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ON THE PROPAGATION OF REGULARITY OF SOLUTIONS TO THE NONLINEAR FIFTH ORDER EQUATION OF KDV TYPE

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Abstract We investigate special regularity of solutions to the initial value problem associated to the nonlinear fifth order equation of KdV type. The main results show that for datum $u_0 \in H^s(\mathbf{R}), F(u) \in C^{s+2}(\mathbf{R})$ with $s \geq 5$, whose restriction belongs to $H^l((x_0, \infty))$ and $H^{l+2}((x_0, \infty))$ respectively, for some $l \in \mathbb{Z}^+$ and $x_0 \in \mathbf{R}$, then the restriction of the corresponding solution $u(\cdot, t)$ belongs to $H^l((b, \infty))$ for any $b \in \mathbf{R}$ and any $t \in (0, T)$. Thus, this type of regularity travels with infinite speed to its left as time evolves. To a certain extent, our results complement the previous studies on the related aspects, and deepen the understanding of such properties for the dispersion equation.

Keywords Nonlinear fifth order equation, Sobolev space, propagation of regularity.

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1. Introduction

In this work we are concerned with the initial value problem (IVP) of the nonlinear fifth order equation of KdV type, which can be written as:

$$\begin{cases} u_t - u_{x^5} - \left(\dfrac{\partial F(u)}{\partial u}\right)_x - \left(\dfrac{\partial G\left(u, u_x\right)}{\partial u}\right)_x + \left(\dfrac{\partial G\left(u, u_x\right)}{\partial u_x}\right)_{xx} = 0, \ x, t \in \mathbf{R}, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

where

$$\begin{cases} G\left(u, u_x\right) = N_1 + N_2 + N_3 + N_4, \\ N_1 = a_1 u u_x^2, \quad N_2 = a_2 u^2 u_x^2, \quad N_3 = a_3 u^3 u_x^2, \quad N_4 = a_4 u_x^3, \end{cases} \tag{1.2}$$

$a_i (i = 1, 2, 3, 4)$ are constants.

Many nonlinear higher order equations of KdV type have been proposed in physical problem. In [1], Guo-Han-Zhou first considered such a model as (1.1), and proved the existence of global smooth solutions with the periodic boundary condition and initial value condition, also they got the local smooth characterization

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of the solution for the initial value problem. More precisely, suppose that $F(u), u_0$ satisfy the following conditions

$$\begin{cases} (1) \quad F(\xi) \in C^{s+2}(\mathbf{R}), \quad |F''(\xi)| \le A_1 \left(1 + |\xi|^7\right), \\ \qquad F'(0) = F(0) = 0, \\ (2) \quad u_0 \in H^s(\mathbf{R}), \quad s \ge 5. \end{cases} \tag{1.3}$$

the following result is proved.

Theorem 1.1. *Under the condition (1.2) and (1.3), the system (1.1) has at least one smooth solution*

$$u(x,t) \in \bigcap_{k=0}^{[s/5]} W_\infty^k \left(0, T; H^{s-5k}(\mathbf{R})\right) \bigcap \left(\bigcap_{k=0}^{[(s+1)/5]} W_\infty^k (0, T; H_{local}^{s+1-5k}(\mathbf{R})) \right).$$

The aim of this work is to study special regularity properties of solutions to the equation (1.1). Firstly, let us briefly recall some works concerned with special regularities and decay properties of some dispersive models. The starting point is a property found by Isaza-Linares-Ponce [3] concerning the propagation of smoothness in solutions to the k-generalized KdV equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad x, t \in \mathbf{R}, k \in \mathbf{Z}^+, \\ u(x,0) = u_0(x). \end{cases}$$

They studied that the propagation of regularity on the right-hand side of the initial value for positive times by Kato's argument, and observed that this regularity travels with infinite speed to its left as time goes by. More precisely, if the initial datum $u_0 \in H^l((x_0, \infty))$, for some $l \in \mathbb{Z}^+$ and $x_0 \in \mathbf{R}$, then the corresponding solution $u(\cdot, t)$ belongs to $H^l((b, \infty))$ for any $b \in \mathbf{R}$ and any $t \in (0, T)$. Later, Kening et al. [7] extended this result to the case where $l > 3/4$. Inspired by [3], Segata-Smith [16] obtained the results regarding the following fifth order dispersive models

$$\partial_t u - \partial_x^5 u + c_1 u^2 \partial_x u + c_2 \partial_x u \partial_x^2 u + c_3 u \partial_x^3 u = 0,$$

where c_1, c_2, c_3 are real constants. In order to show how regularity on the initial data is transferred to the solutions, Linares-Ponce-Smith [9] gave the special regularity properties of the solution to the general quasilinear equation of KdV type, that is

$$\begin{cases} \partial_t u + a\left(u, \partial_x u, \partial_x^2 u\right) \partial_x^3 u + b\left(u, \partial_x u, \partial_x^2 u\right) = 0, \\ u(x,0) = u_0(x), \end{cases}$$

where the functions $a, b : \mathbb{R}^3 \times [0, T] \to \mathbb{R}$ satisfy:

(1) $a(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ are C^∞ with all derivatives bounded in $[-M, M]^3$, for any $M > 0$,

(2) given $M > 0$, there exists $\kappa > 1$ such that

$$1/\kappa \le a(x,y,z) \le \kappa, \qquad \text{for any } (x,y,z) \in [-M, M]^3,$$
$$\partial_z b(x,y,z) \le 0, \qquad \text{for } (x,y,z) \in [-M, M]^3.$$

And they asserted that this depends on the spaces where regularity is measured.

In addition, combining the approach in [3] and [16], there are many similar results have been confirmed for other models, such as the fractional KdV equation [12], Benjamin-Ono equation [5, 10], Benjamin equation [2] and the Intermediate long-wave equation [13]. Interestingly, this property is not only inherent to the one-dimensional nonlinear dispersive equations, but also to the multi-dimensional dispersive models, like Zakharov-Kuznetsov (ZK) equation [8,11], Kadomtsev-Petviashvili II (KP-II) [4], the fifth order Kadomtsev-Petviashvili II (KP5-II) [14], and the Benjamin-Ono-Zakharov-Kuznetsov (BO-ZK) equation [15], etc.

Remark 1.1. Isaza-Linares-Ponce [4] deduced some special regularity properties of solutions to the Kadomtsev-Petviashvili II (KP-II). It should be noted that the method based on energy estimates does not yield a similar result for Kadomtsev-Petviashvili I, because the terms $\partial_x^3 u$ and $-\partial_x^{-1}\partial_y^2 u$ lead to dispersion with opposite sign.

Considering that KdV equation has such a regularity property, naturally, we wonder that whether the propagation of regularity phenomena can be established in the nonlinear fifth order equation of KdV type (1.1). Motivated by the above studies, the objective of this paper is to describe the propagation of one-side regularity exhibited by solutions to the IVP (1.1) provided by Theorem 1.1.

