By (4.20), we have that

$$\int_0^T \|u(t)\|^2_V dt \le M_5(\|u_0\|_{\mathcal{H}}, T, \omega),$$

where

$$M_5(\|u_0\|_{\mathcal{H}}, T, \omega) := \|u_0\|^2_{\mathcal{H}} + 2\int_0^T (K_1(\theta_s\omega) + K_3(\theta_s\omega) + C_1(\|u_0\|_{\mathcal{H}}, T, \omega)K_2(\theta_s\omega))ds.$$

The proof is complete. $\square$

So far, we prove the global existence and uniqueness solution of system (4.1). Let $u(t,\omega,u_0)$ be the solution of system (4.1) at time $t \geq 0$ with initial condition $u_0 \in \mathcal{H}$.

Theorem 4.2. (i) *Suppose (A1)-(A5) hold. For all $\omega \in \Omega$, the solution of system* (4.1) *generates a continuous random dynamical system $\Phi(t,\omega,u_0)$ in $\mathcal{H}$ given by*

$$\Phi(t,\omega,u_0) = u(t,\omega,u_0), \quad u_0 \in \mathcal{H}, t \geq 0, \omega \in \Omega.$$

(ii) *A family $\mathcal{K}(\omega) := \overline{\mathbb{B}_{\mathcal{H}}(0,R(\omega))}$ is a closed positively invariant tempered pullback absorbing set in $\mathcal{H}$ for Φ, centered at* 0 *with radius*

$$R(\omega) := \left(1 + 2\int_{-\infty}^{0} e^{\int_s^0 \mathcal{J}(\theta_r\omega)dr}(K_1(\theta_s\omega) + K_3(\theta_s\omega))ds\right)^{\frac{1}{2}}.$$

Proof. (i) It obviously follows from Theorem 3.1.

(ii) According to (4.21) we have

$$\|\Phi(t,\omega,u_0)\|_{\mathcal{H}}^2 \leq e^{\int_0^t \mathcal{J}(\theta_r\omega)dr}\|u_0\|_{\mathcal{H}}^2 + 2\int_0^t e^{\int_s^t \mathcal{J}(\theta_r\omega)dr}(K_1(\theta_s\omega) + K_3(\theta_s\omega))ds.$$

Replacing ω by $\theta_{-t}\omega$ in Φ we obtain

$$\begin{aligned}
&\|\Phi(t,\theta_{-t}\omega,u_0)\|_{\mathcal{H}}^2 \\
&\leq e^{\int_0^t \mathcal{J}(\theta_{r-t}\omega)dr}\|u_0\|_{\mathcal{H}}^2 + 2\int_0^t e^{\int_s^t \mathcal{J}(\theta_{r-t}\omega)dr}(K_1(\theta_{s-t}\omega) + K_3(\theta_{s-t}\omega))ds \\
&= e^{\int_{-t}^0 \mathcal{J}(\theta_r\omega)dr}\|u_0\|_{\mathcal{H}}^2 + 2\int_{-t}^0 e^{\int_s^0 \mathcal{J}(\theta_r\omega)dr}(K_1(\theta_s\omega) + K_3(\theta_s\omega))ds.
\end{aligned}$$

Since $\mathbb{E}\mathcal{J} < 0$, then for any $u_0 \in B(\theta_{-t}\omega)$,

$$\lim_{t\to\infty} e^{\int_{-t}^0 \mathcal{J}(\theta_r\omega)dr}\|u_0\|_{\mathcal{H}}^2 \leq \lim_{t\to\infty} e^{\int_{-t}^0 \mathcal{J}(\theta_r\omega)dr} d(B(\theta_{-t}\omega))^2 = 0.$$

Then by the temperedness of K_1 and K_3 we have

$$\int_{-\infty}^0 e^{\int_s^0 \mathcal{J}(\theta_r\omega)dr}(K_1(\theta_s\omega) + K_3(\theta_s\omega))ds < \infty.$$

Define

$$R^2(\omega) = 1 + 2\int_{-\infty}^0 e^{\int_s^0 \mathcal{J}(\theta_r\omega)dr}(K_1(\theta_s\omega) + K_3(\theta_s\omega))ds.$$

Then the ball $\mathcal{K}(\omega) := \overline{\mathbb{B}_{\mathcal{H}}(0,R(\omega))}$ is a pullback absorbing set in $\mathcal{H}$. Since $\mathbb{E}\mathcal{J} < 0$ and K_1 and K_3 are tempered, we obtain that the temperedness of $R(\omega)$. It is easy to prove that $\mathcal{K}(\omega)$ is positive invariant. $\square$

