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ANTI-PERIODIC SYNCHRONIZATION OF CLIFFORD-VALUED NEUTRAL-TYPE CELLULAR NEURAL NETWORKS WITH *D* OPERATOR*

Jin Gao[1] and Lihua Dai[2,†]

Abstract This paper explores a class of delayed Clifford-valued neutral-type cellular neural networks with D operator. Considering that the multiplication of Clifford algebras does not satisfy the commutativity, by applying the non-decomposition method, Krasnoselskii's Fixed Point Theorem and the proof by contradiction, we obtain several sufficient conditions for the existence and global exponential synchronization of anti-periodic solutions for Clifford-valued neutral-type cellular neural networks with D operator. Finally, we give one example to illustrate the feasibility and effectiveness of the main results.

Keywords Clifford algebra, synchronization, anti-periodic solutions, cellular neural networks, D operator.

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1. Introduction

The neural network model, which is a cellular neural network model, has attracted the interest of many scholars and has been extensively researched in the past decades. The dynamics for different types of cellular neural networks have been discussed by many scholars, such as the existence, periodic properties, antiperiodic properties and stability, and so on (see [7, 18, 27, 28, 31, 33, 40, 43]). In recent years, the existence and stability of periodic (and anti-periodic) solutions for cellular neural networks have been discussed by some authors (see [13, 19, 20, 34, 45]).

Due to the limit for the switching speed of neurons and amplifiers, time-delay factors are generated in the implementation of neural networks and are inevitable. Because of this fact, in many practical applications for delayed neural networks model, neutral-type neural networks can be described as non-operator-based neutral neural networks and D-operator-based neutral neural networks. However, through

†The corresponding author.
[1]School of Information, Yunnan Communications Vocational and Technical College, 2056 Wanqing Road Chenggong University Town, 650500 Kunming, China
[2]School of Mathematics and Statistics, Southwest University, 2 Tiansheng Road, Beibei District, 400715 Chongqing, China
*This work is supported by the Science Research Fund of Education Department of Yunnan Province of China(No. 2022J0986) and Youth academic and technical leader of Puer College(No. QNRC21-01).
Email: gj123jh@163.com(J. Gao), hlm2136816@163.com(L. Dai)

the comparison between the non-operator-based neutral model and D-operator-based neutral model, we can get that the neutral model with D operator has more general significance than non-operator-based ones, and there are many good results for neutral-type neural networks with D operator (see [1–3, 8, 10, 14–17, 35, 36, 38, 39, 44]).

A new multidimensional neural networks model, which is Clifford-valued neural networks, represents a generalization of the real-valued, complex-valued, and quaternion-valued neural networks. We know that the multiplication of Clifford algebras does not satisfy the commutativity, it does not need to decompose the Clifford-valued neural networks into real-valued neural networks, thus reducing the complexity of the calculation. Recently, the theoretical and applied research of the Clifford-valued neural networks model has become a hot and new topic. A great number of research results have been made by many scholars (see [4, 9, 21–23, 29, 30, 42]).

As well known, anti-periodic synchronization, which is an important dynamic property for differential equations, has played a key role in network control. In recent years, the synchronization for neural networks model has become a new topic and is received many scholars' favor. There's been a lot of research such as exponential synchronization, almost periodic synchronization, anti-periodic synchronization, and so on (see [5, 11, 12, 24–26, 32, 37, 41]).

With the inspiration from the previous research, to fill the gap in the research field of delayed Clifford-valued neutral-type cellular neural networks with D operator, the work of this article comes from three main motivations. (1) Recently, neutral-type neural networks with D operator have been discussed by some authors, but there is little research about Clifford-valued neutral-type cellular neural networks with D operator. (2) Some authors have discussed the anti-periodic synchronization for neural networks with delays. However, there has been no paper about the anti-periodic synchronization for Clifford-valued cellular neural networks. (3) Up to now, in practical applications for neural networks, there has been no paper about the anti-periodic synchronization for delayed Clifford-valued neutral-type cellular neural networks with D operator. Therefore, we will study the anti-periodic synchronization of delayed Clifford-valued neutral-type cellular neural networks with D operator in this paper by using the non-decomposition method, Krasnoselskii's Fixed Point Theorem, and the proof by contradiction.

Compared with the previous kinds of literature, the main contributions of this paper are listed as follows. (1) Firstly, to the best of our knowledge, the introduction of the delayed Clifford-valued neutral-type cellular neural networks with D operator. (2) Secondly, this is the first time to study the anti-periodic synchronization for Clifford-valued neutral-type cellular neural networks with D operator. (3) Thirdly, to avoid the complexity of the calculation, without separating the Clifford-valued neural networks into real-valued neural networks. (4) Fourthly, our method in this paper can be used to discuss the synchronization for other types of Clifford-valued systems with D operator. (5) Finally, we give one example to verify the effectiveness of the conclusion.

Notations: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, +\infty)$ denotes the set of non-negative real numbers, $\mathcal{A}$ denotes the set of Clifford numbers, $\mathcal{A}^n$ denotes the set of n dimensional Clifford numbers, $\|\cdot\|_{\mathcal{A}}$ represents the vector Clifford norm. For $x = \sum\limits_A x^A e_A \in \mathcal{A}$, we define $\|x\|_{\mathcal{A}} = \max\limits_A \Big\{ \mid x^A \mid \Big\}$ and for $x = (x_1, x_2, \cdots, x_n)^T \in$

$\mathcal{A}^n$, we define $\|x\|_{\mathcal{A}^n} = \max\limits_{1\le i\le n}\Big\{\|x_i\|_{\mathcal{A}}\Big\}$. Moreover,

$$
\begin{aligned}
r^- &= \min_{1\le i\le n}\inf_{[0,\omega]} r_i(t), r^+ = \max_{1\le i\le n}\sup_{[0,\omega]} r_i(t),\\
c^- &= \min_{1\le i\le n}\inf_{[0,\omega]} c_i(t), c^+ = \max_{1\le i\le n}\sup_{[0,\omega]} c_i(t),\\
a^+ &= \max_{1\le i,j\le n}\|a_{ij}(t)\|_{\mathcal{A}}, I = \max_{1\le i\le n}\|I_i(t)\|_{\mathcal{A}},\\
b^+ &= \max_{1\le i,j\le n}\|b_{ij}(t)\|_{\mathcal{A}}, d^+ = \max_{1\le i,j\le n}\|d_{ij}(t)\|_{\mathcal{A}}.
\end{aligned}
$$

Above all, we will study the solutions of Clifford-valued neutral-type cellular neural networks with delays and D operator:

$$
\begin{aligned}
&[x_i(t) - r_i(t)x_i(t-\tau_i(t))]'\\
&= -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j\big(x_j(t)\big) + \sum_{j=1}^{n} b_{ij}(t)g_j\big(x_j(t-\gamma_{ij}(t))\big)\\
&\quad + \sum_{j=1}^{n} d_{ij}(t)\int_0^{+\infty} k_{ij}(\theta)h_j\big(x_j(t-\theta)\big)d\theta + I_i(t),
\end{aligned}
\tag{1.1}
$$

where $i = 1, 2, \cdots, n$, $x_i(t) \in \mathcal{A}$ is the state vector of the ith unit at time t, $c_i(t) > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}, b_{ij}, d_{ij} \in \mathcal{A}$ denote the strength of connectivity, the activation functions $f_j, g_j, h_j \in \mathcal{A}$ show how the jth neuron reacts to input, delay factors satisfy that $\tau_i(t), \gamma_{ij}(t) \in \mathbb{R}_+$, $k_{ij}(\theta)$ corresponds to the transmission delay kernel, $I_i \in \mathcal{A}$ denotes the ith component of an external input source introduced from outside the network to the unit i at time t, $r_i(t)$ is a continuous function with respect to t.

The initial value of system (1.1) is the following

$$
x_i(s) = \varphi_i(s), \quad s \in [-\eta, 0], \tag{1.2}
$$

where $\varphi_i(s) \in C\Big([-\eta, 0], \mathcal{A}\Big)$, $i = 1, 2, \cdots, n$, $\eta = \max\Big\{\tau, \gamma\Big\}$, $\tau = \max\limits_{1\le i\le n}\Big\{\sup\limits_{t\in[0,\omega]}\tau_i(t)\Big\}$, $\gamma = \max\limits_{1\le i,j\le n}\Big\{\sup\limits_{t\in[0,\omega]}\gamma_{ij}(t)\Big\}$.

This paper is organized as follows: In Section 2, we introduce some definitions and preliminary lemmas. In Section 3, we establish some sufficient conditions for the existence anti-periodic solutions of system (1.1), global exponential synchronization for system (1.1) and system (3.5). In Section 4, some numerical examples are provided to verify the effectiveness of the theoretical results. Finally, we draw a conclusion in Section 5.

2. Preliminaries

The real Clifford algebra over $\mathbb{R}^m$ is defined as

$$
\mathcal{A} = \Big\{\sum_{A\in\{1,2,\cdots,m\}} u^A e_A, u^A \in \mathbb{R}\Big\},
$$

where $e_A = e_{h_1} \cdots e_{h_\nu}$ with $A = \{h_1 \cdots h_\nu\}$, $1 \leq h_1 < h_2 < \cdots < h_\nu \leq m$ and $1 \leq \nu \leq m$. Moreover, $e_\emptyset = e_0 = 1$ and $e_h, h = 1, 2, \cdots, m$ are said to be Clifford generators and satisfy $e_p^2 = -1$, $p = 1, 2, \cdots, m$, and $e_p e_q + e_q e_p = 0$, $p \neq q, p, q = 1, 2, \cdots, m$. Let $Q = \{\emptyset, 1, 2, ..., A, ..., 12 \cdots m\}$, then it is easy to see that $\mathcal{A} = \{\sum_A u^A e_A, u^A \in \mathbb{R}\}$, where $\sum_A$ is short for $\sum_{A \in Q}$ and dim $\mathcal{A} = 2^m$.

To study the existence of $\frac{\omega}{2}$-anti-periodic solution of system (1.1), we need the following assumptions:

(H_1) For $i, j = 1, 2, \cdots, n$, $\omega > 0$, $r_i, c_i, \tau_i, \gamma_{ij} : \mathbb{R} \to \mathbb{R}_+$ are $\frac{\omega}{2}$-periodic, a_{ij}, b_{ij}, d_{ij}, $f_j, g_j, h_j, I_i : \mathbb{R} \to \mathcal{A}$, and $a_{ij}(t + \frac{\omega}{2}) f_j(u) = -a_{ij}(t) f_j(-u)$, $b_{ij}(t + \frac{\omega}{2}) g_j(u) = -b_{ij}(t) g_j(-u)$, $d_{ij}(t + \frac{\omega}{2}) h_j(u) = -d_{ij}(t) h_j(-u)$,$I_i(t + \frac{\omega}{2}) = -I_i(t)$.

