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CONVERGENCE OF THE TWO-POINT MODULUS-BASED MATRIX SPLITTING ITERATION METHOD*

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Abstract In this paper, we discuss the convergence of the two-point modulus-based matrix splitting iteration method for solving the linear complementarity problem. Some convergence conditions are presented from the spectral radius and the matrix norm when the system matrix is a P-matrix. Besides, the quasi-optimal cases of the method are studied. Numerical experiments are provided to show the presented results.

Keywords Linear complementarity problem, matrix splitting, convergence.

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1. Introduction

In this paper, we consider the following linear complementarity problem (LCP(A,q)) to solve $z \in R^n$ that satisfies

$$z^{\mathrm{T}} r = 0 \quad \text{with} \quad z \geq 0, \quad r = Az + q \geq 0, \tag{1.1}$$

where $A = (a_{ij}) \in R^{n\times n}$ and $q \in R^n$ are given. Many problems can be modeled as (1) under some conditions, for example, the linear and quadratic programming problems, the Nash equilibrium point problem of bi-matrix games, and the free boundary problems of journal bearings (see [1,3,5,6,11,16,18,19] for details).

In order to compute the numerical solution of the LCP(A,q), many iteration methods have been established in recent decades. In all of those iteration methods, the modulus-based type iteration methods attracted many researchers' attention. For this kinds of iteration methods, van Bokhoven presented a modulus method [2],

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Murty discussed the modulus iteration method [18], Hadjidimos and Tzoumas proposed a non-stationary extrapolated modulus algorithms [12], Dong and Jiang studied the modified modulus iteration method [7]. By introducing the parameter matrices and reformulating the LCP(A, q) as an implicit fixed-point equation, Bai presented a class of modulus-based matrix splitting iteration methods [3]. Since then, a series of modulus-based type iteration methods have been presented, including the general modulus-based matrix splitting method [16], the preconditioned modulus-based matrix splitting iteration method [23], the two-sweep modulus-based matrix splitting iteration method [21], the two-step modulus-based matrix splitting iteration method [26], the accelerated modulus-based matrix splitting iteration methods [28], the general two-sweep modulus-based matrix splitting iteration method [19], the relaxation modulus-based matrix splitting iteration method [25], the preconditioned two-step modulus-based matrix splitting iteration method [8]. Most of these iteration methods are very efficient. Except for the modulus-based type iteration methods, there are other iteration methods, such as the interior projection methods and the fixed point method; see [1, 3, 13–16, 20, 22–24, 27] and the references therein.

In this paper, we shall make a further study on the two-sweep modulus-based matrix splitting iteration method. Since the iteration process of this method involves two points, we called it the two-point modulus-based (TPMB) matrix splitting iteration method in our discussion. The numerical experiments in [21] show that the TPMB iteration method is very effective and has larger convergence region than the modulus-based matrix splitting iteration method sometimes. In the present paper, we first explore the general convergence conditions when the system matrix A is a P-matrix. These convergence conditions extend the application range of the TPMB iteration method. Then we focus on the special matrix splittings for the TPMB iteration method, that is, the matrices M and N in the splitting $A = M - N$ satisfy some particular characteristics. In the light of the particular matrix splitting, we study the concrete convergence conditions and the quasi-optimal cases of the TPMB iteration method. Besides, the corresponding numerical experiments are illustrated to show the presented results.

The paper is organized as follows. We introduce the preliminaries in Section 2 and present the main results in Section 3. We show the numerical experiments in Section 4 and end the paper with concluding remarks in Section 5.

2. Preliminary

Let us first review the classical modulus-based matrix splitting iteration method and the two-point modulus-based matrix splitting iteration method for the LCP(A, q).

By utilizing the matrix splitting: $A = M - N$, the LCP(A, q) can be equivalently transformed into an equation

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q \tag{2.1}$$

with

$$z = \frac{1}{\gamma}(|x| + x), \quad r = \frac{1}{\gamma}\Omega(|x| - x),$$

where Ω is a positive diagonal matrix, $M + \Omega$ is nonsingular, and γ is a positive constant. Based on Eq. (2.1), the modulus-based matrix splitting iteration method

[3] and the two-point (two-sweep) modulus-based matrix splitting iteration method [21] are represented as

Method 2.1.

$$(\Omega+M)x^{(k+1)} = Nx^{(k)}+(\Omega-A)|x^{(k)}|-\gamma q \text{ and } z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}|+x^{(k+1)}), \quad (2.2)$$

Method 2.2.

$$(\Omega+M)x^{(k+1)} = Nx^{(k)}+(\Omega-A)|x^{(k-1)}|-\gamma q \text{ and } z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}|+x^{(k+1)}), \quad (2.3)$$

respectively.

The iteration process (2.3) can be reformulated as

$$\begin{pmatrix} x^{(k+1)} \\ x^{(k)} \end{pmatrix} = \begin{pmatrix} (\Omega+M)^{-1}N & (\Omega+M)^{-1}(\Omega-A) \\ I & O \end{pmatrix} \begin{pmatrix} x^{(k)} \\ |x^{(k-1)}| \end{pmatrix} + \begin{pmatrix} -\gamma(\Omega+M)^{-1}q \\ 0 \end{pmatrix}, \quad (2.4)$$

where I is an identity matrix.

Other definitions and notations are reviewed briefly as follows: a real matrix A is called a P-matrix if all of its principal minors are positive; a real matrix A is called an M-matrix (it is called a nonsingular M-matrix in some papers) if $a_{ij} \le 0, i \ne j$ and $A^{-1} \ge O$; the comparison matrix $\langle A\rangle = (\langle a_{ij}\rangle) \in R^{n\times n}$ of a real matrix A is defined as

$$\langle a_{ij}\rangle = \begin{cases} |a_{ij}| & \text{for } i = j, \\ -|a_{ij}| & \text{for } i \ne j, \end{cases} \quad i,j = 1,2,\ldots,n;$$

a real matrix A is called an H-matrix if $\langle A\rangle$ is an M-matrix, and called an H_+-matrix if it has positive diagonals; the absolute value matrix of a real matrix A is represented as $|A| = (|a_{ij}|)$. The splitting $A = M - N$ is called an H-splitting if $\langle M\rangle - |N|$ is an M-matrix and called an H-compatible splitting if $\langle A\rangle = \langle M\rangle - |N|$. For these materials, readers can refer to [4, 9, 10, 16, 21]. From the definitions of H_+-matrix, H-splitting and H-compatible splitting, it is easy to know that the H-compatible splitting of an H_+-matrix is also an H-splitting. In addition, it is well known that the H_+-matrix and the positive definite matrix are two types of P-matrix [3, 5, 7, 13, 18, 22].

3. Main results

In this section, we discuss the convergence of TPMB iteration method from the spectral radius and the matrix norm. From Theorem 4.1 and its proof in [21], we have the following conclusion for the P-matrix linear complementarity problem.