Theorem 1.2. *Let $l \in \mathbb{Z}^+, l \geq 1$. If u_0, $F(u)$ satisfy (1.3), and for some $x_0 \in \mathbb{R}$,*

$$\left\|\partial_x^l u_0\right\|_{L^2(x_0,\infty)}^2 = \int_{x_0}^{\infty} \left(\partial_x^l u_0\right)^2 (x)dx < \infty, \tag{1.4}$$

$$\left\|\partial_x^{l+2} F(u)\right\|_{L^2(x_0,\infty)}^2 = \int_{x_0}^{\infty} \left(\partial_x^{l+2} F(u)\right)^2 (x)dx < \infty, \tag{1.5}$$

then the solution of the IVP (1.1) provided by Theorem 1.1 satisfies that for any $\varepsilon > 0$ and $v > 0$,

$$\sup_{0\leq t\leq T} \int_{x_0+\varepsilon-vt}^{\infty} \left|\partial_x^j u(x,t)\right|^2 dx \leq c_0,$$

for $j = 0, 1, 2, \dots l$, with

$$c_0 = c_0\left(l; \varepsilon; v; T; \|u_0\|_{H^s}; \|F(u)\|_{C^{s+2}}; \left\|\left(\partial_x^l u_0, \partial_x^{l+2} F(u)\right)\right\|_{L^2((x_0,\infty))}\right).$$

In particular, for all $t \in (0, T]$, the restriction of $u(\cdot, t)$ to any interval (x_1, ∞) belongs to $H^l((x_1, \infty))$.

Moreover, for any $\varepsilon > 0, v > 0$, and $R > 0$,

$$\int_0^T \int_{x_0+\varepsilon-vt}^{x_0+R-vt} \left|\partial_x^{l+2} u(x,t)\right|^2 dxdt \leq c_1,$$

where $c_1 = c_1\left(l; \varepsilon; v; R; T; \|u_0\|_{H^s}; \|F(u)\|_{C^{s+2}}; \left\|\left(\partial_x^l u_0, \partial_x^{l+2} F(u)\right)\right\|_{L^2((x_0,\infty))}\right)$.

Several direct consequences can be deduced from Theorem 1.2, for instance (for further outcomes see [3, 16]), since the system (1.1) is time reversible, it indicated

Corollary 1.1. *Let $u \in C\left([0,T]; H^s(\mathbb{R})\right), s \geq 5$, be the solution of the IVP (1.1) provided by Theorem 1.1. If there exist $n \in \mathbb{Z}^+, a \in \mathbb{R}$ and $\hat{t} \in (0, T)$ such that*

$$\partial_x^n u(\cdot, \hat{t}) \notin L^2((a, \infty)),$$

then for any $t \in (0, \hat{t})$ *and any* $b \in \mathbb{R}$,

$$\partial_x^n u(\cdot, t) \notin L^2((b, \infty)).$$

2. Preliminaries

We shall construct our cutoff functions firstly, which are inspired by [3, 16]. For γ be large enough, the constant $\lambda = \lambda(\gamma)$ being chosen to satisfy $\rho(1) = 1$, define the polynomial

$$\rho(x) = \lambda \int_0^x y^\gamma (1-y)^\gamma dy$$

which satisfies

$$\begin{aligned} &\rho(0) = 0, \quad \rho(1) = 1, \\ &\rho'(0) = \rho''(0) = \cdots = \rho^{(\gamma)}(0) = 0, \\ &\rho'(1) = \rho''(1) = \cdots = \rho^{(\gamma)}(1) = 0, \end{aligned}$$

with $0 < \rho, \rho'$ for $0 < x < 1$.

For parameters $\varepsilon, b > 0$, define $\chi \in C^\gamma(\mathbb{R})$ by

$$\chi(x; \varepsilon, b) = \begin{cases} 0 & x \leq \varepsilon, \\ \rho((x-\varepsilon)/b) & \varepsilon < x < b + \varepsilon, \\ 1 & b + \varepsilon \leq x. \end{cases} \tag{2.1}$$

By construction χ is positive for $x \in [\varepsilon, \infty)$ and all derivatives are supported in $[\varepsilon, b+\varepsilon]$. By the definition, χ is positive for $x \in [\epsilon, \infty)$. Computing as Sec. II in [16], we can derive the following properties concerning χ :

(1) $\chi(x; \varepsilon/5, 4\varepsilon/5) = 1$ on $\operatorname{supp} \chi(x; \varepsilon, b) = [\varepsilon, \infty)$,

(2) $\left| \frac{[\chi''(x;\varepsilon,b)]^2}{\chi'(x;\varepsilon,b)} \right| \leq c(\varepsilon, b) \chi'(x; \varepsilon/3, b+\varepsilon)$ on support of χ',

(3) $\left|\chi^{(j)}(x; \varepsilon, b)\right| \leq c(j, b)$, for $j = 1, 2, \ldots, \gamma$,

(4) $\left|\chi^{(j)}(x; \varepsilon, b)\right| \leq c(j, \varepsilon, b) \chi'(x; \varepsilon/3, b+\varepsilon)$ on $[\varepsilon, b+\varepsilon]$, for $j = 1, 2, \ldots, \gamma$.

By interpolation we have the following lemma, which is required to apply the inductive hypothesis.

Lemma 2.1. *Suppose* $\phi \in L^2(\mathbb{R})$ *and for some* $m \in \mathbb{Z}^+, m \geq 2, x_0 \in \mathbb{R}$,

$$\|\partial_x^m \phi\|_{L^2(x_0,\infty)}^2 = \int_{x_0}^\infty |\partial_x^m \phi|^2 dx < \infty.$$

For any $j = 1, 2, \ldots, m-1$ *and* $\delta > 0$,

$$\left\|\partial_x^j \phi\right\|_{L^2(x_0+\delta,\infty)}^2 = \int_{x_0+\delta}^\infty \left|\partial_x^j \phi\right|^2 dx < \infty.$$

Making use of Cauchy-Schwarz, Young's inequality, and Sobolev embedding, we get the following lemma, which is generalized the Lemma 3 of [16].

Lemma 2.2 (Lemma 3, [16]). *Let j_i $(i=1,2,\cdots p+1)\in\mathbb{Z}^+$ and $\varepsilon, b>0$. Suppose nonnegative function $\chi(x;\varepsilon,b)$ has support in $[\varepsilon,\infty)$ and $\chi(x;\varepsilon,b)\geq 1$ whenever $x\geq b+\varepsilon$, as defined in (2.1). Then*

$$\begin{aligned}&\left|\int\prod_{i=1}^{p+1}\partial_x^{j_i}u\chi(x)dx\right|\\ \lesssim&\left\{\int\sum_{m=j_1}^{j_1+p-1}\left(\partial_x^m u\right)^2\chi(x)dx+\int\sum_{m=j_1}^{j_1+p-2}\left(\partial_x^m u\right)^2\chi'(x;\varepsilon/3,b+\varepsilon)dx\right\}\\ &\times\prod_{k=2}^{p}\int\sum_{n=j_k}^{j_k+p-k}\left(\partial_x^n u\right)^2\chi(x;\varepsilon/5,4\varepsilon/5)dx+\int\left(\partial_x^{p+1}u\right)^2\chi(x)dx. \qquad (2.2)\end{aligned}$$

Of course, we can also replace χ with χ'.

