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ON RESONANT FRACTIONAL Q-DIFFERENCE SCHRÖDINGER EQUATIONS

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Abstract The research of fractional q-difference Schrödinger equations has attracted the attention of scholars and abundant results have been obtained in recent years. However, as far as we know, there are no results on resonant fractional q-difference Schrödinger equations. In this paper, we investigate the boundary value problems for fractional q-difference Schrödinger equations at resonance. By virtue of fixed point index theorem and spectral theory of linear operators, we obtain the multiplicity of positive solutions. In addition, we get different stability results, including Ulam-Hyres stability and generalized Ulam-Hyres stability. Give relevant examples to prove the main results.

Keywords Fractional q-difference Schrödinger equations, Resonance, boundary value problems, positive solutions, fixed point index.

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1. Introduction

The q-difference operator proposed by Jackson [11] in 1910 is a bridge between mathematics and physics, which plays an important role in quantum physics, spectral analysis and dynamical systems [5,7]. The study of fractional q-difference was started by Al-Salam [2] and Agarwal [1]. Fractional q-difference theory combines the advantages of discrete mathematics and fractional calculus, and it has been widely concerned by scholars [14,16,17].

The Schrödinger equation was first proposed by Schrödinger [18] in 1926, which describes the motion state of microscopic particles by wave function. It can be applied to quantum semiconductor, electromagnetic wave propagation, seismic migration and many other practical problems [6,19].

We consider the following space-independent fractional q-difference Schrödinger equations

$$D_q^\alpha u(t) + \frac{m}{\hbar}(E - v(t))u(t) = 0, \tag{1.1}$$

where m is the mass of the particle, $\hbar$ is the Planck constant, E is the total energy of the particle, $v(t)$ is the potential energy of particle. Let $\tau = \frac{m}{\hbar}$, $h(t) = E - v(t)$, we obtain the following fractional q-difference equations

$$D_q^\alpha u(t) + \tau h(t) f(u(t)) = 0, \quad t \in (0,1). \tag{1.2}$$

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Li [13] studies the existence of positive solutions for (1.1) and (1.2) by using the fixed point theorem in cones under the following boundary value conditions

$$u(0) = D_q u(0) = D_q u(1) = 0.$$

As far as we know, the existing results of fractional q-difference Schrödinger equation are obtained under non-resonant boundary value conditions [13,15,20], the case of resonance has not been considered at present. Resonance is a phenomenon in which a physical system vibrates at a maximum amplitude at a given frequency. The scattering of the Schrödinger system at resonance is equivalent to that of the Schrödinger equation on the corresponding waveguide manifold [10]. Therefore, it is important to study Schrödinger equation at resonance. Inspired by the above work, we study the multiplicity of positive solutions of boundary value problems for fractional q-difference Schrödinger equation at resonance.

In this paper, we consider boundary value problems for (1.2) satisfying the following boundary value conditions

$$u(0) = 0, u(1) = \eta u(\xi), \tag{1.3}$$

where $1 < \alpha \leq 2$, $\tau > 0$, $0 < \xi < 1$, $\eta\xi^{\alpha-1} = 1$, D_q^α is fractional q-derivative of Riemann-Liouville type, $h : (0, 1) \to (0, +\infty)$ and $f : [0, +\infty) \to (0, +\infty)$ are continuous and h permits singularity.

When $0 < \eta\xi^{\alpha-1} < 1$, boundary value problem (1.2) and (1.3) is non-resonant. In this case, we can use Green function to get the expression of the solution, and then use the fixed point theorem to study the existence of solutions. Many scholars have carried out relevant researches in this situation [4, 9]. However, when $\eta\xi^{\alpha-1} = 1$, boundary value problem (1.2) and (1.3) is resonant. Methods commonly used in the case of non-resonance, such as fixed point theorem, upper and lower solutions, etc., fail in the case of resonance. Therefore, in this paper we need to find other methods to study the positive solutions of resonant boundary value problems.

In addition to the multiplicity of positive solutions, we also study the stability of solutions for boundary value problem (1.2) and (1.3), the stability is an important index for the safe operation of the system. On the premise of the existence of the solution, the condition of stable operation of the system is studied by mathematical method, which can provide theoretical guarantee for the safe operation of the actual system. Therefore, in this paper, we consider different stability results, including Ulam-Hyres stability and generalized Ulam-Hyres stability.

The structure of this paper is as follows. In Section 2, we give the definitions and lemmas of fractional q-derivative and q-integral. In Section 3, by using fixed point index theorem and spectral theory of linear operators, we investigate the multiplicity of positive solutions. In Section 4, we give different stability results, including Ulam-Hyers stability and generalised Ulam-Hyers stability. In Section 5, relevant examples are used to demonstrate the main results.

2. Preliminaries

Let $q \in (0, 1)$ and define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function is given by

$$(a-b)^{(0)}=1, \quad (a-b)^{(n)}=\prod_{k=0}^{n-1}(a-bq^k), \quad n\in\mathbb{N}^+,$$
$$(a-b)^{(\alpha)}=a^\alpha\prod_{k=0}^{+\infty}\frac{a-bq^k}{a-bq^{\alpha+k}}, \quad \alpha\in\mathbb{R}.$$

If $b=0$, then $a^{(\alpha)}=a^\alpha$.

We can get the following properties

$$\begin{aligned}
&[a(t-s)]^{(\alpha)}=a^\alpha(t-s)^{(\alpha)},\\
&{}_tD_q(t-s)^{(\alpha)}=[\alpha]_q(t-s)^{(\alpha-1)},\\
&{}_sD_q(t-s)^{(\alpha)}=-[\alpha]_q(t-qs)^{(\alpha-1)},\\
&\int_a^b D_qf(t)d_qt=f(b)-f(a).
\end{aligned}$$

The q-Gamma function is given by

$$\Gamma_q(x)=\frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x\in\mathbb{R}\setminus\{0,-1,-2,\cdots\},$$

and satisfies $\Gamma_q(x+1)=[x]_q\Gamma_q(x)$.