(H_2) For $j = 1, 2, \cdots, n$, there exist positive constants L_f, L_g, L_h such that

$$\begin{aligned} \|f_j(u) - f_j(v)\|_{\mathcal{A}} &\leq L_f \|u - v\|_{\mathcal{A}}, \\ \|g_j(u) - g_j(v)\|_{\mathcal{A}} &\leq L_g \|u - v\|_{\mathcal{A}}, \\ \|h_j(u) - h_j(v)\|_{\mathcal{A}} &\leq L_h \|u - v\|_{\mathcal{A}}. \end{aligned}$$

(H_3) There exist positive constants r^+, ζ, M_f, M_g, M_h such that

$$\begin{aligned} &0 < r^+ < 1, \ \int_0^{+\infty} k_{ij}(\theta) d\theta \leq \zeta, \\ &\|f_j(u)\|_{\mathcal{A}} \leq M_f, \|g_j(u)\|_{\mathcal{A}} \leq M_g, \|h_j(u)\|_{\mathcal{A}} \leq M_h. \end{aligned}$$

Lemma 2.1 (Krasnoselskii's Fixed Point Theorem [6]). *Let E be a closed convex and nonempty subset of a Banach space $\mathbb{X}$. Let Φ, Ψ be the operators such that*

- (i) $\Phi x + \Psi y \in E$ *for every pair $x, y \in E$;*
- (ii) Φ *is compact and continuous;*
- (iii) Ψ *is a contraction mapping.*

Then there exists $x \in E$ such that $\Phi x + \Psi x = x$.

Definition 2.1. A continuous function $x = \big(x_1, x_2 \cdots, x_n\big)^T : [0, +\infty] \to \mathcal{A}^n$ is said to be a solution of system (1.1), if

- (i) $x_i(s) = \varphi_i(s)$, for $s \in [-\eta, 0]$, $\varphi_i \in C([-\eta, 0], \mathcal{A})$, $i = 1, 2, \cdots, n$;
- (ii) $x(t)$ satisfies system (1.1) for $t \geq 0$.

Definition 2.2. A solution x of system (1.1) is said to be $\frac{\omega}{2}$-anti-periodic solution of system (1.1), if there exists $\omega > 0$ such that

$$x(t + \frac{\omega}{2}) = -x(t).$$

3. Main results

In this section, we will investigate the existence and global exponential synchronization of anti-periodic solutions of delayed Clifford-valued neutral-type cellular neural networks with D operator (1.1), based on the non-decomposition method, Krasnoselskii's Fixed Point Theorem, the proof by contradiction.

Denote

$$\mathbb{X} = \left\{ x \in C\Big([0, \omega], \mathcal{A}^n\Big) : x(t + \frac{\omega}{2}) = -x(t), t \in \mathbb{R} \right\}$$

be a Banach spaces equipped with the norm

$$\|x\|_{\mathbb{X}} = \max_{t \in [0,\omega]} \|x(t)\|_{\mathcal{A}^n}.$$

Let $E \subset \mathbb{X}$ is a closed subset, $E = \Big\{ x : x(t) \in C(\mathbb{R}, \mathcal{A}^n), x(t+\frac{\omega}{2}) = -x(t), \|x\|_{\mathbb{X}} \leq \xi \Big\}$, where

$$\xi := \frac{1}{c^-}\Big[c^- r^+ + c^+ r^+ + \sum_{j=1}^n a^+ L_f + \sum_{j=1}^n b^+ L_g + \sum_{j=1}^n d^+ L_h \zeta \Big] \mid x_i^A \mid_\infty + \frac{1}{c^-}\Big[\sum_{j=1}^n a^+ M_f + \sum_{j=1}^n b^+ M_g + \sum_{j=1}^n d^+ M_h \zeta + I \Big],$$

and

$$\mid x_i^A \mid_\infty = \max_{t \in [0,\omega]} \Big\{ \mid x_i^A(t) \mid \Big\}.$$

Theorem 3.1. *Assume that assumptions* (H_1)-(H_3) *hold. Then system* (1.1) *has at least an* $\frac{\omega}{2}$*-anti-periodic solution.*

Proof. Let $u_i(t) = x_i(t) - r_i(t)x_i(t - \tau_i(t)) \in \mathcal{A}$, then $x_i(t) = u_i(t) + r_i(t)x_i(t - \tau_i(t))$ and system (1.1) can be described as following differential equations

$$\begin{aligned} u_i'(t) = & -c_i(t)u_i(t) - c_i(t)r_i(t)x_i(t - \tau_i(t)) + \sum_{j=1}^n a_{ij}(t) f_j\big(x_j(t)\big) \\ & + \sum_{j=1}^n b_{ij}(t) g_j\big(x_j(t - \gamma_{ij}(t))\big) + \sum_{j=1}^n d_{ij}(t) \\ & \times \int_0^{+\infty} k_{ij}(\theta) h_j\big(x_j(t - \theta)\big) d\theta + I_i(t), \end{aligned} \tag{3.1}$$

where $i = 1, 2, \cdots, n$.

It is well known that an $\frac{\omega}{2}$-anti-periodic solution of system (3.1) is equivalent to find an $\frac{\omega}{2}$-anti-periodic solution of the integral equation

$$\begin{aligned} u_i(t) = & \int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^s c_i(\mu)d\mu}}{1 + e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \Big[- c_i(s)r_i(s)x_i(s - \tau_i(s)) \\ & + \sum_{j=1}^n a_{ij}(s) f_j\big(x_j(s)\big) + \sum_{j=1}^n b_{ij}(s) g_j\big(x_j(s - \gamma_{ij}(s))\big) \\ & + \sum_{j=1}^n d_{ij}(s) \int_0^{+\infty} k_{ij}(\theta) h_j\big(x_j(s - \theta)\big) d\theta + I_i(s) \Big] ds, \end{aligned} \tag{3.2}$$

that is,

$$x_i(t) = r_i(t)x_i(t - \tau_i(t)) + \int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^s c_i(\mu)d\mu}}{1 + e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \Big[- c_i(s)$$

$$\times r_i(s)x_i(s-\tau_i(s)) + \sum_{j=1}^{n} a_{ij}(s) f_j\big(x_j(s)\big) + \sum_{j=1}^{n} b_{ij}(s)$$
$$\times g_j\big(x_j(s-\gamma_{ij}(s))\big) + \sum_{j=1}^{n} d_{ij}(s) \int_0^{+\infty} k_{ij}(\theta) h_j\big(x_j(s-\theta)\big) d\theta + I_i(s)\Big] ds,$$

where $i = 1, 2, \cdots, n$.

Let $E \subset \mathbb{X}$ is a closed subset, $E = \Big\{x : x(t) \in C(\mathbb{R}, \mathcal{A}^n), x(t+\frac{\omega}{2}) = -x(t), \|x\|_{\mathbb{X}} \le \xi\Big\}$, we define one mapping $T = \Phi + \Psi$ as follows

$$(Tx)(t) = \big((\Phi x + \Psi x)_1(t), \cdots, (\Phi x + \Psi x)_n(t)\big)^T,$$

where $(\Phi x + \Psi x)_i(t) \in \mathcal{A}$,

$$\begin{aligned}
(\Phi x)_i(t) = \int_t^{t+\frac{\omega}{2}} & -\frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \Big[- c_i(s) r_i(s) x_i(s-\tau_i(s)) \\
& + \sum_{j=1}^{n} a_{ij}(s) f_j\big(x_j(s)\big) + \sum_{j=1}^{n} b_{ij}(s) g_j\big(x_j(s-\gamma_{ij}(s))\big) \\
& + \sum_{j=1}^{n} d_{ij}(s) \int_0^{+\infty} k_{ij}(\theta) h_j\big(x_j(s-\theta)\big) d\theta + I_i(s)\Big] ds,
\end{aligned} \tag{3.3}$$

and

$$(\Psi x)_i(t) = r_i(t) x_i(t-\tau_i(t)). \tag{3.4}$$

Steep 1: For any $x \in E$ and $t \ge 0$, by (H_1), from (3.3) and (3.4), we have that

$$\begin{aligned}
(\Phi x)_i(t+\frac{\omega}{2}) = & \int_{t+\frac{\omega}{2}}^{t+\frac{\omega}{2}+\frac{\omega}{2}} -\frac{e^{\int_{t+\frac{\omega}{2}}^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \Big[- c_i(s) r_i(s) x_i(s-\tau_i(s)) \\
& + \sum_{j=1}^{n} a_{ij}(s) f_j\big(x_j(s)\big) + \sum_{j=1}^{n} b_{ij}(s) g_j\big(x_j(s-\gamma_{ij}(s))\big) \\
& + \sum_{j=1}^{n} d_{ij}(s) \int_0^{+\infty} k_{ij}(\theta) h_j\big(x_j(s-\theta)\big) d\theta + I_i(s)\Big] ds \\
= & \int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^{\nu} c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \Big[- c_i\Big(\nu+\frac{\omega}{2}\Big) r_i\Big(\nu+\frac{\omega}{2}\Big) \\
& \times x_i\Big(\nu+\frac{\omega}{2}-\tau_i\Big(\nu+\frac{\omega}{2}\Big)\Big) + \sum_{j=1}^{n} a_{ij}\Big(\nu+\frac{\omega}{2}\Big) \\
& \times f_j\Big(x_j\Big(\nu+\frac{\omega}{2}\Big)\Big) + \sum_{j=1}^{n} b_{ij}(\nu+\frac{\omega}{2}) \\
& \times g_j\Big(x_j\Big(\nu+\frac{\omega}{2}-\gamma_{ij}\Big(\nu+\frac{\omega}{2}\Big)\Big)\Big) + \sum_{j=1}^{n} d_{ij}\Big(\nu+\frac{\omega}{2}\Big) \\
& \times h_j\Big(x_j\Big(\nu+\frac{\omega}{2}-\theta\Big)\Big) + I_i\Big(\nu+\frac{\omega}{2}\Big)\Big] d\nu
\end{aligned}$$

$$
\begin{aligned}
&= \int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^{\nu} c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \Big[c_i(\nu) r_i(\nu) x_i(\nu-\tau_i(\nu)) \\
&\quad -\sum_{j=1}^n a_{ij}(\nu) f_j\big(x_j(\nu)\big) - \sum_{j=1}^n b_{ij}(\nu) g_j\big(x_j(\nu-\gamma_{ij}(\nu))\big) \\
&\quad -\sum_{j=1}^n d_{ij}(\nu) \int_0^{+\infty} k_{ij}(\theta) h_j\big(x_j(\nu-\theta)\big) d\theta - I_i(\nu)\Big] d\nu \\
&= -(\Phi x)_i(t),
\end{aligned}
$$

and

$$
\begin{aligned}
(\Psi x)_i(t+\frac{\omega}{2}) &= r_i(t+\frac{\omega}{2}) x_i(t+\frac{\omega}{2}-\tau_i(t+\frac{\omega}{2})) \\
&= -r_i(t) x_i(t+\tau_i(t)) \\
&= -(\Psi x)_i(t),
\end{aligned}
$$

which show that $(Tx)(t)$ is $\frac{\omega}{2}$-anti-periodic.

