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STABILIZATION OF FIXED POINTS IN CHAOTIC MAPS USING NOOR ORBIT WITH APPLICATIONS IN CARDIAC ARRHYTHMIA

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Abstract Controlling chaos through stability in fixed and periodic states is used in various engineering problems such as heat convection, reduction control, spine-wave instability, traffic control models, cardiac arrhythmia, chemical chaos, etc. Traditionally, this process is done in the coordination of chaos and stability in fixed and periodic points by using fixed point iterative procedures. Therefore, this article deals with a novel alliance between stabilization in one-dimensional discrete maps and Noor fixed point iterative procedure. The procedure contains α, β, γ and r, as its four new control parameters due to which the stability rate increases more rapidly than the other existing procedures. The stability theorems and a few time varying examples for fixed and periodic points are studied using Noor control system. Further, the Lyapunov exponent property is also described and the maximum Lyapunov value is determined to examine the stability in fixed and periodic points. Moreover, an improved control-based cardiac arrhythmia model is discussed in the Noor control system. Surprisingly, it is noted that the added new parameters α, β, and γ may increase the stability in chaotic arrhythmia rapidly.

Keywords Chaos, stability, lterative procedure, difference maps, Lyapunov exponent.

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1. Introduction

The chaotic phenomena generated by discrete difference maps and ordinary differential equations have played a vital role in every branch of science, such as physics, chemistry, biology, economics, electronic circuits, and engineering. Among the problems related to chaotic phenomena is chaos control that stabilizes the irregular fixed and periodic points in nonlinear one-dimensional maps. In the last two decades, the chaos control and the stability in aperiodic and irregular system using fixed-point methods applied to control parameters as well as system variables have dominated the researchers and academicians in chaos theory [28]. As far as is concerned, such methods were first established to solve the discrete dynamical systems by Ott et al. [27] and Ushio et al. [43]. Afterward, the study of the stability of fixed

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and periodic points was extended by Vieira et al. [14] in 1996 through a delayed feedback control system. But the modern study of stability and control fully trust on the mathematical as well as computational work established by Pyragas [33] in 1992 through difference and differential equations. Later on, several methods were presented to stabilize the chaos such as nonlinear dynamical inversion [23], oscillating control system [39], constant proportional system [11, 30], predictive control system [32], delayed feedback control [13, 37], entropy control technique [38] and active control technique [34].

In the last few decades, the theory of control is considered at the top of the modern study of nonlinear systems and is used in the processing and modeling of various scientific advancements. In 1991, Ditto et al. [17] established the control of chaos through stability in periodic fixed points of orders one and two. In 1992, Gerfinkel et al. [19] using chaos control property examined the stability in cardiac arrhythmias of rabbit ventricular. Further, Singer and Bau [42] studied theoretically that the control feedback system can be used to examine the change in the properties of the thermal convection in a toroidal loop and also found that the chaos can be suppressed using feedback control. In 1991, Peng et al. [31] studied the isothermal chaos and tried to control it using aperiodic orbits of order 1, 2, and 3. In 1997, Sinha [41] introduced various methods to stabilize the dynamical behavior in nonlinear systems and stabilized the regular as well as chaotic trajectories rapidly to the desired state. Moreover, due to the presence of irregularity in traffic models Jarett and Zhang [21] analyzed the evidence of chaotic phenomena in trip distribution. All the results on traffic control models are influenced by the methodology to detect the irregular motion given by Disbro and Frame [16]. In 2012, Grether et al. [20] demonstrated an efficient transportation problem that depends on two control parameters. In 2014, Ashish et al. [1] introduced an escape criteria for one-dimensional maps using Noor orbit and generated interesting fractal images (see also [12]). Also, for more scientific advancements in the stability and control researchers can follow Noor et al. [25, 26], Jiang et al. [22], Boccaletti et al. [10], Shang et al. [40], Elaydi [18], etc.

In 2017, Baleanu et al. [9] examined the monotonicity and asymptotic stability in the fractional type chaotic maps with Caputo delta property and also proved the stability using the Lyapunov direct method. The minimum entropy control, another interesting property to control the chaotic behavior in chaotic maps was illustrated by Sadeghian et al. [36] using fuzzy algorithm. In 2018 and 2019, in a series of papers, Ashish et al. [2–4] established the chaotic phenomena in chaotic maps using superior control technique and also demonstrated a chaos-based improved application in traffic control system. Recently, in 2021 they also examined the dynamics in modulated chaotic maps in Ashish et al. [5] and discrete hyperbolicity in Ashish et al. [6]. In 2022, Renu et al. [35] studied the dynamical properties of a novel difference equation using the Mann procedure. The dynamical properties like periodicity, fixed-point evolution and Lyapunov exponent were examined using analytical and geometrical interpretations. Further, the dynamic performance for a generalized cubic equation was carried out in [7, 8]. Recently, Panigoro et al. [29] established a Caputo fractional order logistic model to study the preservation of a population with the Allee effect and proportional harvesting. Also, they examined the dynamic property of the model such as fixed points, stability, saddle node and period-doubling bifurcation followed by Cobweb, maximum Lyapunov exponent and bifurcation.

This article shows an advancement in the study of chaos control using the Noor fixed-point feedback procedure, a four-step feedback approach. This article is organized into four sections. Section 1 presents a brief literature review on chaos control and their applications. In Section 2, a few basic definitions and notions are discussed. Further, the main outcomes of the article are presented in Section 3 using the Noor control system. At last, the whole paper is concluded in Section 4.

2. Preliminaries

For the sake of convenience, the several famous definitions and results of the study that plays an important role in further sections are assembled in this region.

Definition 2.1. Let ϕ_r be a self-map defined on a non-empty set V. If $\phi_r(v) = v$, for some $v \in V$, then v is known as a regular point for the map ϕ_r. Also, v is called periodic point of period-p if $\phi_r^p(v) = v$, where $p \in Z^+$ [15].

Definition 2.2. Let ϕ_r be a self-map defined on V and let $\phi_r'(v)$ be the first order derivative of the map ϕ_r, where v is a regular fixed point for the map ϕ_r. Then, for the condition $|\phi_r'(v)| < 1$, v is stable and for the condition $|\phi_r'(v)| > 1$, v is unstable [15].

Surprisingly, the fixed point feedback procedures are assumed as the backbone of the chaos theory in nonlinear dynamics, which are used to solve the dynamical properties in the difference and differential equations. Following is a more efficient feedback procedure:

Definition 2.3. Let $\{v_n\}_{n\in N}$ be a sequence of recursive outcomes of the following system depending on the parameters $\alpha, \beta, \gamma \in (0, 1)$:

$$\begin{aligned} z_n &= (1-\gamma)v_n + \gamma\phi_r(v_n), \\ y_n &= (1-\beta)v_n + \beta\phi_r(z_n), \\ \text{and} \qquad v_{n+1} &= (1-\alpha)v_n + \alpha\phi_r(y_n), \end{aligned}$$

where $\phi_r : V \to V$ is a one-dimensional chaotic map. Then, the whole arrangement is called as Noor feedback procedure [24]. Since the relation contains α, β, γ and r, as its four new control parameters, therefore, it is used as a controlling system in our study. Further, it is seen that for $\alpha = 1$, $\beta = 0 = \gamma$ it reduces into Picard procedure, for $\beta = 0 = \gamma$ it reduces into Mann procedure and for $\gamma = 0$ it reduces into Ishikawa iterative procedure. In further section, we deals with the stability in various orbits using the Noor procedure.