The q-Beta function is given by

$$B_q(x,y)=\int_0^1 t^{x-1}(1-qt)^{(y-1)}d_qt, \quad x,y\in\mathbb{R}^+,$$

and satisfies $B_q(x,y)=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$.

Definition 2.1 ([3]). The fractional q-integral of Riemann-Liouville type of order $\alpha>0$ of a function $u:(0,+\infty)\to\mathbb{R}$ is given by

$$I_q^\alpha u(t)=\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}u(s)d_qs.$$

Definition 2.2 ([3]). The fractional q-derivative of Riemann-Liouville type of order $\alpha>0$ of a function $u:(0,+\infty)\to\mathbb{R}$ is given by

$$D_q^\alpha u(t)=D_q^n(I_q^{n-\alpha}u)(t),$$

where n is the smallest integer greater than or equal to α.

Lemma 2.1 ([3]). *Let $\alpha>0$. If $u\in L_q^1[0,1]$ such that $I_q^{n-\alpha}u\in AC_q^n[0,1]$. Then*

$$I_q^\alpha D_q^\alpha u(t)=u(t)-c_1t^{\alpha-1}-\cdots-c_nt^{\alpha-n},$$

where $c_i\in\mathbb{R}, i=1,\cdots,n$ and n is the smallest integer greater than or equal to α.

Lemma 2.2 ([3]). *Let $\alpha>0$. If $u\in L_q^1[0,1]$. Then*

$$D_q^\alpha I_q^\alpha u(t)=u(t).$$

Lemma 2.3 ([3]). *Let $\alpha > 0$, $\beta > -1$. Then*

$$I_q^\alpha t^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)} t^{\alpha+\beta},$$
$$D_q^\alpha t^{\alpha-1} = 0.$$

Let

$$g(\lambda) = \sum_{k=0}^{+\infty} \frac{\lambda^k}{\Gamma_q(k\alpha+\alpha-2)} = \frac{\alpha-2}{\Gamma_q(\alpha-1)} + \sum_{k=1}^{+\infty} \frac{\lambda^k}{\Gamma_q(k\alpha+\alpha-2)}.$$

It is easy to get that $g'(\lambda) > 0$ on $(0, +\infty)$ and

$$g(0) = \frac{\alpha-2}{\Gamma_q(\alpha-1)} < 0, \quad \lim_{\lambda\to+\infty} g(\lambda) = +\infty.$$

Therefore, there exists a unique root $\lambda^* > 0$ such that

$$g(\lambda^*) = 0.$$

Let $\lambda \in (0, \lambda^*]$. The following boundary value problem is equivalent to (1.2) and (1.3)

$$\begin{cases} -D_q^\alpha u(t) + \lambda u(t) = \tau h(t) f(u(t)) + \lambda u(t), & t \in (0,1), \\ u(0) = 0, u(1) = \eta u(\xi). \end{cases} \tag{2.1}$$

Definition 2.3. Function $u \in C[0,1]$ satisfying (1.2) and (1.3) is called the solution of (2.2).

Theorem 2.1. *If $y \in C[0,1]$. Then the unique solution of problem*

$$\begin{cases} -D_q^\alpha u(t) + \lambda u(t) = y(t), & t \in (0,1), \\ u(0) = 0, u(1) = \eta u(\xi), \end{cases} \tag{2.2}$$

is

$$u(t) = \int_0^1 K(t,s) y(s) d_q s, \tag{2.3}$$

where

$$K(t,s) = H(t,s) + B(t) w(s),$$
$$H(t,s) = \frac{1}{B(1)} \begin{cases} G(1-qs)B(t) - G(t-qs)B(1), & 0 \le qs < t \le 1, \\ G(1-qs)B(t), & 0 \le t \le qs \le 1, \end{cases}$$
$$G(t-qs) = \sum_{k=0}^{+\infty} \frac{\lambda^k (t-qs)^{(k\alpha+\alpha-1)}}{\Gamma_q(k\alpha+\alpha)},$$
$$B(t) = \sum_{k=0}^{+\infty} \frac{\lambda^k t^{k\alpha+\alpha-1}}{\Gamma_q(k\alpha+\alpha)},$$
$$w(s) = \frac{\eta H(\xi,s)}{B(1) - \eta B(\xi)}.$$

Proof. (i) We show that if $u(t)$ is the solution of(2.2), it can be expressed as (2.3). By Lemma 2.1, we obtain

$$-u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \lambda I_q^{\alpha} u(t) = I_q^{\alpha} y(t).$$

From the boundary condition $u(0) = 0$, we get $c_2 = 0$. Therefore

$$(I - \lambda I_q^{\alpha})u(t) = -I_q^{\alpha} y(t) + c_1 t^{\alpha-1}.$$

From Lemma 2.3, we get

$$\begin{aligned}
u(t) =&(I - \lambda I_q^{\alpha})^{-1}(-I_q^{\alpha} y(t) + c_1 t^{\alpha-1})\\
=&\sum_{k=0}^{+\infty} \lambda^k I_q^{k\alpha}(-I_q^{\alpha} y(t) + c_1 t^{\alpha-1})\\
=&-\sum_{k=0}^{+\infty} \lambda^k I_q^{k\alpha+\alpha} y(t) + \sum_{k=0}^{+\infty} c_1 \lambda^k I_q^{k\alpha} t^{\alpha-1}\\
=&-\int_0^t \sum_{k=0}^{+\infty} \frac{\lambda^k (t-qs)^{(k\alpha+\alpha-1)}}{\Gamma_q(k\alpha+\alpha)} y(s) d_q s + \sum_{k=0}^{+\infty} \frac{c_1 \lambda^k \Gamma_q(\alpha) t^{k\alpha+\alpha-1}}{\Gamma_q(k\alpha+\alpha)}\\
=&-\int_0^t G(t-qs)y(s)d_q s + cB(t), \quad c = c_1\Gamma_q(\alpha).
\end{aligned}$$