Steep 2: We show that $\|\Phi x + \Psi x\|_{\mathbb{X}} \le \xi$. For any $x \in E$, $i = 1, 2, \cdots, n$, we have

$$
\begin{aligned}
&\|(\Phi x + \Psi x)(t)\|_{\mathcal{A}^n} = \max_{1\le i\le n} \Big\{ \|(\Phi x + \Psi x)_i(t)\|_{\mathcal{A}} \Big\} \\
&= \max_{1\le i\le n} \Bigg\{ \Bigg\| r_i(t) x_i(t-\tau_i(t)) + \int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^{s} c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \\
&\quad \times \Big[-c_i(s) r_i(s) x_i(s-\tau_i(s)) + \sum_{j=1}^n a_{iij}(s) f_j\big(x_j(s)\big) \\
&\quad + \sum_{j=1}^n b_{ij}(s) g_j\big(x_j(s-\gamma_{ij}(s))\big) + \sum_{j=1}^n d_{ij}(s) \\
&\quad \times \int_0^{+\infty} k_{ij}(\theta) h_j\big(x_j(s-\theta)\big) d\theta + I_i(s)\Big] ds \Bigg\|_{\mathcal{A}} \Bigg\} \\
&\le \max_{1\le i\le n} \Bigg\{ r^+ \|x_i(t-\tau_i(t))\|_{\mathcal{A}} + \int_t^{t+\frac{\omega}{2}} \frac{e^{\int_t^{s} c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} \\
&\quad \times \Big[c^+ r^+ \|x_i(s-\tau_p(s))\|_{\mathcal{A}} + \sum_{j=1}^n \|a_{ij}(s)\|_{\mathcal{A}} \|f_j\big(x_j(s)\big) \\
&\quad - f_j(0)\|_{\mathcal{A}} + \sum_{j=1}^n \|a_{ij}(s)\|_{\mathcal{A}} \|f_j(0)\|_{\mathcal{A}} + \sum_{j=1}^n \|b_{ij}(s)\|_{\mathcal{A}} \\
&\quad \times \|g_j\big(x_j(s-\gamma_{ij}(s))\big) - g_j(0)\|_{\mathcal{A}} + \sum_{j=1}^n \|b_{ij}(s)\|_{\mathcal{A}} \\
&\quad \times \|g_j(0)\|_{\mathcal{A}} + \sum_{j=1}^n \|d_{ij}(s)\|_{\mathcal{A}} \int_0^{+\infty} k_{ij}(\theta) \|h_j\big(x_j(s-\theta)\big) \\
&\quad - h_j(0)\|_{\mathcal{A}} d\theta + \sum_{j=1}^n \|d_{ij}(s)\|_{\mathcal{A}} \int_0^{+\infty} k_{ij}(\theta) \|h_j(0)\|_{\mathcal{A}} d\theta + \|I_i(s)\|_{\mathcal{A}} \Big] ds \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \max_{1\leq i\leq n}\bigg\{r^+\|x_i\|_{\mathcal{A}}+\int_t^{t+\frac{\omega}{2}}\frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}}c_i(\mu)d\mu}}c^+r^+\\
&\quad\times\bigg[\|x_i\|_{\mathcal{A}}+\sum_{j=1}^n a^+L_f\|x_j\|_{\mathcal{A}}+\sum_{j=1}^n b^+L_g\|x_j\|_{\mathcal{A}}\\
&\quad+\sum_{j=1}^n d^+L_h\|x_j\|_{\mathcal{A}}\int_0^{+\infty}k_{ij}(\theta)d\theta+\sum_{j=1}^n a^+M_f\\
&\quad+\sum_{j=1}^n b^+M_g+\sum_{j=1}^n d^+M_h\int_0^{+\infty}k_{ij}(\theta)d\theta+\|I_i(s)\|_{\mathcal{A}}\bigg]ds\bigg\}\\
&\leq \frac{1}{c^-}\bigg[c^-r^++c^+r^++\sum_{j=1}^n a^+L_f+\sum_{j=1}^n b^+L_g+\sum_{j=1}^n d^+L_h\zeta\bigg]\mid x_i^A\mid_\infty\\
&\quad+\frac{1}{c^-}\bigg[\sum_{j=1}^n a^+M_f+\sum_{j=1}^n b^+M_g+\sum_{j=1}^n d^+M_h\zeta+I\bigg]\\
&\leq \xi.
\end{aligned}
$$

Hence, we have $\Phi x+\Psi x\in E$.

Steep 3: We show Ψ is a contraction mapping. For any $x, x^*\in E$, $i=1,2,\cdots,n$, we have

$$
\begin{aligned}
\|\Psi x-\Psi x^*\|_{\mathbb{X}} &= \max_{t\in[0,\omega]}\|(\Psi x-\Psi x^*)(t)\|_{\mathcal{A}^n}\\
&=\max_{t\in[0,\omega]}\max_{1\leq i\leq n}\Big\{\|((\Psi x)_i-(\Psi x^*)_i)(t)\|_{\mathcal{A}}\Big\}\\
&=\max_{t\in[0,\omega]}\max_{1\leq i\leq n}\Big\{\|r_i(t)x_i(t-\tau_i(t))-r_i(t)x_i^*(t-\tau_i(t))\|_{\mathcal{A}}\Big\}\\
&\leq r^+\|x-x^*\|_{\mathbb{X}},
\end{aligned}
$$

where $r_i^+\in(0,1)$. Thus, Ψ is a contraction mapping.

Steep 4: We show Φ is compact and continuous. First, we shall show Φ is continuous. Let $\{x_k\}\in E$ be a convergent sequence of functions such that $x_k(t)\to x(t)$ as $k\to\infty$. Since E is closed, then $x\in E$. for $t\in[0,\frac{\omega}{2}]$, we have that

$$
\begin{aligned}
&\|(\Phi x_k)(t)-(\Phi x)(t)\|_{\mathbb{X}}=\max_{t\in[0,\omega]}\max_{1\leq i\leq n}\Big\{\|(\Phi x_k)_i(t)-(\Phi x)_i(t)\|_{\mathcal{A}}\Big\}\\
&=\max_{t\in[0,\omega]}\max_{1\leq i\leq n}\bigg\{\bigg\|\int_t^{t+\frac{\omega}{2}}-\frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}}c_i(\mu)d\mu}}\bigg[-c_i(s)r_i(s)(x_k)_i(s-\tau_i(s))\\
&\quad+\sum_{j=1}^n a_{ij}(s)f_j\big((x_k)_j(s)\big)+\sum_{j=1}^n b_{ij}(s)g_j\big((x_k)_j(s-\gamma_{ij}(s))\big)\\
&\quad+\sum_{j=1}^n d_{ij}(s)\int_0^{+\infty}k_{ij}(\theta)h_j\big((x_k)_j(s-\theta)\big)d\theta+I_i(s)\bigg]ds\\
&\quad-\int_t^{t+\frac{\omega}{2}}-\frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}}c_i(\mu)d\mu}}\bigg[-c_i(s)r_i(s)x_i(s-\tau_i(s))\\
&\quad+\sum_{j=1}^n a_{ij}(s)f_j\big(x_j(s)\big)+\sum_{j=1}^n b_{ij}(s)g_j\big(x_j(s-\gamma_{ij}(s))\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} d_{ij}(s)\int_{0}^{+\infty} k_{ij}(\theta)h_j\big(x_j(s-\theta)\big)d\theta + I_i(s)\bigg]ds\bigg\|_{\mathcal{A}}\bigg\}\\
&\le \max_{t\in[0,\omega]}\max_{1\le i\le n}\bigg\{\int_t^{t+\frac{\omega}{2}} \frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}}\bigg[c^+r^+\Big\|(x_k)_i(s-\tau_i(s))\\
&\quad -x_i(s-\tau_i(s))\Big\|_{\mathcal{A}} + \sum_{j=1}^{n} a^+L_f\Big\|(x_k)_j(s)-x_j(s)\Big\|_{\mathcal{A}}\\
&\quad +\sum_{j=1}^{n} b^+L_g\Big\|(x_k)_j(s-\gamma_{ij}(s))-x_j(s-\gamma_{ij}(s))\Big\|_{\mathcal{A}}\\
&\quad +\sum_{j=1}^{n} d^+L_h\zeta\Big\|(x_k)_j(s-\theta)-x_j(s-\theta)\Big\|_{\mathcal{A}}\bigg]ds\bigg\}\\
&\le \frac{1}{c^-}\bigg[c^+r^+ + \sum_{j=1}^{n} a^+L_f + \sum_{j=1}^{n} b^+L_g + \sum_{j=1}^{n} d^+L_h\zeta\bigg]\|x_k - x\|_{\mathbb{X}}.
\end{aligned}
$$

Since $x_k(t)\to x(t)$ as $k\to\infty$, thus $\lim_{k\to\infty}\|x_k - x\|_{\mathbb{X}} = 0$, that is

$$
\lim_{k\to\infty}\|(\Phi x_k)(t) - (\Phi x)(t)\|_{\mathbb{X}} = 0.
$$

Therefore Φx is continuous.

