# HOPF BIFURCATION IN THE DELAYED FRACTIONAL LESLIE-GOWER MODEL WITH HOLLING TYPE II FUNCTIONAL RESPONSE\*

Xiaoping Chen $^{1,2,\dagger},$  Chengdai Huang<br/>³, Jinde Cao², Xueying Shi^\$ and An<br/>  ${\rm Luo}^1$ 

**Abstract** In this paper the fractional-order Leslie-Gower model with Holling type II functional response and a single time delay is firstly considered. The stability interval and bifurcation points of developed model are derived via analytic extrapolation by regarding time delay as a bifurcation parameter. Besides, a delayed feedback control is successfully designed to put off the onset of Hopf bifurcation, extend the stability domain, and then the system possesses the stability in a larger parameter range. Some numerical simulations are shown in order to check the availability of the theoretical results.

**Keywords** Stability, Hopf bifurcation, fractional order, Leslie-Gower model, time delay.

MSC(2010) 93C43, 93D20.

## 1. Introduction

It is widely known that predator-prey models are typical and important models in the sphere of ecological systems. In the past few years predator-prey models have been studied extensively in virtue of their theoretical and practical significance. One of the famous predator-prey models is Leslie-Gower model. Because of its significant practical background Leslie-Gower model has attracted lots of applied mathematicians, economists and engineers to study and got many interesting and meaningful results, see, e.g., [1, 2, 4, 13–18, 24, 27, 28].

Email: cxpnuaa@163.com(X. Chen), huangchengdai@163.com(C. Huang) jdcao@seu.edu.cn(J. Cao), shixueying26@163.com(X. Shi)

<sup>&</sup>lt;sup>†</sup>The corresponding author.

<sup>&</sup>lt;sup>1</sup>Department of Mathematics, Taizhou University, Jichuan Road, 225300 Taizhou, China

<sup>&</sup>lt;sup>2</sup>Department of Mathematics, Southeast University, Southeast University Road, 210096 Nanjing, China

<sup>&</sup>lt;sup>3</sup>School of Mathematics and Statistics, Xinyang Normal University, Nanhu road, 464000 Xinyang, China

<sup>\*</sup>The authors were supported by National Natural Science Foundation of China (No. 12001396), Natural Science Foundation of Jiangsu Province (Nos. BK20170591, BK20200268), Natural Science Foundation of Jiangsu Higher Education Institutions of China (Nos. 21KJB110017, 20KJB110005), China Postdoctoral Science Foundation (No. 2018M642130) and Qing Lan Project of the Jiangsu Higher Education Institutions.

excaliburqq@163.com(A. Luo)

In general, biological systems can be described by traditional calculus and fractional calculus. However, Petras [19] pointed out that fractional calculus allows greater degrees of freedom in modeling dynamical systems compared with conventional descriptions of the above systems. Besides, fractional modeling for biological systems can exceedingly boost the capability of discrimination, design and control for dynamic systems since fractional calculus possesses illimitable memory [12, 30, 31]. Therefore, modeling and control of fractional-order biological systems have currently become a research hotspot, and large amounts of significant results have been achieved [7–10, 26, 32]. Recently, the study of fractional-order dynamic systems mainly involves the properties of fractional-order dynamics system, such as stability analysis, undamped oscillations, bifurcations, chaos and so on.

It should be noted that delicate stability results of nonlinear systems can be acquired in the light of puissant bifurcation analysis. Especially, Hopf bifurcation analysis for a nonlinear system is a very effective approach, and has attracted many researchers from various disciplines. At present, bifurcation control is an important research content of bifurcation analysis. We know that the general goal of bifurcation control is to design a controller which is capable of modifying the bifurcation characteristics, thereby to achieve some desirable dynamical behaviors. It is indicated that bifurcation properties of a system can be modified via feedback control methods. Various approaches have been proposed to control bifurcation [3, 5, 21-23, 29].

Based on upwards discussions, we firstly extend the delayed Leslie-Gower model to the fractional case. Then we will study the bifurcation and control of the proposed system. Our contributions of this paper are listed as follows. (1) The first extension of the Leslie-Gower model with time delay to fractional-order case by Caputo fractional derivative. (2) The precise bifurcation conditions of the delayed fractional Leslie-Gower model are shown by taking time delay as bifurcation parameter. (3) A delayed feedback controller which can put off the onset of the Hopf bifurcation is devised for the proposed model.

The structure of this paper is organized as follows. In Section 2, we present some definitions of fractional calculus and the fractional-order Leslie-Gower model with Holling type II functional response. In Section 3, we investigate the linear stability of the positive equilibrium and also give the occurrence of Hopf bifurcation at the positive equilibrium. In Section 4, we establish the essential bifurcation control results via enhancing feedback control method. Some numerical experiments are given in Section 5 to check our theoretical results. Finally, a brief discussion is given to conclude this work.

### 2. Preliminaries

In this section, we will introduce some results about the fractional-order derivatives and the fractional Leslie-Gower model with Holling type II functional response.

It is well known that there are several common definitions of fractional-order derivatives, such as the Grünwald-Letnikov fractional derivative, Riemann-Liouville fractional derivative, the Caputo fractional derivative and so on. Generally speaking, the Caputo derivative is used more at present since the Caputo derivative can represent well-understood features of physical situation and make it more applicable to real world problems. Therefore, in this paper the Caputo derivative is taken into account.

The Caputo derivative for one function g(x) is defined as

$${}_{C}D^{\alpha}_{t_{0},t}g(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_{0}}^{t} (t-s)^{n-\alpha-1} g^{(n)}(s) ds,$$

where  $\alpha$  is the value of fractional order,  $n-1 < \alpha \leq n \in Z^+$ ,  $\Gamma(\cdot)$  is the Gamma function and  $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} ds$ . As a special case, we have  ${}_C D_{0,t}^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} g'(\tau) d\tau$  when  $0 < \alpha \leq 1$ .

As we know, the Laplace transform of the fractional Caputo derivative is given as follows.

$$L\left\{{}_{C}D^{\alpha}_{0,t}g(t);s\right\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}g^{(k)}(0),$$

where  $n-1 < \alpha \leq n \in Z^+$ .

It is easy to see that  $L\left\{ {_C}D_{0,t}^{\alpha}g(t);s \right\} = s^{\alpha}F(s)$  when  $g^{(k)}(0) = 0$  for  $k = 1, 2, \ldots, n$ .

For convenience, we use the notation  $D^{\alpha}g(t)$  to represent the Caputo derivative operator  $_{C}D^{\alpha}_{0,t}g(t)$ .

Next we will introduce the fractional Leslie-Gower model with Holling type II functional response and non-identical orders which is shown by

$$\begin{cases} D^{q_1} x(t) = \left( a_1 - b_1 x(t) - \frac{c_1 y(t-\tau)}{x(t-\tau) + k_1} \right) x(t), \\ D^{q_2} y(t) = \left( a_2 - \frac{c_2 y(t-\tau)}{x(t-\tau) + k_2} \right) y(t), \end{cases}$$
(2.1)

where  $q_i \in (0, 1]$  for i = 1 and 2, x(t) and y(t) stand for the prey population size and the predator population size, respectively,  $\tau \geq 0$  represents time delay and all of the parameters are positive with the ecology meaning as follows.  $a_1$  is the growth rate of prey,  $a_2$  is the growth rate of predator.  $b_1$  measures the strength of competition among individuals of species x(t).  $c_1$  is the maximum value which per capita reduction rate of x(t) can attain.  $c_2$  is the maximum value which per capita reduction rate of y(t) can attain.  $k_1$  measures the extent to which environment provides protection to prey x(t) and  $k_2$  measures the extent to which environment provides protection to predator y(t).