3. Stabilization via Noor Orbit

3.1. Stability in fixed points

Throughout, this section we deal with the stabilization in unstable fixed points, periodic points and chaos using Noor control system. Among the difference and differential dynamical systems that exhibit fixed state, periodic state, and chaos, a well-known one-dimensional family is given by

$$v_{n+1} = \phi_r(v_n), \qquad n \in N. \tag{3.1}$$

In particular, a few one-dimensional maps which are used in the nonlinear dynamics are logistic map $rv(1-v)$, quadratic map $1-rv^2$, cubic map $rv^3-(1-r)v$, Ricker map $ve^{(1-v)r}$ and the generalized map $rv(1-v)^{\alpha}$, where r is a control parameter. Therefore, the following control system is proposed to control the chaos by stabilizing fixed and periodic points embedded into chaos by introducing external parameters. From Definition 2.3, for $v_0 \in [a,b]$, let v_1 be the first output, then, we have

$$\begin{aligned} z_0 &= (1-\gamma)v_0 + \gamma\phi_r(v_0), \\ y_0 &= (1-\beta)v_0 + \beta\phi_r(z_0), \\ \text{and} \quad v_1 &= (1-\alpha)v_0 + \alpha\phi_r(y_0). \end{aligned}$$

Inductively, we can say

$$\begin{aligned} v_{n+1} &= (1-\alpha)v_n + \alpha\phi_r(y_n), \\ y_n &= (1-\beta)v_n + \beta\phi_r(z_n), \\ \text{and} \quad z_n &= (1-\gamma)v_n + \gamma\phi_r(v_n) = N_{\alpha,\beta,\gamma}(v_n) \quad (say), \end{aligned} \tag{3.2}$$

where $n \in N$. The relation (3.2) contains α, β, γ and r, as its four new control parameters and hence the relation is called as Noor control system. Further, we notice that for $\beta = 0 = \gamma$ the Noor control system reduces into Mann control system given by Ashish et al. [4]. Throughout this article, it is proposed to examine the experimental as well as mathematical analysis for chaos into stability for an efficient range of control parameters α, β and γ. The following results are derived:

Theorem 3.1. *Let $N_{\alpha,\beta,\gamma}(v)$ be the Noor control system (3.2) with $z_n = y_n = v^*$ and ϕ_r be an original nonlinear dynamical system (3.1). Then, $v^* \in V$ is the common fixed point for the system (3.1) and (3.2), that is, $\phi_r(v^*) = v^*$ if and only if $N_{\alpha,\beta,\gamma}(v^*) = v^*$, for some $v^* \in V$.*

Proof. Let $v^* \in V$ be a regular fixed point for a nonlinear dynamical system ϕ_r. Then, by using the Noor control system (3.2), we obtain

$$\begin{aligned} N_{\alpha,\beta,\gamma}(v^*) &= \alpha\phi_r(\beta\phi_r(\gamma\phi_r(v^*) + (1-\gamma)v^*) + (1-\beta)v^*) + (1-\alpha)v^*, \\ &= \alpha\phi_r(\beta\phi_r(\gamma v^* + (1-\gamma)v^*) + (1-\beta)v^*) + (1-\alpha)v^*, \\ &= \alpha\phi_r(v^* - \beta v^* + \beta\phi(v^*)) + (1-\alpha)v^*, \\ &= \alpha\phi_r(v^* - \beta v^* + \beta v^*) + (1-\alpha)v^*, \quad (\because\ \phi_r(v^*) = v^*) \\ &= \alpha\phi_r(v^*) + (1-\alpha)v^*, \\ &= \alpha v^* + (1-\alpha)v^*, \quad (\because\ \phi_r(v^*) = v^*) \\ \therefore \qquad N_{\alpha,\beta,\gamma}(v^*) &= v^*. \end{aligned}$$

Thus, the Noor control system $N_{\alpha,\beta,\gamma}(v)$ shares same set of fixed points v^* of the nonlinear dynamical system ϕ_r.

Conversely, let $v^* \in V$ satisfy $N_{\alpha,\beta,\gamma}(v^*) = v^*$ under the given condition $z_n = y_n = v^*$, such that

$$\begin{aligned} &\alpha\phi_r(\beta\phi_r(\gamma\phi_r(v^*) + (1-\gamma)v^*) + (1-\beta)v^*) + (1-\alpha)v^* = v^* \\ &\alpha\phi_r(\beta\phi_r(\gamma\phi_r(v^*) + (1-\gamma)v^*) + (1-\beta)v^*) - \alpha v^* = 0 \\ &\alpha[\phi_r(\beta\phi_r(\gamma\phi_r(v^*) + (1-\gamma)v^*) + (1-\beta)v^*) - v^*] = 0 \end{aligned}$$

$$\begin{aligned}
&\phi_r(\beta\phi_r(\gamma\phi_r(v^*)+(1-\gamma)v^*)+(1-\beta)v^*)=v^* \quad (\because\ \alpha\neq 0)\\
&\phi_r(\beta v_r(\gamma\phi_r(z_n)+(1-\gamma)z_n)+(1-\beta)y_n)=v^* \quad (\because\ z_n=y_n=v^*)\\
&\phi_r(\beta\phi_r(z_n)+y_n-\beta y_n)=v^*\\
&\phi_r(\beta v^*+v^*-\beta v^*)=v^* \quad (\because\ z_n=y_n=v^*)\\
\therefore\quad &\phi_r(v^*)=v^*.
\end{aligned}$$

Hence, the Noor control system $N_{\alpha,\beta,\gamma}(v)$ and the nonlinear dynamical system ϕ_r shares same set of fixed points v^*. This completes the proof. □

Theorem 3.2. *Let $v^*\in V$ be an unstable regular fixed point for a nonlinear dynamical system ϕ_r such that $|\phi_r'(v^*)|>1$. Then, $\ni$ a specific range of parameters α, β and γ such that $\alpha\in(\alpha_{min},\alpha_{max})=\Lambda_\alpha$, $\beta\in(\beta_{min},\beta_{max})=\Lambda_\beta$ and $\gamma\in(\gamma_{min},\gamma_{max})=\Lambda_\gamma$ in the Noor control system (3.2) such that $|N'_{\alpha,\beta,\gamma}(v^*)|<1$.*

Proof. Let $v^*\in V$ be an unstable regular fixed point for a nonlinear dynamical system ϕ_r such that $|\phi_r'(v^*)|>1$. Then, we use the following first order derivative statement of Devaney [15], a point $v^*\in V$ is stable or sink when $|N'_{\alpha,\beta,\gamma}(v^*)|<1$ and is unstable or stretch when $|N'_{\alpha,\beta,\gamma}(v^*)|>1$. Thus, we determine

$$\begin{aligned}
N'_{\alpha,\beta,\gamma}(v^*)&=\alpha\phi_r'[\beta\phi_r(\gamma\phi_r(v^*)+(1-\gamma)v^*)+(1-\beta)v^*].[\beta\phi_r'(\gamma\phi_r(v^*)+(1-\gamma)v^*)\\
&\quad+(1-\beta)].[\gamma\phi_r'(v^*)+1-\gamma]+(1-\alpha),\\
&=\alpha\phi_r'[\beta\phi_r(\gamma v^*+(1-\gamma)v^*)+(1-\beta)v^*].[\beta\phi_r'(\gamma v^*+(1-\gamma)v^*)\\
&\quad+(1-\beta)].[\gamma\phi_r'(v^*)+1-\gamma]+(1-\alpha),\\
&=\alpha\phi_r'[\beta\phi_r(v^*)+(1-\beta)v^*].[\beta\phi_r'(v^*)+(1-\beta)].[\gamma\phi_r'(v^*)+1-\gamma]\\
&\quad+(1-\alpha),\\
&=\alpha\phi_r'(v^*).[\beta\phi_r'(v^*)+(1-\beta)].[\gamma\phi_r'(v^*)+1-\gamma]+(1-\alpha),\\
N'_{\alpha,\beta,\gamma}(v^*)&=\alpha\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]+(1-\alpha). \qquad (3.3)
\end{aligned}$$