Second, we show Φ is compact. On the one hand, we prove the family of functions $\{\Phi x : x\in E\}$ is uniformly bounded. From (3.3), we can get

$$
\begin{aligned}
&\|\Phi x\|_{\mathbb{X}} = \max_{t\in[0,\omega]}\max_{1\le i\le n}\Big\{\|(\Phi x)_i(t)\|_{\mathcal{A}}\Big\}\\
&= \max_{t\in[0,\omega]}\max_{1\le i\le n}\bigg\{\bigg\|\int_t^{t+\frac{\omega}{2}} -\frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}}\bigg[-c_i(s)r_i(s)x_i(s-\tau_i(s))\\
&\quad +\sum_{j=1}^{n} a_{ij}(s)f_j\big(x_j(s)\big) + \sum_{j=1}^{n} b_{ij}(s)g_j\big(x_j(s-\gamma_{ij}(s))\big)\\
&\quad +\sum_{j=1}^{n} d_{ij}(s)\int_0^{+\infty} k_{ij}(\theta)h_j\big(x_j(s-\theta)\big)d\theta + I_i(s)\bigg]ds\bigg\|_{\mathcal{A}}\bigg\}\\
&\le \max_{t\in[0,\omega]}\max_{1\le i\le n}\bigg\{\int_t^{t+\frac{\omega}{2}} \frac{e^{\int_t^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}}\bigg[c^+r^+\|x_i(s-\tau_i(s))\|_{\mathcal{A}}\\
&\quad +\sum_{j=1}^{n} a^+L_f\|x_j(s)\|_{\mathcal{A}} + \sum_{j=1}^{n} b^+L_g\|x_j(s-\gamma_{ij}(s)))\|_{\mathcal{A}}\\
&\quad +\sum_{j=1}^{n} d^+L_h\zeta\|x_j(s-\theta)\|_{\mathcal{A}} + \sum_{j=1}^{n} a^+M_f + \sum_{j=1}^{n} b^+M_g\\
&\quad +\sum_{j=1}^{n} d^+M_h\zeta + \|I_i(s)\|_{\mathcal{A}}\bigg]ds\bigg\}\\
&\le \frac{1}{c^-}\bigg[c^+r^+ + \sum_{j=1}^{n} a^+L_f + \sum_{j=1}^{n} b^+L_g + \sum_{j=1}^{n} d^+L_h\zeta\bigg]\xi
\end{aligned}
$$

$$+\frac{1}{c^-}\Big[\sum_{j=1}^n a^+ M_f + \sum_{j=1}^n b^+ M_g + \sum_{j=1}^n d^+ M_h \zeta + I\Big].$$

On the other hand, we prove Φ is equicontinuous on $[0, \frac{\omega}{2}]$. For $t_1, t_2 \in [0, \frac{\omega}{2}]$, $x \in E$, we obtain

$$\begin{aligned}
&\|(\Phi x)(t_1) - (\Phi x)(t_2)\|_{\mathbb{X}} = \max_{1\le i\le n}\Big\{\|(\Phi x)_i(t_1) - (\Phi x)_i(t_2)\|_{\mathcal{A}}\Big\}\\
&= \max_{1\le i\le n}\Big\{\Big\|\int_{t_1}^{t_1+\frac{\omega}{2}} -\frac{e^{\int_{t_1}^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}}\Big[-c_i(s)r_i(s)x_i(s-\tau_i(s))\\
&\quad+\sum_{j=1}^n a_{ij}(s)f_j\big(x_j(s)\big) + \sum_{j=1}^n b_{ij}(s)g_j\big(x_j(s-\gamma_{ij}(s))\big)\\
&\quad+\sum_{j=1}^n d_{ij}(s)\int_0^{+\infty} k_{ij}(\theta)h_j\big(x_j(s-\theta)\big)d\theta + I_i(s)\Big]ds\\
&\quad-\int_{t_2}^{t_2+\frac{\omega}{2}} -\frac{e^{\int_{t_2}^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}}\Big[-c_i(s)r_i(s)x_i(s-\tau_i(s))\\
&\quad+\sum_{j=1}^n a_{ij}(s)f_j\big(x_j(s)\big) + \sum_{j=1}^n b_{ij}(s)g_j\big(x_j(s-\gamma_{ij}(s))\big)\\
&\quad+\sum_{j=1}^n d_{ij}(s)\int_0^{+\infty} k_{ij}(\theta)h_j\big(x_j(s-\theta)\big)d\theta + I_i(s)\Big]ds\Big\|_{\mathcal{A}}\Big\}\\
&\le \max_{1\le i\le n}\Big\{\int_0^{\frac{\omega}{2}}\Big(\frac{e^{\int_{t_1}^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}} - \frac{e^{\int_{t_2}^s c_i(\mu)d\mu}}{1+e^{\int_0^{\frac{\omega}{2}} c_i(\mu)d\mu}}\Big)\\
&\quad\times\Big[c^+r^+\Big\|x(s-\tau_i(s))\Big\|_{\mathcal{A}} + \sum_{j=1}^n a^+L_f\Big\|x_j(s)\Big\|_{\mathcal{A}}\\
&\quad+\sum_{j=1}^n b^+L_g\Big\|x_j(s-\gamma_{ij}(s))\Big\|_{\mathcal{A}} + \sum_{j=1}^n d^+L_h\zeta\\
&\quad\times\Big\|x_j(s-\theta)\Big\|_{\mathcal{A}} + \sum_{j=1}^n a^+M_f + \sum_{j=1}^n b^+M_g\\
&\quad+\sum_{j=1}^n d^+M_h\zeta + \|I_i(s)\|_{\mathcal{A}}\Big]ds\Big\}\\
&\le \frac{c^+\omega}{2}\Big\{\Big[c^+r^+ + \sum_{j=1}^n a^+L_f + \sum_{j=1}^n b^+L_g + \sum_{j=1}^n d^+L_h\zeta\Big]\xi\\
&\quad+\Big[\sum_{j=1}^n a^+M_f + \sum_{j=1}^n b^+M_g + \sum_{j=1}^n d^+M_h\zeta + I\Big]\Big\}\cdot |\, t_1 - t_2 \,| .
\end{aligned}$$

Therefore, by **Lemma 2.1**, system (1.1) has at least an $\frac{\omega}{2}$-anti-periodic solution. The proof is completed. □

Remark 3.1. In this paper, we show that system (1.1) has an $\frac{\omega}{2}$-anti-periodic solution by applying the different way as that in Theorem 3.1 of [11, 26], namely, by applying the non-decomposition method and Krasnoselskii's Fixed Point Theorem. Compared with Banach fixed point theorem, our proof process of proving the existence of anti-periodic solutions by applying Krasnoselskii's Fixed Point Theorem is very complicated.

Next, to investigate drive-response synchronization, we will consider the neural network system (1.1) as the drive system, and the response system is given by

$$\begin{aligned}&[y_i(t)-r_i(t)y_i(t-\tau_i(t))]'\\&=-c_i(t)y_i(t)+\sum_{j=1}^{n}a_{ij}(t)f_j\big(y_j(t)\big)+\sum_{j=1}^{n}b_{ij}(t)g_j\big(y_j(t-\gamma_{ij}(t))\big)\\&\quad+\sum_{j=1}^{n}d_{ij}(t)\int_0^{+\infty}k_{ij}(\theta)h_j\big(y_j(t-\theta)\big)d\theta+I_i(t)+\varepsilon_i(t),\end{aligned}\tag{3.5}$$

where $i=1,2,\cdots,n$, $y_i(t):\mathbb{R}\to\mathcal{A}$ denotes the state of the response system, $\varepsilon_i(t)\in\mathcal{A}$ is a state-feedback controller, other notations are the same as those in system (1.1).

System (3.5) is supplemented with initial values given by

$$y_i(s)=\psi_i(s),\ \ s\in[-\eta,0],$$

where $\psi_i\in C\Big([-\eta,0],\mathcal{A}\Big), i=1,2,\cdots,n$.

To realize synchronization between (1.1) and (3.5), the controller ε_i is designed as

$$\varepsilon_i(t)=-\sigma_i(t)z_i(t)+\sum_{j=1}^{n}\nu_{ij}(t)\vartheta_j(z_j(t))+\sum_{j=1}^{n}\mu_{ij}(t)\rho_j(z_j(t-\alpha_{ij}(t))),\tag{3.6}$$

where $i=1,2,\cdots,n$, $\sigma_i,\alpha_{ij}:\mathbb{R}\longrightarrow\mathbb{R}_+$, $\nu_{ij},\mu_{ij},\vartheta_j,\rho_j\in\mathcal{A}$.

We are now in a position to discuss the problem of systems (1.1) and (3.5). Let $z_i=y_i-x_i$, $i=1,2,\ldots,n$, $Z_i(t)=z_i(t)-r_i(t)z_i(t-\tau_i(t))$, then the error system is given by

$$\begin{aligned}Z_i'(t)&=-c_i(t)z_i(t)+\sum_{j=1}^{n}a_{ij}(t)\Big(f_j\big(y_j(t)\big)-f_j\big(x_j(t)\big)\Big)\\&\quad+\sum_{j=1}^{n}b_{ij}(t)\Big(g_j\big(y_j(t-\gamma_{ij}(t))\big)-g_j\big(x_j(t-\gamma_{ij}(t))\big)\Big)\\&\quad+\sum_{j=1}^{n}d_{ij}(t)\int_0^{+\infty}k_{ij}(\theta)\Big(h_j\big(y_j(t-\theta)\big)-h_j\big(x_j(t-\theta)\big)\Big)d\theta\\&\quad-\sigma_i(t)z_i(t)+\sum_{j=1}^{n}\nu_{ij}(t)\vartheta_j(z_j(t))+\sum_{j=1}^{n}\mu_{ij}(t)\rho_j(z_j(t-\alpha_{ij}(t))).\end{aligned}\tag{3.7}$$

System (3.7) is supplemented with initial values given by

$$z_i(s)=\psi_i(s)-\varphi_i(s),\ \ s\in[-\eta,0].\tag{3.8}$$

Definition 3.1. The drive system (1.1) and the response system (3.5) are said to be globally exponentially synchronized, if there exist constants $\lambda > 0$ and $M > 0$ such that

$$\|y - x\|_{\mathbb{X}} \leq M\|\psi - \varphi\|_{\mathbb{X}} e^{-\lambda t}, \quad \forall t > 0,$$

where $x = (x_1, x_2, \cdots, x_n)^T$ is a solution of system (1.1) with the initial value $\varphi = \big(\varphi_1, \varphi_2, \cdots, \varphi_n\big)^T$ and $y = \big(y_1, y_2, \cdots, y_n\big)^T$ is a solution of system (3.5) with the initial value $\psi = \big(\psi_1, \psi_2, \cdots, \psi_n\big)^T$.