For example, when $p=2$, (2.2) can be written as

$$\begin{aligned}&\left|\int\partial_x^{j_1}u\partial_x^{j_2}u\partial_x^{j_3}u\chi(x)dx\right|\\ \lesssim&\left\{\int(\partial_x^{j_1+1}u)^2\chi(x)dx+\int(\partial_x^{j_1}u)^2\chi(x)dx+\int(\partial_x^{j_1}u)^2\chi'(x)dx\right\}\\ &\times\int(\partial_x^{j_2}u)^2\chi(x;\varepsilon/5,4\varepsilon/5)dx+\int\left(\partial_x^{j_3}u\right)^2\chi(x)dx. \qquad (2.3)\end{aligned}$$

When $p=3$, (2.2) is written as follows

$$\begin{aligned}&\left|\int\partial_x^{j_1}u\partial_x^{j_2}u\partial_x^{j_3}u\partial_x^{j_4}u\chi(x)dx\right|\\ \lesssim&\int[(\partial_x^{j_2}u)^2+(\partial_x^{j_2+1}u)^2]\chi(x;\varepsilon/5,4\varepsilon/5)dx\int(\partial_x^{j_3}u)^2\chi(x;\varepsilon/5,4\varepsilon/5)dx\\ &\times\left\{\int[(\partial_x^{j_1+2}u)^2+(\partial_x^{j_1+1}u)^2+(\partial_x^{j_1}u)^2]\chi(x)dx+\int[(\partial_x^{j_1}u)^2+(\partial_x^{j_1+1}u)^2]\right.\\ &\left.\times\chi'(x;\varepsilon/3,b+\varepsilon)dx\right\}+\int(\partial_x^{j_4}u)^2\chi(x)dx. \qquad (2.4)\end{aligned}$$

Proof. We take $p=2,3$ as examples to give the proof.

First we observe that $\chi(x;\varepsilon,b)$ is nonnegative, supported on $[\varepsilon,\infty)$ and $\chi(x)\geq 1$ when $x\geq b+\varepsilon$. Thus

$$\begin{aligned}&\left|\int\partial_x^{j_1}u\partial_x^{j_2}u\partial_x^{j_3}u\chi dx\right|\\ \leq&\frac{1}{2}\int\left(\partial_x^{j_1}u\right)^2\left(\partial_x^{j_2}u\right)^2\chi dx+\frac{1}{2}\int\left(\partial_x^{j_3}u\right)^2\chi dx\\ \leq&\frac{1}{2}\left\|\left(\partial_x^{j_1}u\right)^2\chi\right\|_{L_x^\infty}\int_\varepsilon^\infty\left(\partial_x^{j_2}u\right)^2dx+\frac{1}{2}\int\left(\partial_x^{j_3}u\right)^2\chi dx\\ \leq&\frac{1}{2}\left\|\partial_x\left(\left(\partial_x^{j_1}u\right)^2\chi\right)\right\|_{L_x^1}\int\left(\partial_x^{j_2}u\right)^2\chi(x;\varepsilon/5,4\varepsilon/5)dx+\frac{1}{2}\int\left(\partial_x^{j_3}u\right)^2\chi dx, \qquad (2.5)\end{aligned}$$

then Young's inequality yields

$$\left\|\partial_x\left(\left(\partial_x^{j_1}u\right)^2\chi\right)\right\|_{L_x^1}\leq 2\int\left|\partial_x^{j_1}u\partial_x^{1+j_1}u\right|\chi dx+\int\left(\partial_x^{j_1}u\right)^2\chi' dx$$

$$\lesssim \int \left(\partial_x^{1+j_1} u\right)^2 \chi dx + \int \left(\partial_x^{j_1} u\right)^2 \chi dx + \int \left(\partial_x^{j_1} u\right)^2 \chi' dx. \quad (2.6)$$

This completes the proof of (2.3). The inequality involving (2.4) is bounded similarly

$$\begin{aligned}
&\left| \int \partial_x^{j_1} u \partial_x^{j_2} u \partial_x^{j_3} u \partial_x^{j_4} u \chi dx \right| \\
&\leq \frac{1}{2} \int \left(\partial_x^{j_1} u\right)^2 \left(\partial_x^{j_2} u\right)^2 \left(\partial_x^{j_3} u\right)^2 \chi dx + \frac{1}{2} \int \left(\partial_x^{j_4} u\right)^2 \chi dx \\
&\leq \frac{1}{2} \left\| \partial_x \left(\left(\partial_x^{j_1} u\right)^2 \left(\partial_x^{j_2} u\right)^2 \chi \right) \right\|_{L_x^1} \int \left(\partial_x^{j_3} u\right)^2 \chi(x; \varepsilon/5, 4\varepsilon/5) dx + \frac{1}{2} \int \left(\partial_x^{j_4} u\right)^2 \chi dx,
\end{aligned} \quad (2.7)$$

using (2.6) and (2.7), it holds that

$$\begin{aligned}
&\left\| \partial_x \left(\left(\partial_x^{j_1} u\right)^2 \left(\partial_x^{j_2} u\right)^2 \chi \right) \right\|_{L_x^1} \\
\leq & 2 \int \left| \partial_x^{j_1} u \partial_x^{j_1+1} u \right| \left(\partial_x^{j_2} u\right)^2 \chi dx + \int \left(\partial_x^{j_1} u\right)^2 \left(\partial_x^{j_2} u\right)^2 \chi' dx \\
&+ 2 \int \left| \partial_x^{j_2} u \partial_x^{j_2+1} u \right| \left(\partial_x^{j_1} u\right)^2 \chi dx \\
\leq & 4 \int \left(\partial_x^{j_1} u\right)^2 \left(\partial_x^{j_2} u\right)^2 \chi dx + 2 \int \left(\partial_x^{j_1+1} u\right)^2 \left(\partial_x^{j_2} u\right)^2 \chi dx \\
&+ 2 \int \left(\partial_x^{j_2+1} u\right)^2 \left(\partial_x^{j_1} u\right)^2 \chi dx + \int \left(\partial_x^{j_1} u\right)^2 \left(\partial_x^{j_2} u\right)^2 \chi' dx \\
\lesssim & \int [(\partial_x^{j_2} u)^2 + (\partial_x^{j_2+1} u)^2] \chi(x; \varepsilon/5, 4\varepsilon/5) dx \times \Big\{ \int [(\partial_x^{j_1+2} u)^2 + (\partial_x^{j_1+1} u)^2 \\
&+ (\partial_x^{j_1} u)^2] \chi(x) dx + \int [(\partial_x^{j_1} u)^2 + (\partial_x^{j_1+1} u)^2] \chi'(x; \varepsilon/3, b+\varepsilon) dx \Big\}.
\end{aligned} \quad (2.8)$$

This completes the proof of (2.4). Arguing similarly as deriving (2.5)-(2.8), combined with the properties of the cutoff function, it is easy to summarize the inequality (2.2), and we omitted the details here. □

3. Proof of Theorem 1.2

In the rest of this paper, the letter c denotes a generic constant whose exact values may change from line to line, but do not depend on particular solutions or functions. For the sake of calculation, we are going to ignore the coefficients in the following discussion. As the argument is translation invariant, we consider only $x_0 = 0$.