Since v^* is an unstable fixed-point, that is, $|\phi_r'(v^*)|>1$. Therefore, two cases arises for the Noor control system $N_{\alpha,\beta,\gamma}(v)$. Let's prove one by one:

Case-I. When $v^*>0$ is not stable for the system ϕ_r, that is, $\phi_r'(v^*)<-1$. Then, from equation (3.3) if $N'_{\alpha,\beta,\gamma}(v^*)<1$, where $\alpha,\beta,\gamma\in(0,1)$, we can write

$$\begin{aligned}
&\alpha\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]+(1-\alpha)<1,\\
&\alpha\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]-\alpha<0,\\
&\alpha[\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]-1]<0,\\
&\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]-1<0,\\
\text{thus,}\quad &\alpha>0=\alpha_{min}. \qquad (3.4)
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
&\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]-1<0,\\
&\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]<1,\\
&[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]<\frac{1}{\phi_r'(v^*)}. \qquad (3.5)
\end{aligned}$$

Solving (3.5), we obtain

$$\beta<\beta_{max}=\max\left\{\frac{1}{\phi_r'(v^*)-1}\left[\frac{1}{\phi_r'(v^*)}\left(\frac{1}{1+\gamma(\phi_r'(v^*)-1)}\right)-1\right]\right\}$$

$$= \frac{1}{\phi_r'(v^*)-1}\left[\frac{1}{\phi_r'(v^*)}\left(\frac{1}{1+\gamma_{min}(\phi_r'(v^*)-1)}\right)-1\right], \tag{3.6}$$

and

$$\begin{aligned}\gamma < \gamma_{max} &= \max\left\{\frac{1}{\phi_r'(v^*)-1}\left[\frac{1}{\phi_r'(v^*)}\left(\frac{1}{1+\beta(\phi_r'(v^*)-1)}\right)-1\right]\right\}\\ &= \frac{1}{\phi_r'(v^*)-1}\left[\frac{1}{\phi_r'(v^*)}\left(\frac{1}{1+\beta_{min}(\phi_r'(v^*)-1)}\right)-1\right].\end{aligned} \tag{3.7}$$

Now, if $N'_{\alpha,\beta,\gamma}(v^*) > -1$, then, from (3.3), we can say

$$\begin{aligned}\alpha\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]+(1-\alpha) &> -1,\\ \alpha\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]-\alpha &> -2,\\ \alpha[\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]-1] &> -2.\end{aligned} \tag{3.8}$$

Solving (3.8) for the parameters α, β and γ, we obtain

$$\begin{aligned}\alpha<\alpha_{max} &= \max\left\{\frac{-2}{\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]-1}\right\},\\ &= \frac{-2}{\phi_r'(v^*).[1+\beta_{min}(\phi_r'(v^*)-1)].[1+\gamma_{min}(\phi_r'(v^*)-1)]-1},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\beta>\beta_{min} &= \min\left\{\left[\frac{1}{1+\gamma(\phi_r'(v^*)-1)}\right]\left[\frac{1}{\phi_r'(v^*)-1}\left(\frac{1}{\phi_r'(v^*)}\left(1-\frac{2}{\alpha}\right)-1\right)+\gamma\right]\right\}\\ &= \left[\frac{1}{1+\gamma_{max}(\phi_r'(v^*)-1)}\right]\left[\frac{1}{\phi_r'(v^*)-1}\left(\frac{1}{\phi_r'(v^*)}\left(1-\frac{2}{\alpha_{max}}\right)-1\right)+\gamma_{min}\right],\end{aligned} \tag{3.10}$$

$$\begin{aligned}\gamma>\gamma_{min} &= \min\left\{\left[\frac{1}{1+\beta(\phi_r'(v^*)-1)}\right]\left[\frac{1}{\phi_r'(v^*)-1}\left(\frac{1}{\phi_r'(v^*)}\left(1-\frac{2}{\alpha}\right)-1\right)+\beta\right]\right\}\\ &= \left[\frac{1}{1+\beta_{max}(\phi_r'(v^*)-1)}\right]\left[\frac{1}{\phi_r'(v^*)-1}\left(\frac{1}{\phi_r'(v^*)}\left(1-\frac{2}{\alpha_{max}}\right)-1\right)+\beta_{min}\right].\end{aligned} \tag{3.11}$$

Then, from (3.4), (3.6), (3.7), (3.9), (3.10) and (3.11), we find an effective regime $\Lambda_\alpha = (\alpha_{min}, \alpha_{max})$, $\Lambda_\beta = (\beta_{min}, \beta_{max})$ and $\Lambda_\gamma = (\gamma_{min}, \gamma_{max})$ for the control parameters α, β and γ, respectively.

Case-II. When $v^* > 0$ is not stable for the system ϕ_r, that is, $\phi_r'(v^*) > 1$. Then, from (3.3) for $N'_{\alpha,\beta,\gamma}(v^*) < 1$, where $\alpha, \beta, \gamma \in (0, 1)$, we have

$$\begin{aligned}&\alpha\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]+(1-\alpha) < 1,\\ &\alpha[\phi_r'(v^*).[1+\beta(\phi_r'(v^*)-1)].[1+\gamma(\phi_r'(v^*)-1)]-1] < 0,\\ \text{thus,}\quad &\alpha < 0 = \alpha_{max}.\end{aligned} \tag{3.12}$$

Also, if $N'_{\alpha,\beta,\gamma}(v^*) > -1$, then, from (3.8), we get

$$\alpha > \alpha_{min} = \frac{-2}{\phi_r'(v^*).[1+\beta_{max}(\phi_r'(v^*)-1)].[1+\gamma_{max}(\phi_r'(v^*)-1)]-1}. \tag{3.13}$$

Thus, from (3.12) and (3.13) it clear that $\alpha \notin (\alpha_{min}, \alpha_{max})$ which a contradiction because $\alpha \in (0, 1)$. Hence the intervals $\Lambda_\alpha = (\alpha_{min}, \alpha_{max})$, $\Lambda_\beta = (\beta_{min}, \beta_{max})$ and $\Lambda_\gamma = (\gamma_{min}, \gamma_{max})$ determined in Case-I will stabilize the fixed point v^* in an original system under the condition $\phi_r'(v^*) < -1$. □

Remark 3.1. In connection with an unstable regular point $v^* \in V$ and the sign of derivative for $\phi_r(v)$, there exists a prescribed range of control parameters α, β and γ in the form $(\alpha_{min}, \alpha_{max})$, $(\beta_{min}, \beta_{max})$ and $(\gamma_{min}, \gamma_{max})$, respectively through Noor control system (3.2), where the fixed-point v^* get stabilized, that is, $|N'_{\alpha,\beta,\gamma}(v^*)| < 1$.