Theorem 3.2. *Assume that (H_1)-(H_3) hold. If the following conditions are satisfied:*

(H_4) *For $i, j = 1, 2, \cdots, n$, $\sigma_i(t), \alpha_{ij}(t) \in C(\mathbb{R}, \mathbb{R}_+)$, $\nu_{ij}, \mu_{ij}, \vartheta_j, \rho_j \in \mathcal{A}$, there exists positive constant ω such that $\sigma_i(t + \frac{\omega}{2}) = \sigma_i(t), \alpha_{ij}(t + \frac{\omega}{2}) = \alpha_{ij}(t)$, $\nu_{ij}(t + \frac{\omega}{2})\vartheta_j(u) = -\nu_{ij}(t)\vartheta_j(-u)$, $\mu_{ij}(t + \frac{\omega}{2})\rho_j(u) = -\mu_{ij}(t)\rho_j(-u)$.*

(H_5) *For $j = 1, 2, \cdots, n$, $\vartheta_j(0) = \rho_j(0) = 0$, there exist positive constants L_ϑ and L_ρ such that*

$$\|\vartheta_j(u) - \vartheta_j(v)\|_{\mathcal{A}} \leq L_\vartheta \|u - v\|_{\mathcal{A}},$$
$$\|\rho_j(u) - \rho_j(v)\|_{\mathcal{A}} \leq L_\rho \|u - v\|_{\mathcal{A}}.$$

(H_6) *There exist positive constants λ and δ such that*

$$1 - r^+ e^{\lambda\tau} > 0, \quad \int_0^{+\infty} k_{ij}(\theta) e^{\lambda\theta} d\theta \leq \delta,$$

and

$$\begin{aligned} 0 < \frac{1}{c^- + \sigma^- - \lambda} \bigg[&(c^+ + \sigma^+) r^+ e^{\lambda\tau} + \sum_{j=1}^n a^+ L_f \\ &+ \sum_{j=1}^n b^+ L_g e^{\lambda\gamma} + \sum_{j=1}^n d^+ L_h \delta + \sum_{j=1}^n \nu^+ L_\vartheta \\ &+ \sum_{j=1}^n \mu^+ L_\rho e^{\lambda\alpha} \bigg] \frac{1}{1 - r^+ e^{\lambda\tau}} < 1, \end{aligned}$$

where

$$\begin{aligned} &\nu^+ = \max_{1 \leq i,j \leq n} \|\nu_{ij}(t)\|_{\mathcal{A}}, \\ &\sigma^- = \min_{1 \leq i \leq n} \inf_{[0,\omega]} \sigma_i(t), \sigma^+ = \max_{1 \leq i \leq n} \sup_{[0,\omega]} \sigma_i(t), \\ &\mu^+ = \max_{1 \leq i,j \leq n} \|\mu_{ij}(t)\|_{\mathcal{A}}, \alpha = \max_{1 \leq i,j \leq n} \bigg\{ \sup_{t \in [0,\omega]} \alpha_{ij}(t) \bigg\}. \end{aligned}$$

Then the drive system (1.1) *and the response system* (3.5) *are globally exponentially synchronized.*

Proof. Let $Z_i(t) = z_i(t) - r_i(t) z_i(t - \tau_i(t))$, we have that

$$e^{\lambda t}\|z_i(t)\|_{\mathcal{A}} = e^{\lambda t}\|z_i(t) - r_i(t) z_i(t - \tau_i(t)) + r_i(t) z_i(t - \tau_i(t))\|_{\mathcal{A}}$$

$$\leq e^{\lambda t}\|z_i(t) - r_i(t)z_i(t-\tau_i(t))\|_{\mathcal{A}} + r^+ e^{\lambda\tau} e^{\lambda t}\|z_i(t)\|_{\mathcal{A}}.$$

Hence, we have

$$e^{\lambda t}\|z_i(t)\|_{\mathcal{A}} \leq \frac{e^{\lambda t}\|Z_i(t)\|_{\mathcal{A}}}{1 - r^+ e^{\lambda\tau}}.$$

By (H_6), let

$$\begin{aligned} M := (c^- + \sigma^-)\Bigg[&(c^+ + \sigma^+)r^+ e^{\lambda\tau} + \sum_{j=1}^n a^+ L_f \\ &+ \sum_{j=1}^n b^+ L_g e^{\lambda\gamma} + \sum_{j=1}^n d^+ L_h \delta + \sum_{j=1}^n \nu^+ L_\vartheta \\ &+ \sum_{j=1}^n \mu^+ L_\rho e^{\lambda\alpha}\Bigg]^{-1}(1 - r^+ e^{\lambda\tau}) > 1, \end{aligned}$$

then

$$\begin{aligned} \frac{1}{M} &= \frac{1}{c^- + \sigma^-}\Bigg[(c^+ + \sigma^+)r^+ e^{\lambda\tau} + \sum_{j=1}^n a^+ L_f \\ &\quad + \sum_{j=1}^n b^+ L_g e^{\lambda\gamma} + \sum_{j=1}^n d^+ L_h \delta + \sum_{j=1}^n \nu^+ L_\vartheta \\ &\quad + \sum_{j=1}^n \mu^+ L_\rho e^{\lambda\alpha}\Bigg]\frac{1}{1 - r^+ e^{\lambda\tau}} \\ &\leq \frac{1}{c^- + \sigma^- - \lambda}\Bigg[(c^+ + \sigma^+)r^+ e^{\lambda\tau} + \sum_{j=1}^n a^+ L_f \\ &\quad + \sum_{j=1}^n b^+ L_g e^{\lambda\gamma} + \sum_{j=1}^n d^+ L_h \delta + \sum_{j=1}^n \nu^+ L_\vartheta \\ &\quad + \sum_{j=1}^n \mu^+ L_\rho e^{\lambda\alpha}\Bigg]\frac{1}{1 - r^+ e^{\lambda\tau}}. \end{aligned}$$

From (3.7), For $i = 1, 2\cdots, n$, we can have that

$$\begin{aligned} Z_i(t) = Z_i(0)&e^{-\int_0^t (c_i(\xi)+\sigma_i(\xi))d\xi} + \int_0^t e^{-\int_s^t (c_i(\xi)+\sigma_i(\xi))d\xi}\Bigg[-\big(c_i(s) \\ &+\sigma_i(s)\big)r_i(s)z_i(s-\tau_i(s)) + \sum_{j=1}^n a_{ij}(s)\Big(f_j\big(y_j(s)\big) - f_j\big(x_j(s)\big)\Big) \\ &+ \sum_{j=1}^n b_{ij}(s)\Big(g_j\big(y_j(s-\gamma_{ij}(s))\big) - g_j\big(x_j(s-\gamma_{ij}(s))\big)\Big) \\ &+ \sum_{j=1}^n d_{ij}(t)\int_0^{+\infty} k_{ij}(\theta)\Big(h_j\big(y_j(t-\theta)\big) - h_j\big(x_j(t-\theta)\big)\Big)d\theta \\ &+ \sum_{j=1}^n \nu_{ij}(s)\vartheta_j(z_j(s)) + \sum_{j=1}^n \mu_{ij}(s)\rho_j(z_j(s-\alpha_{ij}(s)))\Bigg]ds. \end{aligned}$$

When $t \in [-\eta, 0]$, it is easy to see that there exist two constants $\epsilon > 0$ and $M > 1$ such

$$\|Z_i(0)\|_{\mathcal{A}} < \|\phi\|_{\mathbb{X}} + \epsilon$$

and

$$\|Z(t)\|_{\mathbb{X}} = \max_{1\le i\le n} \Big\{\|Z_i(t)\|_{\mathcal{A}}\Big\} < M(\|\phi\|_{\mathbb{X}} + \epsilon)e^{-\lambda t},$$

that is,

$$\|z(t)\|_{\mathbb{X}} < \frac{M}{1 - r^+ e^{\lambda\tau}}(\|\phi\|_{\mathbb{X}} + \epsilon)e^{-\lambda t},$$

where $\|\phi\|_{\mathbb{X}} = \|\psi - \varphi\|_{\mathbb{X}}$. We claim that

$$\|Z(t)\|_{\mathbb{X}} < M(\|\phi\|_{\mathbb{X}} + \epsilon)e^{-\lambda t}, \;\; t \in [0, +\infty). \tag{3.9}$$

If it is not true, then there must be some $\hat{t} > 0$ such that

$$\|Z(\hat{t})\|_{\mathbb{X}} = \max_{1\le i\le n} \{\|Z_i(\hat{t})\|_{\mathcal{A}}\} = M(\|\phi\|_{\mathbb{X}} + \epsilon)e^{-\lambda \hat{t}} \tag{3.10}$$

and

$$\|Z(t)\|_{\mathbb{X}} < M(\|\phi\|_{\mathbb{X}} + \epsilon)e^{-\lambda t}, \;\; t \in [-\eta, \hat{t}).$$

Hence, we have

$$\begin{aligned}
\|Z_i(\hat{t})\|_{\mathcal{A}} = &\bigg\| Z_i(0)e^{-\int_0^{\hat{t}}(c_i(\xi)+\sigma_i(\xi))d\xi} + \int_0^{\hat{t}} e^{-\int_s^{\hat{t}}(c_i(\xi)+\sigma_i(\xi))d\xi}\bigg[-\big(c_i(s) \\
&+\sigma_i(s)\big)r_i(s)z_i(s-\tau_i(s)) + \sum_{j=1}^n a_{ij}(s)\Big(f_j\big(y_j(s)\big) - f_j\big(x_j(s)\big)\Big) \\
&+\sum_{j=1}^n b_{ij}(s)\Big(g_j\big(y_j(s-\gamma_{ij}(s))\big) - g_j\big(x_j(s-\gamma_{ij}(s))\big)\Big) \\
&+\sum_{j=1}^n d_{ij}(t)\int_0^{+\infty} k_{ij}(\theta)\Big(h_j\big(y_j(t-\theta)\big) - h_j\big(x_j(t-\theta)\big)\Big)d\theta \\
&+\sum_{j=1}^n \nu_{ij}(s)\vartheta_j(z_j(s)) + \sum_{j=1}^n \mu_{ij}(s)\rho_j(z_j(s-\alpha_{ij}(s)))\bigg]ds\bigg\|_{\mathcal{A}} \\
\le\; &\|Z_i(0)\|_{\mathcal{A}}e^{-\int_0^{\hat{t}}(c_i(\xi)+\sigma_i(\xi))d\xi} + \int_0^{\hat{t}} e^{-\int_s^{\hat{t}}(c_i(\xi)+\sigma_i(\xi))d\xi}\bigg[(c^+ + \sigma^+) \\
&\times r^+\|z_i(s-\tau_i(s))\|_{\mathcal{A}} + \sum_{j=1}^n \|a_{ij}(s)\|_{\mathcal{A}}\Big\|\Big(f_j\big(y_j(s)\big) - f_j\big(x_j(s)\big)\Big)\Big\|_{\mathcal{A}} \\
&+\sum_{j=1}^n \|b_{ij}(s)\|_{\mathcal{A}}\Big\|\Big(g_j\big(y_j(s-\gamma_{ij}(s))\big) - g_j\big(x_j(s-\gamma_{ij}(s))\big)\Big)\Big\|_{\mathcal{A}} \\
&+\sum_{j=1}^n \|d_{ij}(t)\|_{\mathcal{A}}\int_0^{+\infty} k_{ij}(\theta)\Big\|\Big(h_j\big(y_j(t-\theta)\big) - h_j\big(x_j(t-\theta)\big)\Big)\Big\|_{\mathcal{A}}d\theta \\
&+\sum_{j=1}^n \|\nu_{ij}(s)\|_{\mathcal{A}}\|\vartheta_j(z_j(s))\|_{\mathcal{A}} + \sum_{j=1}^n \|\mu_{ij}(s)\|_{\mathcal{A}}\|\rho_j(z_j(s-\alpha_{ij}(s)))\|_{\mathcal{A}}\bigg]ds
\end{aligned}$$