We shall use an induction argument. First, for the sake of simplicity of exposition we shall restrict the proofs of theorem to the case of the model equation:

$$\partial_t u - \partial_x^5 u - \left(\frac{\partial F}{\partial u} \right)_x + (\partial_x^2 u)^2 + \partial_x u \partial_x^3 u = 0. \quad (3.1)$$

Additionally, because this nonlinearity has a total of four derivatives, integrating by parts produces a form very similar to $\partial_x u \partial_x^3 u$, we mainly show the estimates of nonlinearity $\partial_x u \partial_x^3 u$.

According to Theorem 1.1, by applying Sobolev embedding theorem, integration by parts, Young's inequality, we can easily get the case $l = 1, 2, 3, 4$

$$\sup_{t\in[0,T]} \int \left(\partial_x^l u\right)^2 (x,t)\chi(x+vt;\varepsilon,b)dx + \int_0^T \int \left(\partial_x^{l+2} u\right)^2 (x,t)\chi'(x+vt;\varepsilon,b)dxdt \leq c_0. \tag{3.2}$$

Case $l = 5$**.** We now prove the case $l = 5$. Applying ∂_x^5 to (3.1) and multiplying the result by $\partial_x^5 u(x,t)\chi(x+vt;\varepsilon,b)$, after some integration by parts, we find

$$\frac{d}{dt}\int \left(\partial_x^5 u\right)^2 \chi(x+vt)dx + \int \left(\partial_x^7 u\right)^2 \chi'(x+vt)dx + \int \left(\partial_x^5 u\right)^2 \chi^{(5)}(x+vt)dx = A_1 + A_2 + A_3, \tag{3.3}$$

where

$$A_1 = v\int \left(\partial_x^5 u\right)^2 \chi'(x+vt)dx + \int \left(\partial_x^6 u\right)^2 \chi'''(x+vt)dx,$$
$$A_2 = \int \partial_x^5 \left(\frac{\partial F}{\partial u}\right)_x \partial_x^5 u\chi(x+vt)dx,$$
$$A_3 = \int \partial_x^5 \left((\partial_x^2 u)^2 + \partial_x u \partial_x^3 u\right) \partial_x^5 u\chi(x+vt)dx.$$

Integrating in the time interval [0,T] and employing (3.2) with $l = 3, 4$, one deduces

$$\begin{aligned}\left|\int_0^T A_1 dt\right| \leq& |v| \int_0^T \int \left(\partial_x^5 u\right)^2 \chi'(x+vt)dxdt \\ &+ \int_0^T \int \left(\partial_x^6 u\right)^2 \chi'(x+vt;\varepsilon/3, b+\varepsilon)dxdt \\ \leq& c_0,\end{aligned} \tag{3.4}$$

since given v, ε, b, T as above there exist $\tilde{c} > 0$ and $R > 0$ such that

$$\chi'(x+vt) \leq \tilde{c}1_{[-R,R]}(x), \quad \forall (x,t) \in \mathbb{R}\times[0,T].$$

Notice that the chain rule yields

$$\partial_x^s(G(u)) = \sum_{1\leq p\leq s} C_s^p \frac{\partial^p G(u)}{\partial u^p} \cdot (\partial_x u)^{\alpha_1} \cdot \left(\partial_x^2 u\right)^{\alpha_2} \dots (\partial_x^s u)^{\alpha_s}, \tag{3.5}$$

with $\sum_{i=1}^s \alpha_i = p, \sum_{i=1}^s i\cdot\alpha_i = s$, and for $\alpha_{[\frac{s}{2}]+1}, \cdots, \alpha_s$, either all of them are zero, or at most one of them is 1 and the rest are zero. Then thanks to (3.5), Young's inequality and Sobolev embedding theorem, we derive

$$\begin{aligned}&\int_0^T A_2 dt \\ =&c\int_0^T \int \big[F^{(7)}(\partial_x u)^6 + F^{(6)}(\partial_x u)^4 \partial_x^2 u + F^{(5)}\left((\partial_x u)^2(\partial_x^2 u)^2 + (\partial_x u)^3 \partial_x^3 u\right) \\ &+ F^{(4)}\left((\partial_x^2 u)^3 + \partial_x u \partial_x^2 u \partial_x^3 u + (\partial_x u)^2 \partial_x^4 u\right) + F^{(3)}\left((\partial_x^3 u)^2 + \partial_x^2 u \partial_x^4 u + \partial_x u \partial_x^5 u\right)\end{aligned}$$

$$
\begin{aligned}
&+ F^{(2)}\partial_x^6 u\big]\partial_x^5 u\chi(x+vt)dxdt\\
\le & c_0 + c\int_0^T\int \left(\partial_x^5 u\right)^2\chi(x+vt)dxdt,
\end{aligned}
\tag{3.6}
$$

where the last term we used integration by parts

$$
\begin{aligned}
&\int F^{(2)}\partial_x^6 u\partial_x^5 u\chi(x+vt)dx\\
=&-\frac{1}{2}\Big(\int F^{(3)}\partial_x u(\partial_x^5 u)^2\chi(x+vt)dx+\int F^{(2)}(\partial_x^5 u)^2\chi'(x+vt)dx\Big).
\end{aligned}
$$

Making use of integrating by parts, it follows

$$
\begin{aligned}
A_3 =& \int \partial_x^2 u\left(\partial_x^6 u\right)^2\chi(x+vt)dx + \int \partial_x u\left(\partial_x^6 u\right)^2\chi'(x+vt)dx\\
&+\int \partial_x^4 u\left(\partial_x^5 u\right)^2\chi(x+vt)dx+\int \partial_x^3 u\left(\partial_x^5 u\right)^2\chi'(x+vt)dx\\
&+\int \partial_x^2 u\left(\partial_x^5 u\right)^2\chi''(x+vt)dx+\int \partial_x u\left(\partial_x^5 u\right)^2\chi'''(x+vt)dx.
\end{aligned}
\tag{3.7}
$$

This expression exhibits a loss of derivatives in that the term

$$
\int \partial_x^2 u\left(\partial_x^6 u\right)^2\chi(x+vt)dx,
\tag{3.8}
$$

can be controlled neither by the well-posedness theory nor by the $l = 4$ case. In [6], Kwon introduced a modified energy to overcome a similar issue. Thus we know that a smooth solution u to the IVP (3.1) satisfies the following identity:

$$
\begin{aligned}
&\frac{d}{dt}\int \partial_x u(\partial_x^4 u)^2\chi dx\\
=&\int \partial_x\partial_t u(\partial_x^4 u)^2\chi dx + 2\int \partial_x u\partial_x^4 u\partial_x^4\partial_t u\chi dx + v\int \partial_x u(\partial_x^4 u)^2\chi' dx\\
=&v\int \partial_x u\left(\partial_x^4 u\right)^2\chi' dx+\int \partial_x\left(\frac{\partial F}{\partial u}\right)_x\left(\partial_x^4 u\right)^2\chi dx+\int \partial_x u\partial_x^4 u\partial_x^4\left(\frac{\partial F}{\partial u}\right)_x\chi dx\\
&+\int \partial_x^2 u\left(\partial_x^6 u\right)^2\chi dx+\int \partial_x u\left(\partial_x^6 u\right)^2\chi' dx+\int\left(\partial_x^4 u+\partial_x u\partial_x^2 u\right)\left(\partial_x^5 u\right)^2\chi dx\\
&+\int(\partial_x^3 u+(\partial_x u)^2)(\partial_x^5 u)^2\chi' dx+\int \partial_x^2 u\left(\partial_x^5 u\right)^2\chi'' dx+\int \partial_x u\left(\partial_x^5 u\right)^2\chi''' dx\\
&+\int(\partial_x^2 u\partial_x^3 u+\partial_x u\partial_x^4 u)\left(\partial_x^4 u\right)^2\chi dx+\int\left(\partial_x u\partial_x^3 u+\left(\partial_x^2 u\right)^2\right)\left(\partial_x^4 u\right)^2\chi' dx\\
&+\int(\partial_x^4 u+\partial_x u\partial_x^2 u)(\partial_x^4 u)^2\chi'' dx+\int(\partial_x^3 u+(\partial_x u)^2)(\partial_x^4 u)^2\chi''' dx\\
&+\int \partial_x^2 u(\partial_x^4 u)^2\chi^{(4)}dx+\int \partial_x u(\partial_x^4 u)^2\chi^{(5)}dx,
\end{aligned}
\tag{3.9}
$$

where $\chi^{(j)}$ denotes $\chi^{(j)}(x+vt)$, we use this identity to eliminate (3.8) from (3.7) yields

$$
A_3 = \frac{d}{dt}\int \partial_x u(\partial_x^4 u)^2\chi dx + v\int \partial_x u\left(\partial_x^4 u\right)^2\chi' dx + \int \partial_x\left(\frac{\partial F}{\partial u}\right)_x\left(\partial_x^4 u\right)^2\chi dx
$$

$$+\int \partial_x u \partial_x^4 u \partial_x^4 \left(\frac{\partial F}{\partial u}\right)_x \chi dx + \int \partial_x u \partial_x^2 u \left(\partial_x^5 u\right)^2 \chi dx + \int (\partial_x u)^2 (\partial_x^5 u)^2 \chi' dx$$
$$+\int (\partial_x^2 u \partial_x^3 u + \partial_x u \partial_x^4 u) \left(\partial_x^4 u\right)^2 \chi dx + \int \left(\partial_x u \partial_x^3 u + \left(\partial_x^2 u\right)^2\right) \left(\partial_x^4 u\right)^2 \chi' dx$$
$$+\int (\partial_x^4 u + \partial_x u \partial_x^2 u)(\partial_x^4 u)^2 \chi'' dx + \int (\partial_x^3 u + (\partial_x u)^2)(\partial_x^4 u)^2 \chi''' dx$$
$$+\int \partial_x^2 u (\partial_x^4 u)^2 \chi^{(4)} dx + \int \partial_x u (\partial_x^4 u)^2 \chi^{(5)} dx. \tag{3.10}$$

Integrating in the time interval [0,T], applying (3.2), (3.5), the hypothesis on the initial and Lemma 2.1, the fundamental theorem of calculus yields

$$\left|\int_0^T A_3 dt\right| \leq c_0 + c\int_0^T \int \left(\partial_x^5 u\right)^2 \chi dx dt. \tag{3.11}$$

Substituting the inequalities (3.4), (3.6), (3.11) in (3.3), Grönwall's inequality leads to

$$\sup_{t\in[0,T]} \int \left(\partial_x^5 u\right)^2 \chi(x+vt) dx + \int_0^T \int \left(\partial_x^7 u\right)^2 \chi'(x+vt) dx dt \leq c_0.$$

This proves the desired result with $l = 5$.

Case $l = 6$**.** Similar to the previous case, for $l = 6$,

$$\frac{d}{dt}\int \left(\partial_x^6 u\right)^2 \chi(x+vt) dx + \int \left(\partial_x^8 u\right)^2 \chi'(x+vt) dx + \int \left(\partial_x^6 u\right)^2 \chi^{(5)}(x+vt) dx$$
$$= A_1 + A_2 + A_3, \tag{3.12}$$

where

$$A_1 = v\int \left(\partial_x^6 u\right)^2 \chi'(x+vt) dx + \int \left(\partial_x^7 u\right)^2 \chi'''(x+vt) dx,$$
$$A_2 = \int \partial_x^6 \left(\frac{\partial F}{\partial u}\right)_x \partial_x^6 u \chi(x+vt) dx,$$
$$A_3 = \int \partial_x^6 \left((\partial_x^2 u)^2 + \partial_x u \partial_x^3 u\right) \partial_x^6 u \chi(x+vt) dx.$$

Integrating in the time interval [0,T] and applying the $l = 4, 5$ results, one deduces

$$\left|\int_0^T A_1 dt\right| \leq |v| \int_0^T \int \left(\partial_x^6 u\right)^2 \chi'(x+vt) dx dt$$
$$+ \int_0^T \int \left(\partial_x^7 u\right)^2 \chi'(x+vt; \varepsilon/3, b+\varepsilon) dx dt$$
$$\leq c_0. \tag{3.13}$$

Then thanks to (3.5), integration by parts, Young's inequality and Sobolev embedding theorem, we derive

$$\int_0^T A_2 dt = c\int_0^T \int \left[F^{(8)}(\partial_x u)^7 + F^{(7)}(\partial_x u)^5 \partial_x^2 u + F^{(6)}\left((\partial_x u)^3 (\partial_x^2 u)^2\right.\right.$$

$$
\begin{aligned}
&+ (\partial_x u)^4 \partial_x^3 u\big) + F^{(5)}\big(\partial_x u(\partial_x^2 u)^3 + (\partial_x u)\partial_x u \partial_x^2 u \partial_x^3 u + (\partial_x u)^3 \partial_x^4 u\big)\\
&+ F^{(4)}\big(\partial_x u(\partial_x^3 u)^2 + (\partial_x^2 u)^2\partial_x^3 u + \partial_x u \partial_x^2 u \partial_x^4 u + (\partial_x u)^2 \partial_x^5 u\big)\\
&+ F^{(3)}\big(\partial_x^3 u \partial_x^4 u + \partial_x^2 u \partial_x^5 u + \partial_x u \partial_x^6 u\big) + F^{(2)}\partial_x^7 u\big]\partial_x^6 u \chi dxdt\\
\le& c_0 + c\int_0^T \int \big[(F^{(8)})^2 + \big(\partial_x^6 u\big)^2\big]\chi(x+vt)dxdt\\
\le& c_0 + c\int_0^T \int \big(\partial_x^6 u\big)^2 \chi dxdt. \qquad (3.14)
\end{aligned}
$$

After integrating by parts, we see

$$
\begin{aligned}
A_3 =& \int \partial_x^2 u \left(\partial_x^7 u\right)^2 \chi dx + \int \partial_x u \left(\partial_x^7 u\right)^2 \chi' dx + \int \partial_x^4 u \left(\partial_x^6 u\right)^2 \chi dx\\
&+ \int \partial_x^3 u \left(\partial_x^6 u\right)^2 \chi' dx + \int \partial_x^2 u \left(\partial_x^6 u\right)^2 \chi'' dx + \int \partial_x u \left(\partial_x^6 u\right)^2 \chi''' dx\\
&+ \int (\partial_x^5 u)^3 \chi' dx. \qquad (3.15)
\end{aligned}
$$