Lemma 3.1. *Suppose $A \in R^{n\times n}$ is a P-matrix with $A = M - N$, and set*

$$W = \begin{pmatrix} (\Omega+M)^{-1}N & (\Omega+M)^{-1}(\Omega-A) \\ I & O \end{pmatrix},$$

then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty}$ *generated by Method 2.2 is convergent for any initial vectors* $x^{(0)}, x^{(1)} \in R^n$ *if any of the following conditions holds:*

(i) $\rho(|W|) < 1$;

(ii) $\rho(|(\Omega+M)^{-1}N| + |(\Omega+M)^{-1}(\Omega-A)|) < 1$;

(iii) $\sigma(\Omega) = \xi(\Omega) + \zeta(\Omega) < 1$;

(iv) $\delta(\Omega) = 2\xi(\Omega) + \eta(\Omega) < 1$,

where

$$\begin{aligned}\xi(\Omega) &= \| \, |(\Omega+M)^{-1}N| \, \|,\\ \eta(\Omega) &= \| \, |(\Omega+M)^{-1}(\Omega-M)| \, \|,\\ \zeta(\Omega) &= \| \, |(\Omega+M)^{-1}(\Omega-A)| \, \|,\end{aligned}$$

and $\|\cdot\|$ *is an induced matrix norm.*

Theorem 3.1. *$A \in R^{n\times n}$ is an M-matrix, and $A = M - N$ is a splitting of A with M being an M-matrix. If $\Omega \geq \operatorname{diag}(A)$ and $(\Omega+M)^{-1}N \geq O$, then the sequence $\{z^{(k)}\}_{k=0}^{+\infty}$ generated by Method 2.2 is convergent for arbitrary initial vectors $x^{(0)}, x^{(1)} \in R^n$.*

Proof. Since A, M are two M-matrices and $\Omega \geq \operatorname{diag}(A)$, we have $\Omega - A \geq O$ and $\Omega + M$ is an M-matrix due to $\Omega + M \geq M$. Then $(\Omega+M)^{-1} \geq O$ and $W \geq O$ in (2.4) under the condition $(\Omega+M)^{-1}N \geq O$. Since the matrix

$$(\Omega+M)^{-1}(\Omega+N-A) = (\Omega+M)^{-1}N + (\Omega+M)^{-1}(\Omega-A) \geq O$$

is associated with the matrix splitting

$$2A = (\Omega+M) - (\Omega+N-A),$$

which is a weak regular splitting (see Definition 3.3 in [10]), we know that $\rho((\Omega+M)^{-1}(\Omega+N-A)) < 1$ holds based on Theorem 3.4 in [10]. Thus from (ii) in Lemma 3.1, we know that Method 2.2 is convergent. □

In the following, we discuss some concrete convergence conditions and the quasi-optimal cases of Method 2.2 based on (ii) and (iv) in Lemma 3.1.

Theorem 3.2. *Suppose $A \in R^{n\times n}$ is a P-matrix, and the splitting $A = M - N$ satisfies that M is a symmetric positive definite M-matrix. Set $\Omega = \omega I$ with $\omega > 0$. Denote the largest and the smallest eigenvalues of M by $\lambda_{\max}$ and $\lambda_{\min}$, respectively, and define $\tau = \| \, |N| \, \|_2$. Then the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty}$ generated by Method 2.2 is convergent for arbitrary initial vectors $x^{(0)}, x^{(1)} \in R^n$ if either of the following two conditions holds:*

(i) If $M = sI (s > 0)$ and $s > \tau$, then $\omega > \tau$. Moreover, $\omega = s$ is a quasi-optimal parameter when $\omega \in (\tau, +\infty)$;

(ii) If $M \neq sI (s > 0)$, $\| \, |(\omega I + M)^{-1}(\omega I - M)| \, \|_2 = ||(\omega I + M)^{-1}(\omega I - M)||_2$ and $\tau < \lambda_{\min}$, then

$$\omega \in (\omega_L, +\infty),$$

where $\omega_L = \dfrac{\tau - \lambda_{\min} + \sqrt{(\lambda_{\min}-\tau)^2 + 4\lambda_{\max}\tau}}{2}$. Moreover, $\omega = \sqrt{\lambda_{\min}\lambda_{\max}}$ is a quasi-optimal parameter.

Proof. Since the matrix A is a P-matrix, the LCP(A, q) has a unique solution. If $\Omega = \omega I$, from (iv) in Lemma 3.1, we know that a sufficient convergence condition of Method 2.2 is

$$\delta(\Omega) = \delta(\omega I) = 2\| \, |(\omega I + M)^{-1}N| \, \|_2 + \| \, |(\omega I + M)^{-1}(\omega I - M)| \, \|_2 < 1.$$

Since M is a symmetric positive definite M-matrix, we have

$$\begin{aligned} \| \, |(\omega I + M)^{-1}N| \, \|_2 &\le \| \, |(\omega I + M)^{-1}| \, \|_2 \cdot \| \, |N| \, \|_2 \\ &= \|(\omega I + M)^{-1}\|_2 \cdot \| \, |N| \, \|_2 \\ &= \max\nolimits_{\lambda \in \mathrm{sp}(M)} \frac{\tau}{\omega + \lambda} = \frac{\tau}{\omega + \lambda_{\min}} \end{aligned}$$

and if $M = sI(s > 0)$ or $M \neq sI(s > 0)$ with $\| \, |(\omega I + M)^{-1}(\omega I - M)| \, \|_2 = ||(\omega I + M)^{-1}(\omega I - M)||_2$, we have

$$\begin{aligned} \| \, |(\omega I + M)^{-1}(\omega I - M)| \, \|_2 &= ||(\omega I + M)^{-1}(\omega I - M)||_2 \\ &= \max\nolimits_{\lambda \in \mathrm{sp}(M)} \frac{|\omega - \lambda|}{\omega + \lambda} \\ &= \max\left\{ \frac{|\omega - \lambda_{\min}|}{\omega + \lambda_{\min}}, \frac{|\omega - \lambda_{\max}|}{\omega + \lambda_{\max}} \right\} \\ &= \begin{cases} \dfrac{|\omega - s|}{\omega + s}, & \lambda_{\max} = \lambda_{\min} = s, (s > 0); \\ \dfrac{\lambda_{\max} - \omega}{\omega + \lambda_{\max}}, \omega \le \sqrt{\lambda_{\min}\lambda_{\max}} \text{ with } \lambda_{\min} \neq \lambda_{\max}; \\ \dfrac{\omega - \lambda_{\min}}{\omega + \lambda_{\min}}, \ \omega \ge \sqrt{\lambda_{\min}\lambda_{\max}} \text{ with } \lambda_{\min} \neq \lambda_{\max}. \end{cases} \end{aligned}$$

If we set

$$\begin{aligned} \bar{\delta}(\Omega) = \bar{\delta}(\omega I) &= 2\|(\omega I + M)^{-1}\|_2 \cdot \| \, |N| \, \|_2 + \|(\omega I + M)^{-1}(\omega I - M)\|_2 \\ &= \begin{cases} \dfrac{|\omega - s|}{\omega + s} + \dfrac{2\tau}{\omega + \lambda_{\min}}, & \lambda_{\max} = \lambda_{\min} = s, (s > 0); \\ \dfrac{\lambda_{\max} - \omega}{\omega + \lambda_{\max}} + \dfrac{2\tau}{\omega + \lambda_{\min}}, \omega \le \sqrt{\lambda_{\min}\lambda_{\max}} \text{ with } \lambda_{\min} \neq \lambda_{\max}; \\ \dfrac{\omega - \lambda_{\min} + 2\tau}{\omega + \lambda_{\min}}, & \omega \ge \sqrt{\lambda_{\min}\lambda_{\max}} \text{ with } \lambda_{\min} \neq \lambda_{\max}, \end{cases} \end{aligned}$$

and solve the inequality

$$\bar{\delta}(\Omega) < 1,$$

we can obtain the convergence region of ω for Method 2.2.