Remark 3.2. It is noticed that if we take $\beta = 0$ and $\gamma = 0$ in the Theorem 3.2, then it reduces into the superior control system given by Ashish et al. [4].

Example 3.1. Let $\phi_r : [0, 1] \to [0, 1]$ be a chaotic map given by $\phi_4(v) = 4v(1 - v)$, where $v \in [0, 1]$ having $v_1 = 0$ and $v_2 = 3/4$ as its two unstable fixed-point, that is, $\phi_4'(v_1) > 1$ and $\phi_4'(v_2) < -1$. Then, using the Noor control system (3.2) examine an effective range of α, β and γ for which the fixed points v_1 and v_2 are stable.

Solution. To examine the prescribed range for the control parameters α, β and γ in the stabilization of the fixed points v_1 and v_2, let us start with the time-series diagram of an original system $4v(1 - v)$. Figure 1(a) shows a complete irregular behavior for an original system $4v(1-v)$, where $v \in [0, 1]$. Therefore, using Theorem 3.2, it is possible to determine a specific regime of the control parameters α, β and γ in the system $N_{\alpha,\beta,\gamma}(v)$, where the fixed points v_1 and v_2 attains its complete stability for an original system. For this, let us start with the derivative of the Noor control system $N_{\alpha,\beta,\gamma}(v)$:

$$N'_{\alpha,\beta,\gamma}(v) = \alpha\phi_r'(v).[1 + \beta(\phi_r'(v) - 1)].[1 + \gamma(\phi_r'(v) - 1)] + (1 - \alpha). \tag{3.14}$$

Now, taking $\phi_4'(v_2) = 4(1 - 2v)$ and $v_2 = \frac{3}{4}$ in (3.14), we get

$$N'_{\alpha,\beta,\gamma}(3/4) = -2\alpha(1 - 3\beta)(1 - 3\gamma) + (1 - \alpha). \tag{3.15}$$

Since $\phi_4'(v_2) = -2 < -1$ at $v_2 = \frac{3}{4}$. Then, the following two cases arises for the stability condition $|N'_{\alpha,\beta,\gamma}(v_2)| < 1$:

Case-I. When $N'_{\alpha,\beta,\gamma}(v_2) < 1$ and $\phi_4'(v_2) = -2 < -1$, then, from (3.15) we get

$$\begin{aligned} & -2\alpha(1 - 3\beta)(1 - 3\gamma) + 1 - \alpha < 1, \\ & -2\alpha(1 - 3\beta)(1 - 3\gamma) - \alpha < 0, \\ & \alpha[-2(1 - 3\beta)(1 - 3\gamma) - 1] < 0, && (3.16) \\ \text{thus,} \quad & \alpha > 0 = \alpha_{min}. && (3.17) \end{aligned}$$

On the other hand from (3.16), we can say

$$\beta < \max\left\{\frac{1}{3}\left(\frac{1}{2(1 - 3\gamma)} - 1\right)\right\} = \frac{1}{3}\left(\frac{1}{2(1 - 3\gamma_{min})} - 1\right), \tag{3.18}$$

$$\text{and,} \quad \gamma < \max\left\{\frac{1}{3}\left(\frac{1}{2(1 - 3\beta)} - 1\right)\right\} = \frac{1}{3}\left(\frac{1}{2(1 - 3\beta_{min})} - 1\right). \tag{3.19}$$

Taking $\gamma_{min} = 0$ and $\beta_{min} = 0$ since $\beta, \gamma \in (0, 1)$, we get

$$\beta < \beta_{max} = \frac{1}{2}, \quad \text{and} \quad \gamma < \gamma_{max} = \frac{1}{2}. \tag{3.20}$$

Case-II. When $N'_{\alpha,\beta,\gamma}(v_2) > -1$ and $\phi'_4(v_2) = -2 < -1$, then, from (3.15) we can write

$$\begin{aligned} &-2\alpha(1-3\beta)(1-3\gamma) + 1 - \alpha > -1, \\ &-2\alpha(1-3\beta)(1-3\gamma) - \alpha > -2. \end{aligned} \tag{3.21}$$

Now, solving (3.21), we obtain

$$\alpha < \max\left\{\frac{2}{2(1-3\beta)(1-3\gamma)-1}\right\} = \frac{2}{2(1-3\beta_{min})(1-3\gamma_{min})-1}. \tag{3.22}$$

Putting $\gamma_{min} = 0$ and $\beta_{min} = 0$, we have

$$\alpha < \alpha_{max} = \frac{2}{3}. \tag{3.23}$$

Similarly, solving (3.21) for β and γ, then, we can say

$$\begin{aligned} \beta &> \min\left\{\left(\frac{1}{1-3\gamma}\right)\left[\frac{1}{-3}\left(\frac{1}{-2}\left(1-\frac{2}{\alpha}\right)-1\right)+\gamma\right]\right\}, \\ &= \left(\frac{1}{1-3\gamma_{max}}\right)\left[\frac{1}{-3}\left(\frac{1}{-2}\left(1-\frac{2}{\alpha_{max}}\right)-1\right)+\gamma_{min}\right], \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \gamma &> \min\left\{\left(\frac{1}{1-3\beta}\right)\left[\frac{1}{-3}\left(\frac{1}{-2}\left(1-\frac{2}{\alpha}\right)-1\right)+\beta\right]\right\} \\ &= \left(\frac{1}{1-3\beta_{max}}\right)\left[\frac{1}{-3}\left(\frac{1}{-2}\left(1-\frac{2}{\alpha_{max}}\right)-1\right)+\beta_{min}\right]. \end{aligned} \tag{3.25}$$

Now, taking $\gamma_{max} = 1$, $\gamma_{min} = 0$, $\beta_{max} = 1$, $\beta_{min} = 0$ and $\alpha_{max} = \frac{2}{3}$ in (3.24) and (3.25), we obtain

$$\beta > \beta_{min} = 0 \quad \text{and} \quad \gamma > \gamma_{min} = 0. \tag{3.26}$$

Thus, from (3.17), (3.20), (3.23) and (3.26) we obtain an effective regime of control parameters: $\Lambda_\alpha = (0, \frac{2}{3})$, $\Lambda_\beta = (0, \frac{1}{2})$ and $\Lambda_\gamma = (0, \frac{1}{2})$ in which the fixed point $v_2 = \frac{3}{4}$ is stabilized globally. While the Figure 1(b) gives a time-series plot for the values $\alpha \in (0, \frac{2}{3})$, $\beta \in (0, \frac{1}{2})$ and $\gamma \in (0, \frac{1}{2})$, where the fixed point v_2 is always stable. Figure 1(c) and 1(d) shows that bifurcation plot in which all the periodic states and chaotic regime approaches to a stable fixed point $v_2 = \frac{3}{4}$.