To facilitate the theoretical analysis and further obtain the main results, we assume that the following condition holds.

$$(\mathbf{H1}) \ \frac{a_2k_2}{c_2} < \frac{a_1k_1}{c_1}.$$

It is obvious that system (2.1) has a unique positive equilibrium point  $E^*(x^*, y^*)$ under the assumption (**H1**), which is defined by:

$$\begin{cases} x^* = \frac{-(c_1a_2 - a_1c_2 + c_2b_1k_1) + \sqrt{\Delta}}{2c_2b_1}, \\ y^* = \frac{a_2(x^* + b_2)}{c_2}, \end{cases}$$
(2.2)

where  $\Delta = (c_1a_2 - a_1c_2 + c_2b_1k_1)^2 - 4c_2b_1(c_1a_2k_2 - c_2a_1k_1).$ 

When  $q_1 = q_2 = 1$ , system (2.1) turns into the following integer-order system

$$\begin{cases} \dot{x}(t) = \left(a_1 - b_1 x(t) - \frac{c_1 y(t - \tau)}{x(t - \tau) + k_1}\right) x(t), \\ \dot{y}(t) = \left(a_2 - \frac{c_2 y(t - \tau)}{x(t - \tau) + k_2}\right) y(t). \end{cases}$$
(2.3)

Through Hopf bifurcation technique, Cao and Yuan [4] observed the existence of Hopf bifurcation phenomena at the positive equilibrium of the above system (2.3). And the analysis indicated that Hopf bifurcation occurs when time delay passes through a certain critical value.

In this paper, our main goal is to seek for the conditions of Hopf bifurcation for system (2.1) by time delay as the bifurcation parameter from the approach of stability analysis [6], and the impact of each order on bifurcation is revealed. Moreover, we intend to design a delayed feedback controller to control the creation of bifurcation and further study the effects of feedback gain on bifurcation control for the proposed system.

# 3. Stability and bifurcation of system (2.1)

In this section, we will discuss the conditions of local stability and Hopf bifurcation for system (2.1) considering time delay  $\tau$  as the bifurcation parameter.

Let  $u(t) = x(t) - x^*$ ,  $v(t) = y(t) - y^*$ . Then system (2.1) can be transformed into

$$\begin{cases} D^{q_1}u(t) = \left(a_1 - b_1(u(t) + x^*) - \frac{c_1(v(t-\tau) + y^*)}{(u(t-\tau) + x^*) + k_1}\right)(u(t) + x^*), \\ D^{q_2}v(t) = \left(a_2 - \frac{c_2(v(t-\tau) + y^*)}{(u(t-\tau) + x^*) + k_2}\right)(v(t) + y^*). \end{cases}$$
(3.1)

Hence, the linearizing system of Eq. (3.1) at (0,0) is

$$\begin{cases} D^{q_1}u(t) = \alpha_1 u(t) + \alpha_2 v(t-\tau) + \alpha_3 u(t-\tau), \\ D^{q_2}v(t) = \beta_1 u(t-\tau) + \beta_2 v(t-\tau), \end{cases}$$
(3.2)

where

$$\begin{aligned} \alpha_1 &= -b_1 x^*, \\ \alpha_2 &= -\frac{c_1 x^*}{x^* + k_1}, \\ \alpha_3 &= \frac{c_1 x^* y^*}{(x^* + k_1)^2}, \\ \beta_1 &= \frac{a_2^2}{c_2}, \\ \beta_2 &= -a_2. \end{aligned}$$

Therefore, the corresponding characteristic equation of system (3.2) is as follows.

$$\det \begin{pmatrix} s^{q_1} - \alpha_1 - \alpha_3 e^{-s\tau} & -\alpha_2 e^{-s\tau} \\ -\beta_1 e^{-s\tau} & s^{q_2} - \beta_2 e^{-s\tau} \end{pmatrix} = 0.$$
(3.3)

Then it follows from Eq. (3.3) that

$$\mathcal{P}_{1}(s) + \mathcal{P}_{2}(s) e^{-s\tau} + \mathcal{P}_{3}(s) e^{-2s\tau} = 0, \qquad (3.4)$$

where

$$\begin{aligned} \mathcal{P}_1(s) &= s^{q_1+q_2} - \alpha_1 s^{q_2}, \\ \mathcal{P}_2(s) &= -\beta_2 s^{q_1} - \alpha_3 s^{q_2} + \alpha_1 \beta_2, \\ \mathcal{P}_3(s) &= -\alpha_2 \beta_1 + \alpha_3 \beta_2. \end{aligned}$$

We know that Eq. (3.4) can be written as the following form

$$\mathcal{P}_{1}\left(s\right)e^{s\tau} + \mathcal{P}_{2}\left(s\right) + \mathcal{P}_{2}\left(s\right)e^{-s\tau} = 0.$$

$$(3.5)$$

Assume that  $s = i\omega$  is one root of Eq. (3.5) where  $\omega > 0$  and *i* denotes the imaginary unit of a complex number. Substituting *s* into Eq. (3.5), we have

$$\begin{cases} (\mathcal{A}_1 + \mathcal{A}_3)\cos w\tau - (\mathcal{B}_1 - \mathcal{B}_3)\sin w\tau = -\mathcal{A}_2, \\ (\mathcal{B}_1 + \mathcal{B}_3)\cos w\tau - (\mathcal{A}_1 - \mathcal{A}_3)\sin w\tau = -\mathcal{B}_2, \end{cases}$$
(3.6)

where  $\mathcal{A}_l$ ,  $\mathcal{B}_l$  are the real part and the imaginary part of  $\mathcal{P}_l(s)$  (l = 1 and 2), respectively, which are shown as follows.

$$\begin{aligned} \mathcal{A}_{1} &= \omega^{q_{1}+q_{2}} \cos \frac{(q_{1}+q_{2})\pi}{2} - \alpha_{1}\omega^{q_{2}} \cos \frac{q_{2}}{2}\pi, \\ \mathcal{B}_{1} &= \omega^{q_{1}+q_{2}} \sin \frac{(q_{1}+q_{2})\pi}{2} - \alpha_{1}\omega^{q_{2}} \sin \frac{q_{2}}{2}\pi, \\ \mathcal{A}_{2} &= -\beta_{2}\omega^{q_{1}} \cos \frac{q_{1}\pi}{2} - \alpha_{3}\omega^{q_{2}} \cos \frac{q_{2}\pi}{2} + \alpha_{1}\beta_{2}, \\ \mathcal{B}_{2} &= -\beta_{2}\omega^{q_{1}} \sin \frac{q_{1}\pi}{2} - \alpha_{3}\omega^{q_{2}} \sin \frac{q_{2}\pi}{2}, \\ \mathcal{A}_{3} &= -\alpha_{2}\beta_{1} + \alpha_{3}\beta_{2}, \\ \mathcal{B}_{3} &= 0. \end{aligned}$$