Now consider some $\kappa \in (0,1)$ satisfying

$$\sum_{i\in\mathbb{Z}} \lambda_i^{\kappa-1} < \infty.$$

Introduce the space

$$V_\kappa = \left\{ u = (u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} \lambda_i^\kappa u_i^2 := \|u\|_{V_\kappa}^2 < \infty \right\}$$

with the inner product

$$(u,v)_{V_\kappa} = \sum_{i\in\mathbb{Z}} \lambda_i^\kappa u_i v_i, \quad u = (u_i)_{i\in\mathbb{Z}}, v = (v_i)_{i\in\mathbb{Z}}.$$

Then we obtain the compact embedding $V_\kappa \subset \mathcal{H}$ (see [20, page 94]). Next, we need to prove the compactness of the random dynamic system Φ.

Lemma 4.3. *There exists a full θ-invariant set (still denoted by Ω) of Ω such that for any $\omega \in \Omega$, and there exists a function $M_6(K,\omega)$, which is bounded for $K \geq 0$ in a bounded set such that*

$$\|\Phi(1,\omega,u_0)\|_{V_\kappa}^2 \leq M_6(\|u_0\|_{\mathcal{H}},\omega).$$

Proof. Notice that

$$\begin{aligned}\frac{d}{dt}\left(t\|u^{(n)}(t)\|_{V_\kappa}^2\right) &= \|u^{(n)}(t)\|_{V_\kappa}^2 + t\frac{d}{dt}\|u^{(n)}(t)\|_{V_\kappa}^2 \\ &= \|u^{(n)}(t)\|_{V_\kappa}^2 + 2t\left(\frac{d}{dt}u^{(n)}(t), u^{(n)}(t)\right)_{V_\kappa}. \end{aligned} \tag{4.22}$$

Integrating (4.22) over the interval $[0,1]$, we have

$$\|u^{(n)}(1)\|_{V_\kappa} = \int_0^1 \|u^{(n)}(t)\|_{V_\kappa}^2 dt + \int_0^1 2t\left(\frac{d}{dt}u^{(n)}(t), u^{(n)}(t)\right)_{V_\kappa} dt. \tag{4.23}$$

The first term on the right-hand side of (4.23) satisfies

$$\int_0^1 \|u^{(n)}(t)\|_{V_\kappa}^2 dt \leq \lambda_0^{\kappa-1}\int_0^1 \|u^{(n)}(t)\|_V^2 dt \leq \lambda_0^{\kappa-1} C_2(\|u_0\|_{\mathcal{H}},1,\omega).$$

To estimate the second term on the right-hand side (4.23), we first have

$$\left(\frac{d}{dt}u^{(n)}(t), u^{(n)}(t)\right)_{V_\kappa} = \left(-A_1 u^{(n)} - A_2 u^{(n)} + F(u^{(n)}) + G, u^{(n)}\right)_{V_\kappa}, \tag{4.24}$$

where

$$\left(-A_1(u^{(n)}), u^{(n)}\right)_{V_\kappa} = -2\|u^{(n)}\|_{V_\kappa}^2, \tag{4.25}$$

$$\left(-A_2(u^{(n)}), u^{(n)}\right)_{V_\kappa} = -\|u^{(n)}\|_{V_{1+\kappa}}^2. \tag{4.26}$$

By (4.2) we obtain

$$\begin{aligned}\left(F(u^{(n)}), u^{(n)}\right)_{V_\kappa} &= \sum_{i\in\mathbb{Z}} \lambda_i^\kappa \left((u_{i-1}^{(n)} + u_{i+1}^{(n)})u_i^{(n)} + \lambda_i (u_i^{(n)})^2\right) \\ &\quad + \sum_{i\in\mathbb{Z}} \lambda_i^\kappa \left(-f_i(u_i^{(n)})u_i^{(n)} + \sigma_i(u_i^{(n)})u_i^{(n)}\zeta_\delta(\omega_i)\right). \end{aligned} \tag{4.27}$$

Now, we estimate each term in the right hand-side of (4.27). By (A2), (A4) and Young's inequality, we have

$$\sum_{i\in\mathbb{Z}} \lambda_i^{\kappa} \left((u_{i-1}^{(n)} + u_{i+1}^{(n)}) u_i^{(n)} \right) = 2\|u^{(n)}\|_{V_\kappa}^2, \tag{4.28}$$

$$\sum_{i\in\mathbb{Z}} \lambda_i^{\kappa} \lambda_i (u_i^{(n)})^2 = \|u^{(n)}\|_{V_{1+\kappa}}^2, \tag{4.29}$$