Similarly, for the fixed point $v_1 = 0$, $\phi'_4(v_1) = 4 > 1$. Then, from Theorem 3.2, we have

$$N'_{\alpha,\beta,\gamma}(0) = 4\alpha(1+3\beta)(1+3\gamma) + (1-\alpha). \tag{3.27}$$

Since $\phi'_4(v_1) = 4 > 1$ at $v_1 = 0$. Then, again for the stability condition $|N'_{\alpha,\beta,\gamma}(v_1)| < 1$ following two cases arises:

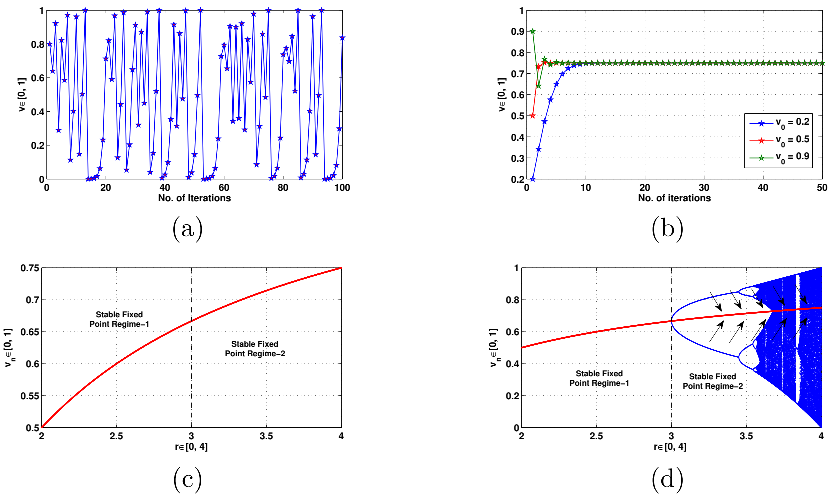


Figure 1. (a) Unstable fixed point plot for an original map $\phi_4(v) = 4v(1-v)$, (b) Stable fixed point plot for an original map $\phi_4(v) = 4v(1-v)$ when $N = 50$, (c) Bifurcation plot for stable fixed points at $\alpha = 0.4, \beta = 0.3$ and $\gamma = 0.3$, (d) Bifurcation plot versus stable fixed points at $\alpha = 0.4, \beta = 0.3$ and $\gamma = 0.3$

Case-I. When $N'_{\alpha,\beta,\gamma}(v_1) < 1$ and $\phi'(v_1) = 4 > 1$, then, from (3.27) we get

$$\begin{aligned} &4\alpha(1+4\beta)(1+4\gamma) + 1 - \alpha < 1, \\ &4\alpha(1+4\beta)(1+4\gamma) - \alpha < 0, \\ \text{thus,}\quad &\alpha < 0 = \alpha_{max}. \end{aligned} \tag{3.28}$$

Case-II. When $N'_{\alpha,\beta,\gamma}(v_1) > -1$ and $\phi'(v_1) = 4 > 1$, then, from (3.27), we get

$$\begin{aligned} \alpha > \alpha_{min} &= \min\left\{\frac{-2}{4(1+3\beta)(1+3\gamma)-1}\right\} \\ &= \frac{-2}{4(1+3\beta_{max})(1+3\gamma_{max})-1}. \end{aligned} \tag{3.29}$$

Substituting $\beta_{max} = 1$ and $\gamma_{max} = 1$, we obtain

$$\alpha > \alpha_{min} = -\frac{2}{63}. \tag{3.30}$$

Thus, from (3.28) and (3.30), we find that $\Lambda_\alpha = (\alpha_{min}, \alpha_{max}) = (-\frac{2}{63}, 0)$, which is a contradiction since $\alpha \in (0,1)$. Therefore, it is clear that the result can not be applied for the fixed point $v_1 = 0$.

Example 3.2. Let $\phi_r : [0,1] \to [0,1]$ be a cubic map given by $\phi_4(v) = 4v^3 - 3v$, where $v \in [0,1]$ having $v_1 = 0$, $v_2 = 1$ and $v_3 = -1$ as its three unstable fixed points, that is, $\phi'_4(v_1) < -1$, $\phi'_4(v_2) > 1$ and $\phi'_4(v_3) > 1$. Then, using the Noor control system (3.2) examine the specific range for the parameters α, β and γ for which the fixed points v_1, v_2 and v_3 are stable.

Solution. Let $\phi_4(v) = 4v^3 - 3v$ be an original cubic map, having three unstable fixed point $v_1 = 0$, $v_2 = 1$ and $v_3 = -1$. Then, using Theorem 3.2 and the Noor control system $N_{\alpha,\beta,\gamma}(v)$, let us determine the stability in v_1, v_2 and v_3. Therefore, we have

$$\phi_4'(v_1) = -3 < -1, \quad \phi_4'(v_2) = 9 > 1 \quad \text{and} \quad \phi_4'(v_3) = 9 > 1. \tag{3.31}$$

Now, first we examine the stabilization in the fixed point v_1. Taking $\phi_4'(v_1) = -3$ at $v_1 = 0$, we get

$$N'_{\alpha,\beta,\gamma}(0) = -3\alpha(1-4\beta)(1-4\gamma) + (1-\alpha). \tag{3.32}$$

Since $\phi_4'(v_1) = -3 < -1$, therefore, for the stabilization of v_1 two case arises, that is, Case-I for $N'_{\alpha,\beta,\gamma}(v_1) < 1$ and Case-II for $N'_{\alpha,\beta,\gamma}(v_1) > -1$. Then, using Theorem 3.2, we obtain the following conditions:

$$\alpha_{min} = 0, \;\; \beta_{min} = 0, \;\; \gamma_{min} = 0, \tag{3.33}$$

$$\alpha_{max} = \frac{2}{3(1-4\beta_{min})(1-4\gamma_{min}) - 1} = \frac{2}{3}, \tag{3.34}$$

$$\beta_{max} = \frac{1}{-4}\left[\frac{1}{-3}\left(\frac{1}{1-4\gamma_{min}}\right) - 1\right] = \frac{1}{3}, \tag{3.35}$$

and,
$$\gamma_{max} = \frac{1}{-4}\left[\frac{1}{-3}\left(\frac{1}{1-4\beta_{min}}\right) - 1\right] = \frac{1}{3}. \tag{3.36}$$

Then, from (3.33)-(3.36), we get the required intervals $\Lambda_\alpha = (0, \frac{2}{3})$, $\Lambda_\beta = (0, \frac{1}{3})$, and $\Lambda_\gamma = (0, \frac{1}{3})$, where the fixed point $v_1 = 0$ get stabilized for the given original system. Similarly, for the unstable fixed points v_2 and v_3 we get $\Lambda_\alpha = (\frac{2}{3}, 0)$, $\Lambda_\beta = (\frac{1}{3}, 0)$, and $\Lambda_\gamma = (\frac{1}{3}, 0)$ which a contradiction since $\alpha, \beta, \gamma \in (0,1)$. Thus, v_2 and v_3 can not be stabilized.