(I) From

$$\begin{cases} \lambda_{\max} = \lambda_{\min} = s, (s > 0); \\ \dfrac{|\omega - s|}{\omega + s} + \dfrac{2\tau}{\omega + \lambda_{\min}} < 1, \end{cases}$$

solving $\dfrac{|\omega - s|}{\omega + s} + \dfrac{2\tau}{\omega + \lambda_{\min}} = \dfrac{|\omega - s|}{\omega + s} + \dfrac{2\tau}{\omega + s} < 1$, we have $s > \tau$ and the convergence region of ω:

$$\omega > \tau.$$

Moreover, the function $\bar{\delta}(\Omega) = \dfrac{|\omega - s| + 2\tau}{\omega + s}$ is decreasing on interval $\omega \in (\tau, s]$ and is increasing on interval $\omega \in [s, +\infty)$, so the minimum value point of $\bar{\delta}(\Omega)$ is $\omega = s$. Since the convergence region $\omega \in (\tau, +\infty)$ is derived from $\bar{\delta}(\Omega) < 1$, thus the minimum value point $\omega = s$ is a quasi-optimal parameter of Method 2.2.

(II) From

$$\begin{cases} \omega \le \sqrt{\lambda_{\min}\lambda_{\max}} \text{ with } \lambda_{\min} \ne \lambda_{\max}; \\ \dfrac{\lambda_{\max} - \omega}{\omega + \lambda_{\max}} + \dfrac{2\tau}{\omega + \lambda_{\min}} < 1, \end{cases}$$

we have $\tau < \lambda_{\min}$ and the convergence region of ω:

$$\omega_L < \omega \le \sqrt{\lambda_{\min}\lambda_{\max}},$$

where $\omega_L = \dfrac{\tau - \lambda_{\min} + \sqrt{(\lambda_{\min} - \tau)^2 + 4\lambda_{\max}\tau}}{2}$. Moreover, the function $\bar{\delta}(\Omega) = \dfrac{\lambda_{\max} - \omega}{\omega + \lambda_{\max}} + \dfrac{2\tau}{\omega + \lambda_{\min}}$ is monotone decreasing on interval $\omega \in (\omega_L, \sqrt{\lambda_{\min}\lambda_{\max}}]$.

(III) From

$$\begin{cases} \omega \ge \sqrt{\lambda_{\min}\lambda_{\max}} \text{ with } \lambda_{\min} \ne \lambda_{\max}; \\ \dfrac{\omega - \lambda_{\min} + 2\tau}{\omega + \lambda_{\min}} < 1, \end{cases}$$

we have $\tau < \lambda_{\min}$ and the convergence region of ω:

$$\omega \ge \sqrt{\lambda_{\min}\lambda_{\max}}.$$

Moreover, the function

$$\bar{\delta}(\Omega) = \bar{\delta}(\omega I) = \frac{\omega - \lambda_{\min} + 2\tau}{\omega + \lambda_{\min}} = 1 + \frac{2(\tau - \lambda_{\min})}{\omega + \lambda_{\min}}$$

is monotone increasing on interval $\omega \in [\sqrt{\lambda_{\min}\lambda_{\max}}, +\infty)$.

Combining with (II) and (III), we obtain the convergence region when $\tau < \lambda_{\min}$, that is

$$\omega \in (\omega_L, +\infty).$$

Moreover, since this convergence region is derived from $\bar{\delta}(\Omega) < 1$, we know that the minimum value point

$$\omega = \sqrt{\lambda_{\min}\lambda_{\max}}$$

is a quasi-optimal parameter of Method 2.2. □

Remark 3.1. (i) Since M is a symmetric positive definite M-matrix in Theorem 3.2, $(\omega I + M)^{-1}(\omega I - M) \ge O$ holds when $\omega \ge \max\{m_{ii}\}$, and $\| \, |(\omega I + M)^{-1}(\omega I - M)| \, \|_2 = ||(\omega I + M)^{-1}(\omega I - M)||_2$ holds. Besides, when $(\omega I + M)^{-1}(\omega I - M)$ is a diagonal matrix, $\| \, |(\omega I + M)^{-1}(\omega I - M)| \, \|_2 = ||(\omega I + M)^{-1}(\omega I - M)||_2$ always holds.

(ii) The particular matrix splitting and the condition $\tau < \lambda_{\min}$ are required in Theorem 3.2. In general, we can set $M = \mathrm{diag}(A)$, then check other conditions. Especially, if A is a symmetric positive definite M-matrix, there is an ordinary matrix splitting, that is $M = A$ and $N = O$, then the condition $\tau = 0 < \lambda_{\min}$ is satisfied.

(iii) Here the quasi-optimal parameter means that it minimizes the upper bound of a convergence condition function.

In the following, we discuss the H_+-matrix, which belongs to the P-matrix and takes the M-matrix as a special case.

Theorem 3.3. *Suppose $A \in R^{n\times n}$ is an H_+-matrix, and the splitting $A = M - N$ satisfies that M is an H_+-matrix and $\langle M\rangle - |N| + \langle A\rangle$ is an M-matrix. If $\Omega \geq \mathrm{diag}(A)$, then the sequence $\{z^{(k)}\}_{k=0}^{+\infty}$ generated by Method 2.2 is convergent for arbitrary initial vectors $x^{(0)}, x^{(1)} \in R^n$.*

Proof. We will prove that the convergence condition (ii) in Lemma 3.1 holds under the assumption. Since M is an H_+-matrix, then $\Omega + \langle M\rangle$ is an M-matrix and $\Omega + M$ is also an H_+-matrix. So,

$$|(\Omega + M)^{-1}| \leq (\langle \Omega + M\rangle)^{-1} = (\Omega + \langle M\rangle)^{-1}$$

holds (see Lemma 3.2 in [10]). Thus

$$|(\Omega + M)^{-1}N| + |(\Omega + M)^{-1}(\Omega - A)| \leq (\Omega + \langle M\rangle)^{-1}(|N| + |\Omega - A|). \tag{3.1}$$

Then, if $\Omega \geq \mathrm{diag}(A)$, $(\Omega + \langle M\rangle)^{-1}(|N| + |\Omega - A|)$ can be looked as the iteration matrix of

$$\langle M\rangle - |N| + \langle A\rangle$$

with the M-splitting:

$$(\Omega + \langle M\rangle) - (|N| + |\Omega - A|).$$

Hence, from Theorem 3.4 in [10], we have

$$\rho((\Omega + \langle M\rangle)^{-1}(|N| + |\Omega - A|)) < 1.$$

Thus, combining with (3.1) and the monotone theory of spectral radius of nonnegative matrices, the inequality

$$\rho(|(\Omega + M)^{-1}N| + |(\Omega + M)^{-1}(\Omega - A)|) < 1$$

holds (see Lemma 3.2 in [10]). Then the conclusion is proved. □

Corollary 3.1. *Suppose $A \in R^{n\times n}$ is an H_+-matrix, and the splitting $A = M - N$ is an H-splitting, i.e., $\langle M\rangle - |N|$ is an M-matrix. If $\Omega \geq \mathrm{diag}(A)$, then the sequence $\{z^{(k)}\}_{k=0}^{+\infty}$ generated by Method 2.2 is convergent for arbitrary initial vectors $x^{(0)}, x^{(1)} \in R^n$.*

Proof. Since $A \in R^{n\times n}$ is an H_+-matrix, $A = M - N$, $\langle M\rangle - |N|$ is an M-matrix and $\langle M\rangle \geq \langle M\rangle - |N|$, we know that M has positive diagonal and $\langle M\rangle$ is an M-matrix, then M is an H_+-matrix.