Therefore

$$\begin{aligned}
u(1) &= -\int_0^1 G(1-qs)y(s)d_q s + cB(1),\\
u(\xi) &= -\int_0^{\xi} G(\xi-qs)y(s)d_q s + cB(\xi).
\end{aligned}$$

By the boundary condition $u(1) = \eta u(\xi)$, we have

$$c = \frac{\int_0^1 G(1-qs)y(s)d_q s - \eta\int_0^{\xi} G(\xi-qs)y(s)d_q s}{B(1) - \eta B(\xi)}.$$

Then $u(t)$ can be expressed as follows

$$\begin{aligned}
u(t) =& -\int_0^t G(t-qs)y(s)d_q s + \frac{\int_0^1 G(1-qs)y(s)d_q s - \eta\int_0^{\xi} G(\xi-qs)y(s)d_q s}{B(1)-\eta B(\xi)} B(t)\\
=&\frac{-\int_0^t G(t-qs)B(1)y(s)d_q s + \int_0^1 G(1-qs)B(t)y(s)d_q s}{B(1)}\\
&-\frac{\int_0^1 G(1-qs)B(t)y(s)d_q s}{B(1)} + \frac{\int_0^1 G(1-qs)y(s)d_q s - \eta\int_0^{\xi} G(\xi-qs)y(s)d_q s}{B(1)-\eta B(\xi)} B(t)\\
=&\int_0^1 H(t,s)y(s)d_q s\\
&+\frac{\eta B(t)[\int_0^1 G(1-qs)B(\xi)y(s)d_q s - \int_0^{\xi} G(\xi-qs)B(1)y(s)d_q s]}{B(1)[B(1)-\eta B(\xi)]}\\
=&\int_0^1 H(t,s)y(s)d_q s + \int_0^1 \frac{\eta B(t)H(\xi,s)}{B(1)-\eta B(\xi)} y(s)d_q s
\end{aligned}$$

$$= \int_0^1 K(t,s)y(s)d_qs.$$

(ii) We present that if $u(t)$ can be expressed as (2.3), then it is the solution of (2.2). In fact,

$$\begin{aligned} u(t) =& (I - \lambda I_q^{\alpha})^{-1}(-I_q^{\alpha}y(t) + c_1 t^{\alpha-1}) \\ =& -\int_0^t G(t-qs)y(s)d_qs + cB(t), \end{aligned}$$

where

$$c = c_1\Gamma_q(\alpha), \quad c = \frac{\int_0^1 G(1-qs)y(s)d_qs - \eta\int_0^{\xi} G(\xi-qs)y(s)d_qs}{B(1) - \eta B(\xi)}.$$

Then, we have

$$(I - \lambda I_q^{\alpha})u(t) = -I_q^{\alpha}y(t) + c_1 t^{\alpha-1}.$$

By Lemmas 2.2 and 2.3, we obtain

$$D_q^{\alpha}u(t) - \lambda u(t) = -y(t).$$

This completes the proof. □

Lemma 2.4. *The following properties hold*
(1) $K(t,s)$ *is continuous on* $[0,1]\times[0,1]$.
(2) $K(t,s) > 0$, $\forall t,s \in (0,1)$.
(3) $m_2(s)t^{\alpha-1} \le K(t,s) \le m_1(s)$, $\forall t,s \in (0,1)$, *where*

$$m_1(s) = G(1-qs) + B(1)w(s), \quad m_2(s) = \frac{w(s)}{\Gamma_q(\alpha)}.$$

Proof. By the definition of $B(t)$ and properties, we get

$$\frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \le B(t) = \sum_{k=0}^{+\infty}\frac{\lambda^k t^{k\alpha+\alpha-1}}{\Gamma_q(k\alpha+\alpha)} \le B(1)t^{\alpha-1},$$

and

$$B'(t) = \sum_{k=0}^{+\infty}\frac{(k\alpha+\alpha-1)\lambda^k t^{k\alpha+\alpha-2}}{\Gamma_q(k\alpha+\alpha)} > 0, \quad t \in (0,1].$$

Since $B(t)$ is continuous at $t = 0$ and increasing on $(0,1]$, monotonicity can be extended to endpoints. Then $B(t)$ is increasing on $[0,1]$ and $B(t) > 0$ on $(0,1]$.

By the definition of $G(t-qs)$ and properties, we have

$$\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \le G(t-qs) = \sum_{k=0}^{+\infty}\frac{\lambda^k(t-qs)^{(k\alpha+\alpha-1)}}{\Gamma_q(k\alpha+\alpha)},$$

and

$${}_tD_q[G(t-qs)] = \sum_{k=0}^{+\infty}\frac{\lambda^k(t-qs)^{(k\alpha+\alpha-2)}}{\Gamma_q(k\alpha+\alpha-1)} > 0, \quad t \in (0,1].$$

Therefore $G(t-qs)$ is increasing with respect to t on $[0,1]$ and $G(t-qs)>0$ on $(0,1]$.