It follows from Eq. (3.6) that

$$\begin{cases} \cos w\tau = \frac{\mathcal{B}_{2}\mathcal{B}_{3} - \mathcal{B}_{1}\mathcal{B}_{2} + \mathcal{A}_{2}\mathcal{A}_{3} - \mathcal{A}_{1}\mathcal{A}_{2}}{\left(\mathcal{A}_{1}^{2} + \mathcal{B}_{1}^{2}\right) - \left(\mathcal{A}_{3}^{2} + \mathcal{B}_{3}^{2}\right)} = \mathcal{G}_{1}\left(w\right),\\ \sin w\tau = \frac{\mathcal{A}_{2}\mathcal{B}_{1} + \mathcal{A}_{2}\mathcal{B}_{3} - \mathcal{B}_{2}\mathcal{A}_{1} - \mathcal{B}_{2}\mathcal{A}_{3}}{\left(\mathcal{A}_{1}^{2} + \mathcal{B}_{1}^{2}\right) - \left(\mathcal{A}_{3}^{2} + \mathcal{B}_{3}^{2}\right)} = \mathcal{G}_{2}\left(w\right). \end{cases}$$
(3.7)

Taking square on both sides of Eq. (3.7) and summing them up, we have

$$\mathcal{G}_1^2(\omega) + \mathcal{G}_2^2(\omega) = 1. \tag{3.8}$$

The following assumption is addressed for ensuring the existence of the roots of Eq. (3.8).

(H2) There exist positive roots for Eq. (3.8). By the fact  $\cos \omega \tau = \mathcal{G}_1(\omega)$  we easily have

$$\tau^{(l)} = \frac{1}{\omega} [\arccos \mathcal{G}_1(\omega) + 2l\pi], \quad l = 0, 1, 2, \dots$$
 (3.9)

Define the bifurcation point

$$\tau_0 = \min\{\tau^{(l)}\}, \quad l = 0, 1, 2, \dots$$
(3.10)

To obtain the bifurcation conditions, we elaborate the following useful hypothesis.

(H3) 
$$\frac{\mathcal{M}_1\mathcal{N}_1 + \mathcal{M}_2\mathcal{N}_2}{\mathcal{N}_1^2 + \mathcal{N}_2^2} \neq 0$$

where  $\mathcal{M}_l$  and  $\mathcal{N}_l$  are described by Appendix A where l = 1 and 2.

**Lemma 3.1.** Assume that  $s(\tau) = \gamma(\tau) + i\omega(\tau)$  is the root of Eq. (3.5) near  $\tau = \tau_0$  satisfying  $\gamma(\tau_0) = 0$  and  $\omega(\tau_0) = \omega_0$ . Then the transversality condition  $\operatorname{Re}\left[\frac{ds}{d\tau}\right]|_{(\tau=\tau_0,\omega=\omega_0)} \neq 0$  holds.

**Proof.** Differentiating both sides of Eq. (3.5) with respect to  $\tau$ , we have

$$\mathcal{P}_{1}'(s) e^{s\tau} \frac{ds}{d\tau} + \mathcal{P}_{1}(s) e^{s\tau} \left(\tau \frac{ds}{d\tau} + s\right) + \mathcal{P}_{2}'(s) \frac{ds}{d\tau} + \mathcal{P}_{3}'(s) e^{-s\tau} \frac{ds}{d\tau} + \mathcal{P}_{3}(s) e^{-s\tau} \left(-\tau \frac{ds}{d\tau} - s\right) = 0,$$

where  $\mathcal{P}'_{l}(s)$  are the derivatives of  $\mathcal{P}_{l}(s)$  with l = 1, 2 and 3.

Therefore,

$$\frac{ds}{d\tau} = \frac{\mathcal{M}(s)}{\mathcal{N}(s)},\tag{3.11}$$

where

$$\mathcal{M}(s) = \left[ \left( -\alpha_2 \beta_1 + \alpha_3 \beta_2 \right) e^{-s\tau} - \left( s^{q_1 + q_2} - \alpha_1 s^{q_2} \right) e^{s\tau} \right] s,$$
  
$$\mathcal{N}(s) = \left[ \left( q_1 + q_2 \right) s^{q_1 + q_2 - 1} - \alpha_1 q_2 s^{q_2 - 1} + s^{q_1 + q_2} - \alpha_1 s^{q_2} \right] e^{s\tau} + \left( -\beta_2 q_1 s^{q_1 - 1} - \alpha_3 q_2 s^{q_2 - 1} \right) - \left( -\alpha_2 \beta_1 + \alpha_3 \beta_2 \right) \tau e^{-s\tau}.$$

It follows from Eq. (3.11) that

$$\operatorname{Re}\left[\frac{ds}{d\tau}\right]|_{(\tau=\tau_0,\omega=\omega_0)} = \frac{\mathcal{M}_1\mathcal{N}_1 + \mathcal{M}_2\mathcal{N}_2}{\mathcal{N}_1^2 + \mathcal{N}_2^2},\tag{3.12}$$

where  $\mathcal{M}_1$ ,  $\mathcal{M}_2$ ,  $\mathcal{N}_1$  and  $\mathcal{N}_2$  are the real and imaginary parts of  $\mathcal{M}(s)$ , the real and imaginary parts of  $\mathcal{N}(s)$ , respectively.

Clearly, the hypothesis (H3) shows that transversality condition holds. Thus, we have proved Lemma 3.1.

Next we further give the following hypotheses to establish the stability of system (2.1) with  $\tau = 0$ .

 $(\mathbf{H4}) \ \alpha_1 + \alpha_3 + \beta_2 < 0,$ 

$$(\mathbf{H5}) \ \alpha_3\beta_1 - \alpha_2\beta_1 + \alpha_1\beta_2 > 0.$$

**Lemma 3.2.** Assume that (H4)-(H5) hold. Then the positive equilibrium point  $E^*(x^*, y^*)$  of system (2.1) with  $\tau = 0$  is asymptotically stable.

**Proof.** If  $\tau = 0$ , then Eq. (3.4) turns to be

$$\lambda^{2} - (\alpha_{1} + \alpha_{3} + \beta_{2})\lambda + \alpha_{3}\beta_{1} - \alpha_{2}\beta_{1} + \alpha_{1}\beta_{2} = 0.$$
(3.13)

Since the assumptions (H4)-(H5) hold, by Routh-Hurwitz criteria we have that the two roots of the above equation (3.13) have negative real parts. This completes the proof.

From the above results and notations, we can easily get the following conclusion.

**Theorem 3.1.** Assume that (H1)-(H5) are satisfied, the following results are available for system (2.1).

(1) When  $\tau \in [0, \tau_0)$ , the positive equilibrium point  $E^*$  of system (2.1) is asymptotically stable.

(2) When  $\tau = \tau_0$  system (2.1) undergoes a Hopf bifurcation at  $E^*$ . Namely, system (2.1) has a branch of periodic solutions bifurcating from the positive equilibrium point  $E^*$  near  $\tau = \tau_0$ .