From the H-splitting condition, we know that

$$\langle A\rangle \geq \langle M\rangle - |N|$$

holds. Thus

$$\langle M\rangle - |N| + \langle A\rangle \geq 2(\langle M\rangle - |N|).$$

Then, the matrix $\langle M\rangle - |N| + \langle A\rangle$ is an M-matrix. According to Theorem 3.3, the conclusion is established. □

In the following, we consider a particular matrix splitting, i.e., the accelerated over-relaxation (AOR) splitting, which is defined as:

$$A = M_{\alpha\beta} - N_{\alpha\beta},\ M_{\alpha\beta} = \frac{1}{\alpha}(D - \beta L),\ N_{\alpha\beta} = \frac{1}{\alpha}[(1-\alpha)D + (\alpha-\beta)L + \alpha U], \quad (3.2)$$

where $D = \mathrm{diag}(A)$, $-L$ and $-U$ are the strictly lower and the upper triangular parts of A, respectively. If A is an H_+-matrix, then the splitting (3.2) is an H-compatible splitting, i.e., $\langle A\rangle = \langle M\rangle - |N|$ holds if

$$0 \leq \beta \leq \alpha \leq 1,\ \text{with } \alpha \neq 0. \quad (3.3)$$

For an H_+-matrix, since the H-compatible splitting is also an H-splitting, we know that Corollary 3.1 holds for the AOR splitting with α and β satisfying (3.3). According to (3.2), there are several particular cases of Method 2.2, that is the two-point modulus-based accelerated over-relaxation (TPMBAOR) iteration method, the two-point modulus-based successive over-relaxation (TPMBSOR) iteration method (for $\alpha = \beta$), the two-point modulus-based Gauss-Seidel (TPMBGS) iteration method (for $\alpha = \beta = 1$) and the two-point modulus-based Jacobi (TPMBJ) iteration method (for $\alpha = 1, \beta = 0$). For the quasi-optimal case of the TPMBAOR iteration method, we have the following conclusion.

Theorem 3.4. *Suppose $A \in R^{n\times n}$ is an H_+-matrix and the AOR splitting satisfies (3.2) and (3.3). If $\Omega \geq D$, then the sequence $\{z^{(k)}\}_{k=0}^{+\infty}$ generated by Method 2.2 is convergent for arbitrary initial vectors $x^{(0)}, x^{(1)} \in R^n$. Moreover,*

(i) the quasi-optimal case is the TPMBGS iteration method, that is $\alpha = \beta = 1$;

(ii) if $\Omega = \omega D$ with $\omega \geq 1$, then the quasi-optimal case is the TPMBGS iteration method with $\omega = 1$.

Proof. From the assumption, the first part of this theorem can be proved from the AOR splitting being an H-splitting and Corollary 3.1.

We prove (i) and (ii) below. Since M has positive diagonals and $\langle M\rangle$ is an M-matrix according to the assumption, we know that M is an H_+-matrix and

$$\rho(|(\Omega+M)^{-1}N| + |(\Omega+M)^{-1}(\Omega - A)|) \leq \rho((\Omega + \langle M\rangle)^{-1}(|N| + |\Omega - A|)) \quad (3.4)$$

holds from Theorem 3.3. We discuss α, β and Ω to make $\rho((\Omega + \langle M\rangle)^{-1}(|N| + |\Omega - A|)) < 1$, and prove (i) and (ii) in three steps.

I) Let α be a fixed nonzero value. If $\Omega \geq D$ and $\beta : 0 \leq \beta \leq \alpha \leq 1$, then, for the TPMBAOR iteration method, we have

$$\begin{aligned}
\widehat{M}_{\alpha\beta} =& \Omega + \langle M_{\alpha\beta}\rangle = \Omega + \langle \frac{1}{\alpha}(D - \beta L)\rangle \\
\geq& \Omega + \langle \frac{1}{\alpha}(D - \alpha L)\rangle = \Omega + \langle M_{\alpha\alpha}\rangle = \widehat{M}_{\alpha\alpha}; \\
\widehat{N}_{\alpha\beta} =& |N_{\alpha\beta}| + |\Omega - A| = \left|\frac{1}{\alpha}[(1-\alpha)D + (\alpha-\beta)L + \alpha U]\right| + |\Omega - A|
\end{aligned}$$

$$\begin{aligned}
&\geq\left|\frac{1}{\alpha}[(1-\alpha)D+(\alpha-\alpha)L+\alpha U]\right|+|\Omega-A|=|N_{\alpha\alpha}|+|\Omega-A|=\widehat{N}_{\alpha\alpha}\\
&=\left|\frac{1}{\alpha}[(1-\alpha)D+\alpha U]\right|+|\Omega-A|;\\
\widehat{A}_{\alpha\beta}=&\widehat{M}_{\alpha\beta}-\widehat{N}_{\alpha\beta}=\Omega+D-|\Omega-D|-2(|L|+|U|)=\widehat{M}_{\alpha\alpha}-\widehat{N}_{\alpha\alpha}=\widehat{A}_{\alpha\alpha}\\
=&2[D-(|L|+|U|)]=2\langle A\rangle.
\end{aligned}$$

Since $2\langle A\rangle$ is an M-matrix, $\widehat{N}_{\alpha\beta}\geq\widehat{N}_{\alpha\alpha}\geq 0$, $\widehat{M}_{\alpha\beta}$ and $\widehat{M}_{\alpha\alpha}$ are M-matrices, thus, there are two regular splittings of $2\langle A\rangle$ [10]. From Theorem 1.1 in [17], we have

$$\rho(\widehat{M}_{\alpha\beta}^{-1}\widehat{N}_{\alpha\beta})\geq\rho(\widehat{M}_{\alpha\alpha}^{-1}\widehat{N}_{\alpha\alpha}),$$

which means that the quasi-optimal parameter β in the TPMBAOR iteration method is $\beta=\alpha$. Thus the quasi-optimal TPMBAOR iteration method is the TPMBSOR iteration method for any $\Omega\geq D$.

II) Set $\alpha=\beta$. If $\Omega\geq D$, then, for the TPMBSOR iteration method, we have

$$\begin{aligned}
\widehat{M}_{\alpha\alpha}=&\langle M_{\alpha\alpha}\rangle+\Omega=\langle\frac{1}{\alpha}(D-\alpha L)\rangle+\Omega\\
&\geq\langle(D-L)\rangle+\Omega=\widehat{M}_{11};\\
\widehat{N}_{\alpha\alpha}=&|N_{\alpha\alpha}|+|\Omega-A|=\left|\frac{1}{\alpha}[(1-\alpha)D+\alpha U]\right|+|\Omega-A|\\
&\geq|U|+|\Omega-A|=|N_{11}|+|\Omega-A|=\widehat{N}_{11};\\
\widehat{A}_{\alpha\alpha}=&\widehat{M}_{\alpha\alpha}-\widehat{N}_{\alpha\alpha}=\Omega+D-|\Omega-D|-2(|L|+|U|)=\widehat{M}_{11}-\widehat{N}_{11}=\widehat{A}_{11}.
\end{aligned}$$

From Theorem 1.1 in [17], we have

$$\rho(\widehat{M}_{\alpha\alpha}^{-1}\widehat{N}_{\alpha\alpha})\geq\rho(\widehat{M}_{11}^{-1}\widehat{N}_{11}),$$

which means that the quasi-optimal case of TPMBSOR iteration method is $\alpha=\beta=1$. Thus the quasi-optimal TPMBSOR iteration method is the TPMBGS iteration method for any $\Omega\geq D$.

Then, collecting I) and II), (i) is proved.

Note that $\Omega=\omega D$ with $\omega\in[1,+\infty)$ satisfies the condition $\Omega\geq D$, thus (i) holds in this case. For (ii), we need to prove that when $\Omega=\omega D$ with $\omega\in[1,+\infty)$, the quasi-optimal parameter in the TPMBGS iteration method is $\omega=1$.