(i) When $0\le t\le qs\le 1$, we get

$$\begin{aligned}K(t,s)=&\frac{G(1-qs)B(t)}{B(1)}+B(t)w(s)\\ \le&G(1-qs)t^{\alpha-1}+B(1)w(s)t^{\alpha-1}\\ \le&G(1-qs)+B(1)w(s)=m_1(s).\end{aligned}$$

When $0\le qs<t\le 1$, we have

$$\begin{aligned}K(t,s)=&\frac{G(1-qs)B(t)-G(t-qs)B(1)}{B(1)}+B(t)w(s)\\ \le&\frac{G(1-qs)B(t)}{B(1)}+B(t)w(s)\\ \le&m_1(s).\end{aligned}$$

(ii) When $0\le t\le qs\le 1$, we get

$$H(t,s)=\frac{G(1-qs)B(t)}{B(1)}\ge 0.$$

Thus,

$$\begin{aligned}K(t,s)&=H(t,s)+B(t)w(s)\ge B(t)w(s)\\ &\ge\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}w(s)=m_2(s)t^{\alpha-1}.\end{aligned}$$

When $0\le qs<t\le 1$, we obtain

$$\begin{aligned}&G(1-qs)B(t)-G(t-qs)B(1)\\ =&\sum_{k=0}^{+\infty}\frac{\lambda^k(1-qs)^{(k\alpha+\alpha-1)}}{\Gamma_q(k\alpha+\alpha)}B(t)-\sum_{k=0}^{+\infty}\frac{\lambda^k(t-qs)^{(k\alpha+\alpha-1)}}{\Gamma_q(k\alpha+\alpha)}B(1).\end{aligned}$$

Therefore,

$$\begin{aligned}&{}_sD_q\big[G(1-qs)B(t)-G(t-qs)B(1)\big]\\ =&\sum_{k=0}^{+\infty}\frac{-\lambda^kq(1-q^2s)^{(k\alpha+\alpha-2)}}{\Gamma_q(k\alpha+\alpha-1)}B(t)+\sum_{k=0}^{+\infty}\frac{\lambda^kq(t-q^2s)^{(k\alpha+\alpha-2)}}{\Gamma_q(k\alpha+\alpha-1)}B(1)\\ \ge&\sum_{k=0}^{+\infty}\frac{\lambda^kq(1-q^2s)^{(k\alpha+\alpha-2)}}{\Gamma_q(k\alpha+\alpha-1)}\big[B(1)-B(t)\big].\end{aligned}\tag{2.4}$$

In fact, denote

$$c(t,s)=\sum_{k=0}^{+\infty}\frac{\lambda^kq(t-q^2s)^{(k\alpha+\alpha-2)}}{\Gamma_q(k\alpha+\alpha-1)}.$$

Then,

$${}_tD_q[c(t,s)]=\sum_{k=0}^{+\infty}\frac{\lambda^kq(t-q^2s)^{(k\alpha+\alpha-3)}}{\Gamma_q(k\alpha+\alpha-2)}<0.$$

Integrating (2.4) from 0 to s, we notice

$$G(1-q\cdot 0)B(t)=G(t-q\cdot 0)B(1).$$

Then we obtain

$$\begin{aligned}
&G(1-qs)B(t)-G(t-qs)B(1)\\
\ge&\int_0^s\sum_{k=0}^{+\infty}\frac{\lambda^k q(1-q^2\tau)^{(k\alpha+\alpha-2)}}{\Gamma_q(k\alpha+\alpha-1)}\big[B(1)-B(t)\big]d\tau\\
=&\Big[\sum_{k=0}^{+\infty}\frac{\lambda^k}{\Gamma_q(k\alpha+\alpha)}-\sum_{k=0}^{+\infty}\frac{\lambda^k(1-qs)^{(k\alpha+\alpha-1)}}{\Gamma_q(k\alpha+\alpha)}\Big]\big[B(1)-B(t)\big]>0.
\end{aligned}$$

Consequently,

$$H(t,s)=\frac{G(1-qs)B(t)-G(t-qs)B(1)}{B(1)}\ge 0.$$

Hence,

$$K(t,s)=H(t,s)+B(t)w(s)\ge m_2(s)t^{\alpha-1}.$$

(iii) By $\eta\xi^{\alpha-1}=1$, we have

$$B(1)-\eta B(\xi)=\sum_{k=0}^{+\infty}\frac{\lambda^k[1-\eta\xi^{k\alpha+\alpha-1}]}{\Gamma_q(k\alpha+\alpha)}>\sum_{k=0}^{+\infty}\frac{\lambda^k[1-\eta\xi^{\alpha-1}]}{\Gamma_q(k\alpha+\alpha)}=0.$$

Combining with $H(t,s)\ge 0$, we obtain

$$w(s)=\frac{\eta H(\xi,s)}{B(1)-\eta B(\xi)}>0.$$

From (ii), we get

$$K(t,s)\ge\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}w(s)>0.$$

(iiii) From the continuity of $B(t)$ and $G(t-qs)$, it can see that $K(t,s)$ is continuous on $[0,1]\times[0,1]$. This completes the proof. □

Let $E=C[0,1]$ with $\|x\|=\max\limits_{t\in[0,1]}|x(t)|$, then E is a Banach space. Denote

$$\chi(t,s)=\frac{m_2(s)}{m_1(s)}t^{\alpha-1},\quad \chi(t)=\min_{s\in[0,1]}\chi(t,s).$$

Define a cone

$$P=\{x\in E: x(t)\ge\chi(t)\|x\|,\quad t\in[0,1]\}.$$

Let $0<a<1$, denote

$$\zeta=\min_{t\in[a,1]}\chi(t),\quad \nu(x)=\min_{t\in[a,1]}x(t),\quad x\in P.$$

For $\forall R\ge r>0$, let

$$\begin{aligned}
&P_r=\{x\in P:\|x\|<r\},\\
&\overline{P}_r=\{x\in P:\|x\|\le r\},\\
&P(\nu,r,R)=\{x\in P: r<\nu(x),\|x\|\le R\},\\
&\overline{P}(\nu,r,R)=\{x\in P: r\le\nu(x),\|x\|\le R\}.
\end{aligned}$$