3.2. Stability in periodic fixed points

Theorem 3.3. *Let $v^* \in V$ be an unstable periodic point for a nonlinear dynamical system ϕ_r, that is, $|\phi_r^{p'}(v^*)| > 1$. Then, $\ni$ a specific range of control parameters α, β and γ such that $\alpha \in (\alpha_{min}, \alpha_{max})$, $\beta \in (\beta_{min}, \beta_{max})$ and $\gamma \in (\gamma_{min}, \gamma_{max})$ in the Noor control system (3.2) such that $|N^{p'}_{\alpha,\beta,\gamma}(v^*)| < 1$.*

Proof. Let $v^* \in V$ be an unstable periodic point for a nonlinear dynamical system ϕ_r. For the stabilization of v^* using Noor control system (3.2), let us start with the derivative of p^{th} iterate of $N_{\alpha,\beta,\gamma}(v)$ and the original system ϕ_r as follows:

$$N^{p'}_{\alpha,\beta,\gamma}(v) = \alpha\phi_r^{p'}(v^*).[1 + \beta(\phi_r^{p'}(v^*) - 1)].[1 + \gamma(\phi_r^{p'}(v^*) - 1)] + (1-\alpha). \tag{3.37}$$

Since v^* is an unstable periodic fixed-point, that is, $|\phi_r^{p'}(v^*)| > 1$. Therefore, two case arises using the Noor control system $N_{\alpha,\beta,\gamma}(v)$. In Case-I, for $\phi_r^{p'}(v^*) < -1$ and $-1 < N^{p'}_{\alpha,\beta,\gamma}(v^*) < 1$, we determine the following effective regime of control parameters α, β and γ as proved in Theorem 3.2 by replacing $\phi_r'(v)$ into $\phi_r^{p'}(v)$:

$$\Lambda^p_\alpha = (\alpha_{min}, \alpha_{max}), \quad \Lambda^p_\beta = (\beta_{min}, \beta_{max}), \quad \text{and} \quad \Lambda^p_\gamma = (\gamma_{min}, \gamma_{max}). \tag{3.38}$$

Similarly, in Case-II, for $\phi_r^{p'}(v^*) > 1$ and $-1 < N^{p'}_{\alpha,\beta,\gamma}(v^*) < 1$, it is examined that $\alpha \notin \Lambda^p_\alpha$, $\beta \notin \Lambda^p_\beta$ and $\gamma \notin \Lambda^p_\gamma$ which is a contradiction as $\alpha, \beta, \gamma \in (0,1)$. Thus, the intervals Λ^p_α, Λ^p_β and Λ^p_γ established in Case-I will be used to determine the stability in aperiodic point v^* of an original system ϕ_r. $\square$

Example 3.3. Let ϕ_r be an original one-dimensional map defined by $\phi_4(v) = 4v(1-v)$, where $v \in [0,1]$. Then, using the Noor control system $N^2_{\alpha,\beta,\gamma}(v)$ determine the stability in periodic fixed-points of order-2 for an effective regime of the control parameters α, β and γ.

Solution. Let $v_1 = \frac{5+\sqrt{5}}{8}$ and $v_2 = \frac{5-\sqrt{5}}{8}$ be the two unstable periodic point for the map $\phi_4(v) = 4v(1-v)$, that is, $|\phi_4^{2'}(v)| > 1$. Therefore, to examine the stability of v_1 and v_2, let us start with the following Noor control system:

$$N^2_{\alpha,\beta,\gamma}(v) = \alpha\phi_4^2(v).[1+\beta(\phi_4^2(v)-1)].[1+\gamma(\phi_4^2(v)-1)] + (1-\alpha). \tag{3.39}$$

Since v_1 and v_2 are the two regular fixed point for the map $\phi_4^2(v)$, that is, $\phi_4^2(v_1) = v_1$ and $\phi_4^2(v_2) = v_2$. Therefore, let us consider

$$\begin{aligned} \phi_4^2(v) &= 16v - 16v^2 - 64v^2(1-v)^2, \\ \text{then,} \quad \phi_4^{2'}(v) &= 16v - 32v - 128v(1-v)^2 + 128v^2(1-v). \end{aligned} \tag{3.40}$$

Substituting $v_1 = \frac{5+\sqrt{5}}{8}$ and $v_2 = \frac{5-\sqrt{5}}{8}$ in (3.40), we get $\phi_4^{2'}(v_1) = -3.99 = \phi_4^{2'}(v_2)$. Then, using Theorem 3.3, we obtain the following effective regime for α, β and γ, where both the fixed points v_1 and v_2 will stabilize:

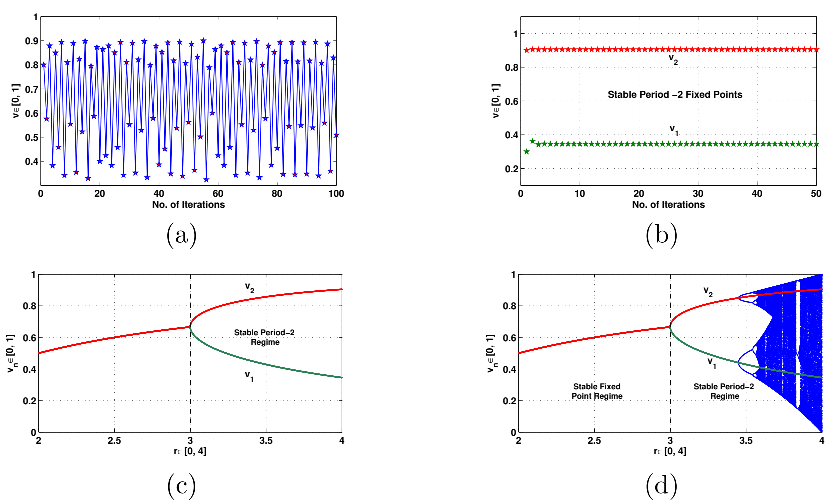


Figure 2. (a) Unstable periodic point plot for an original map $\phi_4(v) = 4v(1-v)$, (b) Stable periodic point plot for an original map $\phi_4^2(v)$ when $N = 50$, (c) Bifurcation plot for stable periodic points at $\alpha = 0.3, \beta = 0.2$ and $\gamma = 0.2$, (d) Bifurcation plot versus stable periodic points at $\alpha = 0.3, \beta = 0.2$ and $\gamma = 0.2$

$$\Lambda^2_\alpha = \left(0, \frac{2}{5}\right), \quad \Lambda^2_\beta = \left(0, \frac{1}{4}\right), \quad \text{and} \quad \Lambda^2_\gamma = \left(0, \frac{1}{4}\right). \tag{3.41}$$

Figure 2(a) presents a time-series plot for unstable periodic fixed points v_1 and v_2 using an original system ϕ_r. While Figures 2(b)-2(d), shows the complete stability behavior. Figure 2(b), represents that all the movement started with $v_0 \in [0,1]$ approaches to v_1 and v_2, respectively. While the bifurcation plot in Figure 2(d) gives a very interesting view of stability of periodicity for the parameter $r \in [0, 4]$. The irregularity and all the periodicity other then v_1 and v_2 reduces to stable periodicity of order 2.