## 4. Stability and bifurcation of the controlled system

We know that various feedback controllers have been designed to control the Hopf bifurcation of fractional-order systems recently, see, e.g., [11, 20, 25]. But the feedback controller has not been carried over into fractional-order Leslie-Gower model with Holling type II functional response to meet the control of the Hopf bifurcation. In this paper a time-delayed force  $m[y(t) - y(t - \tau)]$  is introduced to the second equation of system (2.1). Therefore, the delayed feedback control system can be written as

$$\begin{cases} D^{q_1} x(t) = \left( a_1 - b_1 x(t) - \frac{c_1 y(t - \tau)}{x(t - \tau) + k_1} \right) x(t), \\ D^{q_2} y(t) = \left( a_2 - \frac{c_2 y(t - \tau)}{x(t - \tau) + k_2} \right) y(t) + m \left[ y(t) - y(t - \tau) \right], \end{cases}$$

$$(4.1)$$

where m stands for the feedback gain parameter.

Considering time delay  $\tau$  as a bifurcation parameter in the above system (4.1), we discuss the conditions under which Hopf bifurcation occurs. Let  $u(t) = x(t) - x^*$  and  $v(t) = y(t) - y^*$ , for system (4.1) we have

$$\begin{cases} D^{q_1}u(t) = \left(a_1 - b_1(u(t) + x^*) - \frac{c_1(v(t-\tau) + y^*)}{(u(t-\tau) + x^*) + k_1}\right)(u(t) + x^*), \\ D^{q_2}v(t) = \left(a_2 - \frac{c_2(v(t-\tau) + y^*)}{(u(t-\tau) + x^*) + k_2}\right)(v(t) + y^*) + m[v(t) - v(t-\tau)]. \end{cases}$$

$$(4.2)$$

Then it gains from system (4.2) that the linearized form is

$$\begin{cases} D^{q_1}u(t) = \alpha_1 u(t) + \alpha_2 v(t-\tau) + \alpha_3 u(t-\tau), \\ D^{q_2}v(t) = \beta_1 u(t-\tau) + \beta_2 v(t-\tau) + mv(t) - mv(t-\tau), \end{cases}$$
(4.3)

where  $\alpha_1, \alpha_2, \alpha_3, \beta_1$  and  $\beta_2$  are shown as Eq. (3.2).

Hence, the characteristic equation of system (4.3) is as follows.

$$\det \begin{pmatrix} s^{q_1} - \alpha_1 - \alpha_3 e^{-s\tau} & -\alpha_2 e^{-s\tau} \\ -\beta_1 e^{-s\tau} & s^{q_2} - m + (m - \beta_2) e^{-s\tau} \end{pmatrix} = 0.$$
(4.4)

It follows from Eq. (4.4) that

$$Q_1(s) + Q_2(s)e^{-s\tau} + Q_3(s)e^{-2s\tau} = 0,$$
 (4.5)

where

$$\begin{aligned} \mathcal{Q}_1(s) &= s^{q_1+q_2} - ms^{q_1} - \alpha_1 s^{q_2} + m\alpha_1, \\ \mathcal{Q}_2(s) &= s^{q_1} \left(m - \beta_2\right) - \alpha_3 s^{q_2} - m\alpha_1 + m\alpha_3 + \alpha_1 \beta_2, \\ \mathcal{Q}_3(s) &= -m\alpha_3 - \alpha_2 \beta_1 + \alpha_3 \beta_2. \end{aligned}$$

Assume that  $C_l$  and  $D_l$  are the real and imaginary parts of  $Q_l(s)$  for l = 1, 2 and 3, respectively, which are given as follows.

$$\begin{aligned} \mathcal{C}_{1} &= w^{q_{1}+q_{2}} \cos\left(\frac{q_{1}+q_{2}}{2}\right) \pi - mw^{q_{1}} \cos\frac{q_{1}}{2}\pi - \alpha_{1}w^{q_{2}} \cos\frac{q_{2}}{2}\pi + m\alpha_{1}, \\ \mathcal{D}_{1} &= w^{q_{1}+q_{2}} \sin\left(\frac{q_{1}+q_{2}}{2}\right) \pi - mw^{q_{1}} \sin\frac{q_{1}}{2}\pi - \alpha_{1}w^{q_{2}} \sin\frac{q_{2}}{2}\pi, \\ \mathcal{C}_{2} &= (m-\beta_{2}) w^{q_{1}} \cos\frac{q_{1}}{2}\pi - \alpha_{3}w^{q_{2}} \cos\frac{q_{2}}{2}\pi - m\alpha_{1} + m\alpha_{3} + \alpha_{1}\beta_{2}, \\ \mathcal{D}_{2} &= (m-\beta_{2}) w^{q_{1}} \sin\frac{q_{1}}{2}\pi - \alpha_{3}w^{q_{2}} \sin\frac{q_{2}}{2}\pi, \\ \mathcal{C}_{3} &= -m\alpha_{3} - \alpha_{2}\beta_{1} + \alpha_{3}\beta_{2}, \\ \mathcal{D}_{3} &= 0. \end{aligned}$$

It follows from Eq. (4.5) that

$$Q_1(s)e^{s\tau} + Q_2(s) + Q_3(s)e^{-s\tau} = 0.$$
 (4.6)

Assumed that  $s = i\omega$  is one root of Eq. (4.6) and  $\omega > 0$ . According to Eq. (4.6) we have

$$\begin{cases} (\mathcal{C}_1 + \mathcal{C}_3) \cos w\tau - (\mathcal{D}_1 - \mathcal{D}_3) \sin w\tau = -\mathcal{C}_2, \\ (\mathcal{D}_1 + \mathcal{D}_3) \cos w\tau - (\mathcal{C}_1 - \mathcal{C}_3) \sin w\tau = -\mathcal{D}_2. \end{cases}$$
(4.7)

According to the above equation, it is easy to get

$$\begin{cases} \cos w\tau = \frac{\mathcal{D}_{2}\mathcal{D}_{3} - \mathcal{D}_{1}\mathcal{D}_{2} + \mathcal{C}_{2}\mathcal{C}_{3} - \mathcal{C}_{1}\mathcal{C}_{2}}{\left(\mathcal{C}_{1}^{2} + \mathcal{D}_{1}^{2}\right) - \left(\mathcal{C}_{3}^{2} + \mathcal{D}_{3}^{2}\right)} = \mathcal{H}_{1}(w), \\ \sin w\tau = \frac{\mathcal{C}_{2}\mathcal{D}_{1} + \mathcal{C}_{2}\mathcal{D}_{3} - \mathcal{D}_{2}\mathcal{C}_{1} - \mathcal{D}_{2}\mathcal{C}_{3}}{\left(\mathcal{C}_{1}^{2} + \mathcal{D}_{1}^{2}\right) - \left(\mathcal{C}_{3}^{2} + \mathcal{D}_{3}^{2}\right)} = \mathcal{H}_{2}(w). \end{cases}$$
(4.8)

From Eq. (4.8), we can easily have

$$\mathcal{H}_1^2(\omega) + \mathcal{H}_2^2(\omega) = 1. \tag{4.9}$$

Based on  $\cos \omega \tau = \mathcal{H}_1(\omega)$ , we have

$$\tau_{(l)}^* = \frac{1}{\omega} [\arccos \mathcal{H}_1(\omega) + 2l\pi], \quad l = 0, 1, 2, \dots$$
 (4.10)

To establish the fundamental results of this section, the following assumption is beneficial.