$$\begin{aligned} -\sum_{i\in\mathbb{Z}} \lambda_i^{\kappa} \left(f_i(u_i^{(n)}) u_i^{(n)} \right) &\le \sum_{i\in\mathbb{Z}} \lambda_i^{\kappa} \left(\gamma_i^2 - \frac{3\lambda_i}{4} (u_i^{(n)})^2 \right) \\ &\le \sup_{i\in\mathbb{Z}} \frac{1}{\lambda_i^{1-\kappa}} \|\gamma\|_V^2 - \frac{3}{4} \|u^{(n)}\|_{V_{1+\kappa}}^2, \end{aligned} \tag{4.30}$$

$$\begin{aligned} \sum_{i\in\mathbb{Z}} \lambda_i^{\kappa} \left(\sigma_i(u_i^{(n)}) u_i^{(n)} \zeta_\delta(\theta_t \omega_i) \right) &\le \sum_{i\in\mathbb{Z}} \frac{\zeta_\delta^2(\theta_t\omega_i) u_i^2}{\lambda_i^{1-\kappa}} + \frac{1}{4} \sum_{i\in\mathbb{Z}} \lambda_i^{1+\kappa} \sigma_i^2(u_i^{(n)}) \\ &\le \sup_{i\in\mathbb{Z}} \frac{\zeta_\delta^2(\theta_t\omega_i)}{\lambda_i^{1-\kappa}} \|u\|_{\mathcal{H}}^2 + \frac{1}{4} \|\psi\|_\infty \|u\|_{V_{1+\kappa}}^2 + \frac{1}{4} \|\varphi\|_{V_{1+\kappa}}^2 \\ &\le \sup_{i\in\mathbb{Z}} \frac{\zeta_\delta^2(\theta_t\omega_i)}{\lambda_i^{1-\kappa}} \|u\|_{\mathcal{H}}^2 + \frac{1}{4} \|u\|_{V_{1+\kappa}}^2 + \frac{1}{4} \|\varphi\|_{V_{1+\kappa}}^2, \end{aligned} \tag{4.31}$$

$$\begin{aligned} \left(G, u^{(n)} \right)_{V_\kappa} &\le \eta^2(\theta_t\omega) \sum_{i\in\mathbb{Z}} \frac{g_i^2}{2\lambda_i^{1-\kappa}} + \frac{1}{2} \sum_{i\in\mathbb{Z}} \lambda_i^{1+\kappa} (u_i^{(n)})^2 \\ &\le \eta^2(\theta_t\omega) \sup_{i\in\mathbb{Z}} \frac{1}{2\lambda_i^{1-\kappa}} \|g\|_{\mathcal{H}}^2 + \frac{1}{2} \|u^{(n)}\|_{V_{1+\kappa}}^2. \end{aligned} \tag{4.32}$$

Collecting (4.24)-(4.32), we have

$$\begin{aligned} \left(\frac{d}{dt} u^{(n)}(t), u^{(n)}(t) \right)_{V_\kappa} &\le K_4(\theta_t\omega) \|u\|_{\mathcal{H}}^2 + K_5(\theta_t\omega) \\ &\le K_4(\theta_t\omega) M_4(\|u_0\|_{\mathcal{H}}, 1, \omega) + K_5(\theta_t\omega), \end{aligned}$$

where

$$\begin{aligned} K_4(\omega) &= \sup_{i\in\mathbb{Z}} \frac{\zeta_\delta^2(\omega_i)}{\lambda_i^{1-\kappa}}, \\ K_5(\omega) &= \sup_{i\in\mathbb{Z}} \frac{1}{\lambda_i^{1-\kappa}} \|\gamma\|_V^2 + \eta^2(\omega) \sup_{i\in\mathbb{Z}} \frac{1}{\lambda_i^{1-\kappa}} \|g\|_{\mathcal{H}}^2 + \frac{1}{4} \|\varphi\|_{V_{1+\kappa}}^2. \end{aligned}$$

Since the $\sum_{i\in\mathbb{Z}} \lambda_i^{\kappa-1} < \infty$ and $\zeta_\delta(\omega_i)$ is Gaussian random variable, we have that $\mathbb{E}K_j < \infty$ for $j = 4, 5$. Then, we obtain that there exists a function $M_6(\|u_0\|_{\mathcal{H}}, \omega)$ such that

$$\|u^{(n)}(1)\|_{V_\kappa} \le M_6(\|u_0\|_{\mathcal{H}}, \omega). \tag{4.33}$$

Note that the right-hand side of (4.33) is independent of n, so we have

$$\|u(1)\|_{V_\kappa} \le M_6(\|u_0\|_{\mathcal{H}}, \omega).$$

The proof is complete. □

Finally, we obtain the following main result.