$$
\begin{aligned}
&\le (\|\phi\|_{\mathbb{X}}+\epsilon)e^{-(c^-+\sigma^-)\hat{t}} + M(\|\phi\|_{\mathbb{X}}+\epsilon)\int_0^{\hat{t}} e^{-\int_s^{\hat{t}}(c_i(\xi)+\sigma_i(\xi))d\xi} \\
&\quad\times\Bigg[(c^+ +\sigma^+)r^+e^{\lambda\tau} + \sum_{j=1}^n a^+L_f + \sum_{j=1}^n b^+L_g e^{\lambda\gamma} + \sum_{j=1}^n d^+L_h\delta \\
&\quad + \sum_{j=1}^n \nu^+L_\vartheta + \sum_{j=1}^n \mu^+L_\rho e^{\lambda\alpha}\Bigg]\frac{e^{-\lambda s}}{1-r^+e^{\lambda\tau}}ds \\
&\le M(\|\phi\|_{\mathbb{X}}+\epsilon)e^{-\lambda\hat{t}}\Bigg\{\frac{e^{(\lambda-c^--\sigma^-)\hat{t}}}{M} + \frac{1}{c^-+\sigma^--\lambda}\Bigg[(c^+ +\sigma^+)r^+ \\
&\quad\times e^{\lambda\tau} + \sum_{j=1}^n a^+L_f + \sum_{j=1}^n b^+L_g e^{\lambda\gamma} + \sum_{j=1}^n d^+L_h\delta + \sum_{j=1}^n \nu^+L_\vartheta \\
&\quad + \sum_{j=1}^n \mu^+L_\rho e^{\lambda\alpha}\Bigg]\frac{1-e^{(\lambda-c^--\sigma^-)\hat{t}}}{1-r^+e^{\lambda\tau}}\Bigg\} \\
&\le M(\|\phi\|_{\mathbb{X}}+\epsilon)e^{-\lambda\hat{t}}\Bigg\{e^{(\lambda-c^--\sigma^-)\hat{t}}\Bigg(\frac{1}{M} - \frac{1}{c^-+\sigma^--\lambda}\Bigg[(c^+ +\sigma^+) \\
&\quad\times r^+e^{\lambda\tau} + \sum_{j=1}^n a^+L_f + \sum_{j=1}^n b^+L_g e^{\lambda\gamma} + \sum_{j=1}^n d^+L_h\delta + \sum_{j=1}^n \nu^+L_\vartheta \\
&\quad + \sum_{j=1}^n \mu^+L_\rho e^{\lambda\alpha}\Bigg]\frac{1}{1-r^+e^{\lambda\tau}}\Bigg) + \frac{1}{c^-+\sigma^--\lambda}\Bigg[(c^+ +\sigma^+)r^+ \\
&\quad\times e^{\lambda\tau} + \sum_{j=1}^n a^+L_f + \sum_{j=1}^n b^+L_g e^{\lambda\gamma} + \sum_{j=1}^n d^+L_h\delta + \sum_{j=1}^n \nu^+L_\vartheta \\
&\quad + \sum_{j=1}^n \mu^+L_\rho e^{\lambda\alpha}\Bigg]\frac{1}{1-r^+e^{\lambda\tau}}\Bigg\} \\
&\le M(\|\phi\|_{\mathbb{X}}+\epsilon)e^{-\lambda\hat{t}}\Bigg\{\frac{1}{c^-+\sigma^--\lambda}\Bigg[(c^+ +\sigma^+)r^+e^{\lambda\tau} \\
&\quad + \sum_{j=1}^n a^+L_f + \sum_{j=1}^n b^+L_g e^{\lambda\gamma} + \sum_{j=1}^n d^+L_h\delta + \sum_{j=1}^n \nu^+L_\vartheta \\
&\quad + \sum_{j=1}^n \mu^+L_\rho e^{\lambda\alpha}\Bigg]\frac{1}{1-r^+e^{\lambda\tau}}\Bigg\} \\
&< M(\|\phi\|_{\mathbb{X}}+\epsilon)e^{-\lambda\hat{t}}.
\end{aligned}
$$

Hence,

$$\|Z(\hat{t})\|_{\mathbb{X}} < M(\|\phi\|_{\mathbb{X}}+\epsilon)e^{-\lambda\hat{t}},$$

which contradicts the equality (3.10), and so (3.9) holds. Letting $\epsilon \to 0^+$, then

$$\|Z(t)\|_{\mathbb{X}} \le M\|\phi\|_{\mathbb{X}}e^{-\lambda t},$$

that is,

$$\|z(t)\|_{\mathbb{X}} \le \frac{M}{1-r^+e^{\lambda\tau}}\|\phi\|_{\mathbb{X}}e^{-\lambda t},$$

where

$$\|\phi\|_{\mathbb{X}} = \|\psi - \varphi\|_{\mathbb{X}}.$$

Therefore, the drive system (1.1) and the response system (3.5) are globally exponentially synchronized. The proof is complete. □

Remark 3.2. Compared with Theorem 3.2 of [11], we show the synchronization of anti-periodic solutions for system (1.1) by applying the same way. Unlike the method in literature [12], we gain the synchronization of anti-periodic solutions for system (1.1) by applying the proof by contradiction.

4. Illustrative example

In this section, we give one example to illustrate the feasibility and effectiveness of main results.

Example 4.1. Consider the following delayed Clifford-valued neutral-type cellular neural networks with two neurons as the drive system:

$$\begin{aligned}
&[x_i(t) - r_i(t)x_i(t-\tau_i(t))]' \\
&= -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j\big(x_j(t)\big) + \sum_{j=1}^{n} b_{ij}(t)g_j\big(x_j(t-\gamma_{ij}(t))\big) \\
&\quad + \sum_{j=1}^{n} d_{ij}(t)\int_0^{+\infty} k_{ij}(\theta)h_j\big(x_j(t-\theta)\big)d\theta + I_i(t). \qquad (4.1)
\end{aligned}$$

The corresponding response system is given by

$$\begin{aligned}
&[y_i(t) - r_i(t)y_i(t-\tau_i(t))]' \\
&= -c_i(t)y_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j\big(y_j(t)\big) + \sum_{j=1}^{n} b_{ij}(t)g_j\big(y_j(t-\gamma_{ij}(t))\big) \\
&\quad + \sum_{j=1}^{n} d_{ij}(t)\int_0^{+\infty} k_{ij}(\theta)h_j\big(y_j(t-\theta)\big)d\theta + I_i(t) + \varepsilon_i(t), \qquad (4.2)
\end{aligned}$$

and the controller is as follows:

$$\begin{aligned}
\varepsilon_i(t) = &-\sigma_i(t)z_i(t) + \sum_{j=1}^{n} \nu_{ij}(t)\vartheta_j(z_j(t)) \\
&+ \sum_{j=1}^{n} \mu_{ij}(t)\rho_j(z_j(t-\alpha_{ij}(t))), \qquad (4.3)
\end{aligned}$$

where $n = m = 2$, $i = 1, 2$, $c_1(t) = 3.2 + 0.1\sin 2t$, $c_2(t) = 3.5 + 0.3\sin 2t$, $r_i(t) = \frac{1}{10} + \frac{1}{10}\sin 2t$, $\tau_i(t) = \frac{1}{15} + \frac{1}{15}\sin 2t$, $\gamma_{ij} = \frac{1}{10} + \frac{1}{15}\sin 2t$, $\alpha_{ij} = \frac{1}{16} + \frac{1}{32}\sin 2t$, $\sigma_1(t) = 1.5 + 0.2\sin 2t$, $\sigma_2(t) = 1.3 + 0.1\sin 2t$, $k_{ij}(\theta) = \frac{1}{2}e^{-0.7\theta}$ and

$$\begin{aligned}
a_{11} &= 0.13e_0\sin t + 0.2e_1\sin t + 0.15e_2\sin t + 0.16e_{12}\sin t, \\
a_{12} &= 0.2e_0\sin t + 0.12e_1\sin t + 0.14e_2\sin t + 0.18e_{12}\sin t, \\
a_{21} &= 0.2e_0\sin t + 0.2e_1\sin t + 0.11e_2\sin t + 0.13e_{12}\sin t,
\end{aligned}$$

$$
\begin{aligned}
a_{22} &= 0.14e_0 \sin t + 0.15e_1 \sin t + 0.17e_2 \sin t + 0.3e_{12} \sin t, \\
b_{11} &= 0.11e_0 \sin t + 0.14e_1 \sin t + 0.17e_2 \sin t + 0.2e_{12} \sin t, \\
b_{12} &= 0.22e_0 \sin t + 0.15e_1 \sin t + 0.2e_2 \sin t + 0.1e_{12} \sin t, \\
b_{21} &= 0.21e_0 \sin t + 0.3e_1 \sin t + 0.15e_2 \sin t + 0.17e_{12} \sin t, \\
b_{22} &= 0.2e_0 \sin t + 0.18e_1 \sin t + 0.16e_2 \sin t + 0.14e_{12} \sin t, \\
d_{11} &= 0.12e_0 \sin t + 0.14e_1 \sin t + 0.15e_2 \sin t + 0.2e_{12} \sin t, \\
d_{12} &= 0.2e_0 \sin t + 0.13e_1 \sin t + 0.16e_2 \sin t + 0.18e_{12} \sin t, \\
d_{21} &= 0.21e_0 \sin t + 0.2e_1 \sin t + 0.15e_2 \sin t + 0.14e_{12} \sin t, \\
d_{22} &= 0.2e_0 \sin t + 0.15e_1 \sin t + 0.18e_2 \sin t + 0.13e_{12} \sin t, \\
\nu_{11} &= 0.15e_0 \sin t + 0.16e_1 \sin t + 0.2e_2 \sin t + 0.2e_{12} \sin t, \\
\nu_{12} &= 0.13e_0 \sin t + 0.14e_1 \sin t + 0.18e_2 \sin t + 0.21e_{12} \sin t, \\
\nu_{21} &= 0.21e_0 \sin t + 0.12e_1 \sin t + 0.2e_2 \sin t + 0.11e_{12} \sin t, \\
\nu_{22} &= 0.12e_0 \sin t + 0.2e_1 \sin t + 0.13e_2 \sin t + 0.15e_{12} \sin t, \\
\mu_{11} &= 0.15e_0 \sin t + 0.17e_1 \sin t + 0.18e_2 \sin t + 0.2e_{12} \sin t, \\
\mu_{12} &= 0.22e_0 \sin t + 0.13e_1 \sin t + 0.14e_2 \sin t + 0.17e_{12} \sin t, \\
\mu_{21} &= 0.18e_0 \sin t + 0.19e_1 \sin t + 0.2e_2 \sin t + 0.18e_{12} \sin t, \\
\mu_{22} &= 0.2e_0 \sin t + 0.22e_1 \sin t + 0.15e_2 \sin t + 0.18e_{12} \sin t, \\
I_i &= 0.3e_0 \sin t + 0.2e_1 \sin t + 0.2e_2 \sin t + 0.1e_{12} \sin t, \\
f_j &= \frac{1}{25} \sin x_j^0 e_0 + \frac{1}{25} \sin x_j^1 e_1 + \frac{1}{25} \sin x_j^2 e_2 + \frac{1}{25} \sin x_j^{12} e_{12}, \\
g_j &= \frac{1}{30} \sin x_j^0 e_0 + \frac{1}{30} \sin x_j^1 e_1 + \frac{1}{30} \sin x_j^2 e_2 + \frac{1}{30} \sin x_j^{12} e_{12}, \\
h_j &= \frac{1}{40} \sin x_j^0 e_0 + \frac{1}{40} \sin x_j^1 e_1 + \frac{1}{40} \sin x_j^2 e_2 + \frac{1}{40} \sin x_j^{12} e_{12}, \\
\vartheta_j &= \frac{1}{45} \sin z_j^0 e_0 + \frac{1}{45} i \sin z_j^1 e_1 + \frac{1}{45} \sin z_j^2 e_2 + \frac{1}{45} \sin z_j^{12} e_{12}, \\
\rho_j &= \frac{1}{50} \sin z_j^0 e_0 + \frac{1}{50} i \sin z_j^1 e_1 + \frac{1}{50} \sin z_j^2 e_2 + \frac{1}{50} \sin z_j^{12} e_{12}.
\end{aligned}
$$