3.3. Maximum Lyapunov exponent

The Lyapunov exponent, a well-known characteristics of chaos theory is used to determine the stability and unstability behavior of the fixed and periodic points in nonlinear systems. It is well-known that the negative Lyapunov exponent measures the stability in the system and the positive Lyapunov exponent shows unstability, that is, chaos. In this section, we deal with the definition of maximum Lyapunov exponent to study the stability in fixed and periodic points of chaotic maps using Noor control system (3.2). Therefore, let us start with the following Noor system:

$$N_{\alpha,\beta,\gamma}(v) = \alpha\phi_r(\beta\phi_r(\gamma\phi_r(v) + (1-\gamma)v) + (1-\beta)v) + (1-\alpha)v,$$

where $\phi_r(v)$ denotes an original one-dimensional chaotic map. Now, for the initial values v_0 and $v_0 + h$, where $h \in (0, 1)$, the n^{th} recursive difference in $N_{\alpha,\beta,\gamma}(v)$ is given by

$$N^n_{\alpha,\beta,\gamma}(v_0 + h) - N^n_{\alpha,\beta,\gamma}(v_0) = h.\exp^{n\theta},$$
$$\text{that is,} \quad \frac{N^n_{\alpha,\beta,\gamma}(v_0 + h) - N^n_{\alpha,\beta,\gamma}(v_0)}{h} = \exp^{n\theta}.$$

Taking limit as $n \to \infty$ on both side, we get

$$\theta = \frac{1}{n}\log|N^{n'}_{\alpha,\beta,\gamma}(v_0)|.$$

Then, from Devaney's [15] Definition of derivative for periodic orbits, we obtain

$$\theta = \frac{1}{n}\log|N'_{\alpha,\beta,\gamma}(v_n).N'_{\alpha,\beta,\gamma}(v_{n-1}).\dots N'_{\alpha,\beta,\gamma}(v_0)|,$$
$$\text{that is,} \quad \theta = \frac{1}{n}\sum_{i=0}^{n}\log|N'_{\alpha,\beta,\gamma}(v_i)|, \tag{3.42}$$

where $\alpha, \beta, \gamma \in (0, 1)$, n denotes the number iteration in the orbit and θ is known as the maximum Lyapunov exponent in Noor fixed-point feedback system. Moreover, it is noticed that for the fixed point orbit relation (3.42) reduces into

$$\theta = \log|N'_{\alpha,\beta,\gamma}(v_0)| \tag{3.43}$$

and for the periodic orbits of order-p it reduces into

$$\theta = \frac{1}{p}\sum_{i=0}^{p}\log|N'_{\alpha,\beta,\gamma}(v_i)|. \tag{3.44}$$

Finally, it is observed that for the irregular orbits it is not possible to take all the iteration of the system. Therefore, the finite number of terms of the iterative orbit are taken at a time to determine the maximum Lyapunov exponent of the orbit.

Example 3.4. Let ϕ_r be an original one-dimensional map defined by $\phi_4(v) = 4v(1-v)$, where $v \in [0,1]$. Then, determine the Lyapunov exponent value (θ) for fixed and period-2 points and analyse the stability for a prescribed range of the parameters α, β and γ.

Solution. Since $\phi_4(v) = 4v(1-v)$ is a given logistic map, therefore, let us take $v_1 = 0$ and $v_2 = \frac{3}{4}$ its unstable regular fixed points. Thus, to examine the stabilization in fixed point v_2 it is sufficient to solve the relation (3.43) of maximum Lyapunov exponent. Let us consider

$$N'_{\alpha,\beta,\gamma}(v) = \alpha\phi'_r(v).[1+\beta(\phi'_r(v)-1)].[1+\gamma(\phi'_r(v)-1)] + (1-\alpha). \tag{3.45}$$

Taking $\phi'_r(v) = 4(1-2v)$ and $v = \frac{3}{4}$ in (3.45), we get

$$N'_{\alpha,\beta,\gamma}(3/4) = -2\alpha(1-3\beta)(1-3\gamma) + (1-\alpha). \tag{3.46}$$

Using the relation (3.43) and (3.46), we have

$$\theta = \log|-2\alpha(1-3\beta)(1-3\gamma) + (1-\alpha)|. \tag{3.47}$$

Then, for some particular values $\alpha \in (0, \frac{2}{3})$, $\beta \in (0, \frac{1}{2})$ and $\gamma \in (0, \frac{1}{2})$ in (3.47), we get

$$\theta = \log|-1(1-0.9)(1-0.9) + (1-0.5)|,$$
$$\theta = \log|0.49| = -0.3098.$$

Thus, the maximum Lyapunov exponent is negative for $v_2 = \frac{3}{4}$ and hence from the above Lyapunov definition the fixed point v_2 attains its complete stability in the prescribed range of control parameters α, β and γ.

Similarly, let us consider $v_1 = \frac{5+\sqrt{5}}{8}$ and $v_2 = \frac{5-\sqrt{5}}{8}$ be the two unstable periodic fixed point for the map $\phi_4(v) = 4v(1-v)$. Then, from (3.45), we get

$$N'_{\alpha,\beta,\gamma}(v_1) = \alpha(-3.236).[1+\beta(-3.236-1)].[1+\gamma(-3.236-1)] + (1-\alpha),$$
$$N'_{\alpha,\beta,\gamma}(v_2) = \alpha(+1.236).[1+\beta(+1.236-1)].[1+\gamma(+1.236-1)] + (1-\alpha).$$

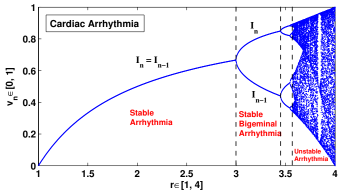
Then, for some particular values $\alpha \in (0, \frac{2}{5})$, $\beta \in (0, \frac{1}{4})$ and $\gamma \in (0, \frac{1}{4})$, we get

$$N'_{\alpha,\beta,\gamma}(v_1) = +0.618, \text{ and } N'_{\alpha,\beta,\gamma}(v_2) = +1.208. \tag{3.48}$$

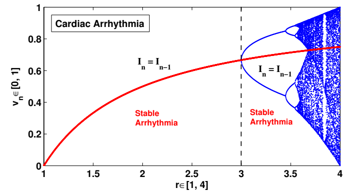
From (3.44) and (3.48), we obtain

$$\theta = \frac{1}{2}[\log|N'_{\alpha,\beta,\gamma}(v_1)| + log|N'_{\alpha,\beta,\gamma}(v_2)|],$$
$$\theta = \frac{1}{2}[\log|0.618| + \log|1.208|],$$
$$\theta = \frac{1}{2}[-0.2090 + 0.0821],$$
$$\theta = -0.0635.$$

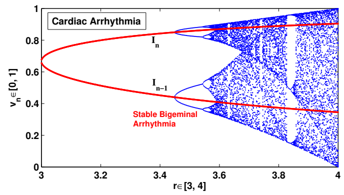
Thus, the maximum Lyapunov is negative for v_1 and v_2 and hence v_1 and v_2 are completely stable for a specific range of controlled parameters α, β and γ.