(H6) Eq. (4.9) has at least one positive real root. Now we define the following bifurcation point

$$\tau_0^* = \min\{\tau_{(l)}^{(*)}\}, \quad l = 0, 1, 2, \dots$$
(4.11)

To capture the bifurcation conditions, the following assumption is addressed. (H7)  $\frac{\mathcal{X}_1\mathcal{Y}_1 + \mathcal{X}_2\mathcal{Y}_2}{\mathcal{Y}_1^2 + \mathcal{Y}_2^2} \neq 0$ , where  $\mathcal{X}_l$  and  $\mathcal{Y}_l$  are shown by Appendix *B* with l = 1 and 2.

**Lemma 4.1.** Assume that  $s(\tau) = \delta(\tau) + i\omega(\tau)$  is the root of Eq. (4.6) near  $\tau = \tau_0^*$  satisfying  $\delta(\tau_0^*) = 0$  and  $\omega(\tau_0^*) = \omega_0$ . Then the transversality condition  $\operatorname{Re}\left[\frac{ds}{d\tau}\right]|_{(\tau=\tau_0^*,\omega=\omega_0^*)} \neq 0$  holds.

**Proof.** Differentiating Eq. (4.6) with respect to  $\tau$ , we have

$$\begin{aligned} \mathcal{Q}_{1}'(s) e^{s\tau} \frac{ds}{d\tau} + \mathcal{Q}_{1}(s) e^{s\tau} \left(\tau \frac{ds}{d\tau} + s\right) + \mathcal{Q}_{2}'(s) \frac{ds}{d\tau} + \mathcal{Q}_{3}'(s) e^{-s\tau} \frac{ds}{d\tau} \\ + \mathcal{Q}_{3}(s) e^{-s\tau} \left(-\tau \frac{ds}{d\tau} - s\right) = 0, \end{aligned}$$

where  $Q'_l(s)$  stands for the derivatives of  $Q_l(s)$  with l = 1, 2 and 3.

Thus,

$$\frac{ds}{d\tau} = \frac{\mathcal{X}(s)}{\mathcal{Y}(s)},\tag{4.12}$$

where

$$\begin{aligned} \mathcal{X}(s) &= \left(\mathcal{Q}_3\left(s\right)e^{-s\tau} - \mathcal{Q}_1\left(s\right)e^{s\tau}\right)s,\\ \mathcal{Y}(s) &= \mathcal{Q}_1'\left(s\right)e^{s\tau} + \mathcal{Q}_1\left(s\right)e^{s\tau} + \mathcal{Q}_2'\left(s\right) + \mathcal{Q}_3'\left(s\right)e^{-s\tau} - \mathcal{Q}_3\left(s\right)\tau e^{-s\tau}. \end{aligned}$$

It follows from (4.12) that

$$\operatorname{Re}\left[\frac{ds}{d\tau}\right]|_{(\tau=\tau_0^*,\omega=\omega_0^*)} = \frac{\mathcal{X}_1\mathcal{Y}_1 + \mathcal{X}_2\mathcal{Y}_2}{\mathcal{Y}_1^2 + \mathcal{Y}_2^2},\tag{4.13}$$

where  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$  and  $\mathcal{Y}_2$  are the real and imaginary parts of  $\mathcal{X}(s)$  and the real and imaginary parts of  $\mathcal{Y}(s)$ , respectively.

Hence, the transversality condition is true through the hypothesis  $(\mathbf{H7})$ . Then we have proved Lemma 4.1. 

Based on Lemma 3.1, Lemma 3.2 and Lemma 4.1, we establish the following theorem.

Theorem 4.1. Assume that (H1) and (H4)-(H7) are met. Then the following statements are true for system (4.1).

(1) When  $\tau \in [0, \tau_0^*)$ , the positive equilibrium point  $E^*$  of system (4.1) is asymptotically stable.

(2) When  $\tau = \tau_0^*$ , system (4.1) undergoes a Hopf bifurcation at  $E^*$ . Namely, system (4.1) has a branch of periodic solutions bifurcating from the positive equilibrium point  $E^*$  near  $\tau = \tau_0^*$ .

### 5. Numerical results

This section provides two numerical examples to demonstrate the effectiveness and feasibility of our theoretical results.

#### 5.1. Example 1

In this example we choose time delay as the bifurcation parameter to examine the stability and bifurcation of system (2.1). For easy to compare, we consider system (2.1) with  $a_1 = 1.5$ ,  $b_1 = 10$ ,  $c_1 = 1.5$ ,  $a_2 = 0.5$ ,  $c_2 = 1.5$  and  $k_2 = 10$ , which is from the literature [4]. Then system (2.1) becomes the following form.

$$\begin{cases} D^{q_1}x(t) = \left(1.5 - 10x(t) - \frac{1.5y(t-\tau)}{x(t-\tau) + k_1}\right)x(t), \\ D^{q_2}y(t) = \left(0.5 - \frac{1.5y(t-\tau)}{x(t-\tau) + 10}\right)y(t). \end{cases}$$
(5.1)

Obviously, the positive equilibrium point of system (5.1) is  $E^* = (0.1, 3.36)$ . To study the impact of time delay on the dynamic characteristics of system (5.1), we should establish the bifurcation point. Then we set  $q_1 = 0.92$  and  $q_2 = 0.95$ . It is easy to find that the critical frequency  $\omega_0$  and the bifurcation point  $\tau_0$  are 0.4838 and 3.4387, respectively. Besides, we can easily check that (H1)-(H5) are true. According to Theorem 3.1, when  $\tau = 3 < \tau_0$  the positive equilibrium point  $E^*$  is asymptotically stable, as shown in Figs. 1–3. On the other hand, as  $\tau$  is increased to pass  $\tau_0$ ,  $E^*$  loses its stability and a Hopf bifurcation occurs. For instance, when  $\tau = 4 > \tau_0$  the positive equilibrium point  $E^*$  is unstable, which is shown in Figs. 4– 6. When  $q_1 = q_2 = 1$ , system (5.1) turns to be the integer-order system. Then we have  $\omega_0 = 0.4990$  and  $\tau_0 = 3.1398$ . Therefore, we can see that fractional-order system can put off the onset of Hopf bifurcation and the stability interval of the corresponding system can be amplified.

Next we will present the different effects of each order on bifurcation for system (5.1). Here, we first set one order, and then we discuss the effects of the second order on the bifurcation of system (5.1).

To be specific, we set  $q_2 = 0.95$  to discuss the effects of  $q_1$  on bifurcation for system (5.1). From Fig. 7 we observe that the bifurcation happens in advance when  $q_1$  decreases.

Besides, to explore the impact of  $q_2$  on bifurcation for system (5.1) we set  $q_1 = 0.92$ . As shown in Fig. 8 we see that the bifurcation takes place prematurely when  $q_2$  increases.

In summary, Hopf bifurcation happens in advance for system (5.1) supposing that  $q_1$  is firstly fixed as  $q_2$  increases. However, the changes of  $q_1$  when  $q_2$  is fixed have little effect on the occurrence of Hopf bifurcation in advance.

#### 5.2. Example 2

In this part, we will design a linear delayed feedback controller to delay the onset of Hopf bifurcation of system (2.1). To better illustrate the effects of bifurcation control through the above proposed controller, all parameters of the system are



Figure 1. The positive equilibrium point  $E^*$  of system (5.1) is asymptotically stable with initial values (0.5,0.5),  $q_1 = 0.92$ ,  $q_2 = 0.95$  and  $\tau = 3 < \tau_0 = 3.4387$ .



