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PERIODIC SOLUTIONS OF SUPERLINEAR PLANAR HAMILTONIAN SYSTEMS WITH INDEFINITE TERMS*

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Abstract Existence of infinitely many periodic solutions for a planar Hamiltonian system $Jz' = \nabla_z H(t, z)$ is proved. We investigate the case in which $\nabla_z H(t, z)$ satisfies a general superlinear condition at infinity via rotation numbers and $x\frac{\partial H}{\partial x}(t, x, y)$ is an indefinite term. Our approach is based on the Poincaré-Birkhoff theorem and the spiral property of large amplitude solutions. Our results generalize the classical result in Jacobowitz [13] and Hartman [12] for second order scalar equations.

Keywords Periodic solution, indefinite term, rotation number, superlinear Hamiltonian system, Poincaré-Birkhoff theorem.

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1. Introduction

We investigate the existence of infinitely many periodic solutions for a planar Hamiltonian system

$$Jz' = \nabla_z H(t, z), \quad \text{for } z = (x, y) \in \mathbb{R}^2. \tag{1.1}$$

We assume $H(t, z) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ to be C^1 in the second variable and T-periodic in the first variable. Here, the symplectic matrix J is defined as $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We investigate the case in which $\nabla_z H(t, z)$ satisfies a general superlinear condition at infinity via rotation numbers and $x\frac{\partial H}{\partial x}(t, x, y)$ is a sign-changing function (named"indefinite term").

In 1976, Jacobowitz [13] investigated the classical superlinear condition

$$(f_1) \qquad \lim_{|x|\to+\infty} \frac{f(t, x)}{x} = +\infty, \quad \text{uniformly in } t \in [0, T],$$

for second order scalar equations

$$x'' + f(t, x) = 0. \tag{1.2}$$

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By using the Poincaré-Birkhoff theorem, he [13] proved the existence of infinitely many periodic solutions for Eq. (1.2). However, an additional sign condition

$$(f_2) \qquad xf(t,x) > 0 \quad \text{for } x \neq 0$$

is needed in [13]. One year later, Hartman [12] refined the results in [13] and removed the sign condition (f_2). However, it is worth noting that the sign condition

$$(f_3) \qquad xf(t,x) > 0 \text{ near } x = \infty$$

is implicit in the condition (f_1). We also refer Fonda and Sfecci [6] for the existence of infinitely many periodic solutions of superlinear system with the sign condition similar to (f_3). In the process of applying the Poincaré-Birkhoff theorem, sign conditions make the problem simple (see for instance [21, pages 2-3]).

Recently, Qian, Torres and Wang [18] introduced a partial superlinear condition

(f_∞) $\quad f(t,x)/x \geq l(t)$ for $|x| \gg 1$ and $t \in [0, 2\pi]$, moreover,

$$\lim_{|x|\to+\infty} \frac{f(t,x)}{x} = +\infty, \quad \text{uniformly for } t \in I \subset [0, 2\pi],$$

where $l(t) \in L^1([0, 2\pi])$, and I is a set of positive measure.
It is clear that $f(t,x)$ may be a sign-changing function in (f_∞). For references related to indefinite weight, please consult the following papers [4,5,22]. Under the condition (f_∞), they investigated the existence of infinitely many periodic solutions for Eq. (1.2). However, the uniqueness of solutions to the associated Cauchy problem is a necessary condition in [18](see [18, Lemma 3.1]). Notice that the uniqueness of solutions is also a crucial condition in [3,12,13].

The existence of periodic solutions for superlinear planar systems (1.1) has also been studied. Various superlinear conditions for a Hamiltonian system like (1.1) have been proposed (see [1,16]). Most of these conditions require two components of the vector field $J\nabla_z H(t,z)$ to be superlinear. Fonda and Sfecci [9] introduced a superlinear condition $\mathcal{A}_2$, which also requires two components of the vector field $J\nabla_z H(t,z)$ to be superlinear, with $\mathcal{R} = \mathbb{R}^2$. However, their results do not cover the superlinear equation (1.2) in Hamiltonian form

$$x' = -y, \quad y' = f(t,x). \tag{1.3}$$

Recently, Boscaggin [3] extended the result of the superlinear Eq. (1.2) to the planar Hamiltonian system (1.1) by introducing the following superlinear condition.