Define functions as follows

$$\Phi_1(t,r,R) = \max\{\tau h(t)f(u(t)) + \lambda u(t) : r\chi(t) \le u(t) \le R\},$$
$$\Phi_2(t,r,R) = \min\{\tau h(t)f(u(t)) + \lambda u(t) : r\chi(t) \le u(t) \le R\}.$$

Define the following two operators

$$Lu(t) = \int_0^1 K(t,s)u(s)d_qs. \tag{2.5}$$

$$Au(t) = \int_0^1 K(t,s)\big[\tau h(s)f(u(s)) + \lambda u(s)\big]d_qs. \tag{2.6}$$

In view of Krein-Rutmann theorem and Lemma 2.4, we have the spectral radius $r(L) > 0$ and L has a positive eigenfunction corresponding to its first eigenvalue $b_1 = (r(L))^{-1}$. Then we have the following lemmas.

Lemma 2.5 ([21]). *If $\lambda \in (0,\lambda^*]$ holds and L is defined as (2.5). Then the first eigenvalue of L is $b_1 = \lambda$ and $u(t) = t^{\alpha-1}$ is the positive eigenfunction corresponding to b_1, that is, $u = \lambda Lu$.*

Lemma 2.6 ([8]). *Let E be a Banach space. P is a cone and P_r is a bounded open set in E. $A : \overline{P}_r \cap P \to P$ is a completely continuous operator.*

(1) *If $\exists u_0 \in P \setminus \{\theta\}$ such that $u - Au \ne \mu u_0$, $\forall \mu \ge 0$, $u \in \partial P_r \cap P$, then $i(A, P_r \cap P, P) = 0$.*

(2) *If $Au \ne \mu u$, $\forall \mu \ge 1$, $u \in \partial P_r \cap P$, then $i(A, P_r \cap P, P) = 1$.*

Lemma 2.7 ([12]). *Let $A : \overline{P}_{r_3} \to P$ be a completely continuous operator. If there exists a concave positive functional $\nu(x) \le \|u\|$ $(u \in P)$ and numbers $r_3 \ge r_2 > r_1 > 0$ satisfying the following conditions*

(1) *$P(\nu, r_1, r_2) \ne \emptyset$ and $\vartheta(Au) > r_1$, if $u \in \overline{P}(\nu, r_1, r_2)$.*

(2) *$Au \in \overline{P}_{r_3}$, if $u \in \overline{P}(\nu, r_1, r_3)$.*

(3) *$\nu(Au) > r_1$, if $u \in \overline{P}(\nu, r_1, r_3)$ with $\|Au\| > r_2$.*

Then, $i(A, \overline{P}(\nu, r_1, r_3), \overline{P}_{r_3}) = 1$.

3. The solvability of fractional q-difference equation

Theorem 3.1. *Assume that there exists numbers $r_5 \ge r_4 > r_3 > r_2 > r_1 > 0$ with $\zeta r_4 \ge r_3$ such that*

(A_1) *$\Phi_1(t, r_1, r_5) \in L[0,1]$.*

(A_2) *$\int_0^1 m_1(s)\Phi_1(s, r_2, r_2)d_qs < r_2$.*

(A_3) *$\int_0^1 m_1(s)\Phi_1(s, r_3, r_5)d_qs < r_5$.*

(A_4) *$\int_0^1 m_1(s)\Phi_2(s, r_3, r_4)d_qs > r_3\zeta^{-1}$.*

Then boundary value problem (1.2) and (1.3) has at least three positive solutions.

Proof. Operator L is defined as (2.6). Firstly, we prove that $A : \overline{P}_{r_5} \to P$ is a completely continuous operator. Obviously, for any $u \in \overline{P}_{r_5}$, $Au \in E$.

(i) For any $u \in \overline{P}_{r_5}$, $t \in [0,1]$, from Lemma 2.4, we get

$$\begin{aligned} Au(t) &= \int_0^1 K(t,s)\big[\tau h(s)f(u(s)) + \lambda u(s)\big]d_q s \\ &\geq \int_0^1 m_2(s)t^{\alpha-1}\big[\tau h(s)f(u(s)) + \lambda u(s)\big]d_q s \\ &= \int_0^1 \chi(t,s)m_1(s)\big[\tau h(s)f(u(s)) + \lambda u(s)\big]d_q s \\ &\geq \chi(t)\int_0^1 m_1(s)\big[\tau h(s)f(u(s)) + \lambda u(s)\big]d_q s \\ &= \chi(t) \parallel Au \parallel . \end{aligned}$$

Then $A(\overline{P}_{r_5}) \subset P$.

(ii) For any $u \in \overline{P}_{r_5}$, $t \in [0,1]$, by Lemma 2.4, we have

$$\begin{aligned} |Au(t)| &= \Big|\int_0^1 K(t,s)[\tau h(s)f(u(s)) + \lambda u(s)]d_q s\Big| \\ &\leq \int_0^1 m_1(s)\big|\tau h(s)f(u(s)) + \lambda u(s)\big|d_q s \\ &\leq \int_0^1 m_1(s)\Phi_1(s,r_1,r_5)d_q s. \end{aligned}$$

Then A is uniformly bounded.

(iii) Since $K(t,s)$ is continuous on $[0,1]\times[0,1]$, for any $\varepsilon > 0$, exists $\delta > 0$ such that for any $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$, we obtain

$$\big|K(t_1,s) - K(t_2,s)\big| < \frac{\varepsilon}{\int_0^1 \Phi_1(s,r_1,r_5)d_q s}.$$

Therefore,

$$\begin{aligned} |Au(t_1) - Au(t_2)| &\leq \int_0^1 \big|K(t_1,s) - K(t_2,s)\big|\big[\tau h(s)f(u(s)) + \lambda u(s)\big]d_q s \\ &\leq \int_0^1 \frac{\varepsilon}{\int_0^1 \Phi_1(s,r_1,r_5)d_q s}\Phi_1(s,r_1,r_5)d_q s = \varepsilon. \end{aligned}$$

Then A is equicontinuous. From Arzela-Ascoli theorem, we know that A is compact.