(a)



(b)



(c)

Figure 3. (a) Bifurcation plot for the Cardiac Arrhythmia model $I_n = rI_{n-1}(1 - I_{n-1})$, (b) Regular cardiac arrhythmia for model (3.50) when $\alpha = 0.4$, $\beta = 0.3$ and $\gamma = 0.3$ and $r \in [1, 4]$, (c) Bigeminal stable cardiac arrhythmia for model (3.50) when $\alpha = 0.3$, $\beta = 0.2$ and $\gamma = 0.2$ and $r \in [3, 4]$

Cardiac Arrhythmia	Original System	Noor Control System
Regular	$1 < r \leq 3$	$\alpha \in (0, \frac{2}{3})$, $\beta, \gamma \in (0, \frac{1}{2})$, $r \in [1, 4]$
Bigeminal	$3 < r \leq 3.45$	$\alpha \in (0, \frac{2}{5})$, $\beta, \gamma \in (0, \frac{1}{4})$, $r \in [3, 4]$
Quadrigeminy	$3.45 < r \leq 3.52$	Bigeminal or regular state
Higher order	$3.52 < r \leq 3.57$	Bigeminal or regular state
Chaotic	$3.57 < r \leq 4$	Bigeminal or regular state

Table 1. Original cardiac arrhythmia versus controlled arrhythmia system when the parameters α, β and γ varies in specific range

3.4. Application of stabilization in Cardiac Arrhythmia

The chaos control of nonlinear dynamical systems has a strong utilization in biology [44]. The first research in biological applications using chaos control was confirmed by Weiss et al. [44] in chaotic cardiac arrhythmia induced by the drug ouabain in rabbit ventricular. But in the last few decades, the study on cardiac arrhythmia is considered at the forefront of medical research because the heartbeat disorder has become a common reason of death in human life. Therefore, we try to introduce an improved chaos control method using the logistic map with a Noor control system that may improve the stability in chaotic arrhythmia. The discussion about the cardiac arrhythmia given by Weiss et al. [44] has properties analogous to the chaotic characteristics of the logistic map. Therefore, the following control model was demonstrated in [44]:

$$\phi_r(I_{n-1}) = I_n = rI_{n-1}(1 - I_{n-1}), \tag{3.49}$$

where I_n is the current interbeat interval, I_{n-1} is the previous interbeat interval, $r \in [0, 4]$ stands for an intracellular Ca level and n is the number of beats during the chaotic phase of cardiac arrhythmia. The intracellular Ca level induced by the ouabain in ventricular tachycardia is taken as the critical parameter (r) of the logistic map, that pushes the heart from fixed to periodic beating and then periodic to irregular beating. Therefore, from this perspective the following assumptions were made:

(i) When $1 < r \leq 3$, the current interbeat interval (I_n) against the previous interbeat (I_{n-1}) are equal, that means, heart is beating with regular stable rate.

(ii) When $3 < r \leq 3.45$, the heart beat begins to approach in bigeminal pattern or period-2 fashion. That means, a bigeminal pattern with a long interbeat interval followed by short interbeat interval in repeating $ABABAB\ldots$ arrangement.

(iii) When $3.45 < r \leq 3.57$, the arrhythmia process develops higher order pattern such as period-4 fashion or quadrigeminy repeating $ABCDABCD\ldots$ arrangement.

(iv) When $3.57 < r \leq 4$, the cardiac arrhythmia process eventually approaches to complete irregularity in interbeats with no repetition. At this stage, 85% heart becomes fully chaotic without any repetition.

Figure 3(a), shows the complete cardiac arrhythmia behavior in which the intracellular Ca level pushes the heart beating from fixed to periodic and periodic to chaotic stage. In 2000, Noor [24] introduced a four-step feedback procedure and proved that it converges speedily then Picard and Mann procedures. Therefore, looking into the strong behavior of Noor control system, it is assumed to introduce a more effective and efficient discrete control model that may reduce the sudden irregularity in cardiac arrhythmia rapidly. Therefore, using the relation (3.2) and the original cardiac arrhythmia model (3.49) the following control system is derived:

$$I_n = I_{n-1} + \alpha(I_{n-1} + \beta(I_{n-1} + \gamma(\phi_r(I_{n-1}) - I_{n-1}) - I_{n-1}) - I_{n-1}), \tag{3.50}$$

where $\phi_r(I_{n-1}) = rI_{n-1}(1 - I_{n-1})$ and r as an intracellular Ca level belongs to $[0, 4]$, I_n represents an interbeat interval in $[0, 1]$ and the system (3.50) is known as discrete cardiac arrhythmia control model with α, β and γ as its controlling parameters. From the previous section it is clear that as the parameters α, β and γ lies in a specific range, the system get stabilized into quadrigeminy, bigeminal and regular states. In particular, When we take $\alpha \in (0, \frac{2}{3})$, $\beta \in (0, \frac{1}{2})$ and $\gamma \in (0, \frac{1}{2})$ at $r = 4$, then the chaotic arrhythmia may be manipulated and reduced into a regular fixed state. Similarly, when we take $\alpha \in (0, \frac{2}{5})$, $\beta \in (0, \frac{1}{4})$ and $\gamma \in (0, \frac{1}{4})$ in the above control model (3.50), then the chaotic arrhythmia may also be reduced in to bigeminal or period-2 state.

Figure 3(b) shows the chaos control in chaotic arrhythmia, which reduces into a regular fixed state, where the current interbeat interval (I_n) is equal to the previous interbeat interval (I_{n-1}) to the whole intracellular Ca level (r). While the Figure 3(c) represents the bigeminal stability in chaotic arrhythmia, that is, the long interbeat interval (I_n) is followed by the short interbeat interval (I_{n-1}) in repeating pattern. Moreover, Table 1, shows how the chaotic arrhythmia reduces into stable, bigeminal, and other higher-order periodic orders such as quadrigeminy or period-4 pattern.

4. Conclusion

By using the Noor control system to the chaotic maps, a few novel results on chaos control, and the improved chaotic cardiac arrhythmia model is studied. Further, it is observed that as compared to Picard (one-step system) and Mann iteration (two-step system), the novel system generally admits four control parameters α, β, γ and r. Especially, we concentrate on the four-step iterative procedure given by Noor [24] in 2000. Further, due to the high convergence rate in the Noor iterative system the stabilization in the fixed and periodic points take place in a more effective and efficient way. Following outcomes are derived:

(i) Theorem 3.1 shows that an original chaotic system and the proposed Noor feedback procedure share the same set of fixed points. Theorem 3.2 is introduced to describe the stabilization in unstable regular points for a family of chaotic maps using the Noor control system. Further, the stability of fixed points in quadratic map $4v(1-v)$ and the cubic map $4v^3 - 3v$ is studied. It is also analyzed that the fixed point always remains in the stable equilibrium state for a specific range of control parameters α, β and γ.

(ii) Theorem 3.3 is proved to examine the stability in unstable periodic fixed points using Noor feedback procedure. Further, the stability in periodic points of quadratic map $4v(1-v)$ is determined in Example 3.3 for a prescribed range of $\alpha \in (0, \frac{2}{5})$, $\beta \in (0, \frac{1}{4})$ and $\gamma \in (0, \frac{1}{4})$ using the Noor control system.

(iii) Maximum Lyapunov exponent, another important characteristic of chaos control is used to examine the stability in chaotic maps. The statement of Lyapunov exponent is derived using the Noor control system and in Example 3.4 the Lyapunov exponent value is established for the stability of the quadratic map.

(iv) Further, an application of chaos control in cardiac arrhythmia is demonstrated in subsection 3.4. The chaotic arrhythmia is first illustrated by using a bifurcation diagram for the logistic map and then the complete dynamical behavior is studied through an interbeat-intracellular-ouabain model. Moreover, a superior cardiac arrhythmia model is proposed using Noor control system and the stability in chaotic arrhythmia is noted for a specific range of control parameters α, β and γ.

It is strongly recommended that future research in chaos control may be expended on various applications in bio-sciences using the Noor control system. Finally, it is noticed that with an original chaotic system and the specific range of control parameters α, β and γ, the chaotic arrhythmia can be stabilized against an ouabain-intracellular-interbeat model. Further, it is observed that such type of control technique may be used in other applications of science and engineering.

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