(H_∞) There exist sequences $(V_n)_n \in \mathcal{P}$ and $(a_n)_n \in L^1(0,T)$ such that, for every $n \in \mathbb{N}$

$$\liminf_{|z|\to+\infty} \frac{\langle \nabla_z H(t,z), z\rangle}{V_n(z)} \geq a_n(t), \quad \text{uniformly for a.e. } t \in [0,T] \tag{1.4}$$

and

$$\lim_{n\to+\infty} \frac{\int_0^T a_n(t)dt}{\tau_{V_n}} = +\infty, \tag{1.5}$$

where $\tau_{V_n} = \int_{\{V_n(x,y)\leq 1\}} dxdy$, and $\mathcal{P}$ is a class of the C^2-functions $V(z) : \mathbb{R}^2 \to \mathbb{R}$ such that

$$0 < V(\lambda z) = \lambda^2 V(z), \quad \forall\ \lambda > 0,\ z \neq 0.$$

Applying the Poincaré-Birkhoff theorem, Boscaggin [3] obtained the existence of infinitely many periodic solutions to the planar Hamiltonian system (1.1).

Comparing the classical results of Jacobowitz [13] and Hartman [12], Boscaggin [3] introduced an additional condition (H_1) to ensure the global continuability of solutions, as described in [3, pages 134-135]. As a result, Boscaggin [3] only achieved a partial generalization of the classical results of Jacobowitz [13] and Hartman [12] for superlinear planar Hamiltonian systems. As is well known, the condition of global continuability of solutions for superlinear equation is important. For instance, it was shown in [6] that there are positive continuous functions $q(t)$ such that the differential equation $x'' + q(t)x^3 = 0$ has a solution which does not exist on $[0, T]$. Consequently, the Poincaré map may not be well defined. To overcome this difficulty, it is necessary to utilize some a priori estimates for the solutions that have a prescribed number of rotations in the phase plane, as shown in Hartman [12] and Fonda and Sfecci [8].

In this paper, we prove the existence of infinitely many periodic solutions for a planar Hamiltonian system (1.1). We investigate the case in which $\nabla_z H(t, z)$ satisfies a general superlinear condition at infinity via rotation numbers and $x\frac{\partial H}{\partial x}(t, x, y)$ is an indefinite term. Moreover, our results is proved without both global continuability and uniqueness of the associated Cauchy problems. Our results generalize the classical result in Jacobowitz [13] and Hartman [12] for second order scalar equations, as well as the results in [3, 9, 18].

A few words about the notations. Let $L(t, z) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be T-periodic in the first variable, differentiable with respect to the second variable, and

$$L(t, \lambda z) = \lambda^2 L(t, z), \quad \text{for every } \lambda > 0.$$

We denote by $\mathcal{Q}$ a set of functions $L(t, z)$. Let $\rho(L_n)$ denote the rotation number of the equation $Jz' = \nabla_z L_n(t, z)$, following [19, page 561].

Theorem 1.1. *Suppose that system* (1.1) *satisfies the following assumptions.*

(h_1) $\limsup_{|z|\to 0} \frac{|\nabla_z H(t,z)|}{|z|} < +\infty$, *uniformly for* $t \in [0, T]$.

(h_2) *For sufficiently large* r_*, *system* (1.1) *has upper and lower spiral functions* $\xi^{\pm}(\cdot)$.

(h_3) *For* $y \neq 0$ *and* $t \in [0, T]$, $sgn(y)\frac{\partial H}{\partial y}(t, 0, y) > 0$.

(h_∞) *There exist* $(L_n)_n \in \mathcal{Q}$ *and* $(V_n)_n \in \mathcal{P}$ *satisfying, for each* $\delta > 0$, *there is* $l_n(t) \in L^1([0, T])$, *with*

$$\langle \nabla_z H(t, z), z\rangle \geq \langle \nabla_z L_n(t, z), z\rangle - \delta V_n(z) - l_n(t), \tag{1.6}$$

for all $z \in \mathbb{R}^2$ *and a.e.* $t \in [0, T]$, *and*

$$\rho(L_n) \to +\infty \quad \text{as } n \to +\infty. \tag{1.7}$$

Then system (1.1) *has infinitely many* mT*-periodic solutions for every integer* $m \geq 1$.

Remark 1.1. The condition (f_4) for Eq. (1.2), as given in [12, 13],

$$(f_4) \qquad \frac{f(t, x)}{x} \ \text{ is bounded near } x = 0, \quad \text{uniformly in } t \in [0, T],$$

implies condition (h_1) with $H(t, z) = \frac{1}{2}y^2 + \int_0^x f(t, s)ds$.

Remark 1.2. The spiral functions are introduced by Wang and Qian (see [19, Definition 1.1]). It is noted in [19, Remark 1.1] that if any solution $z(t)$ is globally defined on $[0, T]$, then assumption (h_2) holds. However, the reverse is not true. For example, consider the equation

$$x'' + l(t)x^3 = 0, \tag{1.8}$$

where $l(t) \geq 0$ for $t \in [0, T]$, and satisfies $\int_0^T l(t)dt > 0$. Clearly, Eq. (1.8) satisfies assumption (h_2) (see [18, Example 4.1]). However, the solutions $x(t)$ of Eq. (1.8) may not be globally defined, and it may not satisfy assumption (H_1) in Boscaggin [3].