Let $\lambda = 0.5$, and by calculating, we have

$$
\begin{aligned}
&c^- = 3.1, \;\; c^+ = 3.8, \;\; r^+ = \frac{1}{5}, \;\; \sigma^- = 1.2, \;\; \sigma^+ = 1.7, \\
&a^+ = 0.3, \;\; b^+ = 0.21, \;\; d^+ = 0.22, \;\; \nu^+ = 0.21, \;\; \mu^+ = 0.22, \\
&L_f = \frac{1}{25}, \;\; L_g = \frac{1}{30}, \;\; L_h = \frac{1}{40}, \;\; L_\vartheta = \frac{1}{45}, \;\; L_\rho = \frac{1}{50}, \;\; \omega = 2\pi, \\
&1 - r^+ e^{\lambda\tau} \approx 0.7862 > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
0 &< \frac{1}{c^- + \sigma^- - \lambda} \Bigg[(c^+ + \sigma^+) r^+ e^{\lambda\tau} + \sum_{j=1}^n a^+ L_f + \sum_{j=1}^n b^+ L_g e^{\lambda\gamma} + \sum_{j=1}^n d^+ L_h \delta \\
&\quad + \sum_{j=1}^n \nu^+ L_\vartheta + \sum_{j=1}^n \mu^+ L_\rho e^{\lambda\alpha} \Bigg] \frac{1}{1 - r^+ e^{\lambda\tau}} \\
&\approx 0.4221 < 1,
\end{aligned}
$$

It is not difficult to verify that all conditions (H_1)-(H_6) are satisfied. Therefore, by Theorem 3.1, from Figures 1-4, we have that system (4.1) has a unique π-anti-periodic solution, which is globally exponentially stable. From Figures 5-8, we have that system (4.2) has a unique π-anti-periodic solution, which is globally exponentially stable.

When applying a nonlinear controller (4.3), Theorem 3.2 implies that system (4.1) and system (4.2) are globally exponentially synchronized, namely, from Figures 9, we can see the drive and response system can reach globally exponentially synchronized.

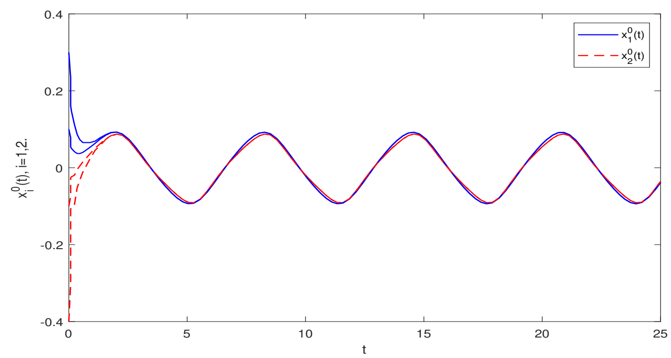

Figure 1. Transient states of the solutions x_i^0, $i = 1, 2$.

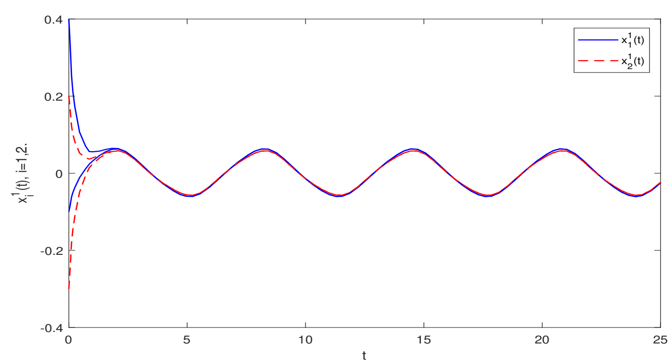

Figure 2. Transient states of the solutions x_i^1, $i = 1, 2$.

Remark 4.1. When $m = 2$, the considered Clifford-valued neural network is a quaternion-valued neural network. Compared with [11,12,26], by applying the non-decomposition method, Krasnoselskii¡¯s Fixed Point Theorem and the proof by contradiction, we get the main result. By using the Simulink toolbox in MATLAB, from Figures 1-9, we get the effectiveness and efficiency of the proposed method.

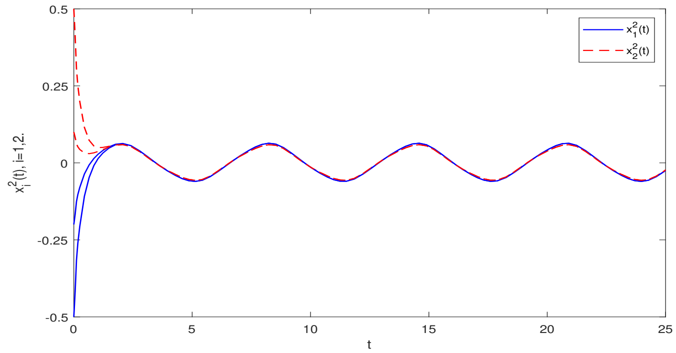

Figure 3. Transient states of the solutions x_i^2, $i = 1, 2$.

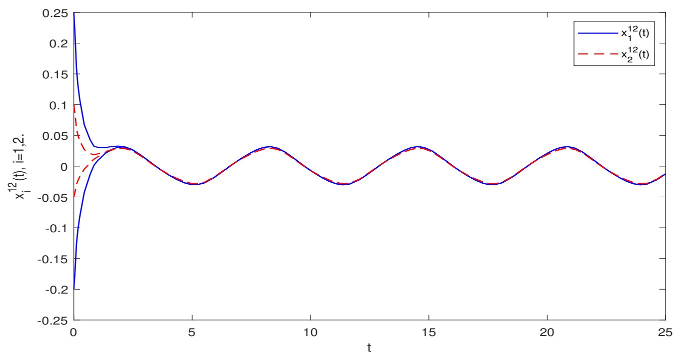

Figure 4. Transient states of the solutions x_i^{12}, $i = 1, 2$.

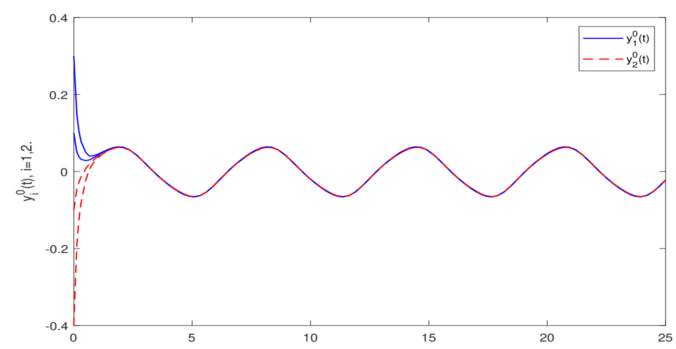

Figure 5. Transient states of the solutions y_i^0, $i = 1, 2$.

5. Conclusion

This paper discusses a class of Clifford-valued neutral-type cellular neural networks with D operator and delays. To overcome the complexity of the calculation, we obtain several sufficient conditions for the existence of anti-periodic solutions for

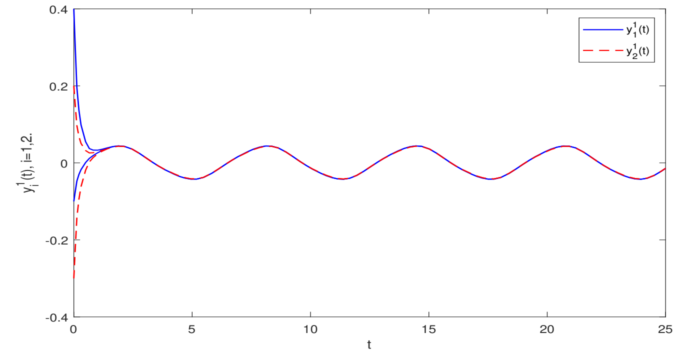

Figure 6. Transient states of the solutions y_i^1, $i = 1, 2$.

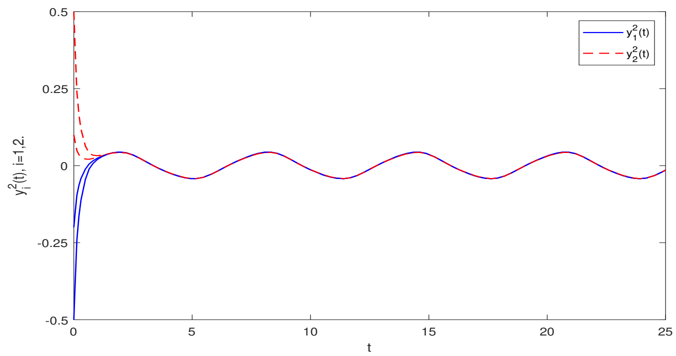

Figure 7. Transient states of the solutions y_i^2, $i = 1, 2$.

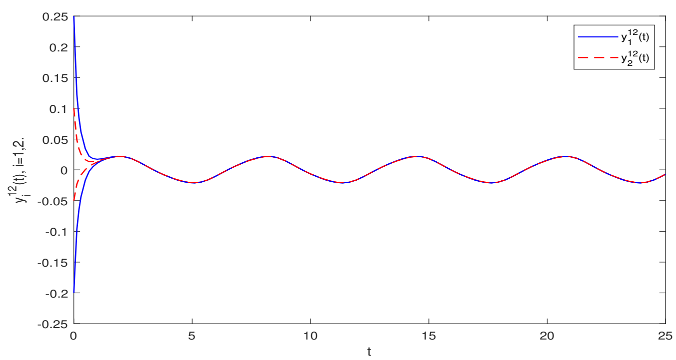

Figure 8. Transient states of the solutions y_i^{12}, $i = 1, 2$.