Theorem 4.3. *Let Φ be the continuous random dynamical system generated by* (4.1)*, then Φ has a unique random attractor.*

Proof. From Theorem 4.2, we know that Φ is a continuous random dynamical system with positive invariant absorbing set $\mathcal{K} \in \mathcal{D}$. Define

$$B(\omega) := \overline{\Phi(1, \theta_{-1}\omega, \mathcal{K}(\theta_{-1}\omega))}^{\mathcal{H}} \subset \mathcal{K}(\omega).$$

This inclusion relationship ensures that $B \in \mathcal{D}$. Then by Lemma 4.3, we obtain the set B is a compact absorbing set. From Proposition 2.1, we know that Φ possesses a random global attractor. □

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References

[1] C. Aliprantis and K. Border, *Infinite Dimensional Analysis: A Hitchhikers Guide*, Springer, Berlin, 2007.

[2] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Berlin, 1998.

[3] P. Bates, B. Wang and K. Lu, *Attractors for lattice dynamical systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2001, 11, 143–153.

[4] P. Bates, H. Lisei and K. Lu, *Attractors for stochastic lattice dynamical systems*, Stoch. Dyn., 2006, 6, 1–21.

[5] H. Bessaih, M. Garrido-Atienza, X. Han and B. Schmalfuss, *Stochastic lattice dynamical systems with fractional noise*, SIAM J. Math. Anal., 2017, 49, 1495–1518.

[6] T. Caraballo, M. Garrido-Atienza, B. Schmalfuss and J. Valero, *Non–autonomous and random attractors for delay random semilinear equations without uniqueness*, Discrete Contin. Dyn. Syst., 2008, 21, 415–443.

[7] T. Caraballo, X. Han, B. Schmalfuss and J. Valero, *Random attractors for stochastic lattice dynamical systems with infinite multiplicative white noise*, Nonlinear Anal., 2016, 130, 255–278.

[8] T. Caraballo and K. Lu, *Attractors for stochastic lattice dynamical systems with a multiplicative noise*, Front. Math. China, 2008, 3, 317–335.

[9] S. N. Chow, *Lattice Dynamical Systems,* Lecture Notes in Math., 1822, Springer, Berlin, 2003, 1–102.

[10] S. N. Chow and J. Mallet-Paret, *Pattern formation and spatial chaos in lattice dynamical systems*, IEEE Trans. Circuits Syst., 1995, 42, 746–751.

[11] L. O. Chua and T. Roska, *The CNN paradigm*, IEEE Trans. Circuits Syst., 1993, 40, 147–156.

[12] F. Flandoli and B. Schmalfuss, *Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise*, Stoch. Stoch. Rep., 1996, 59, 21–45.

[13] A. Gu, *Asymptotic behavior of random lattice dynamical systems and their Wong-Zakai approximations*, Discrete Contin. Dyn. Syst. Ser. B, 2019, 24, 5737–5767.

[14] A. Gu and B. Wang, *Asymptotic behavior of random Fitzhugh-Nagumo systems driven by colored noise*, Discrete Contin. Dyn. Syst. Ser. B, 2018, 23, 1689–1720.

[15] M. Hilbert, *A solid-solution model for inhomogeneous systems*, Acta Metall., 1961, 9, 525–535.

[16] O. Kallenberg, *Foundations of Modern Probability*, Springer-Verlag, New York, 1997.

[17] R. Kapral, *Discrete models for chemically reacting syetems*, J. Math. Chem., 1991, 6, 113–163.

[18] J. P. Keener, *Propagation and its failure in coupled systems of discrete excitable cells*, SIAM J. Appl. Math., 1987, 47, 556–572.

[19] L. Ridolfi, P. D'Odorico and F. Laio, *Noise-Induced Phenomena in the Environmental Sciences*, Cambridge University Press, New York, 2011.

[20] G. Sell and Y. You, *Dynamics of Evolutionary Equations*, Springer-Verlag, New York, 2002.

[21] G. Uhlenbeck and L. Ornstein, *On the theory of Brownian motion*, Phys. Rev., 1930, 36, 823–841.

[22] M. Wang and G. Uhlenbeck, *On the theory of Brownian motion. II*, Rev. Modern Phys., 1945, 17, 323–342.

[23] X. Wang, J. Shen, K. Lu and B. Wang, *Wong-Zakai approximations and random attractors for non-autonomous stochastic lattice systems*, J. Differential Equations, 2021, 280, 477–516.

[24] M. Zgurovsky, P. Kasyanov, O. Kapustyan, J. Valero and N. Zadoianchuk, *Attractors for Lattice Dynamical Systems*, In: Evolution Inclusions and Variation Inequalities for Earth Data Processing III. Advances in Mechanics and Mathematics, Springer, Berlin, 2012, 27.