Remark 1.3. Superlinear condition (h_∞) is a general definition for planar Hamiltonian systems via rotation numbers (see Section 4 for details).

Now, we give a example, which can be proved by Theorem 1.1 (see Section 4 for more details).

Example 1.1. Suppose a planar system

$$\begin{cases} x' = y + \dfrac{\partial \mathcal{U}}{\partial y}(t, x, y), \\ y' = -\dfrac{\partial \mathcal{U}}{\partial x}(t, x, y) - f(t, x), \end{cases} \tag{1.9}$$

such that $\mathcal{U}(t, x, y)$ is a C^1-function and 2π-periodic in t, $|\frac{\partial \mathcal{U}}{\partial x}(t, x, y)| \leq a|x|$ and $|\frac{\partial \mathcal{U}}{\partial y}(t, x, y)| \leq b|y|$ with $a, b \in (0, 1)$, $f(t, x) = |\sin t|x^3 - x\cos t$. Then, for every integer $m \geq 1$, system (1.9) has infinitely many $2m\pi$-periodic solutions.

Remark 1.4. The Hamiltonian function of system (1.9) is

$$H(t, x, y) = \frac{y^2}{2} + \int_0^x f(t, s)ds + \mathcal{U}(t, x, y).$$

It is clear that no functions $\kappa_{2,1}(y)$ and $\kappa_{1,1}(x)$ satisfy assumption $\mathcal{A}_2$ in [9], with $a_{i,j} = \pm\infty$, $i, j \in \{1, 2\}$. Moreover, the functions $\frac{\partial \mathcal{U}}{\partial x}(t, x, y)$ and $\frac{\partial \mathcal{U}}{\partial y}(t, x, y)$ may be unbounded for sufficiently large x and y. Therefore, system (1.9) does not satisfy assumption $\mathcal{A}_3$ in [9], with $\mathcal{R} = \mathbb{R}^2$.

Remark 1.5. For any $t \in \{2k\pi, k \in \mathbb{Z}\}$, one has

$$\frac{\partial \mathcal{U}}{\partial x}(t, x, y) + f(t, x) \leq (a-1)x < 0, \quad \text{for } x \gg 1 \text{ and } y \in \mathbb{R}.$$

On the other hand, for any $t \in \{2k\pi + \pi/2, k \in \mathbb{Z}\}$, we have

$$\frac{\partial \mathcal{U}}{\partial x}(t, x, y) + f(t, x) \geq x^3 - ax > 0, \quad \text{for } x \gg 1 \text{ and } y \in \mathbb{R}.$$

Therefore,

$$x\left(\frac{\partial \mathcal{U}}{\partial x}(t, x, y) + f(t, x)\right)$$

is an indefinite term for system (1.9).

The remaining sections of this paper are organized as follows. In Section 2, we introduce a modified Hamiltonian system represented by (2.2) and provide a proof of Theorem 1.1. In Section 3, we conduct a comparative analysis between the superlinear conditions discussed in this paper and those already existing in the literature. Furthermore, we include a proof for Example 1.1 as an application of Theorem 1.1.

2. Existence of infinitely many periodic solutions

We recall that, if $z(s) \neq 0$ for every $s \in [0,t]$, the t-rotation number of $z(t)$ is defined as

$$\mathrm{Rot}(z(t);[0,t]) = \frac{1}{2\pi}\int_0^t \frac{\langle Jz'(t), z(t)\rangle}{|z(t)|^2}dt.$$

Precisely, writing $z(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t))$, we have

$$\mathrm{Rot}(z(t);[0,t]) = \frac{\theta_0 - \theta(t)}{2\pi}, \quad \text{for all } t \in [0,T],$$

where $(r_0, \theta_0) = (r(0), \theta(0))$, as given in ([19, pages 560-561]). We also refer [7,11,14,15,17,20,23,24] for the nice applications via rotation numbers. Following [19], we will denote by $\mathrm{Rot}^H(t;z)$ the t-rotation number of $z(t)$, which is a solution of system (1.1). For any $t \in [0,T]$, we will write the t-rotation number $\mathrm{Rot}^L(t;z)$ of $Jz' = \nabla_z L(t,z)$ as $\mathrm{Rot}^L(t;v)$ with $v = z(0)/|z(0)|$.