(iiii) Assume that $\{u_n\} \subset \overline{P}_{r_5}$ and $\parallel u_n - u_0 \parallel \to 0 (n \to +\infty)$. For $\forall \varepsilon > 0$, $\exists \delta \in (0, \frac{1}{2})$ such that

$$\int_0^\delta m_1(s)\Phi_1(s,r_1,r_5)d_q s < \frac{\varepsilon}{6},$$
$$\int_{1-\delta}^1 m_1(s)\Phi_1(s,r_1,r_5)d_q s < \frac{\varepsilon}{6}.$$

Function $h(t)$ is continuous on $[\delta, 1-\delta]$, f is continuous and $\parallel u_n - u_0 \parallel \to 0 (n \to +\infty)$, exists $N > 0$ such that for any $n > N$, we get

$$\big|\tau h(t)f(u_n(t)) - \tau h(t)f(u_0(t))\big| < \frac{\varepsilon}{3\int_0^1 m_1(s)d_q s}, \quad t \in [\delta, 1-\delta].$$

Therefore,

$$\begin{aligned}\|Au_n - Au_0\| &\leq \max_{t\in[0,1]} \int_0^1 K(t,s)\big|\tau h(s)f(u_n(s)) - \tau h(s)f(u_0(s))\big|d_qs\\ &\leq \int_0^1 m_1(s)\big|\tau h(s)f(u_n(s)) - \tau h(s)f(u_0(s))\big|d_qs\\ &\leq 2\int_0^\delta m_1(s)\Phi_1(s,r_1,r_5)d_qs + 2\int_{1-\delta}^1 m_1(s)\Phi_1(s,r_1,r_5)d_qs\\ &\quad + \int_\delta^{1-\delta} m_1(s)\big|\tau h(s)f(u_n(s)) - \tau h(s)f(u_0(s))\big|d_qs\\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Then A is continuous. Thus $A : \overline{P}_{r_5} \to P$ is a completely continuous operator.

Next, we present that A has three fixed points.

For any $u \in \partial P_{r_2}$, $t \in [0,1]$, we have $r_2\chi(t) \leq u(t) \leq r_2$. By Lemma 2.4 and (A_2), we obtain

$$Au(t) \leq \int_0^1 m_1(s)\Phi_1(s,r_2,r_2)d_qs < r_2,$$

which implies that $Au \neq \mu u$, $\forall \mu \geq 1$, $u \in \partial P_{r_2}$. From Lemma 2.6, we get

$$i(A, P_{r_2}, P) = 1. \tag{3.1}$$

Thus we show that A has a positive fixed point on P_{r_2}.

Then we present A has no fixed point on P_{r_1}. We just need to prove exists $u_0 \in P \setminus \{\theta\}$ such that

$$u - Au \neq \mu u_0, \quad \forall \mu \geq 0, \quad u \in \partial P_{r_1}.$$

Otherwise, suppose that exists $\mu_1 > 0$ and $u_1 \in \partial P_{r_1}$ such that

$$u_1 - Au_1 = \mu_1 u_0.$$

Thus, $u_1 \geq \mu_1 u_0$. Let $\mu^* = \sup\{\mu : u_1 \geq \mu u_0\}$.

$$Au_1(t) = \int_0^1 K(t,s)\big[\tau h(s)f(u_1(s)) + \lambda u_1(s)\big]d_qs \geq \int_0^1 K(t,s)\lambda u_1(s)d_qs = \lambda Lu_1(t).$$

From Lemma 2.5, we know that the spectral radius of L is $r(L) = \lambda^{-1}$ and the corresponding eigenfunction is $u(t) = t^{\alpha-1}$. That is, $Lu = \lambda^{-1}u$. Therefore,

$$u_1 = Au_1 + \mu_1 u_0 \geq \lambda Lu_1 + \mu_1 u_0 \geq \lambda L(\mu^* u_0) + \mu_1 u_0 = (\mu^* + \mu_1)u_0,$$

which contradicts the definition of μ^*. By Lemma 2.6, we obtain

$$i(A, P_{r_1}, P) = 0. \tag{3.2}$$

From (3.1) and (3.2), we get A has a fixed point $u_1 \in P_{r_2} \setminus P_{r_1}$.

For any $u \in \partial P_{r_5}$, $t \in [0,1]$, we obtain $r_3\chi(t) \leq u(t) \leq r_5$. From Lemma 2.4 and (A_3), we have

$$Au(t) \leq \int_0^1 m_1(s)\Phi_1(s,r_3,r_5)d_qs < r_5.$$

Therefore,

$$i(A, P_{r_5}, P) = 1. \tag{3.3}$$

Finally, we present that $i(A, \overline{P}(\nu, r_3, r_5), \overline{P}_{r_5}) = 1$.