This expression exhibits a loss of derivatives in that the term

$$
\int \partial_x^2 u \left(\partial_x^7 u\right)^2 \chi dx. \qquad (3.16)
$$

A smooth solution u to the IVP (3.1) satisfies the following identity:

$$
\begin{aligned}
&\frac{d}{dt}\int \partial_x u (\partial_x^5 u)^2 \chi dx\\
=& v\int \partial_x u \left(\partial_x^5 u\right)^2 \chi' dx + \int \partial_x \left(\frac{\partial F}{\partial u}\right)_x \left(\partial_x^5 u\right)^2 \chi dx + \int \partial_x u \partial_x^5 u \partial_x^5 \left(\frac{\partial F}{\partial u}\right)_x \chi dx\\
&+ \int \partial_x^2 u \left(\partial_x^7 u\right)^2 \chi dx + \int \partial_x u \left(\partial_x^7 u\right)^2 \chi' dx + \int \left(\partial_x^4 u + \partial_x u \partial_x^2 u\right) \left(\partial_x^6 u\right)^2 \chi dx\\
&+ \int (\partial_x^3 u + (\partial_x u)^2)(\partial_x^6 u)^2 \chi' dx + \int \partial_x^2 u \left(\partial_x^6 u\right)^2 \chi'' dx + \int \partial_x u \left(\partial_x^6 u\right)^2 \chi''' dx\\
&+ \int (\partial_x^2 u \partial_x^3 u + \partial_x u \partial_x^4 u) \left(\partial_x^5 u\right)^2 \chi dx + \int \left(\partial_x^5 u + \partial_x u \partial_x^3 u + \left(\partial_x^2 u\right)^2\right) \left(\partial_x^5 u\right)^2 \chi' dx\\
&+ \int (\partial_x^4 u + \partial_x u \partial_x^2 u)(\partial_x^5 u)^2 \chi'' dx + \int (\partial_x^3 u + (\partial_x u)^2)(\partial_x^5 u)^2 \chi''' dx\\
&+ \int \partial_x^2 u (\partial_x^5 u)^2 \chi^{(4)} dx + \int \partial_x u (\partial_x^5 u)^2 \chi^{(5)} dx. \qquad (3.17)
\end{aligned}
$$

We use this identity to eliminate (3.16) from (3.15), the term A_3 can be written as

$$
\begin{aligned}
A_3 =& \frac{d}{dt}\int \partial_x u (\partial_x^5 u)^2 \chi dx + v\int \partial_x u \left(\partial_x^5 u\right)^2 \chi' dx + \int \partial_x \left(\frac{\partial F}{\partial u}\right)_x \left(\partial_x^5 u\right)^2 \chi dx\\
&+ \int \partial_x u \partial_x^5 u \partial_x^5 \left(\frac{\partial F}{\partial u}\right)_x \chi dx + \int \partial_x u \partial_x^2 u \left(\partial_x^6 u\right)^2 \chi dx + \int (\partial_x u)^2 (\partial_x^6 u)^2 \chi' dx\\
&+ \int (\partial_x^2 u \partial_x^3 u + \partial_x u \partial_x^4 u) \left(\partial_x^5 u\right)^2 \chi dx + \int (\partial_x^4 u + \partial_x u \partial_x^2 u)(\partial_x^5 u)^2 \chi'' dx\\
&+ \int \left(\partial_x^5 u + \partial_x u \partial_x^3 u + \left(\partial_x^2 u\right)^2\right) \left(\partial_x^5 u\right)^2 \chi' dx + \int (\partial_x^3 u + (\partial_x u)^2)(\partial_x^5 u)^2 \chi''' dx
\end{aligned}
$$

$$+\int \partial_x^2 u(\partial_x^5 u)^2\chi^{(4)}dx+\int \partial_x u(\partial_x^5 u)^2\chi^{(5)}dx. \tag{3.18}$$

Integrating in the time interval [0,T], applying the $l=5$ result, the formula (3.5), and hypothesis on the initial, the fundamental theorem of calculus yields

$$\left|\int_0^T A_3 dt\right| \le c_0 + c\int_0^T \int \left(\partial_x^6 u\right)^2 \chi dxdt. \tag{3.19}$$

Substituting the inequalities (3.13), (3.14), (3.19) in (3.12), applying Grönwall's inequality produces

$$\sup_{t\in[0,T]}\int \left(\partial_x^6 u\right)^2 \chi(x+vt)dx+\int_0^T\int \left(\partial_x^8 u\right)^2 \chi'(x+vt)dxdt \le c_0.$$

Similar to the treatment of $l=5,6$, we know that the inequality (3.2) is true for the cases $l=7,8$, the analysis is omitted.

Remark 3.1. It is worth mentioning that as l gets bigger, some terms in A_2 and A_3 are difficult to handled directly even by integration by parts, so (2.3) in Lemma 2.2 can be used. For example, when $l=8$, the following term appears in A_2.

$$\begin{aligned}
&|\int F^{(3)}\partial_x^4 u\partial_x^5 u\partial_x^8 u\chi(x+vt)dx| \\
\le&\|F^{(3)}\|_{L_x^\infty}\Big\{\Big[\int(\partial_x^5 u)^2\chi(x+vt)dx+\int(\partial_x^4 u)^2\chi(x+vt)dx+\int(\partial_x^4 u)^2\chi'(x+vt)dx\Big] \\
&\times\int(\partial_x^5 u)^2\chi(x;\varepsilon/5,4\varepsilon/5)dx+\int(\partial_x^8 u)^2\chi(x+vt)dx\Big\} \\
\le& c_0+c\int(\partial_x^8 u)^2\chi(x+vt)dx.
\end{aligned}$$

Case $l\ge 9$**.** We prove the general case $l\ge 9$ by induction. In detail, we assume that if $u_0,\ F(u)$ satisfy (1.4), (1.5), then

$$\sup_{t\in[0,T]}\int \left(\partial_x^j u\right)^2 \chi(x+vt)dx+\int_0^T\int \left(\partial_x^{j+2} u\right)^2 \chi'(x+vt)dxdt \le c_0,$$

for $j=1,2,3,...,l,\ l\ge 9$, and for any $v,\varepsilon,b>0$.