Let $H_0(z)$ be a C^1-function satisfying

$$H_0(z) \geq \max_{0\leq t\leq T} H(t,0,y) + x^2, \tag{2.1}$$

and

$$H_0(z) \to +\infty \iff |z| \to +\infty.$$

Let us define a truncated function

$$\lambda(s) = \begin{cases} 1, & |s| \leq R_*; \\ \text{smooth connection}, & R_* < |s| < R_{**}; \\ 0, & |s| \geq R_{**}, \end{cases}$$

and $\lambda'(s) \leq 0$ for $s \geq 0$, where R_*, R_{**} are positive parameters.

We now state the Hamiltonian system

$$Jz' = \nabla_z H_\lambda(t,z), \quad z = (x,y) \in \mathbb{R}^2, \tag{2.2}$$

where $H_\lambda(t,z) = \lambda(x^2)\lambda(y^2)[H(t,z) - H_0(z)] + H_0(z)$. For the associated Cauchy problem of (2.2), there is global continuability on $[0,T]$. In fact, if $|z|^2 \geq 2R_{**}$, system (2.2) is equivalent to $Jz' = \nabla_z H_0(z)$. So that the global existence of $z(t)$ is guaranteed.

By (h_1), there are two numbers $M > 0$ and $r_M > 0$ satisfying

$$\frac{|\nabla_z H(t,z)|}{|z|} \leq M, \quad \text{for every } t \in [0,T] \text{ and } 0 < |z| < r_M.$$

Lemma 2.1. *Let $R_* > r_M^2$. There exist a integer $J > 0$ and a constant $\delta > 0$ such that, for a solution $z(t)$ of (2.2) satisfies $0 < |z_0| < \delta$, it holds that*

$$0 < |z(t)| < r_M, \quad \text{for } t \in [0,T]$$

and

$$\mathrm{Rot}^{H_\lambda}(z(t);[0,T]) < J.$$

Proof. Let $z(t)$ be a solution of (2.2) satisfying $0 < |z(t)| < r_M$ in a certain interval I. So that $H_\lambda(t,z) = H(t,z)$ for $t \in I$. For $t \in I$, one has

$$\begin{aligned}|r'(t)| &= \frac{|x'(t)x(t) + y'(t)y(t)|}{r(t)} \\ &\leq \frac{x^2(t) + y^2(t) + |\frac{\partial H}{\partial y}(t,x,y)|^2 + |\frac{\partial H}{\partial x}(t,x,y)|^2}{2r(t)} \\ &\leq (1+M^2)r(t).\end{aligned}$$

Choosing $\delta \in (0, r_M e^{-(1+M^2)T})$, and assume that $0 < |z_0| < \delta$. Let I be the maximal interval of time containing $t = 0$ in which $0 < |z(t)| < r_M$. Then, we have

$$|z_0| \exp\left(-(1+M^2)|t|\right) \leq |z(t)| \leq |z_0| \exp\left((1+M^2)|t|\right), \quad \text{for } t \in I.$$

Hence, $[0,T] \subseteq I$ and

$$0 < |z(t)| < r_M, \quad \text{for } t \in [0,T].$$

So that

$$-\theta'(t) = \frac{\langle \nabla_z H_\lambda(t,z), z\rangle}{|z|^2} \leq \frac{|\nabla_z H_\lambda(t,z)||z|}{|z|^2} \leq M.$$

Hence, choosing the integer J with $J > MT/2\pi$, it holds that

$$\mathrm{Rot}^{H_\lambda}(z(t);[0,T]) < J.$$

□

Proof of Theorem 1.1. The proof will be divided into four steps.

Step 1. By Lemma 2.1, for every integer $m \geq 1$, we can find $R_1 > 0$ sufficiently small such that if $|z_0| = R_1$ then

$$0 < |z(t)| < r_M, \quad \text{for } t \in [0, mT],$$

and

$$\mathrm{Rot}^{H_\lambda}(mT; z) < j_m^*, \quad \text{if } |z_0| = R_1, \tag{2.3}$$

where $j_m^* = mJ$.

Step 2. Let j be an integer such that $j \geq j_m^*$. We will prove that there exists R_2 satisfying

$$\mathrm{Rot}^{H_\lambda}(mT; z) > j, \quad \text{if } |z_0| = R_2. \tag{2.4}$$

Indeed, from (1.7), there exists $n \in \mathbb{N}$ satisfying

$$\rho(L_n) > j/m.$$

Then, by (1.6) and Lemma 2.4 in [19], there exists $R > r_M$ satisfying, for each solution $z(t)$ of system (1.1) such that $|z(t)| \geq R$ for all $t \in [0, mT]$, it holds that

$$\mathrm{Rot}^H(mT; z) > j. \tag{2.5}$$

Take

$$R_2 = (\xi_{j+1}^-)^{-1}(R), \quad R_2' = \xi_{j+1}^+(R_2) \quad \text{and} \quad \sqrt{R_*} \geq R_2',$$

where $\xi_{j+1}^{\pm}(\cdot)$ are functions introduced in [19, Lemma 3.3].