Figure 3. The positive equilibrium point  $E^*$  of system (5.1) is asymptotically stable with initial values (0.5,0.5),  $q_1 = 0.92$ ,  $q_2 = 0.95$  and  $\tau = 3 < \tau_0 = 3.4387$ .



Figure 5. The positive equilibrium point  $E^*$ of system (5.1) is unstable and a periodic oscillation bifurcates from the positive equilibrium point  $E^*$  with initial values (0.5,0.5),  $q_1 = 0.92$ ,  $q_2 = 0.95$  and  $\tau = 4 > \tau_0 = 3.4387$ .



Figure 2. The positive equilibrium point  $E^*$  of system (5.1) is asymptotically stable with initial values (0.5,0.5),  $q_1 = 0.92$ ,  $q_2 = 0.95$  and  $\tau = 3 < \tau_0 = 3.4387$ .



**Figure 4.** The positive equilibrium point  $E^*$  of system (5.1) is unstable and a periodic oscillation bifurcates from the positive equilibrium point  $E^*$  with initial values (0.5,0.5),  $q_1 = 0.92$ ,  $q_2 = 0.95$  and  $\tau = 4 > \tau_0 = 3.4387$ .



Figure 6. The positive equilibrium point  $E^*$ of system (5.1) is unstable and a periodic oscillation bifurcates from the positive equilibrium point  $E^*$  with initial values (0.5,0.5),  $q_1 = 0.92$ ,  $q_2 = 0.95$  and  $\tau = 4 > \tau_0 = 3.4387$ .

same as those in Example 1. That is, we consider the following system.

$$\begin{cases} D^{q_1} x(t) = \left( 1.5 - 10x(t) - \frac{1.5y(t-\tau)}{x(t-\tau) + k_1} \right) x(t), \\ D^{q_2} y(t) = \left( 0.5 - \frac{1.5y(t-\tau)}{x(t-\tau) + 10} \right) y(t) + m[y(t) - y(t-\tau)]. \end{cases}$$
(5.2)





Figure 7. Illustration of bifurcation point  $\tau_0$  versus the order  $q_1$  for system (5.1) with  $q_2 = 0.95$ .

**Figure 8.** Illustration of bifurcation point  $\tau_0$  versus the order  $q_2$  for system (5.1) with  $q_1 = 0.92$ .

According to Example 1 we know that system (5.1) loses its stability and Hopf bifurcation occurs if  $\tau > 3.4387$ . In the following, we will discuss the impact of feedback gain m on the bifurcation for system (5.2). To control bifurcation of system (5.2) and achieve desirable dynamic characteristics, we set the feedback gain m = -0.1. Then we have the critical frequency  $\omega_0^* = 0.3614$  and the bifurcation point  $\tau_0^* = 5.2428$ . As shown in Fig. 9 we can see that the stability domain becomes smaller when the feedback gain m increases. Thus, the proposed system is controlled, but the corresponding integer order one is not controlled under the determined system parameters and feedback gain as shown in Figs. 10–11. Besides, according to Figs. 12–13 we observe that the effects of bifurcation control are much better when feedback gain decreases.



Figure 9. Illustration of bifurcation point  $\tau_0^*$  versus gain *m* for controlled system (5.2) with  $q_1 = 0.92$  and  $q_2 = 0.95$ . The bifurcation points are gradually getting smaller and smaller as feedback gain increases, which shows that the Hopf bifurcation occurs in advance.

Figure 10. The positive equilibrium point of controlled system (5.2) is unstable with initial value (0.5,0.5), m = -0.1,  $\tau = 5.24$ ,  $q_1 = 1$  and  $q_2 = 1$ .

## 6. Conclusion

We have discussed the dynamic characteristics of the delayed fractional Leslie-Gower model with Holling type II functional response. We have firstly studied the stability



Figure 11. Controlled system (5.2) converges to the positive equilibrium point with initial value (0.5,0.5), m = -0.1,  $\tau = 5.24$ ,  $q_1 = 0.92$  and  $q_2 = 0.95$ .



Figure 12. The effect of bifurcation control for system (5.2) becomes better as feedback gain decreases with initial value (0.5,0.5),  $q_1 = 0.92$ ,  $q_2 = 0.95$ ,  $\tau = 5.24$  and feedback gain m = -0.2, m = -0.15, m = -0.08.



Figure 13. The effect of bifurcation control for system (5.2) becomes better as feedback gain decreases with initial value (0.5,0.5),  $q_1 = 0.92$ ,  $q_2 = 0.95$ ,  $\tau = 5.24$  and feedback gain m = -0.2, m = -0.15, m = -0.08.

of the proposed Leslie-Gower model, and some sufficient conditions for the existence of Hopf bifurcation are derived by means of analyzing the corresponding characteristic equation. The results obtained show that time delay has a significant impact on the stability. Furthermore, we have found that the onset of bifurcation can be put off as the other order decreases when one order is fixed. Secondly, a delayed feedback controller has been devised for system (2.1) to defer the Hopf bifurcation. We find that the feedback gain is excellent for controlling the dynamical behaviors. In addition, we can easily see that the obtained conditions of the control method proposed in this paper are accurate, simple and apt to be verified. Some numerical examples have been provided to confirm the validity and effectiveness of the developed results.

# Appendix A

Computation of the expressions  $\mathcal{M}_1$ ,  $\mathcal{M}_2$ ,  $\mathcal{N}_1$  and  $\mathcal{N}_2$  in Eq. (3.12)

$$\mathcal{M}_1 = \omega_0 \left[ \left( -\alpha_2 \beta_1 + \alpha_3 \beta_2 \right) + \left( \omega^{q_1 + q_2} \cos \frac{(q_1 + q_2)}{2} \pi - \alpha_1 \omega^{q_2} \cos \frac{q_2}{2} \pi \right) \right] \sin \omega_0 \tau_0$$