III) Set $\alpha=\beta=1$. If $\Omega=\omega D,\omega\in[1,+\infty)$, then, for the TPMBGS iteration method, we have

$$\begin{aligned}
\widehat{M}_{11\omega}=&\langle M_{11}\rangle+\Omega=\langle(D-L)\rangle+\omega D\geq\langle(D-L)\rangle+D=\widehat{M}_{111};\\
\widehat{N}_{11\omega}=&|N_{11}|+|\Omega-A|=|U|+|\omega D-A|=|\omega-1|D+|L|+2|U|\\
&\geq|L|+2|U|=|N_1|+|1D-A|=\widehat{N}_{111};\\
\widehat{A}_{11\omega}=&\widehat{M}_{11\omega}-\widehat{N}_{11\omega}=(\omega+1-|\omega-1|)D-2(|L|+|U|)\\
&=2(D-|L|-|U|)=\widehat{A}_{111}=2\langle A\rangle.
\end{aligned}$$

From Theorem 1.1 in [17], we also have

$$\rho(\widehat{M}_{11\omega}^{-1}\widehat{N}_{11\omega})\geq\rho(\widehat{M}_{111}^{-1}\widehat{N}_{111}).$$

Then the quasi-optimal parameter ω in the TPMBGS iteration method is $\omega=1$.

Thus the conclusion (ii) is proved. □

4. Numerical experiments

In this section, we show several examples to illustrate the presented results. Example 4.1 is for the convergence region, the quasi-optimal parameter matrix Ω and the comparison of the different convergence conditions, and it is connected with Lemma 3.1 and Theorem 3.2. Example 4.2 and Example 4.3 are for the TPMBAOR iteration method, and both examples are connected with Theorem 3.4. Some notations are described as follows: the number of iteration steps is denoted by IT, and the norm of the residual vector is denoted by RES. Here, RES is defined as:

$$\mathrm{RES}(z^{(k)}) = ||\min(z^{(k)}, Az^{(k)} + q)||_2,$$

where $z^{(k)}$ is the kth approximate solution. The iteration stops when $\mathrm{RES}(z^{(k)}) < 1.0e-5$ or k reaches 250. The system matrix A in the first three examples is given by $A(\mu,\theta,\varphi) = \widehat{A} + \mu I + \theta B + \varphi C$, where μ, θ and φ are three constants,

$$\widehat{A} = \mathrm{Tridiag}(-I, S, -I) \in R^{n\times n}$$

is a block-tridiagonal matrix, $I \in R^{n\times n}$ is the identity matrix, $B = \mathrm{tridiag}(0,0,1) \in R^{n\times n}$ and $S = \mathrm{tridiag}(-1,4,-1) \in R^{m\times m}$ are two tridiagonal matrices with different orders, $C = \mathrm{diag}([1,2,1,2,...]) \in R^{n\times n}$ is a diagonal matrix, m is a positive integer and $n = m^2$. For convenience, we set $m = 30$,

$$q = (1,-1,1,-1,\ldots,1,-1,\ldots)^{\mathrm{T}} \in R^n$$

and all initial iteration vectors are chosen to be

$$x^{(0)} = x^{(1)} = (1,0,1,0,\ldots,1,0,\ldots)^{\mathrm{T}} \in R^n$$

in our experiments. By changing the values of μ, θ and φ in $A(\mu,\theta,\varphi)$, we test many cases in our experiments, such as $A(2,0,1)$, $A(1,0,1)$, $A(1,0,2)$, $A(0,0,1)$, $A(1,1,0)$, $A(1,1,1)$, $A(0,1,1)$, $A(0,2,1)$, $A(2,2,-1)$, $A(0,1,2)$, $A(1,2,1)$, and so on. The first four matrices are symmetric positive definite matrices, and the rest are H_+-matrices, so they are all P-matrices. For the construction way of A and the choosing way of q, readers can refer to [3,19]. We also consider another case in our experiments, that is $A(\mu,\theta,\varphi) = \widehat{A} + \mu I + \theta B + \varphi C$ with

$$\widehat{A} = \mathrm{Tridiag}(-0.5I, S, -0.5I) \in R^{n\times n}.$$

Since the numerical results are similar, we omit this part. In Example 4.4, we consider a practical linear complementarity problem, which is derived from the Black-Scholes American option pricing problem. All the tests are performed in MATLAB R2016b on a Dell Laptop (Intel(R) Core(TM) i5-7200U CPU @ 2.50GHz 2.70GHz, 4.00GB RAM).

Example 4.1. In this example, we test Lemma 3.1 and Theorem 3.2. We consider four cases: $A(2,0,1)$, $A(1,1,0)$, $A(1,0,1)$ and $A(1,1,1)$. For the case $A(2,0,1)$, we set $M = sI = \max(\mathrm{diag}(A))I = 8I$, then $\tau = 4.5413$; for the case $A(1,1,0)$, we set $M = sI = \max(\mathrm{diag}(A))I = 5I$, then $\tau = 3.0507$, here 'max(diag(A))' denotes the maximum value of the diagonal elements of A. So, both cases satisfy (i) in Theorem 3.2, that is $M = sI(s > 0)$ and $s > \tau$. We set $\Omega = \omega I$ with

$$\omega \in (\tau, +\infty).$$

For convenience, we denote the lower bound τ and the particular parameter value s by ω_L and ω_S, respectively. For the case $A(1,0,1)$, we set $M = \text{triu}(A,-1) - \text{triu}(A,2)$, which is the tridiagonal part of A, where 'triu' is an operator in MATLAB, which can generate the upper triangle parts of A. Then $\tau = 1.9897$ and $\lambda_{\min} = 4.4484$. For the case $A(1,1,1)$, we set $M = \text{diag}(A)$, which is the diagonal part of A. Then $\tau = 3.0507$ and $\lambda_{\min} = 6$. So, both cases satisfy (ii) in Theorem 3.2, that is $M \neq sI(s > 0)$ and $\tau < \lambda_{\min}$. We set $\Omega = \omega I$ and consider