We now estimate a solution of (2.2) with $|z_0| = R_2$ as follows.

(i) Note that $H_\lambda(t, z) = H(t, z)$ when $R \leq |z(t)| \leq R_2'$ for all $t \in [0, mT]$. From (2.5), we find that (2.4) holds.

(ii) If there is $t_1 \in (0, mT)$ such that $|z(t_1)| > R_2'$, then there is $t_1' \in (0, t_1)$ satisfying $|z(t_1')| = R_2'$ and

$$\xi_{j+1}^-(R_2) \leq |z(t)| \leq \xi_{j+1}^+(R_2)$$

for $t \in [0, t_1']$. In this case, $H_\lambda(t, z) = H(t, z)$ for $t \in [0, t_1']$. So that the solution $z(t)$ of system (2.2) is also a solution of system (1.1) when $t \in [0, t_1']$. By Lemma 3.3 in [19], we know that

$$\theta(t_1') - \theta_0 = -2(j+1)\pi. \tag{2.6}$$

Considering the solutions of system (2.2), we have

$$\begin{aligned} x'y &= \frac{\partial H_\lambda}{\partial y}(t, 0, y)y \\ &= y[2y\lambda'(y^2)(H(t, 0, y) - H_0(0, y)) \\ &\quad + (1 - \lambda(y^2))\frac{\partial H_0}{\partial y}(0, y) + \lambda(y^2)\frac{\partial H}{\partial y}(t, 0, y)] \end{aligned}$$

for $x = 0$ and $t \in [0, mT]$. By (2.1) and (h_3), one has, for $x = 0$, $y \neq 0$ and $t \in [0, mT]$,

$$x'y > 0,$$

which implies that

$$\theta(mT) - \theta(t_1') < \pi. \tag{2.7}$$

Combining (2.6) with (2.7), we have

$$\theta(mT) - \theta_0 < -2j\pi,$$

which implies that (2.4) holds.

(iii) When there is $t_2 \in (0, mT)$ satisfying $|z(t_2)| < R$, we can proceed analogously to the proof of (2.4) as (ii).

Step 3. We construct an annular $\Omega = \overline{B}_{R_2} \backslash B_{R_1}$. By (2.3), (2.4) and the Poincaré-Birkhoff theorem (the version in [8, 10, 19]), system (2.2) has at least two distinct mT-periodic solutions $z_m^i(t)$ satisfying $z_m^i(0) \in \Omega$ and

$$\mathrm{Rot}^{H_\lambda}(mT; z_m^i(t)) = j, \quad i = 1, 2. \tag{2.8}$$

Step 4. What is left is to prove that $z_m^i(t)$ satisfy $|z_m^i(t)| \leq \sqrt{R_*}$, $i = 1, 2$. That is, system (1.1) has two distinct mT-periodic solutions $z_m^i(t)$, $i = 1, 2$. It is clear that $z_m^i(0) \in \Omega$. If there exists $t_3 \in (0, mT)$ satisfying $|z_m^1(t_3)| > \sqrt{R_*}$, we have $t_3' \in (0, t_3)$ such that $|z_m^1(t_3')| = R_2'$ and $|z_m^1(t)| \leq R_2' \leq \sqrt{R_*}$ for $t \in [0, t_3']$. For clarity, let $\theta_m^1(t)$ denote the argument function of $z_m^1(t)$. By use of an argument similar to Step 2, one has

$$\theta_m^1(t_3') - \theta_m^1(0) = -2(j+1)\pi$$

and

$$\theta_m^1(mT) - \theta_m^1(0) < -2j\pi.$$

So that $\mathrm{Rot}^{H_\lambda}(mT; z_m^1(t)) > j$, a contradiction with (2.8). Therefore, for all $t \in [0, mT]$, one has $|z_m^1(t)| \leq \sqrt{R_*}$. Similarly, $|z_m^2(t)| \leq \sqrt{R_*}$ holds for all $t \in [0, mT]$. □

3. Applications

Firstly, we show that superlinear conditions (H_∞) in [3] and (f_∞) in [18] are special cases of (h_∞), which is introduced in Theorem 1.1. The discussions will be divided into the following two cases.