$$\begin{split} &+\omega_{0}\left[\omega^{q_{1}+q_{2}}\sin\frac{(q_{1}+q_{2})}{2}\pi-\alpha_{1}\omega^{q_{2}}\sin\frac{q_{2}}{2}\pi\right]\cos\omega_{0}\tau_{0},\\ \mathcal{M}_{2}=\omega_{0}\left[(-\alpha_{2}\beta_{1}+\alpha_{3}\beta_{2})-\left(\omega^{q_{1}+q_{2}}\cos\frac{(q_{1}+q_{2})}{2}\pi-\alpha_{1}\omega^{q_{2}}\cos\frac{q_{2}}{2}\pi\right)\right]\cos\omega_{0}\tau_{0}\\ &+\omega_{0}\left[\omega^{q_{1}+q_{2}}\sin\frac{(q_{1}+q_{2})}{2}\pi-\alpha_{1}\omega^{q_{2}}\sin\frac{q_{2}}{2}\pi\right]\sin\omega_{0}\tau_{0},\\ \mathcal{N}_{1}=\left[(q_{1}+q_{2})\omega^{q_{1}+q_{2}-1}\cos\frac{(q_{1}+q_{2}-1)}{2}\pi\\ &-\alpha_{1}q_{2}\omega^{q_{2}-1}\cos\frac{(q_{2}-1)}{2}\pi+\omega^{q_{1}+q_{2}}\cos\frac{(q_{1}+q_{2})}{2}\pi\right]\cos\omega_{0}\tau_{0}\\ &+\left[-\alpha_{1}\omega^{q_{2}}\cos\frac{q_{2}}{2}\pi+\tau(\alpha_{2}\beta_{1}-\alpha_{3}\beta_{2})\right]\cos\omega_{0}\tau_{0}-\beta_{2}q_{1}\omega^{q_{1}-1}\cos\frac{(q_{1}-1)}{2}\pi\\ &-\left[(q_{1}+q_{2})\omega^{q_{1}+q_{2}-1}\sin\frac{(q_{1}+q_{2}-1)}{2}\pi-\alpha_{1}q_{2}\omega^{q_{2}-1}\sin\frac{(q_{2}-1)}{2}\pi\right]\sin\omega_{0}\tau_{0}\\ &-\left[\omega^{q_{1}+q_{2}}\sin\frac{(q_{1}+q_{2})}{2}\pi-\alpha_{1}\omega^{q_{2}}\cos\frac{q_{2}}{2}\pi\right]\sin\omega_{0}\tau_{0}-\alpha_{3}q_{2}\omega^{q_{2}-1}\cos\frac{(q_{2}-1)}{2}\pi,\\ \mathcal{N}_{2}=\left[(q_{1}+q_{2})\omega^{q_{1}+q_{2}-1}\cos\frac{(q_{1}+q_{2}-1)}{2}\pi+\omega^{q_{1}+q_{2}}\cos\frac{(q_{1}+q_{2})}{2}\pi\right]\sin\omega_{0}\tau_{0}\\ &+\left[-\alpha_{1}\omega^{q_{2}}\cos\frac{q_{2}}{2}\pi+\tau(-\alpha_{2}\beta_{1}+\alpha_{3}\beta_{2})\right]\sin\omega_{0}\tau_{0}-\beta_{2}q_{1}\omega^{q_{1}-1}\sin\frac{(q_{1}-1)}{2}\pi\\ &+\left[(q_{1}+q_{2})\omega^{q_{1}+q_{2}-1}\sin\frac{(q_{1}+q_{2}-1)}{2}\pi-\alpha_{1}q_{2}\omega^{q_{2}-1}\sin\frac{(q_{2}-1)}{2}\pi\right]\sin\omega_{0}\tau_{0}\\ &+\left[\omega^{q_{1}+q_{2}}\sin\frac{(q_{1}+q_{2})}{2}\pi-\alpha_{1}\omega^{q_{2}}\sin\frac{q_{2}}{2}\pi\right]\sin\omega_{0}\tau_{0}-\alpha_{3}q_{2}\omega^{q_{2}-1}\sin\frac{(q_{2}-1)}{2}\pi.\\ \end{split}$$

# Appendix B

Computation of the expressions  $\mathcal{X}_1$ ,  $\mathcal{X}_2$ ,  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  in Eq. (4.13)

$$\begin{aligned} \mathcal{X}_{1} &= w_{0}^{*} \Biggl\{ \Biggl[ \left( -m\alpha_{3} - \alpha_{2}\beta_{1} + \alpha_{3}\beta_{2} \right) + \left( w^{q_{1}+q_{2}}\cos\left(\frac{q_{1}+q_{2}}{2}\right)\pi \right. \\ &- mw^{q_{1}}\cos\frac{q_{1}}{2}\pi - \alpha_{1}w^{q_{2}}\cos\frac{q_{2}}{2}\pi + m\alpha_{1} \Biggr) \Biggr] \sin w_{0}^{*}\tau_{0}^{*} \\ &+ \Biggl[ w^{q_{1}+q_{2}}\sin\left(\frac{q_{1}+q_{2}}{2}\right)\pi - mw^{q_{1}}\sin\frac{q_{1}}{2}\pi - \alpha_{1}w^{q_{2}}\sin\frac{q_{2}}{2}\pi \Biggr] \Biggr\}, \\ \mathcal{X}_{2} &= w_{0}^{*} \Biggl\{ \Biggl[ \left( -m\alpha_{3} - \alpha_{2}\beta_{1} + \alpha_{3}\beta_{2} \right) - \left( w^{q_{1}+q_{2}}\cos\left(\frac{q_{1}+q_{2}}{2}\right)\pi \right] \Biggr\}, \end{aligned}$$

$$\begin{split} &-mw^{q_1}\cos\frac{q_1}{2}\pi-\alpha_1w^{q_2}\cos\frac{q_2}{2}\pi+m\alpha_1\bigg)\bigg]\cos w_0{}^*\tau_0{}^*\\ &+\bigg[w^{q_1+q_2}\sin\left(\frac{q_1+q_2}{2}\right)\pi-mw^{q_1}\sin\frac{q_1}{2}\pi-\alpha_1w^{q_2}\sin\frac{q_2}{2}\pi\bigg]\sin w_0{}^*\tau_0{}^*\bigg\},\\ \mathcal{Y}_1&=\bigg[\left(q_1+q_2\right)w^{q_1+q_2-1}\cos\left(\frac{q_1+q_2-1}{2}\right)\pi-mq_1w^{q_1-1}\cos\left(\frac{q_1-1}{2}\right)\pi\\ &-\alpha_1q_2w^{q_2-1}\cos\left(\frac{q_2-1}{2}\right)\pi+w^{q_1+q_2}\cos\left(\frac{q_1+q_2}{2}\right)\pi-mw^{q_1}\cos\frac{q_1}{2}\pi\\ &-\alpha_1w^{q_2}\cos\frac{q_2}{2}\pi+m\alpha_1+\tau\left(m\alpha_3+\alpha_2\beta_1-\alpha_3\beta_2\right)\bigg]\cos w_0{}^*\tau_0{}^*\\ &-\bigg[\left(q_1+q_2\right)w^{q_1+q_2-1}\sin\left(\frac{q_1+q_2-1}{2}\right)\pi-mq_1w^{q_1-1}\sin\left(\frac{q_1-1}{2}\right)\pi\\ &-\alpha_1q_2w^{q_2-1}\sin\left(\frac{q_2-1}{2}\right)\pi+w^{q_1+q_2}\sin\left(\frac{q_1+q_2}{2}\right)\pi\\ &-mw^{q_1}\sin\frac{q_1}{2}\pi-\alpha_1w^{q_2}\sin\frac{q_2}{2}\pi\bigg]\sin w_0{}^*\tau_0{}^*\\ &+\left(m-\beta_2\right)q_1w^{q_1-1}\cos\left(\frac{q_1-1}{2}\right)\pi-mq_1w^{q_1-1}\cos\left(\frac{q_1-1}{2}\right)\pi\\ &-\alpha_1q_2w^{q_2-1}\cos\left(\frac{q_2-1}{2}\right)\pi+w^{q_1+q_2}\cos\left(\frac{q_1+q_2}{2}\right)\pi-mw^{q_1}\cos\frac{q_1}{2}\pi\\ &-\alpha_1w^{q_2}\cos\frac{q_2}{2}\pi+m\alpha_1+\tau\left(-m\alpha_3-\alpha_2\beta_1+\alpha_3\beta_2\right)\bigg]\sin w_0{}^*\tau_0{}^*\\ &+\bigg[\left(q_1+q_2\right)w^{q_1+q_2-1}\sin\left(\frac{q_1+q_2-1}{2}\right)\pi-mq_1w^{q_1-1}\sin\left(\frac{q_1-1}{2}\right)\pi\\ &-\alpha_1q_2w^{q_2-1}\sin\left(\frac{q_2-1}{2}\right)\pi+w^{q_1+q_2}\sin\left(\frac{q_1+q_2}{2}\right)\pi\\ &-\alpha_1q_2w^{q_2-1}\sin\left(\frac{q_2-1}{2}\right)\pi+w^{q_1+q_2}\sin\left(\frac{q_1+q_2}{2}\right)\pi\\ &-\alpha_1q_2w^{q_2-1}\sin\left(\frac{q_2-1}{2}\right)\pi+w^{q_1+q_2}\sin\left(\frac{q_1+q_2}{2}\right)\pi\\ &-mw^{q_1}\sin\frac{q_1}{2}\pi-\alpha_1w^{q_2}\sin\frac{q_2}{2}\pi\bigg]\cos w^*\tau_0{}^*\\ &+\bigg(m-\beta_2)q_1w^{q_1-1}\cos\left(\frac{q_1-1}{2}\right)\pi-\alpha_3q_2w^{q_2-1}\cos\left(\frac{q_2-1}{2}\right)\pi. \end{split}$$