(i) It is obvious that $P(\nu, r_3, r_4) \neq \emptyset$. For any $u \in \overline{P}(\nu, r_3, r_4)$, we get $r_3 \leq u(t) \leq r_4$, $t \in [a, 1]$. By Lemma 2.6 and (A_4), we have

$$\begin{aligned}
\nu(Au) &= \min_{t\in[a,1]} Au(t) = \min_{t\in[a,1]} \int_0^1 K(t,s)\big[\tau h(s) f(u(s)) + \lambda u(s)\big] d_q s \\
&\geq \min_{t\in[a,1]} \int_0^1 m_2(s) t^{\alpha-1}\big[\tau h(s) f(u(s)) + \lambda u(s)\big] d_q s \\
&= \min_{t\in[a,1]} \int_0^1 \chi(t,s) m_1(s)\big[\tau h(s) f(u(s)) + \lambda u(s)\big] d_q s \\
&\geq \min_{t\in[a,1]} \int_0^1 \chi(t) m_1(s)\big[\tau h(s) f(u(s)) + \lambda u(s)\big] d_q s \\
&\geq \zeta \int_0^1 m_1(s) \Phi_2(s, r_3, r_4) d_q s > r_3.
\end{aligned}$$

(ii) For any $u \in \overline{P}(\nu, r_3, r_5)$, we obtain $r_3 \leq u(t) \leq r_5$, $t \in [0, 1]$. By Lemma 2.6 and (A_3), we get

$$\begin{aligned}
Au(t) &= \int_0^1 K(t,s)\big[\tau h(s) f(u(s)) + \lambda u(s)\big] d_q s \\
&\leq \int_0^1 m_1(s)\big[\tau h(s) f(u(s)) + \lambda u(s)\big] d_q s \\
&\leq \int_0^1 m_1(s) \Phi_1(s, r_3, r_5) d_q s \leq r_5.
\end{aligned}$$

Therefore $Au \in \overline{P}_{r_5}$.

(iii) For any $u \in \overline{P}(\nu, r_3, r_5)$ with $\|Au\| > r_4$, $\zeta r_4 \geq r_3$, $t \in [a, 1]$, we have

$$\nu(Au) = \min_{t\in[a,1]} Au(t) \geq \min_{t\in[a,1]} \chi(t)\|Au\| = \zeta\|Au\| > \zeta r_4 \geq r_3.$$

By Lemma 2.7, we obtain

$$i(A, \overline{P}(\nu, r_3, r_5), \overline{P}_{r_5}) = 1. \tag{3.4}$$

From (3.1), (3.3) and (3.4), we get

$$i(A, P_{r_5} \setminus (\overline{P}(\nu, r_3, r_5) \cup P_{r_2}), \overline{P}_{r_5}) = -1. \tag{3.5}$$

By (3.4) and (3.5), A has two fixed points $u_2 \in \overline{P}(\nu, r_3, r_5)$ and $u_3 \in P_{r_5} \setminus (\overline{P}(\nu, r_3, r_5) \cup P_{r_2})$. Therefore these fixed points are three positive solutions for (1.2) and (1.3). This completes the proof. □

4. The stability of fractional q-difference equation

Definition 4.1. The solution is Ulam-Hyers (UH) stable, if there exist a constant $M_1 \geq 0$ and $\varepsilon > 0$, for each solution $u \in C([0, 1], \mathbb{R})$,

$$\big|D_q^\alpha u(t) + \tau h(t) f(u(t))\big| \leq \varepsilon, \quad t \in [0, 1], \tag{4.1}$$

and a solution $u^* \in C([0,1],\mathbb{R})$, such that $|u-u^*| \leq M_1\varepsilon$. The solution is generalised Ulam-Hyers (GUH) stable, if there exists $\psi \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\psi(0)=0$, such that $|u-u^*| \leq M_1\psi(\varepsilon)$.

Remark 4.1. Function $u \in C([0,1],\mathbb{R})$ is the solution of (4.1), iff there exists a function $\varpi \in C([0,1],\mathbb{R})$ depends on u, such that

(i) $\big|\varpi(t)\big| \leq \varepsilon$, $t \in [0,1]$.

(ii) $D_q^\alpha u(t) + \tau h(t) f(u(t)) - \varpi(t) = 0$.

Proof. If (i) and (ii) hold, we obtain,

$$\big|D_q^\alpha u(t) + \tau h(t) f(u(t))\big| = \big|\varpi(t)\big| \leq \varepsilon.$$

If $u \in C([0,1],\mathbb{R})$ is the solution of (4.1), we get,

$$-\varepsilon \leq D_q^\alpha u(t) + \tau h(t) f(u(t)) \leq \varepsilon.$$

Hence, there exists $\varpi(t) \in [-\varepsilon,\varepsilon]$, such that

$$D_q^\alpha u(t) + \tau h(t) f(u(t)) = \varpi(t).$$

This completes the proof. □

Lemma 4.1. *If $u \in C([0,1],\mathbb{R})$ is the solution of*

$$\begin{cases} D_q^\alpha u(t) + \tau h(t) f(u(t)) = \varpi(t), \quad t \in (0,1), \\ u(0)=0, u(1) = \eta u(\xi). \end{cases} \tag{4.2}$$

Then, u satisfies the following inequality

$$\big|u(t) - Au(t)\big| \leq w_1\varepsilon,$$

where

$$w_1 = \int_0^1 m_1(s) d_q s.$$

Proof. If $u \in C([0,1],\mathbb{R})$ is the solution of (4.2), then

$$u(t) = Au(t) - \int_0^1 K(t,s)\varpi(s) d_q s.$$

In view of Lemma 2.4 and Remark 4.1, we obtain

$$\begin{aligned} \big|u(t) - Au(t)\big| \leq& \int_0^1 K(t,s)\big|\varpi(s)\big| d_q s \\ \leq& \varepsilon \int_0^1 m_1(s) d_q s \\ =& w_1\varepsilon. \end{aligned}$$

This completes the proof. □

Theorem 4.1. *If the following conditions hold.*

(B_1) $f : [0,+\infty) \to (0,+\infty)$ *is continuous.*

(B_2) $\big|f(u_1) - f(u_2)\big| \leq l\big|u_1 - u_2\big|$, $t \in [0,1]$, $u_1, u_2 \in \mathbb{R}$, $l > 0$.