(1) The superlinear condition (H_∞) is a special case of (h_∞) for planar system (1.1). Indeed, from (1.4), for each $\delta > 0$, there exists $l_n(t) \in L^1([0,T])$ such that, for every $n \in \mathbb{N}$,

$$\langle \nabla_z H(t,z), z\rangle \geq (a_n(t) - \delta)V_n(z) - l_n(t),$$

for all $z \in \mathbb{R}^2$ and a.e. $t \in [0,T]$. Taking $L_n(t,z) = a_n(t)V_n(z)/2$, by (1.5), one has

$$\mathrm{Rot}_{V_n}^{L_n}(T; v) := \frac{1}{2\tau_{V_n}} \int_0^T \frac{\langle \nabla_z L_n(t,z), z\rangle}{V_n(z(t))} dt = \frac{\int_0^T a_n(t)dt}{2\tau_{V_n}} \to +\infty$$

as $n \to \infty$. Using Proposition 2.2 in [2], we have $\mathrm{Rot}^{L_n}(T; v) \to +\infty$ $(n \to +\infty)$. Therefore, we can apply Lemma 2.2 in [19] to conclude that $\rho(L_n) \to +\infty$ $(n \to +\infty)$, and thus (h_∞) is satisfied.

(2) We claim that the partial superlinear condition (f_∞) is a special case of (h_∞) for second order equation (1.2). Indeed, by (f_∞), we have

$$f(t,x)x \geq a_n(t)x^2, \quad \text{for large enough } x \text{ and } t \in [0, 2\pi],$$

where

$$a_n(t) = \begin{cases} n^2, & t \in I; \\ l(t), & t \in [0,2\pi]\backslash I. \end{cases}$$

Then, for large enough x and $t \in [0, 2\pi]$, one has

$$\langle \nabla_z H(t,z), z\rangle = y^2 + f(t,x)x \geq y^2 + a_n(t)x^2,$$

where $H(t,z) = \int_0^x f(t,s)ds + \frac{y^2}{2}$. Taking

$$L_n(t,z) = \frac{a_n(t)x^2 + y^2}{2} \quad \text{and} \quad V_n(z) = x^2 + y^2,$$

inequality (1.6) is satisfied.

Consider the systems

$$Jz' = \nabla_z L_n(t, z). \tag{3.1}$$

Using a general polar coordinate

$$x = \frac{r}{n}\cos\varphi, \quad y = r\sin\varphi,$$

we have

$$-\varphi'(t) = \frac{n(x'y - xy')}{n^2x^2 + y^2} = \frac{n(a_n(t)x^2 + y^2)}{n^2x^2 + y^2}.$$

It follows that for $t \in I$, $-\varphi'(t) = n$; and for $t \in [0, 2\pi]\backslash I$,

$$-\varphi'(t) \geq \frac{-nl^-(t)x^2}{n^2x^2 + y^2} \geq -\frac{\frac{l^-(t)}{n}(n^2x^2 + y^2)}{n^2x^2 + y^2} = -\frac{l^-(t)}{n},$$

where $l^-(t) = \max\{0, -l(t)\}$. Then the general argument function $\varphi(t)$ satisfies

$$\varphi(0) - \varphi(2\pi) \geq n\mathrm{mes}(I) - \frac{\int_0^{2\pi}|l(t)|dt}{n} \to +\infty\ (n \to \infty). \tag{3.2}$$

Let us denote by $\mathrm{Rot}^{L_n}(2\pi; v)$ the 2π-rotation number of solution $z(t)$ of (3.1). From (3.2), we have $\mathrm{Rot}^{L_n}(2\pi; v) \to +\infty$ as $n \to \infty$. By Lemma 2.2 in [19], we have $\rho(L_n) \to +\infty$ as $n \to \infty$.

Secondly, we give the proof of Example 1.1.

Proof of Example 1.1. The proof falls naturally into two steps.

Step 1. We will check that (h_∞) holds for system (1.9). Since $f(t, x) = |\sin t|x^3 - x\cos t$, it follows that

$$\liminf_{|x|\to+\infty} \frac{f(t, x)}{x} \geq \liminf_{|x|\to+\infty} (\sqrt{2}x^2/2 - \sqrt{2}/2) = +\infty, \quad \text{for } t \in [\pi/4, \pi/2],$$

and for $t \in [0, 2\pi]$,

$$\frac{f(t, x)}{x} = |\sin t|x^2 - \cos t \geq -\cos t.$$

Then, for fixed $n \in \mathbb{N}$, one has

$$\liminf_{|x|\to+\infty} \frac{f(t, x)}{x} \geq c_n(t), \quad \text{uniformly a.e. in } t \in [0, 2\pi],$$

where

$$c_n(t) = \begin{cases} n^2 + 1 + a, & t \in [\pi/4, \pi/2]; \\ -\cos t, & t \in [0, 2\pi]\backslash[\pi/4, \pi/2], \end{cases}$$