Now we have that

$$u_0|_{(0,\infty)}\in H^{l+1}((0,\infty)),\ F(u)|_{(0,\infty)}\in H^{l+3}((0,\infty)).$$

Similar to the case above, for $l\ge 9$,

$$\begin{aligned}
&\frac{d}{dt}\int \left(\partial_x^{l+1}u\right)^2\chi(x+vt)dx+\int \left(\partial_x^{l+3}u\right)^2\chi'(x+vt)+\left(\partial_x^{l+1}u\right)^2\chi^{(5)}(x+vt)dx \\
=&A_1+A_2+A_3,
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
A_1&=v\int \left(\partial_x^{l+1}u\right)^2\chi'(x+vt)dx+\int \left(\partial_x^{l+2}u\right)^2\chi'''(x+vt)dx, \\
A_2&=\int \partial_x^{l+1}\left(\frac{\partial F}{\partial u}\right)_x \partial_x^{l+1}u\chi(x+vt)dx, \\
A_3&=\int \partial_x^{l+1}\left((\partial_x^2 u)^2+\partial_x u\partial_x^3 u\right)\partial_x^{l+1}u\chi(x+vt)dx.
\end{aligned}$$

Integrating in the time interval [0,T], and from the induction cases $l-1, l$, one deduces

$$|\int_0^T A_1 dt| \le c_0. \tag{3.21}$$

Thanks to (3.5), we have

$$\partial_x^{l+2}(F'(u)) = \sum_{1\leqslant p\leqslant l+2} C_{l+2}^p F^{(p+1)}(u)\cdot(\partial_x u)^{\alpha_1}\cdot\left(\partial_x^2 u\right)^{\alpha_2}\dots\left(\partial_x^{l+2}u\right)^{\alpha_{l+2}},$$

with $\sum_{i=1}^{l+2}\alpha_i = p, \sum_{i=1}^{l+2} i\cdot\alpha_i = l+2$, and for $\alpha_{[\frac{l+2}{2}]+1},\cdots,\alpha_{l+2}$, either all of them are zero, or at most one of them is 1 and the rest are zero. Notice that the above equation ignores coefficients yields

$$\begin{aligned}
&\partial_x^{l+1}(\frac{\partial F}{\partial u})_x\\
=&F^{(l+3)}(\partial_x u)^{l+2} + F^{(l+2)}(\partial_x u)^l\partial_x^2 u + F^{(l+1)}[(\partial_x u)^{l-1}\partial_x^3 u + (\partial_x u)^{l-2}(\partial_x^2 u)^2]\\
&+ F^{(l)}[(\partial_x u)^{l-4}(\partial_x^2 u)^3 + (\partial_x u)^{l-3}\partial_x^2 u\partial_x^3 u + (\partial_x u)^{l-2}\partial_x^4 u]\\
&+ F^{(l-1)}[(\partial_x u)^{l-6}(\partial_x^2 u)^4 + (\partial_x u)^{l-5}(\partial_x^2 u)^2\partial_x^3 u + (\partial_x u)^{l-4}\partial_x^2 u\partial_x^4 u\\
&+ (\partial_x u)^{l-4}(\partial_x^3 u)^2 + (\partial_x u)^{l-3}\partial_x^5 u]\\
&+ F^{(l-2)}[(\partial_x u)^{l-8}(\partial_x^2 u)^5 + (\partial_x u)^{l-7}(\partial_x^2 u)^2\partial_x^3 u + (\partial_x u)^{l-6}(\partial_x^2 u)^2\partial_x^4 u\\
&+ (\partial_x u)^{l-6}\partial_x^2 u(\partial_x^3 u)^2 + (\partial_x u)^{l-5}\partial_x^3 u\partial_x^4 u + (\partial_x u)^{l-5}\partial_x^2 u\partial_x^5 u + (\partial_x u)^{l-4}\partial_x^6 u]\\
&\vdots\\
&+ F^{(l-i)}[(l-i-1) \text{ terms of } u \text{ to divide the } (l+2) \text{ derivative}]\\
&\vdots\\
&+ F^{(3)}\left[\partial_x u\partial_x^{l+1}u + \partial_x^2 u\partial_x^l u + \partial_x^3 u\partial_x^{l-1}u + \cdots + \partial_x^{[\frac{l+2}{2}]}u\partial_x^{l+2-[\frac{l+2}{2}]}u\right]\\
&+ F^{(2)}\partial_x^{l+2}u.
\end{aligned}$$

By Young's inequality, Sobolev embedding theorem and (1.5), the first term can be estimated as

$$\begin{aligned}
&\int F^{(l+3)}(\partial_x u)^{l+2}\partial_x^{l+1}u\chi(x+vt)dx\\
\le& c\int \left[(F^{(l+3)})^2 + \left(\partial_x^{l+1}u\right)^2\right]\chi(x+vt)dx\\
\le& c_0 + c\int(\partial_x^{l+1}u)^2\chi(x+vt)dx.
\end{aligned}$$

Making use of integration by parts, the last term holds that

$$\begin{aligned}
&\int F^{(2)}\partial_x^{l+2}u\partial_x^{l+1}u\chi(x+vt)dx\\
\le& c\left|\int F^{(3)}\partial_x u(\partial_x^{l+1}u)^2\chi(x+vt)dx + \int F^{(2)}(\partial_x^{l+1}u)^2\chi'(x+vt)dx\right|
\end{aligned}$$

$$\leq c_0 + c\int (\partial_x^{l+1}u)^2\chi(x+vt)dx.$$

The middle terms can be processed by Young's inequality, Sobolev embedding theorem and (2.2) in Lemma 2.2, we omit the calculation details here. In summary, we have

$$\left|\int_0^T A_2 dt\right| \leq c_0 + c\int_0^T \int \left(\partial_x^{l+1}u\right)^2 \chi dxdt. \tag{3.22}$$

We note that when $l \geq 9$, the identity similar to (3.9), (3.17) can not completely contain the term A_3 (in fact, only the first eight items are included, see (3.26)). After integrating by parts, for positive numbers $\alpha_1, \alpha_2, \alpha_3$, we arrive at

$$\begin{aligned} A_3 =& \int \partial_x^2 u \left(\partial_x^{l+2}u\right)^2 \chi dx + \int \partial_x u \left(\partial_x^{l+2}u\right)^2 \chi' dx + \int \partial_x^4 u \left(\partial_x^{l+1}u\right)^2 \chi dx \\ &+ \int \partial_x^3 u \left(\partial_x^{l+1}u\right)^2 \chi' dx + \int \partial_x^2 u \left(\partial_x^{l+1}u\right)^2 \chi'' dx + \int \partial_x u \left(\partial_x^{l+1}u\right)^2 \chi''' dx \\ &+ \int \partial_x^6 u \left(\partial_x^{l}u\right)^2 \chi dx + \int \partial_x^5 u \left(\partial_x^{l}u\right)^2 \chi' dx + \int \sum_{*} \partial_x^{\alpha_1} u \left(\partial_x^{\alpha_2}u\right)^2 \chi^{(\alpha_3)} dx, \end{aligned} \tag{3.23}$$

where $*$ indicates the following conditions, for $i \in \mathbb{Z}^+$,

$$(*)\begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 2l+6, \qquad \alpha_1 \leq \alpha_2. \\ \text{Depending on the value of } \alpha_2, \text{ we have} \\ \alpha_2 = l-1, \quad \alpha_1+\alpha_3 = 8, \quad \alpha_1 \geq 6, \\ \alpha_2 = l-2, \quad \alpha_1+\alpha_3 = 10, \quad \alpha_1 \geq 7, \\ \alpha_2 = l-3, \quad \alpha_1+\alpha_3 = 12, \quad \alpha_1 \geq 8, \\ \vdots \ \vdots \\ \alpha_2 = l-i, \quad \alpha_1+\alpha_3 = 2i+6, \quad \alpha_1 \geq i+5. \end{cases}$$