$$\omega \in (\omega_L, +\infty),$$

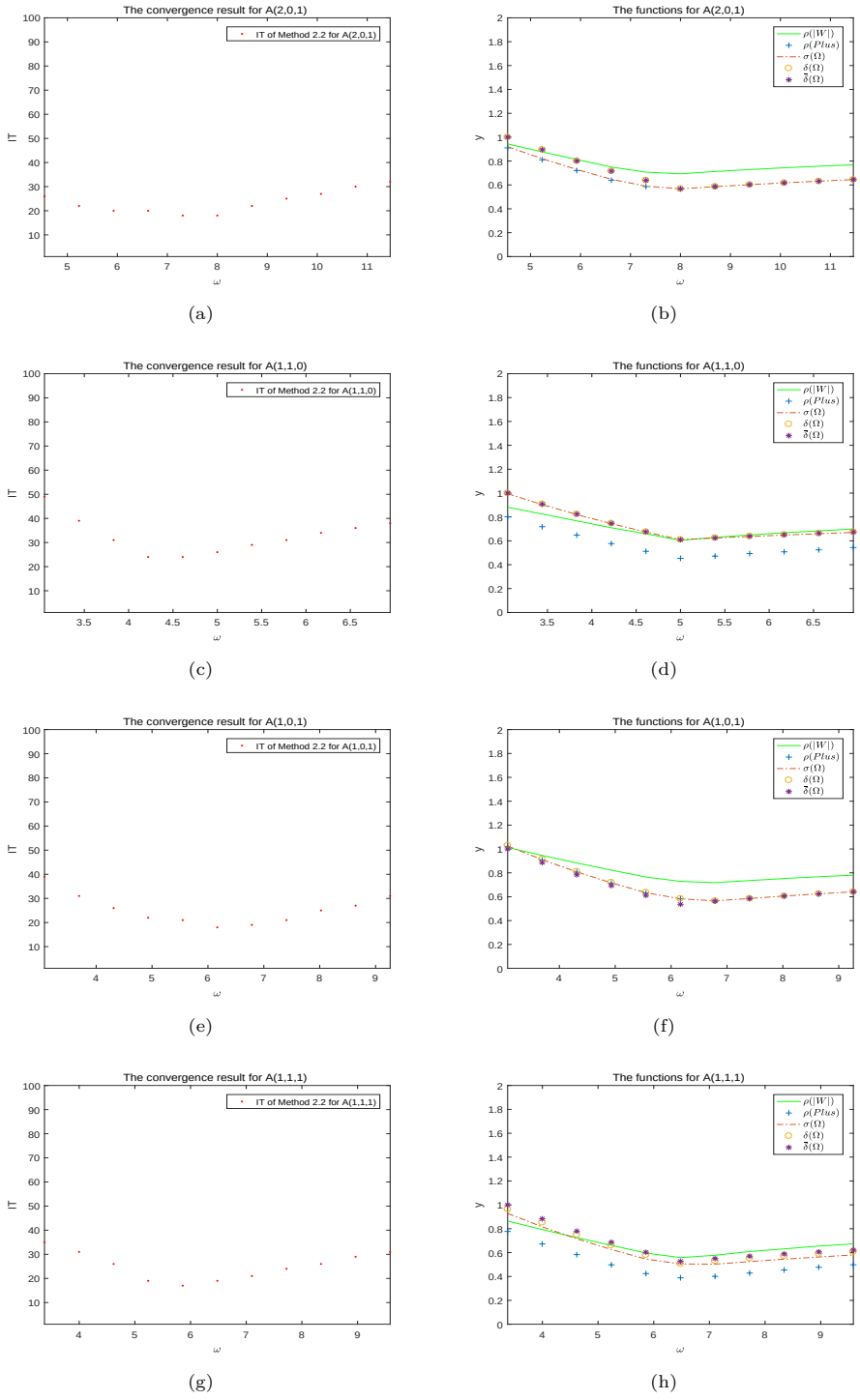
where ω_L is defined as that in Theorem 3.2. We denote $\sqrt{\lambda_{\min}\lambda_{\max}}$ by ω_S. Then, we have the same notation ω_S for the four cases. We set some values of ω in $[\omega_L, +\infty)$, that is

$$\omega = \omega_L : \frac{\omega_S - \omega_L}{5} : 2\omega_S - \omega_L,$$

where ':' is an operator in MATLAB. Then ω_L is the first value and ω_S is in the middle of the 11 values. We consider $\| \, |(\omega I + M)^{-1}(\omega I - M)| \, \|_2 = ||(\omega I + M)^{-1}(\omega I - M)||_2$, $\rho(|(\Omega+M)^{-1}N| + |(\Omega+M)^{-1}(\Omega - A)|) < 1$ and IT for Theorem 3.2. Meanwhile, we compare the four convergence conditions given in Theorem 3.1, that is, $\rho(|W|) < 1$, $\rho(|(\Omega + M)^{-1}N| + |(\Omega + M)^{-1}(\Omega - A)|) < 1$, $\sigma(\Omega) < 1$ and $\delta(\Omega) < 1$. The numerical results are shown in Table 1 and Figure 1.

Table 1. Numerical results of the TPMB iteration method

$A(2,0,1)$											
ω	ω_L	ω_2	ω_3	ω_4	ω_5	ω_S	ω_7	ω_8	ω_9	ω_{10}	ω_{11}
D_N	0	0	0	0	0	0	0	0	0	0	0
$\rho(Plus)$	0.91	0.81	0.72	0.63	0.58	0.56	0.58	0.60	0.61	0.63	0.64
IT	26	22	20	20	18	18	22	25	27	30	32
$A(1,1,0)$											
ω	ω_L	ω_2	ω_3	ω_4	ω_5	ω_S	ω_7	ω_8	ω_9	ω_{10}	ω_{11}
D_N	0	0	0	0	0	0	0	0	0	0	0
$\rho(Plus)$	0.80	0.71	0.64	0.57	0.51	0.45	0.47	0.49	0.50	0.52	0.54
IT	49	39	31	24	24	26	29	31	34	36	38
$A(1,0,1)$											
ω	ω_L	ω_2	ω_3	ω_4	ω_5	ω_S	ω_7	ω_8	ω_9	ω_{10}	ω_{11}
D_N	0.02	0.02	0.02	0.02	0.02	0.04	0.03	0	0	0	0
$\rho(Plus)$	1.02	0.90	0.80	0.71	0.63	0.58	0.56	0.58	0.60	0.62	0.64
IT	39	31	26	22	21	18	19	21	25	27	31
$A(1,1,1)$											
ω	ω_L	ω_2	ω_3	ω_4	ω_5	ω_S	ω_7	ω_8	ω_9	ω_{10}	ω_{11}
D_N	0	0	0	0	0	0	0	0	0	0	0
$\rho(Plus)$	0.68	0.67	0.58	0.49	0.42	0.38	0.40	0.42	0.45	0.47	0.49
IT	35	31	26	19	17	19	21	24	26	29	31



(a) (b) (c) (d) (e) (f) (g) (h)

Figure 1. The numerical results about IT and the convergence conditions

In Table 1, D_N and $\rho(Plus)$ represent $\| \, |(\omega I + M)^{-1}(\omega I - M)| \, \|_2 - ||(\omega I + M)^{-1}(\omega I - M)||_2$ and $\rho(|(\Omega + M)^{-1}N| + |(\Omega + M)^{-1}(\Omega - A)|)$, respectively. From Table 1, we can see that for the cases $A(2,0,1)$, $A(1,1,0)$ and $A(1,1,1)$, the convergence domain of ω can extend to ω_2-ω_{11} since the condition $D_N = 0$ always holds from Theorem 3.2. The convergence domain of ω can extend to ω_L since $\rho(Plus) < 1$ on ω_L from (ii) in Lemma 3.1. For the case $A(1,0,1)$, the convergence domain of ω can not extend to ω_2-ω_7 since $D_N = 0$ does not hold from Theorem 3.2. However, the convergence domain of ω can extend to ω_2-ω_7 since $\rho(Plus) < 1$ from (ii) in Lemma 3.1. This example verifies the convergence conditions in Lemma 3.1 and Theorem 3.2, and it also shows that the convergence domain obtained by Lemma 3.1 is larger than that of Theorem 3.2.

The left side of Figure 1 corresponds to the data of ω and IT in Table 1. For the cases $A(2,0,1)$ and $A(1,1,0)$, the numerical results show that $\omega = \omega_S = s$ is a good parameter when $M = sI$ with $s > 0$. For the cases $A(1,0,1)$ and $A(1,1,1)$, the former does not accord with Theorem 3.2 since that $D_N = 0$ does not hold, and the latter accords with Theorem 3.2. However, both numerical results show that $\omega = \omega_S = \sqrt{\lambda_{\min}\lambda_{\max}}$ is a good parameter when $M \neq sI$. Therefore, this example verifies the conclusion of Theorem 3.2 for the quasi-optimal parameters. The right side of Figure 1 is for the convergence functions. The y-axis represents the function value, the lines '$+ + +$' and '$* * *$' represent $\rho(Plus)$ and $\bar{\delta}(\Omega)$, respectively. The numerical results show that $\rho(Plus)$ is the lowest in the five condition functions: $\rho(|W|)$, $\rho(Plus)$, $\sigma(\Omega)$, $\delta(\Omega)$ and $\bar{\delta}(\Omega)$, and they have the similar monotone property.