Hence that $f(t, x)x \geq (c_n(t) - 1)x^2$ for a.e. $t \in [0, 2\pi]$ and sufficiently large $|x|$, and finally that

$$\begin{aligned} \langle \nabla_z H(t, z), z\rangle &= \left(\frac{\partial \mathcal{U}}{\partial x}(t, x, y) + f(t, x)\right)x + \left(y + \frac{\partial \mathcal{U}}{\partial y}(t, x, y)\right)y \\ &\geq (c_n(t) - 1 - a)x^2 + (1 - b)y^2 - \delta(x^2 + y^2) \end{aligned}$$

for a.e. $t \in [0, 2\pi]$ and sufficiently large $|x|$, where

$$H(t,z) = \int_0^x f(t,s)ds + \frac{y^2}{2} + \mathcal{U}(t,z).$$

Taking

$$L_n(t,z) = (c_n(t) - 1 - a)\frac{x^2}{2} + \frac{(1-b)y^2}{2} \quad \text{and} \quad V_n(z) = x^2 + y^2,$$

inequality (1.6) holds.

Next, we will prove that $\rho(L_n) \to +\infty$ as $n \to +\infty$. Writing $(x(t), y(t))$ in general polar coordinate

$$x(t) = \frac{r(t)}{n}\cos(\varphi(t)), \quad y(t) = \frac{r(t)}{\sqrt{1-b}}\sin(\varphi(t)), \tag{3.3}$$

one has

$$-\varphi'(t) = \frac{n\sqrt{1-b}(x'y - xy')}{n^2x^2 + (1-b)y^2} = \frac{n\sqrt{1-b}[(c_n(t) - 1 - a)x^2 + (1-b)y^2]}{n^2x^2 + (1-b)y^2}.$$

It follows that

$$-\varphi'(t) = n\sqrt{1-b} \quad \text{for } t \in [\pi/4, \pi/2],$$

and

$$\begin{aligned}
-\varphi'(t) &\geq -\frac{n\sqrt{1-b}[(\cos t + 1 + a)x^2]}{n^2x^2 + (1-b)y^2} \\
&\geq -\frac{n(2+a)\sqrt{1-b}x^2}{n^2x^2 + (1-b)y^2} \\
&\geq -\frac{(2+a)\sqrt{1-b}}{n} \cdot \frac{n^2x^2 + (1-b)y^2}{n^2x^2 + (1-b)y^2} = -\frac{(2+a)\sqrt{1-b}}{n}
\end{aligned}$$

for $t \in [0, 2\pi] \backslash [\pi/4, \pi/2]$. Therefore, we have

$$-\int_0^{2\pi} \varphi'(t)dt \geq \frac{n\pi\sqrt{1-b}}{4} - \frac{2\pi(2+a)\sqrt{1-b}}{n} \to +\infty \ (n \to \infty),$$

which implies that $\frac{1}{2\pi}(\varphi(0) - \varphi(2\pi)) \to +\infty$ as $n \to \infty$. Since the generalized polar coordinate (3.3) is really a kind of elliptic coordinates, we have $\mathrm{Rot}^{L_n}(t; v) \to +\infty$ as $n \to \infty$. Hence $\rho(L_n) \to +\infty$ as $n \to +\infty$ by Lemma 2.2 in [19].

Step 2. We will check that conditions (h_1), (h_2) and (h_3) hold for system (1.9). It is easy to see that system (1.9) satisfies (h_3). We conclude that

$$\lim_{|x| \to 0} \left| \frac{f(t,x)}{x} \right| \leq \lim_{|x| \to 0} (|\sin t| x^2 + 1) = 1, \quad \text{for } t \in [0, 2\pi],$$

hence that

$$|f(t,x)| < 2|x|, \quad \text{for sufficiently small } x \text{ and } t \in [0, 2\pi],$$

and finally that (h_1) holds.