# References

- [1] J. F. Andrews, A mathematical model for the continuous culture of microorganisms utilizing inhibitory substrates, Biotechnol. Bioeng., 1968, 10, 707–723.
- [2] M. A. Aziz-Alaoui and M. Daher Okiye, Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-Type II schemes, Appl. Math. Lett., 2003, 16, 1069–1075.

- [3] E. H. Abed and J. Fu, Local feedback stabilization and bifurcation control: II. Stationary bifurcation, Syst. Control Lett., 1987, 8, 467–473.
- [4] J. Cao and R. Yuan, Bifurcation analysis in a modified Lesile-Gower model with Holling type II functional response and delay, Nonlinear Dyn., 2016, 84, 1341–1352.
- [5] G. Chen, J.L. Moiola and H. Wang, Bifurcation control: theories, methods and applications, Int. J. Bifurcat. Chaos, 2000, 10, 511–548.
- [6] W. Deng, C. Li and J. Lü, Stability analysis of linear fractional differential system with multiple time delays, Nonlinear Dyn., 2007, 48, 409–416.
- [7] Y. Fan, X. Huang, Z. Wang and Y. Li, Improved quasi-synchronization criteria for delayed fractional-order memristor-based neural networks via linear feedback control, Neurocomputing, 2018, 306, 68–79.
- [8] Y. Fan, X. Huang, Z. Wang and Y. Li, Global dissipativity and quasisynchronization of asynchronous updating fractional-order memristor-based neural networks via interval matrix method, J. Franklin Inst., 2018, 355, 5998– 6025.
- [9] Y. Fan, X. Huang, Z. Wang and Y. Li, Nonlinear dynamics and chaos in a simplified memristor-based fractional-order neural network with discontinuous memductance function, Nonlinear Dynam., 2018, 93, 611–627.
- [10] X. Huang, Y. Fan, J. Jia, Z. Wang and Y. Li, Quasi-synchronization of fractional-order memristor-based neural networks with parameter mismatches, IET Control Theory Appl., 2017, 11, 2317–2327.
- [11] C. Huang, J. Cao and M. Xiao, Hybrid control on bifurcation for a delayed fractional gene regulatory network, Chaos Solitons Fract., 2016, 87, 19–29.
- [12] J. Jia, X. Huang, Y. Li, J. Cao and A. Alsaedi, Global stabilization of fractionalorder memristor-based neural networks with time delay, IEEE Trans. Neural Netw. Learn. Syst., 2020, 31, 997–1009.
- [13] A. Korobeinikov, A Lyapunov function for Leslie-Gower predator-prey models, Appl. Math. Lett., 2001, 14, 697–699.
- [14] P. H. Leslie, Some further notes on the use of matrices in population mathematics, Biometrika, 1948, 35, 213–245.
- [15] P. H. Leslie, A stochastic model for studying the properties of certain biological systems by numerical methods, Biometrika, 1958, 45, 16–31.
- [16] P. H. Leslie and J.C. Gower, The properties of a stochastic model for the predator-prey type of interaction between two species, Biometrika, 1960, 47, 219–234.
- [17] Y. Li and D. Xiao, Bifurcations of a predator-prey system of Holling and Leslie types, Chaos Soliton. Fract., 2007, 34, 606–620.
- [18] A. F. Nindjin, et al., Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay, Nonlinear Anal. RWA., 2006, 7, 1104–1118.
- [19] I. Petras, Fractional-order nonlinear systems: modeling, analysis and simulation, Springer-Verlag, New York, 2011.

- [20] F. Padula, S. Alcantara, R. Vilanova and A. Visioli,  $H_{\infty}$  control of fractional linear systems, Automatica, 2013, 49, 2276–2280.
- [21] Y. Pan, H. Yu and M. Er, Adaptive neural pd control with semiglobal asymptotic stabilization guarantee, IEEE Trans. Neural Netw. Learn Syst., 2014, 25, 2264– 2274.
- [22] Y. Pan, Y. Liu, B. Xu and H. Yu, Hybrid feedback feedforward: an efficient design of adaptive neural network control, Neural Networks, 2016, 76, 122–134.
- [23] Y. Pan and H. Yu, Composite learning from adaptive dynamic surface control, IEEE Trans. Autom. Control, 2016, 61, 2603–2609.
- [24] Y. Song, S. Yuan and J. Zhang, Bifurcation analysis in the delayed Leslie-Gower predator-prey system, Appl. Math. Model., 2009, 33, 4049–4061.
- [25] M. Shi and Z. Wang, Stability and Hopf bifurcation control of a fractional-order small world network model, Sci. China Phys. Mech., 2013, 43, 467–477.
- [26] P. Sopasakis and H. Sarimveis, Stabilising model predictive control for discretetime fractional-order systems, Automatica, 2017, 75, 24–31.
- [27] R. Yafia, F. Adnani and H. Alaoui, Limit cycle and numerical simulations for small and large delays in a predator-prey model with modified Leslie-Gower and Holling-type II schemes, Nonlinear Anal. RWA., 2008, 9, 2055–2067.
- [28] S. Yuan and Y. Song, Stability and Hopf bifurcations in a delayed Leslie-Gower predator-prey system, J. Math. Anal. Appl., 2009, 355, 82–100.
- [29] P. Yu and G. Chen, Hopf bifurcation control using nonlinear feedback with polynomial functions, Int. J. Bifurcat. Chaos, 2004, 14, 1683–1704.
- [30] L. Zhang and Y. Yang, Finite time impulsive synchronization of fractional order memristive BAM neural networks, Neurocomputing, 2020, 384, 213–224.
- [31] L. Zhang and Y. Yang, Optimal quasi-synchronization of fractional-order memristive neural networks with PSOA, Neural Comput. Appl., 2020, 32, 9667– 9682.
- [32] A. A. Zamani, S. Tavakoli and S. Etedali, Fractional order PID control design for semi-active control of smart base-isolated structures: A multi-objective cuckoo search approach, ISA Trans., 2017, 67, 222–232.