This expression exhibits a loss of derivatives in that the term

$$\int \partial_x^2 u \left(\partial_x^{l+2}u\right)^2 \chi dx. \tag{3.24}$$

For positive numbers $\beta_1, \beta_2, \beta_3$, a smooth solution u to the IVP (3.1) satisfies the following identity:

$$\begin{aligned} &\frac{d}{dt}\int \partial_x u(\partial_x^l u)^2\chi dx \\ =& v\int \partial_x u(\partial_x^l u)^2\chi' dx + \int \partial_x(\frac{\partial F}{\partial u})_x(\partial_x^l u)^2\chi dx + \int \partial_x u\partial_x^l u\partial_x^l(\frac{\partial F}{\partial u})_x\chi dx \\ &+ \int \partial_x^2 u(\partial_x^{l+2}u)^2\chi dx + \int \partial_x u\left(\partial_x^{l+2}u\right)^2\chi' dx + \int \left(\partial_x^4 u + \partial_x u\partial_x^2 u\right)\left(\partial_x^{l+1}u\right)^2\chi dx \\ &+ \int (\partial_x^3 u + (\partial_x u)^2)(\partial_x^{l+1}u)^2\chi' dx + \int \partial_x^2 u\left(\partial_x^{l+1}u\right)^2\chi'' dx + \int \partial_x u\left(\partial_x^{l+1}u\right)^2\chi''' dx \end{aligned}$$

$$+\int(\partial_x^6 u+\partial_x^2 u\partial_x^3 u+\partial_x u\partial_x^4 u)(\partial_x^l u)^2\chi dx+\int(\partial_x^4 u+\partial_x u\partial_x^2 u)(\partial_x^l u)^2\chi'' dx$$
$$+\int(\partial_x^5 u+\partial_x u\partial_x^3 u+(\partial_x^2 u)^2)(\partial_x^l u)^2\chi' dx+\int(\partial_x^3 u+(\partial_x u)^2)(\partial_x^l u)^2\chi''' dx$$
$$+\int\partial_x^2 u(\partial_x^l u)^2\chi^{(4)}dx+\int\partial_x u(\partial_x^l u)^2\chi^{(5)}dx+\int\sum_{**}\partial_x^{\beta_1}u\partial_x^{\beta_2}u(\partial_x^{\beta_3}u)^2\chi^{(\beta_4)}dx, \tag{3.25}$$

where $**$ indicates the following conditions

$$(**)\quad\begin{cases}\beta_1+\beta_2+2\beta_3+\beta_4=2l+5,\\ \beta_1<\beta_2\leqslant\beta_3,\\ l+4-\left[\dfrac{l+4}{2}\right]\leqslant\beta_3\leqslant l-1,\\ 0\leqslant\beta_4\leqslant\left[\dfrac{l+4}{2}\right]-4.\end{cases}$$

Using the identity (3.25) eliminate (3.24) from (3.23), we find

$$\begin{aligned}A_3=&\frac{d}{dt}\int\partial_x u(\partial_x^l u)^2\chi dx+v\int\partial_x u(\partial_x^l u)^2\chi' dx+\int\partial_x(\frac{\partial F}{\partial u})_x(\partial_x^l u)^2\chi dx\\&+\int\partial_x u\partial_x^l u\partial_x^l(\frac{\partial F}{\partial u})_x\chi dx+\int\partial_x u\partial_x^2 u\left(\partial_x^{l+1}u\right)^2\chi dx+\int(\partial_x u)^2(\partial_x^{l+1}u)^2\chi' dx\\&+\int(\partial_x^2 u\partial_x^3 u+\partial_x u\partial_x^4 u)(\partial_x^l u)^2\chi dx+\int(\partial_x u\partial_x^3 u+(\partial_x^2 u)^2)(\partial_x^l u)^2\chi' dx\\&+\int(\partial_x^4 u+\partial_x u\partial_x^2 u)(\partial_x^l u)^2\chi'' dx+\int(\partial_x^3 u+(\partial_x u)^2)(\partial_x^l u)^2\chi''' dx\\&+\int\partial_x^2 u(\partial_x^l u)^2\chi^{(4)}dx+\int\partial_x u(\partial_x^l u)^2\chi^{(5)}dx+\int\sum_{**}\partial_x^{\beta_1}u\partial_x^{\beta_2}u(\partial_x^{\beta_3}u)^2\chi^{(\beta_4)}dx\\&+\int\sum_{*}\partial_x^{\alpha_1}u\left(\partial_x^{\alpha_2}u\right)^2\chi^{(\alpha_3)}dx,\end{aligned}\tag{3.26}$$

where the treatment of $\int\partial_x u\partial_x^l u\partial_x^l(\frac{\partial F}{\partial u})_x\chi dx$ is similar to the term A_2. Applying the inequalities (2.3), (2.4) in Lemma 2.2, one obtains

$$\begin{aligned}&\int\sum_{*}\partial_x^{\alpha_1}u(\partial_x^{\alpha_2}u)^2\chi^{(\alpha_3)}dx\\&\leq\sum_{*}\left\{\int(\partial_x^{\alpha_1+1}u)^2\chi^{(\alpha_3)}dx+\int(\partial_x^{\alpha_1}u)^2\chi^{(\alpha_3)}dx+\int(\partial_x^{\alpha_1}u)^2\chi^{(\alpha_3+1)}dx\right\}\\&\quad\times\int(\partial_x^{\alpha_2}u)^2\chi(x;\varepsilon/5,4\varepsilon/5)dx+\sum_{*}\int(\partial_x^{\alpha_2}u)^2\chi^{(\alpha_3)}dx,\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}&\int\sum_{**}\partial_x^{\beta_1}u\partial_x^{\beta_2}u(\partial_x^{\beta_3}u)^2\chi^{(\beta_4)}dx\\&\leq\int[(\partial_x^{\beta_2+1}u)^2+(\partial_x^{\beta_2}u)^2]\chi(x;\varepsilon/5,4\varepsilon/5)dx\int(\partial_x^{\beta_3}u)^2\chi(x;\varepsilon/5,4\varepsilon/5)dx\end{aligned}$$

$$\times \sum_{**} \Big\{ \int [(\partial_x^{\beta_1+2}u)^2 + (\partial_x^{\beta_1+1}u)^2 + (\partial_x^{\beta_1}u)^2]\chi^{(\beta_4)}dx + \int [(\partial_x^{\beta_1+1}u)^2 + (\partial_x^{\beta_1}u)^2]\chi^{(\beta_4+1)}dx \Big\} + \sum_{**} \int (\partial_x^{\beta_3}u)^2 \chi^{(\beta_4)}dx. \tag{3.28}$$

Hence, from our induction hypothesis, combining the inequalities (3.27) and (3.28), the fundamental theorem of calculus leads to

$$\left| \int_0^T A_3 dt \right| \le c_0 + c \int_0^T \int \left(\partial_x^{l+1}u\right)^2 \chi dxdt. \tag{3.29}$$

Inserting the inequalities (3.21), (3.22), (3.29) in (3.20), employing Grönwall's inequality produces

$$\sup_{t\in[0,T]} \int \left(\partial_x^{l+1}u\right)^2 \chi(x+vt)dx + \int_0^T \int \left(\partial_x^{l+3}u\right)^2 \chi'(x+vt)dxdt \le c_0.$$

This closes our induction and completes the proof of Theorem 1.2.

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