Example 4.2. In this example, we investigate the function $\rho((\Omega + \langle M\rangle)^{-1}(|N| + |\Omega - A|))$ utilized in Theorem 3.4 (see expression (3.4)) and the influence of α, β in the TPMBAOR iteration method. We consider $\alpha, \beta \in [0, 1 + \frac{2}{l}]$ with $l > 0$, which is larger than the interval $[0,1]$ discussed in Theorem 3.4. Specifically, we set

$$\alpha = \frac{1}{l} : \frac{1}{l} : (1 + \frac{2}{l}) \quad \text{and} \quad \beta = \frac{1}{l} : \frac{1}{l} : (1 + \frac{2}{l})$$

with $l = 9$, and $\alpha_9 = \beta_9 = 1$. Then, both $\beta \leq \alpha$ and $\alpha \leq \beta$ are included in our experiments. We consider four cases: $A(0,0,1), A(0,1,1), A(0,2,1)$ and $A(2,2,-1)$, all of which are H_+-matrices. Setting $\Omega = D$, which is the diagonal part of A, then we obtain Figure 2 and Table 2.

Since the numerical results are similar, we only show the data of case $A(0,0,1)$ in Table 2, which correspond to the first two of Figure 2. The notation ρ in Table 2 represents $\rho((\Omega + \langle M\rangle)^{-1}(|N| + |\Omega - A|))$ utilized in Theorem 3.4. From Figure 2 and Table 2, we have the observations as follows. (i) When the AOR splitting is the H-compatible splitting, that is, $0 \leq \beta \leq \alpha \leq 1$ with $\alpha \neq 0$, the TPMBGS iteration method ($\alpha = \beta = 1$) is good. (ii) When the spitting is not an H-compatible splitting, that is, α and β do not satisfy the inequality $0 \leq \beta \leq \alpha \leq 1$, the TPMBAOR iteration method is still convergent since $\rho((\Omega + \langle M\rangle)^{-1}(|N| + |\Omega - A|)) < 1$. This indicates that the convergence region of α and β discussed in Theorem 3.4 is sufficient. Besides, we can find that the TPMBGS iteration method has good performance in a large convergence range. (iii) The spectral radius $\rho((\Omega + \langle M\rangle)^{-1}(|N| + |\Omega - A|))$ is not exactly consistent with the number of iteration steps.

Example 4.3. In this example, we consider the parameter ω with $\Omega = \omega D$, $\omega \in [1, +\infty)$. The parameters α and β in the TPMBAOR iteration method are set to

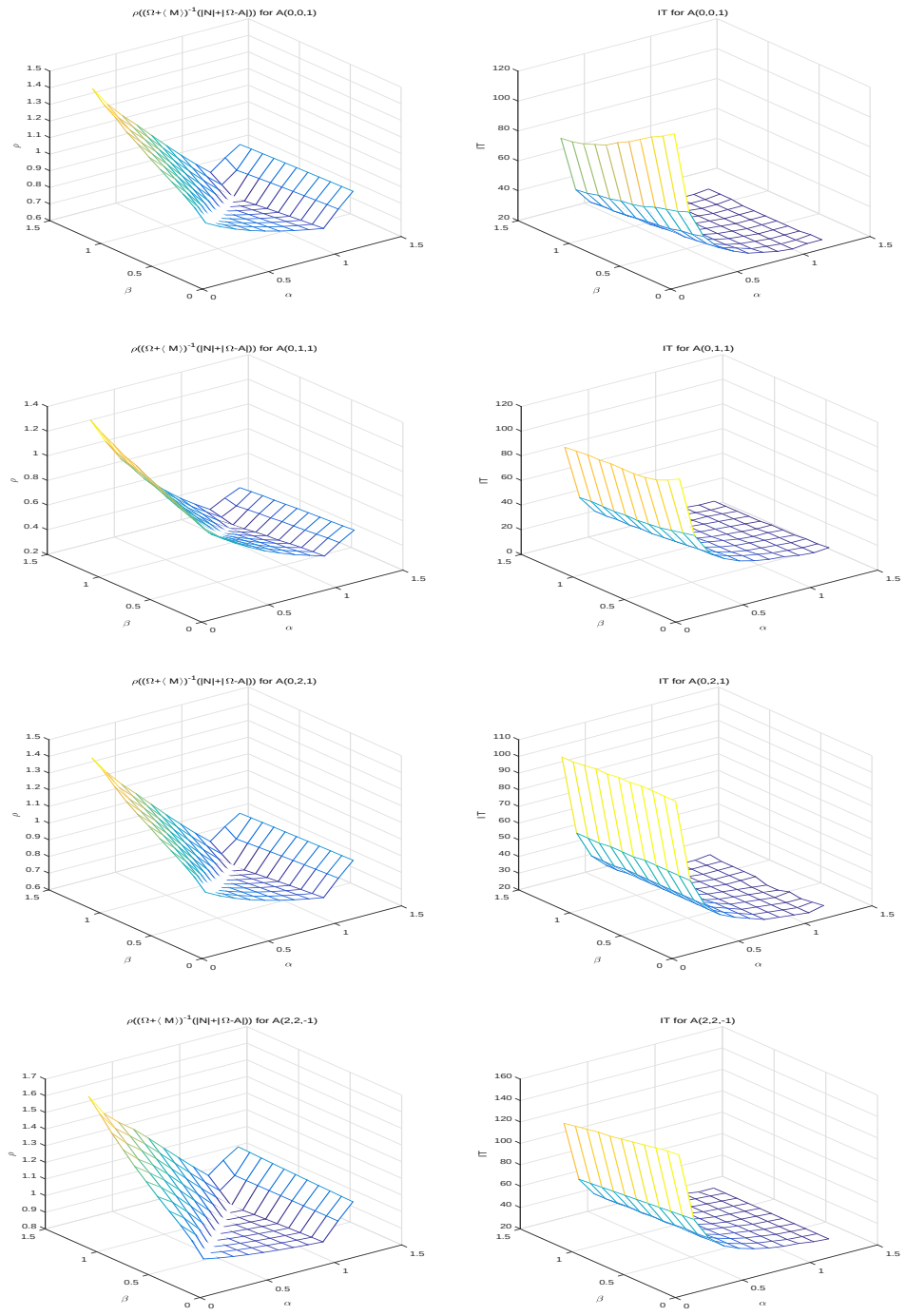


Figure 2. The comparison between $\rho((\Omega+\langle M\rangle)^{-1}(|N|+|\Omega-A|))$ and IT for the TPMBAOR iteration method

be same as Example 4.2. For each of $A(0,0,1)$, $A(0,1,1)$, $A(0,2,1)$ and $A(2,2,-1)$,