Next, we check (h_2). Divide $\mathbb{R}^2$ into four regions as

$$\mathcal{D}_1 = \{(x,y)|x \geq 0, y > 0\}; \qquad \mathcal{D}_2 = \{(x,y)|x > 0, y \leq 0\};$$
$$\mathcal{D}_3 = \{(x,y)|x \leq 0, y < 0\}; \qquad \mathcal{D}_4 = \{(x,y)|x < 0, y \geq 0\}.$$

Let $F_+(x) = \int_0^x f_+(s)ds$, where $f_+(x) = sgn(x)\max_{0\leq t\leq 2\pi}(a|x| + |f(t,x)|)$. Consider two functions

$$V(x,y) = \begin{cases} R(x,y), & \text{for } (x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3, \\ I(x,y), & \text{for } (x,y) \in \mathcal{D}_2 \cup \mathcal{D}_4, \end{cases}$$

and

$$U(x,y) = \begin{cases} I(x,y), & \text{for } (x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3, \\ R(x,y), & \text{for } (x,y) \in \mathcal{D}_2 \cup \mathcal{D}_4, \end{cases}$$

where

$$I(x,y) = \frac{1-b}{2}y^2 + F_+(x), \quad R(x,y) = \frac{x^2}{2} + \frac{y^2}{2}.$$

Since

$$\frac{f(t,x)}{x} = |\sin t|x^2 - \cos t \geq -1, \quad \text{for } x \neq 0 \text{ and } t \in [0,2\pi],$$

we have $-f(t,x)y \leq xy$ for $(x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3$ and $t \in [0,2\pi]$. Since $|\frac{\partial \mathcal{U}}{\partial x}(t,x,y)| \leq a|x|$ and $|\frac{\partial \mathcal{U}}{\partial y}(t,x,y)| \leq b|y|$, we get

$$x\left(y + \frac{\partial \mathcal{U}}{\partial y}(t,x,y)\right) \leq (1+b)xy$$

and

$$-y\frac{\partial \mathcal{U}}{\partial x}(t,x,y) \leq axy$$

for $(x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3$. It follows that

$$\begin{aligned}\frac{dR(x(t),y(t)}{dt} &= xx' + yy' = x\left(y + \frac{\partial \mathcal{U}}{\partial y}(t,x,y)\right) - \left(\frac{\partial \mathcal{U}}{\partial x}(t,x,y) + f(t,x)\right)y \\ &\leq (a+b+2)xy \leq (a+b+2)R(x(t),y(t))\end{aligned}$$

for $(x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3$. Similarly, we can deduce that

$$\begin{aligned}\frac{dR(x(t),y(t)}{dt} &= x\left(y + \frac{\partial \mathcal{U}}{\partial y}(t,x,y)\right) - \left(\frac{\partial \mathcal{U}}{\partial x}(t,x,y) + f(t,x)\right)y \\ &\geq (1+b)xy + axy + xy \geq -(a+b+2)R(x(t),y(t))\end{aligned}$$

for $(x,y) \in \mathcal{D}_2 \cup \mathcal{D}_4$.

Observe that

$$\begin{aligned}\frac{dI(x(t),y(t)}{dt} &= (1-b)yy' + f_+(x)x' \\ &= (b-1)y\left(\frac{\partial \mathcal{U}}{\partial x}(t,x,y) + f(t,x)\right) + f_+(x)\left(y + \frac{\partial \mathcal{U}}{\partial y}(t,x,y)\right) \\ &= (1-b)y\left(f_+(x) - \frac{\partial \mathcal{U}}{\partial x}(t,x,y) - f(t,x)\right)\end{aligned}$$

$$+ f_+(x)\left(\frac{\partial \mathcal{U}}{\partial y}(t,x,y) + by\right).$$

For $(x,y) \in \mathcal{D}_2$, we get $f_+(x) \geq 0$, $f_+(x) - \frac{\partial \mathcal{U}}{\partial x}(t,x,y) - f(t,x) \geq 0$ and $\frac{\partial \mathcal{U}}{\partial y}(t,x,y) + by \leq 0$. Moreover, we have $f_+(x) \leq 0$, $f_+(x) - \frac{\partial \mathcal{U}}{\partial x}(t,x,y) - f(t,x) \leq 0$ and $\frac{\partial \mathcal{U}}{\partial y}(t,x,y) + by \geq 0$ for $(x,y) \in \mathcal{D}_4$. Hence, $I'(x(t),y(t)) \leq 0$ for $(x,y) \in \mathcal{D}_2 \cup \mathcal{D}_4$. Using a similar argument as above, we have $I'(x(t),y(t)) \geq 0$ for $(x,y) \in \mathcal{D}_1 \cup \mathcal{D}_3$.

For $y = 0$ and $t \in [0,2\pi]$, we have $\frac{\partial H}{\partial y}(t,x,y) = 0$. Moreover, for $y \neq 0$ and $t \in [0,2\pi]$, we have $sgn(y)\frac{\partial H}{\partial y}(t,x,y) > 0$. By Lemma 4.1 in [19] with $\varphi(x) = 0$, we can prove that (h_2) holds for system (1.9). System (1.9) has infinitely many $2m\pi$-periodic solutions by Theorem 1.1. $\square$

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