Clifford-valued neutral-type cellular neural networks with D operator and delays by using the non-decomposition method and the Krasnoselskii's Fixed Point Theorem. We obtain the anti-periodic synchronization for Clifford-valued neutral-type cellular neural networks with D operator by using the proof by contradiction, one example is given. Our method can be extended to discuss the existence and synchronization (or

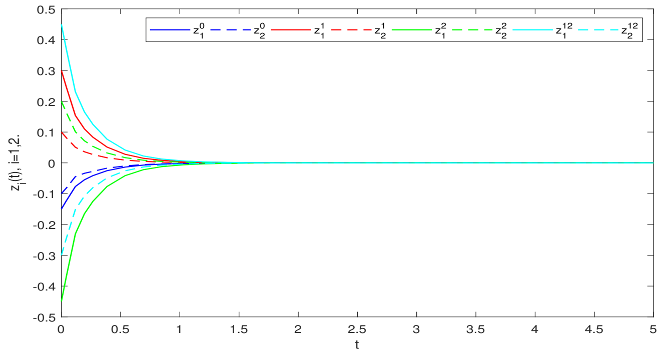

Figure 9. State response curve of four parts of synchronization error.

stability) of periodic (or anti-periodic) solutions for other types of Clifford-valued neural networks.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] C. Aouiti, E. Assali and I. Gharbia, *Pseudo Almost Periodic Solution of Recurrent Neural Networks with D Operator on Time Scales*, Neural Process. Lett., 2019. DOI: 10.1007/s11063-019-10048-2.

[2] C. Aouiti and F. Dridi, *Weighted pseudo almost automorphic solutions for neutral type fuzzy cellular neural networks with mixed delays and D operator in Clifford algebra*, Int. J. Syst. Sci., 2020, 1–23.

[3] C. Aouiti, F. Dridi, Q. Hui and E. Moulay, *(μ,ν)-Pseudo Almost Automorphic Solutions of Neutral Type Clifford-Valued High-Order Hopfield Neural Networks with D Operator*, Neural Process. Lett., 2021, 53, 799–828.

[4] E. Assali, *A spectral radius-based global exponential stability for Clifford-valued recurrent neural networks involving time-varying delays and distributed delays*, Comput. Appl. Math., 2023, 42, 48.

[5] S. Ali, M. Usha, Q. Zhu and S. Shanmugam, *Synchronization Analysis for Stochastic T-S Fuzzy Complex Networks with Markovian Jumping Parameters and Mixed Time-Varying Delays via Impulsive Control*, Math. Probl. Eng., 2020, 1–27.

[6] R. Agarwal, S. Grace and D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic, 2020.

[7] J. Cao, *New results concerning exponential stability and periodic solutions of delayed cellular neural networks*, Phys. Lett. A, 2003, 307, 136–147.

[8] Z. Chen, *Global exponential stability of anti-periodic solutions for neutral type CNNs with D operator*, Int. J. Mach. Learn. Cyb., 2017. DOI: 10.1007/s13042-016-0633-9.

[9] A. Chaouki and F. Touati, *Global dissipativity of Clifford-valued multidirectional associative memory neural networks with mixed delays*, Comput. Appl. Math., 2020, 39(4), 310–330.

[10] B. Du, *New results on stability of periodic solution for CNNs with proportional delays and D operator*, Kybernetika, 2019, 55, 852–869.

[11] J. Gao and L. Dai, *Anti-Periodic Synchronization of Clifford-Valued Neutral-Type Recurrent Neural Networks With D Operator*, IEEE Access, 2022, 10, 9519–9528.

[12] J. Gao and L. Dai, *Anti-periodic synchronization of quaternion-valued high-order Hopfield neural networks with delays*, AIMS Math., 2022, 7(8), 14051–14075.

[13] C. Huang, S. Wen and L. Huang, *Dynamics of anti-periodic solutions on shunting inhibitory cellular neural networks with multi-proportional delays*, Neurocomputing, 2019, 357, 47–52.

[14] C. Huang, R. Su, J. Cao and S. Xiao, *Asymptotically stable of high-order neutral cellular neural networks with proportional delays and D operators*, Math. Comput. Simulat., 2019. DOI: 10.1016/j.matcom.2019.06.001.

[15] C. Huang, H. Yang and J. Cao, *Weighted pseudo almost periodicity of multi-proportional delayed shunting inhibitory cellular neural networks with D operator*, Discrete Cont. Dyn. S, 2020, 1259–1272.

[16] R. Jia and S. Gong, *Convergence of neutral type SICNNs involving proportional delays and D operators*, Adv. Differ. Equations, 2018. DOI: 10.1186/s13662-018-1830-5.

[17] F. Kong, Q. Zhu, K. Wang and J. Nieto, *Stability analysis of almost periodic solutions of discontinuous BAM neural networks with hybrid time-varying delays and D operator*, J. Franklin I., 2019. DOI: 10.1016/j.jfranklin.2019.09.030.

[18] Y. Li, L. Zhao and X. Chen, *Existence of periodic solutions for neutral type cellular neural networks with delays*, Appl. Math. Model., 2012, 36, 1173–1183.

[19] Y. Li and J. Qin, *Existence and global exponential stability of periodic solutions for quaternion-valued cellular neural networks with time-varying delays*, Neurocomputing, 2018, 292, 91–103.

[20] Y. Li and J. Xiang, *Existence and global exponential stability of anti-periodic solutions for quaternion-valued cellular neural networks with time-varying delays*, Adv. Differ. Equations, 2020, 47.

[21] Y. Liu, P. Xu, J. Lu and J. Liang, *Global stability of Clifford-valued recurrent neural networks with time delays*, Nonlinear Dynam., 2016, 84(2), 767–777.

[22] Y. Li and J. Xiang, *Existence and global exponential stability of anti-periodic solution for Clifford-valued inertial Cohen-Grossberg neural networks with delays*, Neurocomputing, 2018. DOI: 10.1016/j.neucom.2018.12.064.

[23] Y. Li and S. Shen, *Almost automorphic solutions for Clifford-valued neutral-type fuzzy cellular neural networks with leakage delays on time scales*, Neurocomputing, 2020, 417, 23–35.

[24] X. Li, J. Fang and H. Li, *Master-slave exponential synchronization of delayed complex-valued memristor-based neural networks via impulsive control*, Neural Networks, 2017. DOI: 10.1016/j.neunet.2017.05.008.

[25] Y. Li, X. Meng and Y. Ye, *Almost Periodic Synchronization for Quaternion-Valued Neural Networks with Time-Varying Delays*, Complexity, 2018, 1–13.

[26] Y. Li, Y. Fang and J. Qin, *Anti-periodic Synchronization of Quaternion-valued Generalized Cellular Neural Networks with Time-varying Delays and Impulsive Effects*, Int. J. Control Autom., 2019. DOI: 10.1007/s12555-018-0385-2.

[27] G. Peng and L. Huang, *Anti-periodic solutions for shunting inhibitory cellular neural networks with continuously distributed delays*, Nonlinear Anal. Real., 2009, 10, 2434–2440.

[28] L. Peng and W. Wang, *Anti-periodic solutions for shunting inhibitory cellular neural networks with time-varying delays in leakage terms*, Neurocomputing, 2013, 111, 27–33.

[29] G. Rajchakit, R. Sriraman, N. Boonsatit, P. Hammachukiattikul, C. Lim and P. Agarwal, *Global exponential stability of Clifford-valued neural networks with time-varying delays and impulsive effects*, Adv. Differ. Equations, 2021. DOI: 10.1186/s13662-021-03367-z.

[30] G. Rajchakit, R. Sriramanb, P. Vigneshc and C. Lim, *Impulsive effects on Clifford-valued neural networks with time-varying delays: An asymptotic stability analysis*, Appl. Math. Comput., 2021, 407, 126309.

[31] J. Shao, *Anti-periodic solutions for shunting inhibitory cellular neural networks with time-varying delays*, Phys. Lett. A, 2008, 372, 5011–5016.

[32] Z. Wang, J. Cao, Z. Cai and L. Rutkowski, *Anti-synchronization in fixed time for discontinuous reaction-diffusion neural networks with time-varying coefficients and time delay*, IEEE Trans. Cybern, 2019, 50, 2758–276.

[33] C. Xu and Y. Wu, *Anti-periodic solutions for high-order cellular neural networks with mixed delays and impulses*, Adv. Differ. Equations, 2015, 161.

[34] C. Xu, *Anti-periodic oscillations in fuzzy cellular neural networks with time-varying delays*, J. Exp. Theor. Artif. In., 2019, 621–635.

[35] C. Xu and P. Li, *On anti-periodic solutions for neutral shunting inhibitory cellular neural networks with time-varying delays and D operator*, Neurocomputing, 2017. DOI: 10.1016/j.neucom.2017.08.030.

[36] Y. Xu, *Exponential Stability of Pseudo Almost Periodic Solutions for Neutral Type Cellular Neural Networks with D Operator*, Neural Process. Lett., 2017. DOI: 10.1007/s11063-017-9584-8.

[37] Z. Xu, X. Li and P. Duan, *Synchronization of complex networks with time-varying delay of unknown bound via delayed impulsive control*, Neural Networks, 2020. DOI: 10.1016/j.neunet.2020.02.003.

[38] L. Yao, *Global convergence of CNNs with neutral type delays and D operator*, Neural Comput. Appl., 2016. DOI: 10.1007/s00521-016-2403-8.

[39] G. Yang and W. Wang, *New Results on Convergence of CNNs with Neutral Type Proportional Delays and D Operator*, Neural Process. Lett., 2018. DOI: 10.1007/s11063-018-9818-4.

[40] K. Yuan, J. Cao and J. Deng, *Exponential stability and periodic solutions of fuzzy cellular neural networks with time-varying delays*, Neurocomputing, 2006, 69, 1619–1627.

[41] B. Zhang, F. Deng, S. Xie and S. Luo, *Exponential synchronization of stochastic time-delayed memristor-based neural networks via distributed impulsive control*, Neurocomputing, 2018. DOI: 10.1016/j.neucom.2018.01.051.

[42] J. Zhu and J. Sun, *Global exponential stability of clifford-valued recurrent neural networks*, Neurocomputing, 2015. DOI: 10.1016/j.neucom.2015.08.016.

[43] Q. Zhou, *Anti-periodic solutions for cellular neural networks with oscillating coefficients in leakage terms*, Int. J. Mach. Learn. Cyb., 2017, 8, 1607–1613.

[44] A. Zhang, *Pseudo almost periodic solutions for neutral type SICNNs with D operator*, J. Exp. Theor. Artif. In., 2016, 1–13.

[45] Q. Zhang, F. Lin, G. Wang and Z. Long, *Existence and stability of periodic solutions for stochastic fuzzy cellular neural networks with time-varying delay on time scales*, Dynam. Syst. Appl., 2018, 27, 851–871.