(B_3) $\int_0^1 m_1(s)\big(l\tau h(s) + \lambda\big) d_q s < 1$.

Then boundary value problem (1.2) *and* (1.3) *is UH stable and GUH stable.*

Proof. For each solution $u \in C([0,1],\mathbb{R})$ of (4.2) and solution u^* of (1.2) and (1.3), by Lemmas 2.4 and 4.1, we get

$$\begin{aligned}\|u-u^*\| &= \|u-Au^*\| = \|u-Au+Au-Au^*\| \\ &\leq \|u-Au\| + \|Au-Au^*\| \\ &\leq w_1\varepsilon + \int_0^1 K(t,s)\Big[\tau h(s)\big|f(u(s))-f(u^*(s))\big| + \lambda\big|u(s)-u^*(s)\big|\Big]d_qs \\ &\leq w_1\varepsilon + \int_0^1 m_1(s)\Big[l\tau h(s)\big|u(s)-u^*(s)\big| + \lambda\big|u(s)-u^*(s)\big|\Big]d_qs \\ &\leq w_1\varepsilon + \|u-u^*\|\int_0^1 m_1(s)\big(l\tau h(s)+\lambda\big)d_qs \\ &:= w_1\varepsilon + k\|u-u^*\|.\end{aligned}$$

Therefore,

$$\|u-u^*\| \leq \frac{w_1}{1-k}\varepsilon := M_1\varepsilon.$$

Then boundary value problem (1.2) and (1.3) is UH stable. For $\psi(\varepsilon)=\varepsilon$, boundary value problem (1.2) and (1.3) is GUH stable, which can be seen in Figure 1. □

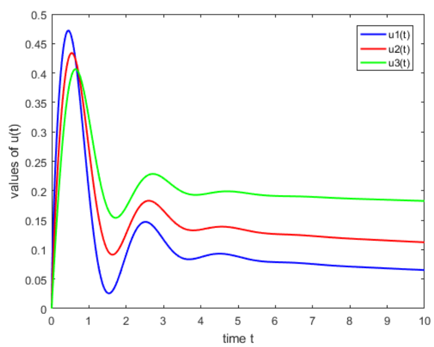

Figure 1. Illustration of proof.

5. Example

Example 5.1. Consider the following boundary value problem:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{5}{2}}u(t) + \frac{1}{5}t\sin u = 0, \quad t\in(0,1), \\ u(0) = D_{\frac{1}{2}}u(0) = 0, u(1) = 8u(\frac{1}{4}), \end{cases} \tag{5.1}$$

where $q=\frac{1}{2}$, $\alpha=\frac{5}{2}$, $\tau=\frac{1}{5}$, $h(t)=t$, $f(u(t))=\sin u$, $\eta=8$, $\xi=\frac{1}{4}$, $\eta\xi^{\alpha-1}=1$.

In Theorem 3.1, let $\lambda=\frac{1}{5}$ and $a=\frac{1}{2}$. Therefore, we obtain $G(1)=0.7076$, $B(\xi)=0.0885$, $\chi(t)=0.9583t^{1.5}$, $\int_0^1 m_1(s)d_qs = 2.6425$, $\zeta^{-1}=0.3388$.

Let $r_1=0.8$, $r_2=1.5$, $r_3=3.2$, $r_4=12$, $r_5=20$. Then

$$\int_0^1 m_1(s)\Phi_1(s,r_2,r_2)d_qs \approx 1.3213 < r_2,$$

$$\int_0^1 m_1(s)\Phi_1(s,r_3,r_5)d_qs \approx 11.0985 < r_5,$$
$$\int_0^1 m_1(s)\Phi_2(s,r_3,r_4)d_qs \approx 1.1627 > r_3\zeta^{-1}.$$

From Theorem 3.1, we get (5.1) has at least three positive solutions.

In Theorem 4.1, we have,

$$\big|f(u_1)-f(u_2)\big| = \big|\sin u_1 - \sin u_2\big| \le \big|u_1-u_2\big|,$$
$$\int_0^1 m_1(s)\big(l\tau h(s)+\lambda\big)d_qs \approx 0.78 < 1.$$

From Theorem 4.1, we obtain (5.1) is UH stable and GUH stable.

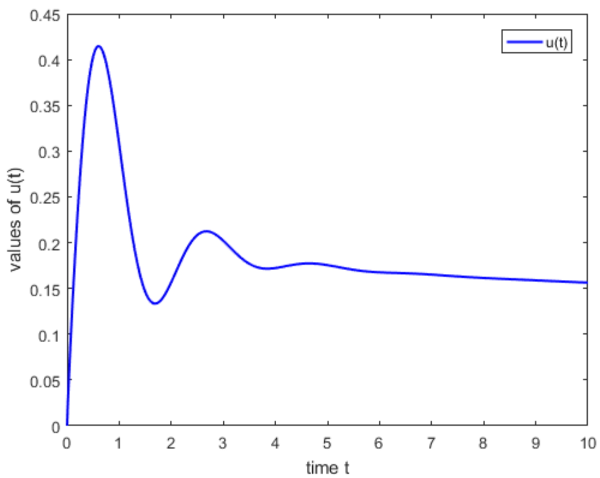

Figure 2. Illustration of proof.

6. Conclusion

In this paper, we investigate the boundary value problems for fractional q-difference Schrödinger equations at resonance. In view of fixed point index theorem and spectral theory of linear operators, we obtain the multiplicity of positive solutions. In addition, we give different stability results, including UH stability and GUH stability. The results of solvability and stability of solutions obtained in this paper can also be applied to other types of equations, for example, fractional delay q-difference equations and fractional advanced q-difference equations. Moreover, we can further consider the fractional q-difference equations under different boundary value conditions, such as multi-point boundary value problems, integral boundary value problems and so on.

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