Table 2. Numerical results of the TPMBAOR iteration method

$A(0,0,1)$											
ρ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}
β_1	0.94	0.89	0.86	0.82	0.80	0.77	0.75	0.73	0.72	0.81	0.90
β_2	1.00	0.89	0.85	0.82	0.79	0.77	0.75	0.73	0.71	0.81	0.89
β_3	1.05	0.95	0.85	0.81	0.79	0.76	0.74	0.72	0.71	0.80	0.89
β_4	1.09	1.01	0.91	0.81	0.78	0.75	0.73	0.72	0.70	0.80	0.89
β_5	1.14	1.06	0.96	0.87	0.77	0.75	0.73	0.71	0.69	0.80	0.89
β_6	1.18	1.10	1.01	0.92	0.83	0.74	0.72	0.70	0.69	0.79	0.89
β_7	1.23	1.15	1.06	0.97	0.88	0.79	0.71	0.69	0.68	0.79	0.88
β_8	1.27	1.19	1.11	1.02	0.93	0.84	0.76	0.69	0.67	0.78	0.88
β_9	1.33	1.23	1.15	1.06	0.98	0.89	0.81	0.73	0.66	0.78	0.88
β_{10}	1.38	1.27	1.19	1.11	1.02	0.94	0.86	0.78	0.71	0.77	0.88
β_{11}	1.44	1.32	1.23	1.15	1.07	0.98	0.90	0.82	0.75	0.82	0.87
IT	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}
β_1	118	63	44	34	29	25	23	21	20	21	21
β_2	113	60	42	33	28	25	22	21	20	20	21
β_3	109	58	41	32	27	24	22	21	20	20	20
β_4	104	56	39	31	27	24	22	21	20	20	20
β_5	99	53	38	31	26	24	22	21	20	20	20
β_6	95	51	37	30	26	23	22	21	20	20	20
β_7	90	50	36	29	26	23	22	21	20	20	20
β_8	87	48	35	29	25	23	22	21	20	20	20
β_9	84	47	34	28	25	23	22	21	20	20	21
β_{10}	82	45	34	28	25	23	22	21	20	20	21
β_{11}	81	44	33	27	24	23	22	21	20	20	21

we consider three cases: $\omega = 1, 2, 3$. The numerical results are shown in Figure 3.

In each sub-figure of Figure 3, the value of ω is $1, 2, 3$ in turn. From Figure 3, we can see that when $\omega = 1$, i.e., $\Omega = D$, the number of iteration steps is minimal.

Example 4.4. In this example, we consider the linear complementarity problems derived from the Black-Scholes American option pricing problem. Based on the Black-Scholes model, we can obtain the LCP(A, q) with $A = \text{tridiag}(-\tau, 1 + 2\tau, -\tau) \in R^{n\times n}$ being an M-matrix when $\tau > 0$. For the detailed materials, readers can refer to [20, 21]. We set $q = Ae(e = (1, 1, 1, ..., 1)^{\mathrm{T}} \in R^n)$ with $n = 1000$ and consider the TPMBSOR iteration method (that is $\alpha = \beta$ in the TPMBAOR iteration method) with $\alpha = \beta = \frac{1}{l} : \frac{1}{l} : 1 + \frac{2}{l}$ and $l = 10$, then we obtain Table 3 as follows.

Example 4.4 shows that the TPMBSOR iteration method can deal with the linear complementarity problem effectively and the TPMBGS iteration method, i.e., $\alpha = \beta = 1$, is a quasi-optimal case.

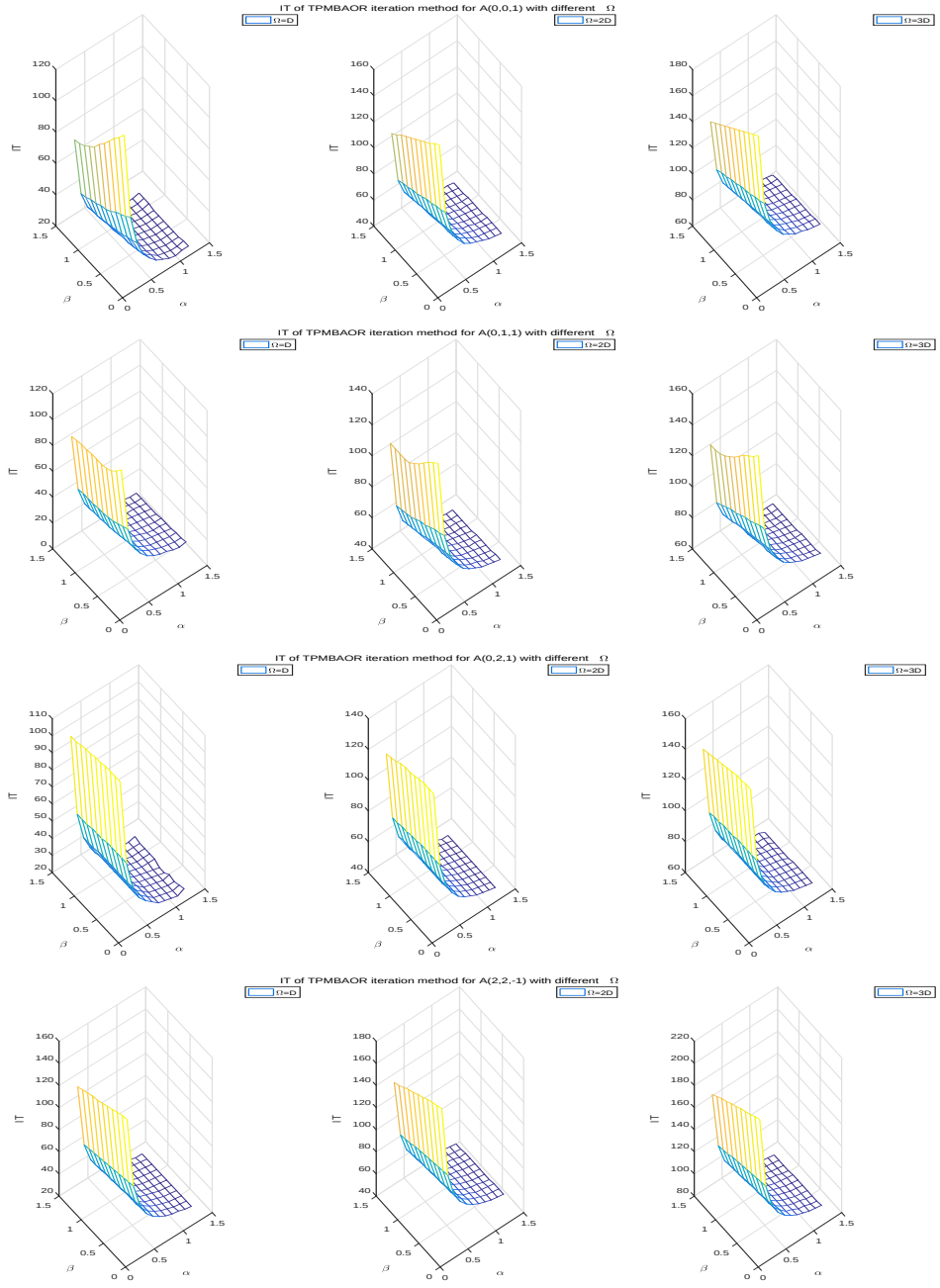


Figure 3. The comparison of IT for the TPMBAOR iteration method with different Ωs

5. Concluding remarks

In this paper, we studied the convergence problems of the TPMB matrix splitting iteration method for solving the LCP(A, q) with a P-matrix. Some convergence theories are provided, including the general convergence conditions and the quasi-

Table 3. Numerical results of the TPMBSOR iteration method

$A = \text{tridiag}(-\tau, 1+2\tau, -\tau)$												
IT	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	$\alpha_{10}=1$	α_{11}	α_{12}
$\tau = 0.5$	9	5	3	3	3	3	2	2	2	2	2	2
$\tau = 1$	12	7	5	4	4	3	3	3	3	3	3	3
$\tau = 1.5$	16	9	7	6	5	5	4	4	4	4	4	4
$\tau = 2$	20	11	8	7	6	5	5	5	5	5	4	4
$\tau = 2.5$	24	13	10	8	7	7	6	6	6	6	6	6
$\tau = 3$	28	16	12	10	8	8	7	7	6	6	6	6

optimal cases from two respects: the spectral radius and the matrix norm. The numerical examples are illustrated to show the presented results.

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